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Resolvent expansion for the Schrödinger operator on a graph with infinite rays

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In this article we report on the authors' recent work [IJ3] on an expansion of the resolvent for the Schrödinger operator on a graph with rays. We obtain precise expressions for the first few coefficients of the expansion around the threshold 0 in terms only of the generalized eigenfunctions. This in particular justifies the natural definition of threshold resonances for the generalized eigenfunctions solely by the growth rate at infinity.

1 The free operator

In this section we define a *graph with rays*, and fix our free operator H_0 on it. Here we denote the set of vertices by G, and the set of edges by E_G, hence we consider the graph (G, E_G). We sometimes call it simply the graph G. The free operator H_0 is defined as a direct sum of the free Dirichlet Schrödinger operators on a finite part and rays, being different from the graph Laplacian $-\Delta_G$.

Let (K, E_0) be a connected, finite, undirected and simple graph, without loops or multiple edges, and let (L_α, E_α), $\alpha = 1, \ldots, N$, be N copies of the discrete half-line, i.e.

$$L_\alpha = \mathbb{N} = \{1, 2, \ldots\}, \quad E_\alpha = \bigl\{\{n, n+1\};\ n \in L_\alpha\bigr\}.$$

We construct the graph (G, E_G) by jointing (L_α, E_α) to (K, E_0) at a vertex $x_\alpha \in K$ for $\alpha = 1, \ldots, N$:

$$\begin{aligned} G &= K \cup L_1 \cup \cdots \cup L_N, \\ E_G &= E_0 \cup E_1 \cup \ldots E_N \cup \bigl\{\{x_1, 1^{(1)}\}, \ldots, \{x_N, 1^{(N)}\}\bigr\}. \end{aligned}$$

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Here we distinguished 1 of L_α by a superscript: $1^{(\alpha)} \in L_\alpha$. Note that two different half-lines (L_α, E_α) and (L_β, E_β), $\alpha \neq \beta$, could be jointed to the same vertex $x_\alpha = x_\beta \in K$.

Let h_0 be the free *Dirichlet* Schrödinger operators on K: For any function $u\colon K \to \mathbb{C}$ we define

$$(h_0 u)[x] = \sum_{\{x,y\}\in E_0} (u[x] - u[y]) + \sum_{\alpha=1}^{N} s_\alpha[x]u[x] \quad \text{for } x \in K,$$

where $s_\alpha[x] = 1$ if $x = x_\alpha$ and $s_\alpha[x] = 0$ otherwise. Note that the Dirichlet boundary condition is considered being set on the *boundaries* $1^{(\alpha)} \in L_\alpha$ outside K. Similarly, for $\alpha = 1, \ldots, N$ let h_α be the free Dirichlet Schrödinger operators on L_α: For any function $u\colon L_\alpha \to \mathbb{C}$ we define

$$(h_\alpha u)[n] = \begin{cases} 2u[1] - u[2] & \text{for } n = 1, \\ 2u[n] - u[n+1] - u[n-1] & \text{for } n \geq 2. \end{cases}$$

Then we define the free operator H_0 on G as a direct sum

$$H_0 = h_0 \oplus h_1 \oplus \cdots \oplus h_N, \tag{1.1}$$

according to a direct sum decomposition

$$F(G) = F(K) \oplus F(L_1) \oplus \cdots \oplus F(L_N),$$

where $F(X) = \{u\colon X \to \mathbb{C}\}$ denotes the set of all the functions on a space X.

In the definition (1.1) interactions between K and L_α are absent, and the free operator H_0 does not coincide with the graph Laplacian $-\Delta_G$ defined as

$$(-\Delta_G u)[x] = \sum_{\{x,y\}\in E_G} (u[x] - u[y]).$$

In fact, we can write

$$-\Delta_G = H_0 + J, \quad J = -\sum_{\alpha=1}^{N} \Big(|s_\alpha\rangle\langle f_\alpha| + |f_\alpha\rangle\langle s_\alpha| \Big), \tag{1.2}$$

where $f_\alpha[x] = 1$ if $x = 1^{(\alpha)}$ and $f_\alpha[x] = 0$ otherwise. The operator H_0 is actually simpler and more useful than $-\Delta_G$, since it does not have a zero eigenvalue or a zero resonance, and the asymptotic expansion of its resolvent around 0 does not have a singular part. This fact effectively simplifies the expansion procedure for the perturbed resolvent, and enables us to obtain more precise expressions for the coefficients than those in [IJ1]. The interaction J is a special case of general perturbations considered in Assumption 2.1, see Proposition 2.2. Hence the graph Laplacian $-\Delta_G$ can be treated as a perturbation of the free operator H_0.

