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ON KATO'S PAPER "ON THE CAUCHY PROBLEM FOR THE (GENERALIZED) KORTEWEG-DE VRIES EQUATION"

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ABSTRACT. In this talk, we shall illustrate the decisive influence that the seminal paper by Professor Tosio Kato

"On the Cauchy problem for the (generalized) Korteweg-de Vries equation", Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8** (1983), 93-128

has had in the extraordinary development on the study of nonlinear dispersive equations of the last thirty years.

1. INTRODUCTION

The paper is concerned with the so called generalized Korteweg-de Vries (gKdV) equation.

$$\begin{cases} \partial_t u + \partial_x^3 u + a(u)\partial_x u = 0, & x,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

To simplify, we shall consider $a(u) = u^k$, $k \in \mathbb{Z}^+$.

After Scott Russell (1830's) observation and experiments, and Boussinesq (1860's), it was deduced by Korteweg and de Vries (1895), for the case $k = 1$ (KdV).

In 1967 Gardner-Greene-Kruskal-Miura introduced a method to solve it, the inverse scattering method. This method also applies to the modified KdV $k = 2$ (mKdV).

For $k = 1, 2$ real solutions satisfy infinitely many conservation laws. For general k we have three:

$$I_1(u) = \int u(x,t)dx, \qquad I_2(u) = \int u^2(x,t)dx$$

and

$$I_3(u) = \int ((\partial_x u)^2 - \frac{1}{(k+1)(k+2)} u^{k+2})(x,t)dx$$

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From Kato's paper :

(Section 1 : Introduction)

"....... Our main object is to show that the Cauchy problem for (KdV) and (gKdV) are well posed.

It appears that there is no precise definition of well-posedness,...."

"Consider the abstract Cauchy problem

$$\frac{du}{dt} = f(u), \qquad t > 0, \qquad u(0) = \phi. \tag{1.2}$$

Suppose there are two Banach spaces $Y \subset X$, with the injection continuous, such that f is continuous on Y to X.

Suppose that for each $\phi \in Y$ there is a real number $T > 0$ and a unique function

$$u \in C([0,T] : Y) \tag{1.3}$$

[hence $du/dt \in C([0,T] : X)$] satisfying (1.2) for $t \in (0,T]$.

Suppose, moreover, that the map $\phi \to u$ is continuous from Y $C([0,T] : Y)$. Then we may say that the problem (1.2) is locally well posed in Y: If T can be taken arbitrarily large, then the problem is globally well posed in Y.

This notion of well-posedness is rather strong and is not always realized, or at least not always proved in its full strength in the literature."

In section 3 of the manuscript Kato wrote :

(Section 3 : Review of the H^s Theory)

"Local well-posedness for (gKdV) with $Y = H^s$, $s \geq 3$, $X = H^{s-3}$ was proved in ....

The same proof works for $s \geq 2$. In fact, local well-posedness has almost nothing to do with the special structure of the KdV equation......

The local result for (1.1) has been extended to $s > 3/2$."

The key ingredient for the solution of the optimal regularity for the IVP (1.1) was given in Section 6 of the paper:

(Section 6 : The Smoothing Effects)

Theorem 1. *(Kato (1983))*

Let $s > 3/2$, $0 < T < \infty$. If $u \in C([0,T] : H^s(\mathbb{R}))$ is the solution of the IVP for the gKdV for $u_0 \in H^s(\mathbb{R})$, then

$$u \in L^2([0,T] : H^{s+1}(-R,R)) \qquad \forall\, R > 0.$$

with the associated norm depending only on $\|u_0\|_{s,2}$, R, T.

Proof (linear case): Since

$$\partial_t u + \partial_x^3 u = 0, \qquad u(x,0) = u_0 \in L^2(\mathbb{R}), \tag{1.4}$$

multiplying by $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' \in C_0$, $\varphi'(x) \geq 0$ one gets

$$\frac{d}{dt}\int u^2(x,t)\varphi dx + 3\int (\partial_x u)^2(x,t)\varphi' dx - \int u^2(x,t)\varphi^{(3)} dx = 0.$$

Hence, using the preservation of the L^2-norm of the solution integration in time the last identity yields the result.

