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Riccati equation for positive semidefinite matrices

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1 Introduction

For given positive definite matrices A and B, and an arbitrary matrix T, the matrix equation

$$X^*A^{-1}X - T^*X - X^*T = B$$

is said to be an algebraic Riccati equation. In particular, the case $T = 0$ in above, that is,

$$X^*A^{-1}X = B.$$

is called a Riccati equation.

In the preceding paper [3], we discussed them. In this paper, we extend them by the use of the Moore-Penrose generalized inverse. Precisely, we consider the following matrix equation;

$$X^*A^{\dagger}X - T^*X - X^*T = B,$$

where $A^{\dagger}$ is the Moore-Penrose generalized inverse of A. So the Riccati equation is of form

$$X^*A^{\dagger}X = B.$$

We call them *a generalized algebraic Riccati equation* and *a generalized Riccati equation*, respectively.

In this note, we first show that every generalized algebraic Riccati equation is reduced to a generalized Riccati equation, and that solutions of a generalized Riccati equation are analyzed. Next we show that under the kernel inclusion $\ker A \subset \ker B$, the geometric mean $A\#B$ is a solution of a generalized Riccati equation $XA^{\dagger}X = B$. As an application, we give another proof to equality conditions of matrix Cauchy-Schwarz inequality due to J. I. Fujii [2]: Let X and Y be $k \times n$ matrices and

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$Y^*X = U|Y^*X|$ a polar decomposition of an $n \times n$ matrix Y^*X with unitary U. Then

$$|Y^*X| \leq X^*X \# U^*Y^*YU.$$

Finally we discuss an order relation between $A\#B$ and $A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^\dagger)^{1/2}A^{1/2}$ for positive semidefinite matrices A and B.

2 Solutions of generalized algebraic Riccati equation

Following after [3], we discuss solutions of a generalized algebraic Riccati equation. Throughout this note, P_X means the projection onto the range of a matrix X.

Lemma 2.1. *Let A and B be positive semidefinite matrices and T a arbitrary matrix. Then W is a solution of a generalized Riccati equation*

$$W^*A^\dagger W = B + T^*AT$$

if and only if $X = W + AT$ is a solution of a generalized algebraic Riccati equation

$$X^*A^\dagger X - T^*P_AX - X^*P_AT = B.$$

Proof. Put $X = W + AT$. Then it follows that

$$X^*A^\dagger X - T^*P_AX - X^*P_AT = W^*A^\dagger W - T^*AT,$$

so that we have the conclusion.

Theorem 2.2. *Let A and B be positive semidefinite matrices. Then W is a solution of a generalized Riccati equation*

$$W^*A^\dagger W = B \quad \text{with} \quad \text{ran } W \subseteq \text{ ran } A$$

if and only if $W = A^{\frac{1}{2}}UB^{\frac{1}{2}}$ for some partial isometry U such that $U^*U \geq P_B$ and $UU^* \leq P_A$.

Proof. Suppose that $W^*A^\dagger W = B$ and ran $W \subseteq$ ran A. Since $\|(A^{\frac{1}{2}})^\dagger Wx\| = \|B^{\frac{1}{2}}x\|$ for all vectors x, there exists a partial isometry U such that $UB^{\frac{1}{2}} = (A^{\frac{1}{2}})^\dagger W$ with $U^*U = P_B$ and $UU^* \leq P_A$. Hence we have

$$A^{\frac{1}{2}}UB^{\frac{1}{2}} = P_AW = W.$$

The converse is easily checked: If $W = A^{\frac{1}{2}}UB^{\frac{1}{2}}$ for some partial isometry U such that $U^*U \geq P_B$ and $UU^* \leq P_A$, then ran $W \subseteq$ ran A and

$$W^*A^\dagger W = B^{\frac{1}{2}}U^*P_AUB^{\frac{1}{2}} = B^{\frac{1}{2}}U^*UB^{\frac{1}{2}} = B.$$

Corollary 2.3. *Notation as in above. Then X is a solution of a generalized algebraic Riccati equation*

$$X^*A^\dagger X - T^*X - X^*T = B$$

with ran $X \subseteq$ ran A if and only if $X = A^{\frac{1}{2}}U(B + T^*AT)^{\frac{1}{2}} + AT$ for some partial isometry U such that $U^*U \geq P_{B+T^*AT}$ and $UU^* \leq P_A$.

