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On decomposable generic graphs

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Abstract

There are many results about the existence of universal graphs for classes of countable graphs. In this note, we try to construct a generic graph for some class of finite graphs.

1. Introduction and some definitions

There are many results showing the existence of universal graphs for classes of countable graphs. Most of them treat the classes of countable graphs which forbid some kind of subgraphs. In particular, R.Diestel and et al. proved the existence of decomposable universal graph for the class $\mathbf{\Gamma}$ of countable graphs which forbid having a member of $H(K_4)$ or $T(K_4)$ (see definitions below) as subgraph.

But it is easily checked that their class $\mathbf{\Gamma}$ has no amalgamation property. In this note, we try to alter the class $\mathbf{\Gamma}$ to have amalgamation property and construct some generic structures.

We recall some definitions at first. In this note, we define graph structures as follows.

Definition 1 Let the language $L = \{R(x, y)\}$ and $R(x, y)$ be a binary relation symbol.

An $R-$structure G is said to be a *graph* if
$R(x, y)$ is symmetric, $G \models \forall x \forall y[R(x, y) \longrightarrow R(y, x)]$,
$R(x, y)$ is irreflexive, $G \models \forall x[\neg R(x, x)]$.

Definition 2 Let $\mathcal{G}$ be a class of countable graphs.

A member G of $\mathcal{G}$ is called (*strongly*) *universal in* $\mathcal{G}$ if every $G' \in \mathcal{G}$ is isomorphic to some (induced) subgraph of G.

There are some results on the existence of universal graphs for classes $\mathcal{G}$ characterized by the notions, subdivision and minor of graphs. We recall the definitions of them.

Definition 3 A *subdivision* of a graph X, denoted by TX, is any graph arising from X by replacing its edges with independent paths.

Definition 4 Let G be a graph and $V(G)$ be its vertex set. And let X be another graph and $\{V_x : x \in V(X)\}$ is a partition of $V(G)$ into connected subsets such that ;

for any two vertices $x, y \in V(X)$, there is a $V_x - V_y$ edge in G if and only if x and y are adjacent in X.

In this situation, we say that there exists a *contractive homomorphism from G onto X* and denote $G = HX$.

And we call X is a *minor* of G if G has a subgraph G' such that $G' = HX$.

Theorem 5 *(R.Diestel, R.Halin and W.Vogler* [1] *)*

For Γ a class of countable graphs, we denote $\mathcal{G}(\Gamma)$ the class of all countable graphs that do not contain any subgraph isomorphic to a member of Γ.

Then $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has a strongly universal element, and for any n with $5 \leq n \leq \aleph_0$, $\mathcal{G}(TK_n) = \mathcal{G}(HK_n)$ has no universal element.

It is known that 2−connected graphs are constructed from a cycle by successively adding paths. Some refined argument of it is used to show the existence of universal graph in 2−connected members of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$.

2. 2−connected generic graphs

The strongly universal graph G of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ in the previous theorem has homogeneity to some degree, but G is not a generic graph.

In model theory many important examples of generic structures have been constructed. And most of them are graph structures constructed by amalgamation property. In this section, we try to characterize some generic graphs for variations of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$.

We begin with the definitions of amalgamation property and Fraïssé limit (generic structure).

In the following, for sets $A \subset B$, we denote $B \setminus A = \{b \in B : b \notin A\}$.

Definition 6 Let L be a (finite relational) language and let $\mathbf{K}$ be a class of finite L−structures.

We say that $\mathbf{K}$ has *amalgamation property* if for any $A \subset B_1 \in \mathbf{K}$ and $A \subset B_2 \in \mathbf{K}$, there are $C \in \mathbf{K}$ and ${B_1}' \subset C$, and ${B_2}' \subset C$ such that $A \subset C$ and ${B_1}' \cong_A B_1$, and ${B_2}' \cong_A B_2$.

In particular, we say that $\mathbf{K}$ has *free amalgamation property* if for any $A \subset B_1 \in \mathbf{K}$ and $A \subset B_2 \in \mathbf{K}$, there are $C = B_1 \otimes_A B_2 \in \mathbf{K}$ and ${B_1}' \subset C$, and ${B_2}' \subset C$ such that $A \subset C$ and ${B_1}' \cong_A B_1$, and ${B_2}' \cong_A B_2$ satisfying that there is no relation between ${B_1}' \setminus A$ and ${B_2}' \setminus A$.