2 The perturbed operator

In this section we introduce our class of perturbations. We also provide a simple classification of threshold types in terms of the growth rate of the generalized eigenfunctions. This classification will be justified by our main results presented in Section 3.

Set for $s \in \mathbb{R}$

$$\mathcal{L}^s = \ell^{1,s}(G) = \big(\ell^1(K)\big) \oplus \big(\ell^{1,s}(L_1)\big) \oplus \cdots \oplus \big(\ell^{1,s}(L_N)\big),$$
$$(\mathcal{L}^s)^* = \ell^{\infty,-s}(G) = \big(\ell^\infty(K)\big) \oplus \big(\ell^{1,s}(L_1)\big) \oplus \cdots \oplus \big(\ell^{1,s}(L_N)\big),$$

where for $\alpha = 1, \dots, N$

$$\ell^{1,s}(L_\alpha) = \Big\{x\colon L_\alpha \to \mathbb{C};\ \sum_{n\in L_\alpha} (1+n^2)^{s/2}|x[n]| < \infty\Big\},$$
$$\ell^{\infty,-s}(L_\alpha) = \Big\{x\colon L_\alpha \to \mathbb{C};\ \sup_{n\in L_\alpha} (1+n^2)^{-s/2}|x[n]| < \infty\Big\}.$$

We consider the following class of perturbations, cf. [JN1, IJ1, IJ2].

Assumption 2.1. Assume that $V \in \mathcal{B}(\mathcal{H})$ is self-adjoint, and that there exist an injective operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^\beta)$ with $\beta \geq 1$ and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{K})$, both defined on some abstract Hilbert space $\mathcal{K}$, such that

$$V = vUv^* \in \mathcal{B}\big((\mathcal{L}^\beta)^*, \mathcal{L}^\beta\big).$$

We note that V is compact on $\mathcal{H}$ under Assumption 2.1. Let us provide a criterion for Assumption 2.1 in terms of weighted ℓ^2-spaces. We use the standard weighted space notation such as $\ell^{2,s}(G)$, $s \in \mathbb{R}$.

Proposition 2.2. *Assume that $V \in \mathcal{B}(\mathcal{H})$ is self-adjoint, and that it extends to an operator in $\mathcal{B}\big(\ell^{2,-\beta-1/2-\epsilon}(G), \ell^{2,\beta+1/2+\epsilon}(G)\big)$ for some $\beta \geq 1$ and $\epsilon > 0$. Then V satisfies Assumption 2.1 for the same β.*

By this criterion we can see that the interaction J from (1.2) satisfies Assumption 2.1. For another criterion for Assumption 2.1 we refer to [IJ1, Appendix B].

Under Assumption 2.1 we let

$$H = H_0 + V,$$

and consider the solutions to the zero eigen-equation $H\Psi = 0$ in the largest space where it can be defined. Define the *generalized zero eigenspace* $\widetilde{\mathcal{E}}$ as

$$\widetilde{\mathcal{E}} = \big\{\Psi \in (\mathcal{L}^\beta)^*;\ H\Psi = 0\big\}.$$

Let $\mathbf{n}^{(\alpha)} \in (\mathcal{L}^1)^*$, $\mathbf{1}^{(\alpha)} \in (\mathcal{L}^0)^*$ be the functions defined as

$$\mathbf{n}^{(\alpha)}[x] = \begin{cases} m & \text{for } x = m \in L_\alpha, \\ 0 & \text{for } x \in G \setminus L_\alpha, \end{cases} \qquad \mathbf{1}^{(\alpha)}[x] = \begin{cases} 1 & \text{for } x \in L_\alpha, \\ 0 & \text{for } x \in G \setminus L_\alpha, \end{cases}$$

respectively, and abbreviate the spaces spanned by these functions as

$$\mathbb{C}\mathbf{n} = \mathbb{C}\mathbf{n}^{(1)} \oplus \cdots \oplus \mathbb{C}\mathbf{n}^{(N)}, \quad \mathbb{C}\mathbf{1} = \mathbb{C}\mathbf{1}^{(1)} \oplus \cdots \oplus \mathbb{C}\mathbf{1}^{(N)}.$$

We can show that under Assumption 2.1 with $\beta \geq 1$ the generalized eigenfunctions have specific asymptotics:

$$\widetilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}.$$

With this asymptotics we consider the following subspaces:

$$\mathcal{E} = \widetilde{\mathcal{E}} \cap (\mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}), \quad \mathsf{E} = \widetilde{\mathcal{E}} \cap \mathcal{L}^{\beta-2}.$$

A function in $\widetilde{\mathcal{E}} \setminus \mathcal{E}$ should be called a *non-resonance eigenfunction*, one in $\mathcal{E} \setminus \mathsf{E}$ a *resonance eigenfunction*, and one in E a *bound eigenfunction*, but we shall often call them *generalized eigenfunctions* or simply *eigenfunctions.*

Let us introduce the same classification of threshold as in [IJ1, Definition 1.6].