Proof. By Lemma 2.1, X is a solution of a generalized algebraic Riccati equation $X^*A^\dagger X - T^*P_AX - X^*P_AT = B$ if and only if $W = X - AT$ is a solution of $W^*A^\dagger W = B + T^*AT$. Since ran $X \subseteq$ ran A if and only if ran $W \subseteq$ ran A, we have the conclusion by Theorem 2.2.

3 Solutions of a generalized Riccati equation

Since $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ for invertible A, the geometric mean $A\#B$ is the unique solution of the Riccati equation $XA^{-1}X = B$ if $A > 0$, see [5] for an early work. So we consider it for positive semidefinite matrices by the use of the Moore-Penrose generalized inverse, that is,

$$XA^\dagger X = B$$

for positive semidefinite matrices A, B.

Theorem 3.1. *Let A and B be positive semidefinite matrices satisfying the kernel inclusion $\ker A \subset \ker B$. Then $A\#B$ is a solution of a generalized Riccati equation*

$$XA^\dagger X = B.$$

Moreover, the uniqueness of its solution is ensured under the additional assumption $\ker A \subset \ker X$.

proof. We first note that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$ and $P_A = P_{A^\dagger}$. Putting $X_0 = A\#B$, a recent result due to Fujimoto-Seo [4, Lemma 2.2] says that

$$X_0 = A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2}.$$

Therefore we have

$$\begin{aligned} X_0A^\dagger X_0 &= A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}P_A[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2} \\ &= A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger] \\ &= P_ABP_A = B \end{aligned}$$

Since ran $X_0 \subset$ ran $A^{1/2}$, X_0 is a solution of the equation.
The second part is proved as follows: If X is a solution of $XA^\dagger X = B$, then

$$(A^{1/2})^\dagger XA^\dagger X(A^{1/2})^\dagger = (A^{1/2})^\dagger B(A^{1/2})^\dagger,$$

so that

$$(A^{1/2})^\dagger X(A^{1/2})^\dagger = [(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}.$$

Hence we have

$$P_AXP_A = A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2} = X_0.$$

Since $P_AXP_A = X$ by the assumption, $X = X_0$ is obtained.

As an application, we give a simple proof of the case where the equality holds in matrix Cauchy-Schwarz inequality, see [4, Lemma 2.5].

Corollary 3.2. *Let X and Y be $k \times n$ matrices and $Y^*X = U|Y^*X|$ a polar decomposition of an $n \times n$ matrix Y^*X with unitary U. If $\ker X \subset \ker YU$, then*

$$|Y^*X| = X^*X\#U^*Y^*YU$$

if and only if $Y = XW$ for some $n \times n$ matrix W.

proof. Since $\ker X^*X \subset \ker U^*Y^*YU$, the preceding theorem implies that $|Y^*X|$ is a solution of a generalized Riccati equation, i.e.,

$$U^*Y^*YU = |Y^*X|(X^*X)^\dagger|Y^*X| = U^*Y^*X(X^*X)^\dagger X^*YU,$$

or consequently

$$Y^*Y = Y^*X(X^*X)^\dagger X^*Y.$$

Noting that $X(X^*X)^\dagger X^*$ is the projection P_X, we have $Y^*Y = Y^*P_XY$ and hence $Y = P_XY = X(X^*X)^\dagger X^*Y$ by $(Y - P_XY)^*(Y - P_XY) = 0$, so that $Y = XW$ for $W = (X^*X)^\dagger X^*Y$.

