数理解析研究所講究録
第2078巻 2018年 154-159

集合に対する劣線形的スカラー化スキームとファジィ集合への応用
(Sublinear-like scalarization scheme for sets and its application to fuzzy sets)

新潟大学大学院自然科学研究科　池 浩一郎 (Koichiro Ike)
田中 環 (Tamaki Tanaka)
Graduate School of Science and Technology, Niigata University

1 Introduction

A linear functional on a real vector space is a bilinear form as a function of two variables of the original space and its dual space; it is an inner product of two vectors in the case of a finite-dimensional space. Also, it is one of the most useful tools for evaluation with respect to some index of the adequacy of efficiency in multiobjective programming or vector optimization.

Generally speaking, a totally ordered space like the real field is very useful for prefenrece, evaluation, computation, or comparison on the values of real-valued functions. On the other hand, multiobjective programming and vector optimization are based on multicriteria like some partial ordering, and minimal and maximal notions like Pareto optimal solution or efficient solution are defined with respect to a certain ordering cone (i.e., a dominance cone); see [7].

From the viewpoint of scalarization, we know several approaches related to order-monotonicity in an ordered vector space. For example, the notion of weighted sum is a good tool for the scalarization of vectors in multicriteria problems, and it is regarded as the projection (i.e., inner product with the weight vector d) in $\mathbb{R}^n$. The average of elements is also a special case of weighted sum with the weight $d=(1/n,\ldots,1/n)^{\mathrm{T}}$. They all are linear scalarization methods, and they can be regarded as a special case of a certain sublinear scalarization (introduced by Tammer [1, 2]):

$$h_C(v;d):=\inf\{t\in\mathbb{R}\mid v\in td-C\}$$

where C is a convex cone in a real topological vector space and $d \in \operatorname{int} C$. If the ordering cone is a half space $C = \{v \mid \langle d, v\rangle \geq 0\}$ where $\langle d, d\rangle = 1$, then $h_C(v; d) = \langle d, v\rangle$. This approach can be applied for the case of set optimization, and we get a basic idea of sublinear-like scalarization and its generalization as unifications of several scalarizations for sets; see [3, 6].

Fuzzy set is a concept initiated by Zadeh [9] to formulate unusual sets containing uncertainty or vagueness. In this paper, we apply the scalarization scheme mentioned above to fuzzy set theory on the basis of [4]. We introduce two important notions, which we call fuzzy-set relations and difference evaluation functions for fuzzy sets, and show some relationship between them. Also, we provide a calculation method for the functions in a certain polyhedral case.

2 Preliminaries

Let V be a real normed vector space and $\mathcal{P}(V)$ denote the set of all subsets of V. For given $A,\ B \in \mathcal{P}(V)$, the algebraic sum $A + B$ is defined by $A + B := \{a + b \mid a \in A,\ b \in B\}$. Let C be a convex cone in V. The relation $\leq_C$ is induced by C as follows: For $v_1,\ v_2 \in V$, $v_1 \leq_C v_2 \ :\Longleftrightarrow\ v_2 - v_1 \in C$.

We give a definition of certain binary relations between sets, called set relations. This is a modified version of the original one proposed in [5].

Definition 1. The eight types of set relations are defined by

$$\begin{aligned}
A \leq_C^{(1)} B \ &:\Longleftrightarrow\ \forall a \in A,\ \forall b \in B,\ a \leq_C b;\\
A \leq_C^{(2L)} B \ &:\Longleftrightarrow\ \exists a \in A,\ \forall b \in B,\ a \leq_C b;\\
A \leq_C^{(2U)} B \ &:\Longleftrightarrow\ \exists b \in B,\ \forall a \in A,\ a \leq_C b;\\
A \leq_C^{(2)} B \ &:\Longleftrightarrow\ A \leq_C^{(2L)} B \text{ and } A \leq_C^{(2U)} B;\\
A \leq_C^{(3L)} B \ &:\Longleftrightarrow\ \forall b \in B,\ \exists a \in A,\ a \leq_C b;\\
A \leq_C^{(3U)} B \ &:\Longleftrightarrow\ \forall a \in A,\ \exists b \in B,\ a \leq_C b;\\
A \leq_C^{(3)} B \ &:\Longleftrightarrow\ A \leq_C^{(3L)} B \text{ and } A \leq_C^{(3U)} B;\\
A \leq_C^{(4)} B \ &:\Longleftrightarrow\ \exists a \in A,\ \exists b \in B,\ a \leq_C b
\end{aligned}$$

for $A,\ B \in \mathcal{P}(V) \setminus \{\emptyset\}$.

