A relation between definable G vector bundles and definable fiber bundles

Tomohiro Kawakami

Department of Mathematics, Wakayama University
Sakaedani, Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

1 Introduction

Let G be a compact Lie group. It is well-known that the set of isomorphism classes of G vector bundle over a G space space with free action corresponds bijectively to the set of isomorphism classes of vector bundles over the orbit space [1].

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Everything is considered in $\mathcal{N}$ and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous.

General references on o-minimal structures are [2], [3], also see [6].

In this paper we prove that the set of isomorphism classes of definable G vector bundles over a definable G set X is in one–to–one correspondence to that of definable vector bundles over a definable set X/G when the action on X is free.

2 Our result

A field $(R, +, \cdot, <)$ with a dense linear order $<$ without endpoints is an *ordered field* if it satisfies the following two conditions.

(1) For any $x, y, z \in R$, if $x < y$, then $x + z < y + z$.

(2) For any $x, y, z \in R$, if $x < y$ and $z > 0$, then $xz < yz$.

An ordered field $(R, +, \cdot, <)$ is a *real field* if for any $y_1, \ldots, y_m \in R$, $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.

A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.

(1) For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.

(2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal* (*order minimal*) if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.

If $(R, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.

The topology of R is the interval topology and the topology of R^n is the product topology.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f $(\subset X \times Y \subset R^n \times R^m)$ is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ) and gx instead of $\phi(g, x)$.

A definable map $f : X \to Y$ between definable G sets is a *definable G map* if for any $x \in X, g \in G, f(gx) = gf(x)$. A definable G map is a *definable G homeomorphism* if it is a homeomorphism.

A definable set X is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in X. If $R = \mathbb{R}$, then for any definable subset X of $\mathbb{R}^n$,

X is compact if and only if it is definably compact. In general a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \le x \le 1\}$ is definably compact but not compact.

Theorem 2.1 ([5]). *Let X be a definable subset of R^n. Then X is definably compact if and only if X is closed and bounded.*

Theorem 2.2 (Existence of definable quotient ([2])). *Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

Let X be a definable G set. The action on X is *free* if for any x in X, the isotropy subgroup $G_x = \{g \in G | gx = x\}$ of x is the trivial group.

Definition 2.3. *A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a definable fiber bundle over X with fiber F and structure group K if the following two conditions are satisfied:*

(1) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(2) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \ne \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations definable.

Definable fiber bundles with compatible definable local trivializations are identified.

Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(1) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(2) For any i, j such that $U_i \cap f^{-1}(V_j) \ne \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

We say that a bijective definable fiber bundle morphism $\overline{f} : E \to E'$ is a *definable fiber bundle equivalence* if it covers a definable homeomorphism $f : X \to X'$ and $(\overline{f})^{-1} : E' \to E$ is a definable fiber bundle morphism covering $f^{-1} : X' \to X$. A definable fiber bundle equivalence $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$.

A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K. We write (E, p, X, K) for (E, p, X, F, K).

Definition 2.4. *(1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a definable vector bundle if $F = R^n, K = GL(n, R)$.*

(2) Let G be a definable group. A definable vector bundle $\eta = (E, p, X)$ is a definable G vector bundle if E, X are definable G sets, $p : E \to X$ is a definable G map, and G acts on E by definable vector bundle isomorphism.

Our result is the following.

Theorem 2.5 ([4])**.** *Let G be a definably compact definable group and X a definable G set. If G acts on X freely, then the set of isomorphism classes of definable G vector bundles over X corresponds bijectively to the set of isomorphism classes of definable vector bundles over X/G.*

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