Kato smoothing effect (homogeneous version) was generalized and extended by Kruzhkov-Faminskii (1984), Sjölin (1987), Vega (1988), Constantin-Saut (1989).......

Ginibre-Y. Tsutsumi (1989) were the first ones to use Kato smoothing effect to improve the uniqueness results of solution of the KdV and mKdV in weighted spaces.

Consider the linear problem

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x) \end{cases} \tag{1.5}$$

whose solution is given by the group $\{V(t)\}_{-\infty}^{\infty}$

$$V(t)v_0(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} e^{8\pi^3 it\xi^3}\widehat{v_0}(\xi)d\xi.$$

Theorem (Kenig-P.-Vega (1993)) $\exists\, c_0,\, c > 0 \quad \forall x \in \mathbb{R}$

$$\|\, \partial_x V(t)v_0(x)\|_{L_t^2} = c_0\|v_0\|_2,$$

$$\|\, \partial_x^2 \int_0^t V(t-t')f(\cdot,t')dt'\|_{L_x^\infty L_t^2} \leq c\|f\|_{L_x^1 L_t^2}.$$

Proof (homogeneous case): by changing variables $\xi^3 = \eta$ one has

$$\begin{aligned} \partial_x V(t)v_0(x) &= \int_{-\infty}^{\infty} 2\pi\xi i\, e^{2\pi i x\xi} e^{8\pi^3 it\xi^3}\widehat{v_0}(\xi)d\xi \\ &= c\int_{-\infty}^{\infty} e^{2\pi i x\eta^{1/3}} e^{8\pi^3 it\eta}\widehat{v_0}(\eta^{1/3})\,\eta^{-1/3}\,d\eta. \end{aligned}$$

Now using Plancherel's theorem in the t variable one gets the desired result.

Notice that this optimal one-dimensional version of Kato smoothing effect involved a $L_x^\infty L_T^2$-norm, first in time and then in space.

It needs to be combined with estimates in the $L_x^p L_T^\infty$-norm, which correspond to the maximal function associated to the group $\{V(t)\}$.

Roughly, the smoothing effect for a dispersive operator with real symbol A of order m

$$\partial_t u + iA(D)u = 0,$$

provides a gain of $m-2$ derivatives in the homogeneous case and a gain of $m-1$ derivatives in the inhomogeneous case. (It cannot hold in hyperbolic equations.)

Strichartz estimates provide a gain of$(m-2)/4$ derivatives in the homogeneous case and a gain of $(m-2)/2$ derivatives in the inhomogeneous case.

The solution of the IVP

$$\partial_t v + \partial_x^3 v = f(x,t), \qquad v(x,0) = 0,$$

is given by the formula:

$$v(x,t) = c \int\int \frac{e^{it\tau} - e^{it\xi^3}}{\tau - \xi^3}\, e^{ix\xi}\, \widehat{\widehat{f}}(\xi,\tau) d\xi d\tau.$$

This motivates Bourgain (1993) to define the spaces $X_{s,b}$, $s,\, b \in \mathbb{R}$,

$$\|f\|_{X_{s,b}} = (\int\int (1+|\tau-\xi^3|)^{2b}(1+|\xi|^2)^{2s}|\widehat{\widehat{f}}(\xi,\tau)|^2 d\xi d\tau)^{1/2}.$$

In the context of the wave equation they were previously introduced by Rauch-Reed (1982) and M. Beals (1983).

Kato Smoothing Effect allows us to consider the integral equation version of the problem (1.1)

$$u(t) = V(t)u_0 + \int_0^t V(t-t')(u^k\partial_x u)(t')dt',$$

using the contraction principle. As a byproduct one gets that the map data-solution $u_0 \to u(t)$ is smooth.