A fuzzy set $\tilde{A}$ is a pair $(V, \mu_{\tilde{A}})$ where $\mu_{\tilde{A}}$ is a function from V to $[0,1]$ and called the membership function of $\tilde{A}$. For each $\alpha \in [0,1]$, the α-level set of $\tilde{A}$ is defined by

$[\tilde{A}]_\alpha := \{v \in V \mid \mu_{\tilde{A}}(v) \geq \alpha\}$ ($\alpha \in (0,1]$) and $[\tilde{A}]_0 := \mathrm{cl}\{v \in V \mid \mu_{\tilde{A}}(v) > 0\}$. For $d \in V$, the translation $\tilde{A} + d$ is defined by $\mu_{\tilde{A}+d}(v) := \mu_{\tilde{A}}(v - d)$ for $v \in V$.

Let Ω be a nonempty subset of $[0,1]$. A fuzzy set $\tilde{A}$ is said to be

(i) Ω-normal if $[\tilde{A}]_\alpha$ is nonempty for all $\alpha \in \Omega$;
(ii) Ω-compact if $[\tilde{A}]_\alpha$ is compact for all $\alpha \in \Omega$.

We denote by $\mathcal{F}^{\Omega}_{\mathrm{nor}}(V)$ the set of all Ω-normal fuzzy sets in V.

3 Comparison Criteria and Difference Evaluation

We newly introduce the following two key concepts.

Definition 2. For each $\xi = 1$, $2L$, $2U$, 2, $3L$, $3U$, 3, 4, the fuzzy-set relation $\preceq^{\Omega(\xi)}_C$ is defined by

$$\tilde{A} \preceq^{\Omega(\xi)}_C \tilde{B} \;:\Longleftrightarrow\; \forall \alpha \in \Omega,\; [\tilde{A}]_\alpha \preceq^{(\xi)}_C [\tilde{B}]_\alpha$$

for $\tilde{A}$, $\tilde{B} \in \mathcal{F}^{\Omega}_{\mathrm{nor}}(V)$.

Definition 3. Let $d \in \operatorname{int} C$. For each $\xi = 1$, $2L$, $2U$, 2, $3L$, $3U$, 3, 4, the difference evaluation function $\varphi^{\Omega(\xi)}_{C,d} \colon \mathcal{F}^{\Omega}_{\mathrm{nor}}(V) \times \mathcal{F}^{\Omega}_{\mathrm{nor}}(V) \to \overline{\mathbb{R}}$ is defined by

$$\varphi^{\Omega(\xi)}_{C,d}(\tilde{A}, \tilde{B}) := \sup\left\{t \in \mathbb{R} \;\middle|\; \tilde{A} + td \preceq^{\Omega(\xi)}_C \tilde{B}\right\}$$

for $\tilde{A}$, $\tilde{B} \in \mathcal{F}^{\Omega}_{\mathrm{nor}}(V)$.

Note that the design of the above functions is based on that of the scalarizing functions for sets proposed in [6], which originates from the idea of Gerstewitz's (Tammer's) scalarizing function for vectors (e.g., see [1, 2]).

4 Biconditional Statements

For a, $b \in \mathbb{R}$, it naturally holds $a \leq b \iff b - a \geq 0$ and $a < b \iff b - a > 0$. Here we extend such equivalences to the case of fuzzy sets.

Let X be a topological space and $B_\varepsilon := \{v \in V \mid \|v\| < \varepsilon\}$ for each $\varepsilon > 0$. A set-valued map $F \colon X \to \mathcal{P}(V)$ is said to be

(i) Hausdorff upper continuous (H-u.c.) at $x_0 \in X$ if for any $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $F(x) \subset F(x_0) + B_\varepsilon$ for all $x \in U$;

(ii) Hausdorff lower continuous (H-l.c.) at $x_0 \in X$ if for any $\varepsilon > 0$ there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + B_\varepsilon$ for all $x \in U$;

(iii) Hausdorff upper (or lower) continuous if it is so at every $x \in X$.

Let $\tilde{A}$ be a fuzzy set in V.

Definition 4. The fuzzy set $\tilde{A}$ is said to be

(i) stable to Ω-level decrease if the map $\Omega \ni \alpha \mapsto [\tilde{A}]_\alpha \in \mathcal{P}(V)$ is H-u.c.;

(ii) stable to Ω-level increase if the map $\Omega \ni \alpha \mapsto [\tilde{A}]_\alpha \in \mathcal{P}(V)$ is H-l.c.