Theorem 7 *Let L be a language and $\mathbf{K}$ be a class of (isomorphism types of) finite $L-$structures.*

Suppose that $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ is closed under substructures, and $\mathbf{K}$ has amalgamation property,

then there is a countable $L-$structure M with the following properties ;

1. *Any finite $X \subset M$ is a member of $\mathbf{K}$,*
2. *If $A \subset B \in \mathbf{K}$ and $A \subset M$, then there is a copy $B' \subset M$ such that $B' \cong_A B$.*

A countable $L-$structure having the properties 1 and 2 above is called a *Fraïssé Limit* (*generic structure*) of $\mathbf{K}$.

It is easily checked that $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has no amalgamation property.

Example 8 *Let A be a graph with vertices $\{a_i : i < 9\}$ such that ;*

$\{a_0, a_2, a_3\}$ and $\{a_1, a_3, a_4\}$ are triangles and $\{a_i : 2 \leq i \leq 8\}$ is a cycle, and there is no other edge in A,

and let B and C be extensions of A such that ;

B is the extension of A with an $A-A$ path of length 3 whose endvertices are $\{a_2, a_4\}$, and

C is also the extension of A with an $A-A$ path of length 4 whose endvertices are $\{a_3, a_5\}$.

Then there is no amalgam of B and C over A in $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$.

In this section, we try to construct a 2−connected generic graph for some class $\mathbf{K}$ of finite graphs. We settle notions to fix the class $\mathbf{K}$. They are some modification in [1].

Definition 9 Let G be a graph and $\mathcal{P}$ a set of finite paths in G. Call another set $L = L(\mathcal{P})$ of finite paths in G a *labelling* of $\mathcal{P}$ if each path in L is contained in some path of $\mathcal{P}$.

Let H be a graph and $G \subset H$, and $\mathcal{P}$ a labelled set of finite paths in G. We call H an *extension* of G with respect to $\mathcal{P}$ if there exists a labelled set $\mathcal{P_H}$ of independent $G-G$ paths in H such that

$$H = G \cup \bigcup_{P \in \mathcal{P_H}} P$$

and the endvertices of each $P \in \mathcal{P_H}$ coincide with the endvertices of some $T \in L(\mathcal{P})$.

Definition 10 Let G be a graph and G_0 be an edge of G.

G is *constructible from G_0 with respect to labels* if G can be expressed as $G = \bigcup_{i<n} G_i$ with G_i for $i < n$ in such a way that there exists a set $\mathcal{P}_0$ and $\mathcal{P}_i$ of independent $G_i - G_i$ paths in G for $i < n-1$ such that

1) $G_0 \cong K_2$,
2) $G_{i+1} = G_i \cup \bigcup_{P \in \mathcal{P}_i} P$,
3) G_{i+1} is an extension of G_i with respect to $\mathcal{P}_{i-1}$.

G is *constructible with respect to labels* if for any edge G_0 of G, G is constructible from G_0 with respect to labels.

In the definition above, we take $\mathcal{P}_i$ maximally at each stage.

Here we define a set of labelling as the set of chordless paths satisfying the next conditions.

Definition 11 Let a finite graph G be constructible with respect to labels such that $G = \bigcup_{i<n} G_i$, and $\mathcal{P}_i$ is independent $G_i - G_i$ paths in G for $i < n-1$.

We define a labelling $L(\mathcal{P}_{i-1})$ as the set of all those subpaths T of some $P' \in \mathcal{P}_{i-1}$ that form a cycle together with some $P \in \mathcal{P}_i$. For $P \in \mathcal{P}_i$, we take its labelling T with the minimal length.

We say that a graph G has a labelling with *length* n if every labelling T of G (in all stages of construction) has the length at most n.

In the following, we consider the easiest case that the length of labels is fewer than 3.

Notation 12 *We settle the class of* $2-$*connected finite graphs which are constructible w.r.t. labels and their labels are fewer than* 3*, and we denote the class* $\mathbf{K_2}$.

It is easily checked that there are many $2-$connected graphs which do not belong to $\mathbf{K_2}$.