Notice that if $u(x,t)$ solves $\partial_t u + \partial_x^3 + u^k \partial_x u = 0$, then $u_\lambda(x,t) = \lambda^{2/k}u(\lambda x, \lambda^3 t)$ solves the same equation, with data $u_\lambda(x,0) = \lambda^{2/k}u_0(\lambda x)$. Hence,

$$\|D^s u_\lambda(\cdot,0)\|_2 = \lambda^{2/k+s-1/2}\|D^s u_0\|_2.$$

This suggests that the optimal Sobolev index s should be

$$s_k = 1/2 - 2/k = (k-4)/2k.$$

WELL POSEDNESS (WP) IN $H^s(\mathbb{R})$:

For $k \in \mathbb{Z}^+$, $s > 3/2$ local WP (LWP) Bona-Smith (1976), Kato (1979)

For $k = 1$ (based on contraction principle) (scaling $s_1 = -3/2$) :
$s > 3/4$ LWP Kenig-P.-Vega (1993),
$s \geq 0$ GWP Bourgain (1993),
$s > -3/4$ LWP Kenig-P.-Vega (1996),
$s > -3/4$ GWP Colliander-Keel-Staffilani-Takaoka-Tao (2003),
$s = -3/4$ LWP Christ-Colliander-Tao (2003), GWP Kishimoto (2009) and Guo (2009).

For $k=2$ (scaling $s_2=-1/2$):
$s \geq 1/4$ LWP Kenig-P.-Vega (1993),
$s > 1/4$ GWP Colliander-Keel-Staffilani-Takaoka-Tao (2003),
$s \geq 1/4$ GWP Kishimoto (2009)

For $k=3$ (scaling $s_3=-1/6$):
$s > -1/6$ (scaling) LWP Grunrock (2005),
$s = -1/6$ (critical) WP Tao (2007),
$s > -1/42$ GWP Grunrock-Panthee-Drumond Silva (2007).

For $k \geq 4$:
$s \geq (k-4)/2k$ (scaling/critical) LWP Kenig-P.-Vega (1993).

For $k=4$: there exist H^1-solutions which blow up in finite, Martel-Merle (2002) (Martel-Merle-Raphael (2014)).

A similar result for the powers $k=5,6,....$ remains as an open problem.

These results have shown to be "optimal" : Bourgain (mKdV), Kenig-P.-Vega (2001)-(2003), Nakanishi-Takaoka-Tsutsumi (2001), Christ-Colliander-Tao (2003),........

Next, we continue with Kato's paper in section 8.

(Section 8: The $H^{2r}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ theory).

THEOREM (Kato) : Let $u_0 \in H^{2r}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, $r \in \mathbb{Z}^+$.
There is $T>0$, depending only on the $H^{2r}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ norm of u_0, and a unique solution $u=u(x,t)$ to the IVP for the gKdV such that

$$u \in C([0,T] : H^{2r}(\mathbb{R}) \cap L^2(|x|^{2r}dx)).$$

The map $u_0 \to u$ is continuous.

The main idea is that the operators $\Gamma = x + 3t\partial_x^2$ and $\partial_t + \partial_x^3$ commute.

Corollary (i): The result holds in $H^{2s}(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, $s \geq r$.

Corollary (ii): The result holds in $\mathcal{S}(\mathbb{R})$.

THEOREM (Isaza-Linares-P. (2015))
Let

$$u \in C(\mathbb{R} : L^2(\mathbb{R}))$$

be a solution of the IVP for the KdV. If there exist $\alpha > 0$ and two different times t_0, $t_1 \in \mathbb{R}$ such that

$$|x|^\alpha u(x,t_0), \quad |x|^\alpha u(x,t_1) \in L^2(\mathbb{R}),$$

then

$$u \in C(\mathbb{R} : H^{2\alpha}(\mathbb{R}) \cap L^2(|x|^{2\alpha}\mathbb{R})).$$

Next, we continue with Kato's paper in sections 10-11.

(Section 10-11: The $H^s(\mathbb{R}) \cap L^2(e^{\beta x}dx)$ theory-Regularity.)