Definition 2.3. The threshold $z = 0$ is said to be

1. a *regular point*, if $\mathcal{E} = \mathsf{E} = \{0\}$;
2. an *exceptional point of the first kind*, if $\mathcal{E} \supsetneq \mathsf{E} = \{0\}$;
3. an *exceptional point of the second kind*, if $\mathcal{E} = \mathsf{E} \supsetneq \{0\}$;
4. an *exceptional point of the third kind*, if $\mathcal{E} \supsetneq \mathsf{E} \supsetneq \{0\}$.

It should be noted here that there is a dimensional relation:

$$\dim(\widetilde{\mathcal{E}}/\mathcal{E}) + \dim(\mathcal{E}/\mathsf{E}) = N, \quad 0 \leq \dim \mathsf{E} < \infty,$$

the former of which reflects a certain topological stability of the non-decaying eigenspace under small perturbations.

We can also show that for any $\Psi_1 \in \widetilde{\mathcal{E}}$ and $\Psi_2 \in \mathcal{E}$, if we let

$$\Psi_1 - \sum_{\alpha=1}^N c_\alpha^{(1)} \mathbf{n}^{(\alpha)} \in \mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}, \quad \Psi_2 - \sum_{\alpha=1}^N c_\alpha^{(2)} \mathbf{1}^{(\alpha)} \in \mathcal{L}^{\beta-2},$$

then these coefficients are orthogonal:

$$\sum_{\alpha=1}^{N} \overline{c}_{\alpha}^{(2)} c_{\alpha}^{(1)} = 0.$$

By this fact it would be natural to introduce *orthogonality* in $\widetilde{\mathcal{E}}$ in terms of the asymptotics, and accordingly define the *generalized orthogonal projections.* We use $\langle \cdot, \cdot \rangle$ to denote the duality between between $\mathcal{L}^s$ and $(\mathcal{L}^s)^*$. If $\beta \geq 2$ then $\langle \Phi, \Psi \rangle$ is defined for $\Phi \in \mathsf{E}$ and $\Psi \in \mathcal{E}$. If we only assume $\beta \geq 1$ then we must assume $\Phi \cdot \Psi \in \mathcal{L}^0$ to justify the notation $\langle \Phi, \Psi \rangle$. Here $(\Phi \cdot \Psi)[n] = \Psi[n]\Phi[n]$, $n \in G$, is the pointwise product.

Definition 2.4. We call a subset $\{\Psi_\gamma\}_\gamma \subset \mathcal{E}$ a *resonance basis*, if the set $\{[\Psi_\gamma]\}_\gamma$ of representatives forms a basis in $\mathcal{E}/\mathsf{E}$. It is said to be *orthonormal*, if

1. for any γ and $\Psi \in \mathsf{E}$ one has $\overline{\Psi} \cdot \Psi_\gamma \in \mathcal{L}^0$ and$\langle \Psi, \Psi_\gamma \rangle = 0$;
2. there exists an orthonormal system $\{c^{(\gamma)}\}_\gamma \subset \mathbb{C}^N$ such that for any γ

$$\Psi_\gamma - \sum_{\alpha=1}^{N} c_{\alpha}^{(\gamma)} \mathbf{1}^{(\alpha)} \in \mathcal{L}^{\beta-2}.$$

The *orthogonal resonance projection* $\mathcal{P}$ is defined as

$$\mathcal{P} = \sum_{\gamma} |\Psi_\gamma\rangle\langle\Psi_\gamma|.$$

Definition 2.5. We call a basis $\{\Psi_\gamma\}_\gamma \subset \mathsf{E}$ a *bound basis* to distinguish it from a resonance basis. It is said to be *orthonormal*, if for any γ and γ' one has $\overline{\Psi}_{\gamma'} \cdot \Psi_\gamma \in \mathcal{L}^0$ and

$$\langle \Psi_{\gamma'}, \Psi_\gamma \rangle = \delta_{\gamma\gamma'}.$$

The *orthogonal bound projection* P is defined as

$$\mathsf{P} = \sum_{\gamma} |\Psi_\gamma\rangle\langle\Psi_\gamma|.$$

We remark that the above orthogonal projections $\mathcal{P}$ and P are independent of choice of orthonormal bases.

