数理解析研究所講究録
第2074巻 2018年 9-10

PROPAGATION OF SINGULARITIES AT A NON ANALYTIC HYPERBOLIC FIXED POINT AND APPLICATIONS TO THE QUANTIZATION OF RESONANCES

SETSURO FUJIIÉ (RITSUMEIKAN UNIVERSITY)

The talk will be a review of the joint works [1, 2, 3] with Jean-François Bony (Bordeaux), Thierry Ramond (Paris XI) and Maher Zerzeri (Paris XIII) about the asymptotic distribution of resonances of the semiclassical Schrödinger operator in $\mathbb{R}^n$,

$$P=-h^2\Delta+V(x),$$

with real-valued smooth compactly supported potential $V(x)$. The cut-off resolvent $\chi(P-E)^{-1}\chi$, defined on the upper half complex plane of the spectral parameter E, is extended meromorphically to the lower one beyond the continuous spectrum on the real positive axis. Its poles are called *resonances*. Resonances near the real axis are the most interesting. In fact, on one hand, the imaginary part of resonances are the reciprocal of the decay rate of the quantum state and, on the other hand, their asymptotic distribution in the semiclassical limit are closely related to the geometry of the underlying classical dynamics described by the Hamiltonian

$$p(x,\xi)=|\xi|^2+V(x),$$

where $\xi=(\xi_1,\dots,\xi_n)$ is the momentum.

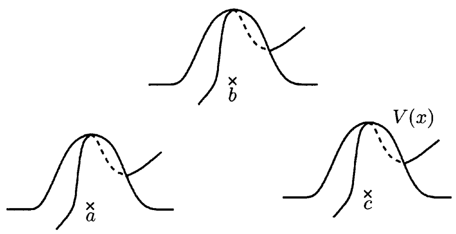
Let E_0 be a positive energy. The *trapped set* $K(E_0)$ is defined as the set of (x,ξ) on the energy surface $p^{-1}(E_0)$ such that the integral curve $\exp tH_p(x,\xi)$ of the Hamiltonian vector field $H_p=\partial_\xi p\cdot\partial_x-\partial_x p\cdot\partial_\xi$ stays bounded.

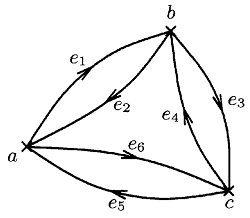
The aim is to give the semiclassical distribution of resonances near an energy E_0 such that the trapped set consists of a finite set of hyperbolic fixed points and associated homoclinic/heteroclinic trajectories. It is typically the case when the potential V has one or many bumps of the same height and the energy E_0 is their (non-degenerate) maximum level. The trapped set is a finite oriented *graph* if the hyperbolic fixed points are regarded as vertices and homo/heteroclinic trajectories as edges.

We give the quantization rule of resonances in the form

$$\det(I-Q(E,h))=1,$$

where $Q(E,h)$, a matrix whose size is the number of edges in the graph, is determined by the geometry of the trapped set. More precisely, the $(e,\tilde e)$ element of this matrix for two edges $e,\tilde e$ is given by $h^{\alpha_v-i\beta_vE_1}e^{iA_e/h}q_{e,\tilde e}$ when the origin of e and the endpoint of $\tilde e$ are the same vertex v and by 0 otherwise. Here $E_1:=(E-E_0)/h$ is supposed to be bounded, $\alpha_v=\sum_{j=2}^n\lambda_j^v/(2\lambda_1^v)$, $\beta_v=1/\lambda_1^v$ where $\lambda_1^v\le\cdots\le\lambda_n^v$ are the positive characteristic exponents at v, A_e is the action integral along e and $q_{e,\tilde e}$ is a function only of $\alpha_v-i\beta_vE_1$ which can be explicitly given in terms of the geometry along e and $\tilde e$.

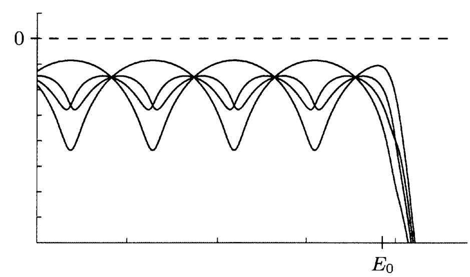




Our main theorem states that the set of resonances is "close" to the set of roots of the quantization rule in a suitably chosen h-dependent complex neighborhood of E_0. From this theorem, we deduce a precise asymptotic distribution of resonances.

The first approximation of the imaginary part of the resonances closest to the real axis is $-D_0 h$. The coefficient D_0, that we call *minimal damping index* is the minimum over all cycles γ in the graph of the quantity $\alpha(\gamma)/\beta(\gamma)$ where $\alpha(\gamma), \beta(\gamma)$ are the sum of α_v, β_v respectively over all vertices in γ.

More precise asymptotics of order $h/|\log h|$ can be deduced from the above quantization rule, and it is at this level that a variety of distributions appear depending on the geometry of the classical dynamics. The resonances are described as roots of an exponential sum and, in general, form a *cloud.* But in many typical cases, they accumulate along several curves. The figure below gives the accumulation curves of resonances near the barrier top of three isotropic bumps located at vertices of an equilateral triangle as the figure above. They can be explicitly computed from our quantization rule.



References

[1] Bony, J.-F., Fujiié, S., Ramond, T. and Zerzeri, M., Microlocal kernel of pseudodifferential operators at a hyperbolic fixed point, *J. Funct. Anal.* **252** (2007), no. 1, 68–125.

[2] Bony, J.-F., Fujiié, S., Ramond, T. and Zerzeri, M., Resonances for homoclinic trapped sets. *arXiv:1603.07517.*

[3] Bony, J.-F., Fujiié, S., Ramond, T. and Zerzeri, M., Barrier-top resonances for non globally analytic potentials, *arXiv:1610.06384*, to appear in J. Spectral Theory.