THEOREM (Kato) : Let $u \in C([0,\infty) : H^2(\mathbb{R}))$ be a solution of the IVP for the KdV with

$$u_0 \in H^2(\mathbb{R}) \cap L^2(e^{\beta x}dx), \quad \text{for some } \beta > 0,$$

then

$$e^{\beta x}u \in C([0,\infty) : L^2(\mathbb{R})) \cap C((0,\infty) : H^\infty(\mathbb{R})),$$

with

$$\|e^{\beta x}u(t)\|_2 \le e^{Kt}\,\|e^{\beta x}u_0\|_2, \qquad t > 0,$$

$K = K(\beta, \|u_0\|_2)$.

The map data-solution $u_0 \to u(t)$ is continuous

$$L^2(\mathbb{R}) \cap L^2(e^{\beta x}dx) \;\to\; C([0,T] : L^2(e^{\beta x}dx)),$$

for any $T > 0$.

The main idea is that formally in $L^2(e^{\beta x}dx)$ the operator

$$\partial_t + \partial_x^3$$

becomes

$$\partial_t + (\partial_x - \beta)^3 = \partial_t + \partial_x^3 - 3\beta\partial_x^2 + 3\beta^2\partial_x - \beta^3$$

so the equation exhibits a parabolic behavior for $t > 0$.

THEOREM (S. Tarama (2004)) : If

$$u_0 \in L^2(e^{\delta|x|^{1/2}}dx), \;\; \delta > 0,$$

then the solution of the KdV becomes analytic in x for each $t \neq 0$.

His proof based on the Inverse Scattering Method.

All these ideas and techniques (Kato smoothing effects, Strichartz estimates, Bourgain spaces, maximal functions,.....) no only provide sharp well-posedness results.

The inhomogeneous smoothing effect provides the local existence theory for "small" data $u_0 \in H^s(\mathbb{R}^n) \cap L^2(|x|^k)$ for the equation

$$\partial_t u + i\Delta u = P(u, \bar{u}, \nabla_x u, \nabla_x \bar{u}), \tag{1.6}$$

with $P : \mathbb{C}^{2n+2} \to \mathbb{C}$ a polynomial without constant or linear terms (Kenig-P.-Vega (1993)).

N. Hayashi-T. Ozawa (1994) removed the "smallness" assumption on the data in the 1-dimensional case.

H. Chihara (1995) removed the "smallness" assumption on the data in all dimensions.

The weight condition is related with the so-called Mizohata condition: for the linear IVP

$$\begin{cases} \partial_t u = i\Delta u + \vec{b}(x) \cdot \nabla u, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.7}$$

where $\vec{b} = (b_1, \ldots, b_n)$ with $b_j : \mathbb{R} \to \mathbb{C}$, $j = 1, \ldots, n$ smooth functions, the hypothesis

$$\operatorname*{Sup}_{\hat{\xi} \in \mathbb{S}^{n-1}} \ \operatorname*{Sup}_{\substack{x \in \mathbb{R}^n \\ l \in \mathbb{R}}} \left| \int_0^l \operatorname{Im} b_j(x + r\hat{\xi})\, \hat{\xi}_j \, dr \right| < \infty \tag{1.8}$$

is a necessary condition for the L^2-well-posedness of (1.7).

Consider the IVP associated to the general quasi-linear Schrödinger equations:

$$\begin{cases} \partial_t u = i a_{jk}(x,t,u,\bar{u},\nabla_x u, \nabla_x \bar{u}) \partial^2_{jk} u \\ \qquad + b_{jk}(x,t,u,\bar{u},\nabla_x u, \nabla_x \bar{u}) \partial^2_{jk} \bar{u} \\ \qquad + \vec{b}_1(x,t,u,\bar{u},\nabla_x u, \nabla_x \bar{u}) \cdot \nabla_x u \\ \qquad + \vec{b}_2(x,t,u,\bar{u},\nabla_x u, \nabla_x \bar{u}) \cdot \nabla_x \bar{u} \\ \qquad + c_1(x,t,u,\bar{u},\nabla_x u) u + c_2(x,t,u,\bar{u},\nabla_x u) \bar{u} \\ \qquad + f(x,t), \\ \\ u(x,0) = u_0(x). \end{cases} \tag{1.9}$$