4 Geometric mean in operator Cauchy-Schwarz inequality

The origin of Corollary 3.2 is the operator Cauchy-Schwarz inequality due to J.I.Fujii [2], which says as follows:

OCS inequality *If $X, Y \in B(H)$ and $Y^*X = U|Y^*X|$ is a polar decomposition of Y^*X, then*

$$|Y^*X| \leq X^*X \# U^*Y^*YU.$$

In his proof of it, the following well-known fact due to Ando [1] is used: For $A, B \geq 0$, the geometric mean $A\#B$ is given by

$$A\#B = \max\left\{X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}$$

First of all, we discuss the case $Y^*X \geq 0$ in (OCS). That is,

$$Y^*X \leq X^*X\#Y^*Y$$

is shown: Noting that $Y^*X = X^*Y \geq 0$, we have

$$\begin{pmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{pmatrix} = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \geq 0,$$

which means $Y^*X \leq X^*X\#Y^*Y$.

The proof for a general case is presented by applying the above: Noting that $(YU)^*X = |Y^*X| \geq 0$, it follows that

$$|Y^*X| = (YU)^*X \leq X^*X\#(YU)^*YU.$$

Remark 1. We can give a direct proof to the general case:

$$\begin{pmatrix} X^*X & |Y^*X| \\ |Y^*X| & U^*Y^*YU \end{pmatrix} = \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix} \geq 0.$$

Remark 2. An equivalent condition which the equality holds in the matrix C-S inequality is known by Fujimoto-Seo [4]: Under the assumption $\ker X \subset \ker YU$, the equality holds if and only if $YU = XW$ for some W. In the proof, they use

(1) If $\ker A \subset \ker B$, then $A\#BA^{\dagger}B = B$.

(2) If $A\#B = A\#C$ and $\ker A \subset \ker B \cap \ker C$, then $B = C$.

Related to matrix Cauchy-Schwarz inequality, the following result is obtained by Fujimoto-Seo [4]:

Let $\mathbb{A} = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ be positive definite matrix. Then $B \geq C^*A^{-1}C$ holds. Furthermore it is known by them:

Theorem 4.1. *Let $\mathbb{A}$ be as in above and $C = U|C|$ a polar decomposition of C with unitary U. Then*

$$|C| \leq U^*AU \; \# \; C^*A^{-1}C.$$

Proof. It can be also proved as similar as in above : Since $|C| = U^*C = C^*U$, we have

$$\begin{pmatrix} U^*AU & |C| \\ |C| & C^*A^{-1}C \end{pmatrix} = \begin{pmatrix} A^{1/2}U & A^{-1/2}C \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A^{1/2}U & A^{-1/2}C \\ 0 & 0 \end{pmatrix} \geq 0.$$

The preceding result is generalized a bit by the use of the Moore-Penrose generalized inverse, for which we note that $(A^{1/2})^{\dagger} = (A^{\dagger})^{1/2}$ for $A \geq 0$:

Theorem 4.2. *Let $\mathbb{A}$ be of form as in above and positive semidefinite, and $C = U|C|$ a polar decomposition of C with unitary U. If ran $C \subseteq$ ran A, then*

$$|C| \leq U^*AU \; \# \; C^*A^{\dagger}C.$$

Proof. Let P_A be the projection onto the range of A. Since $P_AC = C$ and $C^*P_A = C^*$, we have $|C| = U^*P_AC = C^*P_AU$. Hence it follows that

$$\begin{pmatrix} U^*AU & |C| \\ |C| & C^*A^{\dagger}C \end{pmatrix} = \begin{pmatrix} A^{1/2}U & (A^{\dagger})^{1/2}C \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A^{1/2}U & (A^{\dagger})^{1/2}C \\ 0 & 0 \end{pmatrix} \geq 0.$$

5 A generalization of formulae for geometric mean

Since $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ for invertible A, the geometric mean $A\#B$ for positive semidefinite matrices A and B might be expected the same formulae as for positive definite matrices, i.e.,

$$A\#B = A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2}.$$

As a matter of fact, the following result is known by Fujimoto and Seo:

Theorem 5.1. *Let A and B be positive semidefinite matrices. Then*

$$A\#B \le A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2},$$

If the kernel inclusion $\ker A \subset \ker B$ *is assumed, then the equality holds in above.*