3 Main results

In this section we present the main theorems of [IJ3] classifying the resolvent expansions according to threshold types given in Definition 2.3. In the statements below we have to impose different assumptions on the parameter β depending on threshold types. For simplicity we state the results only for integer values of β, but an extension to general β is straightforward.

We set

$$R(\kappa) = (H + \kappa^2)^{-1} \text{ for } -\kappa^2 \notin \sigma(H), \quad \mathcal{B}^s = \mathcal{B}(\mathcal{L}^s, (\mathcal{L}^s)^*).$$

Theorem 3.1. *Assume that the threshold* 0 *is a* regular *point, and that Assumption 2.1 is fulfilled for some integer* $\beta \geq 2$. *Then*

$$R(\kappa) = \sum_{j=0}^{\beta-2} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-1}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+1}$ *for* j *even, and* $G_j \in \mathcal{B}^j$ *for* j *odd. The coefficients* G_j *can be computed explicitly. In particular,*

$$G_{-2} = \mathsf{P} = 0, \quad G_{-1} = \mathcal{P} = 0.$$

Theorem 3.2. *Assume that the threshold* 0 *is an exceptional point of the* first kind, *and that Assumption 2.1 is fulfilled for some integer* $\beta \geq 3$. *Then*

$$R(\kappa) = \sum_{j=-1}^{\beta-4} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-3}) \quad \textit{in } \mathcal{B}^{\beta-1}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. The coefficients* G_j *can be computed explicitly. In particular,*

$$G_{-2} = \mathsf{P} = 0, \quad G_{-1} = \mathcal{P} \neq 0.$$

Theorem 3.3. *Assume that the threshold* 0 *is an exceptional point of the* second kind, *and that Assumption 2.1 is fulfilled for some integer* $\beta \geq 4$. *Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. The coefficients* G_j *can be computed explicitly. In particular,*

$$G_{-2} = \mathsf{P} \neq 0, \quad G_{-1} = \mathcal{P} = 0.$$

Theorem 3.4. *Assume that the threshold* 0 *is an exceptional point of the* third kind, *and that Assumption 2.1 is fulfilled for some integer* $\beta \geq 4$. *Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. The coefficients* G_j *can be computed explicitly. In particular,*

$$G_{-2} = \mathsf{P} \neq 0, \quad G_{-1} = \mathcal{P} \neq 0.$$

Theorems 3.1–3.4 justify the classification of threshold types only by the growth properties of eigenfunctions:

Corollary 3.5. *The threshold type determines and is determined by the coefficients* G_{-2} *and* G_{-1} *from Theorems 3.1–3.4.*

We can also compute the coefficients G_0 and G_1. They can be considered as part of the main results of [IJ3]. However, their expressions are very long, and we omit them in this article, see [IJ3, Appendix B]. These results are generalizations of [IJ1] on the discrete full line $\mathbb{Z}$ and [IJ2] on the discrete half-line $\mathbb{N}$. The strategy for proofs is also similar to [IJ1, IJ2], implementing the expansion scheme of [JN1, JN2] in its full generality. However, due to our choice of the free operator the expansion procedure gets simplified.

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References

[IJ1] K. Ito and A. Jensen, A complete classification of threshold properties for one-dimensional discrete Schrödinger operators. Rev. Math. Phys. **27** (2015), no. 1, 1550002, 45 pp.

[IJ2] K. Ito and A. Jensen, Resolvent expansions for the Schrödinger operator on the discrete half-line. J. Math. Phys. **58** (2017), no. 5, 052101, 24 pp.

[IJ3] K. Ito and A. Jensen, Resolvent expansion for the Schrodinger operator on a graph with infinite rays. Preprint 2017. `https://arxiv.org/abs/1712.01592`

[JN1] A. Jensen and G. Nenciu, A unified approach to resolvent expansions at thresholds. Rev. Math. Phys., **13** (2001), 717–754.

[JN2] A. Jensen and G. Nenciu, Erratum: "A unified approach to resolvent expansions at thresholds" [Rev. Math. Phys. 13(6), 717–754 (2001)]. Rev. Math. Phys. 16(5), 675–677 (2004).