Hypotheses: (to simplify consider $b_{jk} = 0$)

(H1) Ellipticity: $\forall M > 0 \ \exists \gamma_M > 0$

$$a_{jk}(x,t,\vec{z})\xi_j \cdot \xi_k \geq \gamma_M |\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall\, \vec{z} \in \mathbb{C}^{2n+2} \quad |\vec{z}| \leq M.$$

(H2) Asymptotic Flatness: $\exists c > 0 \ \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}$

$$|\partial_{x_l} a_{jk}(x,t,\vec{0})| + |\partial^2_{x_l x_m} a_{jk}(x,t,\vec{0})| \leq \frac{c}{1 + |x|^2}.$$

(H3) Nontrapping condition: for the data $u_0 \in H^s(\mathbb{R}^n)$ the Hamiltonian flow (bicharateristics) associated to the symbol

$$h(u_0) = -a_{jk}(x,0,u_0,\bar{u_0},\nabla_x u_0, \nabla_x \bar{u_0})\xi_j \xi_k$$

is non-trapping.

(H4) Growth of the cofficients of the first order coefficients + (H5) Regularity.

Based on the artificial viscosity method, using classical pseudo-differential operators and other techniques including S. Doi's argument of establishing Kato smoothing effect in solutions of Schrödinger equation with variable coefficients:

THEOREM (Kenig-P.-Vega (2004)) :Under the hypotheses (H1)-(H5) there exist

$$s,\ s_1 \in \mathbb{Z}^+,\ s > s_1 + 4$$

such that the IVP for the quasi-linear Schrödinger equation is "locally well-posed" for

$$u_0 \in H^s(\mathbb{R}^n), \quad |x|^2 \partial_x^\alpha u_0 \in L^2(\mathbb{R}^n), \ |\alpha| \leq s_1$$

and

$$f \in L^1(\mathbb{R} : H^s(\mathbb{R}^n)), \quad |x|^2 \partial_x^\alpha f \in L^1(\mathbb{R} : L^2(\mathbb{R}^n)), \ |\alpha| \leq s_1.$$

"locally well-posed" means

$$u \in C([0,T] : H^{s-\epsilon}(\mathbb{R}^n)) \cap L^\infty([0,T] : H^s(\mathbb{R}^n)).......$$

In a latter work, Kenig-Ponce-Rolvung-Vega the ellipticity assumption (H1) was replaced by the more general one $(a_{jk}(\cdot)))$ is a non-degenerated matrix.

Further consequence of Kato Smoothing Effects : Propagation of regularity in solutions of the IVP for the k-gKdV.

Theorem (Isaza-Linares-P. (2015))

If $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $m \in \mathbb{Z}^+$, $m \geq 1$ and $x_0 \in \mathbb{R}$

$$\| \partial_x^m u_0 \|^2_{L^2((x_0,\infty))} = \int_{x_0}^\infty |\partial_x^m u_0(x)|^2 dx < \infty, \tag{1.10}$$

then the solution $u = u(x,t)$ of the IVP for the gKdV satisfies : $\forall \nu > 0$

$$\sup_{0\leq t\leq T} \int_{x_0-\nu t}^\infty (\partial_x^m u)^2(x,t)\, dx < c, \tag{1.11}$$

with $c = c(m; \|u_0\|_{3/4^+,2}; \| \partial_x^m u_0 \|_{L^2((x_0,\infty))}; \nu; T)$.

Moreover, for any $\nu \geq 0$ and $R > 0$

$$\int_0^T \int_{x_0-\nu t}^{x_0+R-\nu t} (\partial_x^{m+1} u)^2(x,t)\, dx\, dt < c, \tag{1.12}$$

with $c = c(m; \|u_0\|_{3/4^+,2}; \| \partial_x^m u_0 \|_{L^2((x_0,\infty))}; \nu; R; T)$.

Thus, this kind of regularity moves with infinite speed to its left as time evolves.

This result has been extended to solutions of quasi-linear KdV equations (Linares-P.-Smith (2016)).

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