Proof. For the first half, it suffices to show that if $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \ge 0$, then

$$X \le A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2}$$

because of Ando's definition of the geometric mean. We here use the facts that $(A^{1/2})^{\dagger} = (A^{\dagger})^{1/2}$, and that if $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \ge 0$ for positive semdefinite X, then $X = AA^{\dagger}X = P_AX$ and $B \ge XA^{\dagger}X$.
Now, since $B \ge XA^{\dagger}X$, we have

$$(A^{1/2})^{\dagger}B(A^{1/2})^{\dagger} \ge [(A^{1/2})^{\dagger}X(A^{1/2})^{\dagger}]^2,$$

so that Löwner-Heinz inequality implies

$$[(A^{1/2})^{\dagger}B(A^{1/2})^{\dagger}]^{1/2} \ge (A^{1/2})^{\dagger}X(A^{1/2})^{\dagger}.$$

Hence it follows from $X = P_AX$ that

$$A^{1/2}[(A^{1/2})^{\dagger}B(A^{1/2})^{\dagger}]^{1/2}A^{1/2} \ge X.$$

Next suppose that $\ker A \subset \ker B$. Then we have $\operatorname{ran} B \subset \operatorname{ran} A$ and so

$$A^{1/2}(A^{1/2})^{\dagger}B(A^{1/2})^{\dagger}A^{1/2} = B.$$

Therefore, putting $C = (A^{1/2})^{\dagger}B(A^{1/2})^{\dagger}$ and

$$Y = A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2} = A^{1/2}C^{1/2}A^{1/2},$$

we have

$$\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} I & C^{1/2} \\ C^{1/2} & C \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \ge 0,$$

which implies that $Y \le A\#B$ and thus $Y = A\#B$ by combining the result in the first half.

By checking the proof carefully, we have an improvement:

Theorem 5.2. *Let A and B be positive semidefinite matrices. Then*

$$A\#B \leq A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2},$$

In particular, the equality holds in above if and only if $P_A = AA^{\dagger}$ commutes with B.

Proof. Notation as in above. If $P_A = AA^{\dagger}(= A^{1/2}(A^{1/2})^{\dagger})$ commutes with B, we have $P_ABP_A \leq B$. Therefore we have

$$\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} \geq \begin{pmatrix} A & Y \\ Y & P_ABP_A \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} I & C^{1/2} \\ C^{1/2} & C \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \geq 0,$$

Conversely assume that the equality holds. Then $\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} \geq 0$. Hence we have

$$B \geq YA^{\dagger}Y = A^{1/2}CA^{1/2} = P_ABP_A,$$

which means P_A commutes with B.

Finally we cite the following lemma which we used in the proof of Theorem 5.1.

Lemma 5.3. *If $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, then $X = AA^{\dagger}X = P_AX$ and $B \geq XA^{\dagger}X$.*

Proof. The assumption implies that

$$\begin{pmatrix} (A^{1/2})^{\dagger} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} (A^{1/2})^{\dagger} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_A & (A^{1/2})^{\dagger}X \\ X^*(A^{1/2})^{\dagger} & B \end{pmatrix} \geq 0.$$

Moreover, since

$$\begin{aligned} 0 &\leq \begin{pmatrix} 1 & -(A^{1/2})^{\dagger}X \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} P_A & (A^{1/2})^{\dagger}X \\ X^*(A^{1/2})^{\dagger} & B \end{pmatrix} \begin{pmatrix} 1 & -(A^{1/2})^{\dagger}X \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_A & 0 \\ 0 & B - X^*A^{\dagger}X \end{pmatrix}, \end{aligned}$$

we have $B \geq X^*A^{\dagger}X$.

Next we show that $X = P_AX$. It is equivalent to $\ker A \subseteq \ker X^*$. Suppose that $Ax = 0$. Putting $y = -\frac{1}{\|B\|}X^*x$, we have

$$\begin{aligned} 0 &\leq \left(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= (Xy, x) + (X^*x, y) + (By, y) \\ &= -\frac{2}{\|B\|}\|X^*x\|^2 + \frac{1}{\|B\|^2}(BX^*x, X^*x) \\ &\leq -\frac{\|X^*x\|^2}{\|B\|} \leq 0. \end{aligned}$$

Hence we have $X^*x = 0$, that is, $\ker A \subseteq \ker X^*$ is shown.

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