Proposition 1. *If $\tilde{A}$ is Ω-compact, then $\tilde{A}$ is stable to Ω-level decrease.*

Theorem 1. *Let $\tilde{A}$, $\tilde{B} \in \mathcal{F}^{\Omega}_{\mathrm{nor}}(V)$. For each $\xi = 1,\ 2L,\ 2U,\ 2,\ 3L,\ 3U,\ 3,\ 4$, the following statements hold:*

(i)
- *Without any additional conditions in the case of $\xi = 1$;*
- *If $\tilde{A}$ is Ω-compact in the case of $\xi = 2L,\ 3L$;*
- *If $\tilde{B}$ is Ω-compact in the case of $\xi = 2U,\ 3U$;*
- *If $\tilde{A}$, $\tilde{B}$ are Ω-compact in the case of $\xi = 2,\ 3$;*
- *If Ω has a maximum and $\tilde{A}$, $\tilde{B}$ are $\{\max \Omega\}$-compact in the case of $\xi = 4$,*

$$\tilde{A} \leq^{\Omega(\xi)}_{\mathrm{cl}\, C} \tilde{B} \iff \varphi^{\Omega(\xi)}_{C,d}(\tilde{A}, \tilde{B}) \geq 0;$$

(ii)
- *If Ω has a minimum and $\tilde{A}$, $\tilde{B}$ are $\{\min \Omega\}$-compact in the case of $\xi = 1$;*
- *If Ω is closed, $\tilde{A}$ is stable to Ω-level increase, and $\tilde{B}$ is Ω-compact in the case of $\xi = 2L,\ 3L$;*
- *If Ω is closed, $\tilde{A}$ is Ω-compact, and $\tilde{B}$ is stable to Ω-level increase in the case of $\xi = 2U,\ 3U$;*
- *If Ω is closed and $\tilde{A}$, $\tilde{B}$ are both Ω-compact and stable to Ω-level increase in the case of $\xi = 2,\ 3$;*
- *If Ω has a maximum in the case of $\xi = 4$,*

$$\tilde{A} \leq^{\Omega(\xi)}_{\mathrm{int}\, C} \tilde{B} \iff \varphi^{\Omega(\xi)}_{C,d}(\tilde{A}, \tilde{B}) > 0.$$

5 Numerical Calculation Method

We deal with a calculation method for the difference evaluation functions for fuzzy sets by generalizing the one discussed in [8]. The following theorem reveals that values of the functions can be calculated simply by solving a finite number of linear programming problems and finding the minimum of a finite number of optimal values.

For $n \in \mathbb{N}$, let $\Delta^n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i = 1,\ \lambda_i \geq 0\ (i = 1, \dots, n)\}$.

Theorem 2. *Let $\tilde{A}$, $\tilde{B} \in \mathcal{F}^{\Omega}_{\mathrm{nor}}(V)$. Suppose the following conditions are satisfied:*

- $C = \bigcap_{l=1,\dots,q}\{v \in V \mid \langle p_l, v\rangle \geq 0\}$ *where $p_1, \dots, p_q$ are nonzero continuous linear functionals on V;*
- $\Omega = \{\alpha_1, \dots, \alpha_\omega\}$ *where $\alpha_1, \dots, \alpha_\omega \in [0,1]$ and $\alpha_1 < \cdots < \alpha_\omega$;*
- *For each $h = 1, \dots, \omega$, $[\tilde{A}]_{\alpha_h} = \mathrm{co}\{a_1^h, \dots, a_{m_h}^h\}$ and $[\tilde{B}]_{\alpha_h} = \mathrm{co}\{b_1^h, \dots, b_{n_h}^h\}$ where $a_1^h, \dots, a_{m_h}^h, b_1^h, \dots, b_{n_h}^h \in V$.*

Then,

(i) $\varphi_{C,d}^{\Omega(1)}(\tilde{A}, \tilde{B}) = \min_{i=1,\dots,m_1} \min_{j=1,\dots,n_1} \min_{l=1,\dots,q} \left\langle \frac{p_l}{\langle p_l, d\rangle}, b_j^1 - a_i^1 \right\rangle$;

(ii) $\varphi_{C,d}^{\Omega(2L)}(\tilde{A}, \tilde{B}) = \min_{h=1,\dots,\omega} \sup\big\{t \in \mathbb{R} \bigm| \langle p_l, d\rangle t + \sum_{i=1}^{m_h} \langle p_l, a_i^h\rangle \lambda_i \leq \min_{j=1,\dots,n_h} \langle p_l, b_j^h\rangle\ (l = 1, \dots, q)$ *for some* $\lambda \in \Delta^{m_h}\big\}$;

(iii) $\varphi_{C,d}^{\Omega(2U)}(\tilde{A}, \tilde{B}) = \min_{h=1,\dots,\omega} \sup\big\{t \in \mathbb{R} \bigm| \langle p_l, d\rangle t + \sum_{j=1}^{n_h} \langle p_l, -b_j^h\rangle \mu_j \leq \min_{i=1,\dots,m_h} \langle p_l, -a_i^h\rangle\ (l = 1, \dots, q)$ *for some* $\mu \in \Delta^{n_h}\big\}$;

(iv) $\varphi_{C,d}^{\Omega(2)}(\tilde{A}, \tilde{B}) = \min\left\{\varphi_{C,d}^{\Omega(2L)}(\tilde{A}, \tilde{B}),\ \varphi_{C,d}^{\Omega(2U)}(\tilde{A}, \tilde{B})\right\}$;

(v) $\varphi_{C,d}^{\Omega(3L)}(\tilde{A}, \tilde{B}) = \min_{h=1,\dots,\omega} \min_{j=1,\dots,n_h} \sup\big\{t \in \mathbb{R} \bigm| \langle p_l, d\rangle t + \sum_{i=1}^{m_h} \langle p_l, a_i^h\rangle \lambda_i \leq \langle p_l, b_j^h\rangle\ (l = 1, \dots, q)$ *for some* $\lambda \in \Delta^{m_h}\big\}$;

(vi) $\varphi_{C,d}^{\Omega(3U)}(\tilde{A}, \tilde{B}) = \min_{h=1,\dots,\omega} \min_{i=1,\dots,m_h} \sup\big\{t \in \mathbb{R} \bigm| \langle p_l, d\rangle t + \sum_{j=1}^{n_h} \langle p_l, -b_j^h\rangle \mu_j \leq \langle p_l, -a_i^h\rangle\ (l = 1, \dots, q)$ *for some* $\mu \in \Delta^{n_h}\big\}$;

(vii) $\varphi_{C,d}^{\Omega(3)}(\tilde{A}, \tilde{B}) = \min\left\{\varphi_{C,d}^{\Omega(3L)}(\tilde{A}, \tilde{B}),\ \varphi_{C,d}^{\Omega(3U)}(\tilde{A}, \tilde{B})\right\}$;

(viii) $\varphi_{C,d}^{\Omega(4)}(\tilde{A}, \tilde{B}) = \sup\big\{t \in \mathbb{R} \bigm| \langle p_l, d\rangle t + \sum_{i=1}^{m_\omega} \langle p_l, a_i^\omega\rangle \lambda_i + \sum_{j=1}^{n_\omega} \langle p_l, -b_j^\omega\rangle \mu_j \leq 0\ (l = 1, \dots, q)$ *for some* $\lambda \in \Delta^{m_\omega}$ *and* $\mu \in \Delta^{n_\omega}\big\}$.

参考文献

[1] Gerstewitz, C. (Tammer, C.): Nichtkonvexe dualität in der vektoroptimierung. Wiss. Z. Tech. Hochsch. Leuna-Merseburg **25**(3), 357–364 (1983)

[2] Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)

[3] Hamel, A.H., Löhne, A.: Minimal element theorems and Ekeland's principle with set relations. J. Nonlinear Convex Anal. **7**, 19–37 (2006)

[4] Ike, K., Tanaka, T.: Convex-cone-based comparisons of and difference evaluations for fuzzy sets. Optimization, doi: 10.1080/02331934.2018.1432608

[5] Kuroiwa, D., Tanaka, T., Ha, T.X.D.: On cone convexity of set-valued maps. Nonlinear Anal. **30**(3), 1487–1496 (1997)

[6] Kuwano, I., Tanaka, T., Yamada, S.: Unified scalarization for sets and set-valued Ky Fan minimax inequality. J. Nonlinear Convex Anal. **11**(3), 513–525 (2010)

[7] Sawaragi, Y., Nakayama, H., Tanino, T.: Theory of Multiobjective Optimization. Academic Press, Orlando, Florida (1985)

[8] Yu, H., Ike, K., Ogata, Y., Saito, Y., Tanaka, T.: Computational methods for set-relation-based scalarizing functions. Nihonkai Math. J. **28**(2), 139–149 (2017)

[9] Zadeh, L.A.: Fuzzy sets. Information and Control **8**, 338–353 (1965)