Example 13 *The next graphs are not in* $\mathbf{K_2}$ *;*
1. G_1 *is a graph whose vertices* $V(G_1) = \{v_0, v_1, \cdots, v_7\}$. *The vertices* $\{v_0, v_1, v_2, v_3, v_4\}$ *and* $\{v_1, v_2, v_3, v_5, v_6, v_7\}$ *are cycles.*

If a finite graph G which is constructed by two cycles and these cycles have a common path with the length more than 2, then $G \notin \mathbf{K_2}$.
2. G_2 *is a graph whose vertices* $V(G_2) = \{v_0, v_1, \cdots, v_5\}$. *The vertices* $\{v_0, v_1, v_2\}$ *and* $\{v_0, v_5, v_4\}$ *are paths, and* (v_0, v_3), (v_2, v_4), (v_3, v_4) *are edges.*
3. G_3 *and* G_4 *are graphs whose vertices are* $V(G_3) = \{v_0, v_1, \cdots, v_6\}$ *and* $V(G_4) = \{v_0, v_1, \cdots, v_7\}$. G_3 *is the graph that the edge* (v_0, v_3) *in* G_2 *is replaced by the path* $\{v_0, v_6, v_3\}$, *and* G_4 *is the graph that the edge* (v_3, v_4) *in* G_3 *is replaced by the path* $\{v_3, v_7, v_4\}$.
4. G_5 *is the graph whose vertices* $V(G_5) = \{v_0, v_1, \cdots, v_5\}$. *The vertices* $\{v_0, v_1, v_2\}$ *is a path and* $\{v_2, v_3, v_4, v_5\}$ *is a cycle, and* (v_0, v_3), (v_0, v_5), (v_1, v_4) *are edges.*
5. G_6 *is the graph whose vertices* $V(G_6) = \{v_0, v_1, \cdots, v_6\}$. *The vertices* $\{v_0, v_1, v_2\}$ *and* $\{v_0, v_6, v_4\}$ *are paths and* $\{v_2, v_3, v_4, v_5\}$ *is a cycle, and* (v_0, v_3), (v_0, v_5) *are edges.*

There are many examples except these above. And variations of graphs above by replacing paths with longer ones are not in $\mathbf{K_2}$.

There is some characterization of these graphs. For a constructible graph G, the length of labels for G is denoted by $ln(G)$ in the following.

Lemma 14 *Let $G' \subset G$ be $2-$connected finite graphs (G' be an induced subgraph of G). And let G_0 be an edge of G'.*

If G' is not constructible from G_0, then G is also not constructible from G_0.

Sketch of proof ;

Assume that G is constructible from G_0, that is, $G = \bigcup_{i \leq k} G_i$ for some $k < \aleph_0$ where G_i is an induced subgraph of G which is constructed until $i-$step. In particular, $G' \subset \bigcup_{i \leq n} G_i$ and n is the minimal such number with $n \leq k$.

Case.1 There is a path P appeared at $n-$step such that some subpath of P in G' has a label in G' with the length more than 3.

Now the length of label $ln(P) \leq 2$. P is partitioned into several subpaths of G' and those of $G \setminus G'$ in general. For the sake of convenience, let $P = P_0 \cup P_1$ where $P_0 \subset G \setminus G'$ and $P_1 \subset G'$. And let the endvertices of P be $\{v_0, v_{m-1}\}$ and that of P_1 be $\{v_l, v_{m-1}\}$. Moreover let the label of P be a path $P_2 = \{v_0, v, v_{m-1}\}$ for some $v \in G \setminus G'$, and the label of P_1 in G' be a path $P_3 = \{v_l, v_0', v_1', \cdots, v_{m-1}\}$. It is easily checked that the endvertices of P and the label P_2 appeared at $(n-1)-$step. By the same reason, the label P_3 appeared until $(n-1)-$step. Then by the way of construction, the path P is not taken as a one path at $n-$step. When $P \subset G'$, we can also deduce a contradiction.

Case.2 Otherwise, in this case, we repeat the same argument as Case.1 at $(n-1)-$step. ∎

We must check that $\mathbf{K_2}$ has free amalgamation property at first.

Fact 15 *The class $\mathbf{K_2}$ has no free amalgamation property.*

Example 16 *Let A be a graph with vertices $\{a_i : i < 6\}$ such that ;*

$\{a_0, a_1, a_2, a_5\}$ is a cycle and $\{a_2, a_3, a_4, a_5\}$ is a path,

and let B and C be extensions of A such that ;

B is the extension of A with an $A-A$ path of length 2 whose endvertices are $\{a_0, a_4\}$, and

C is also the extension of A with an $A-A$ path of length 2 whose endvertices are $\{a_0, a_3\}$.

Then there is no free amalgam of B and C over A in $\mathbf{K_2}$.

Problem 17 *What additional conditions are necessary for $\mathbf{K_2}$ to have free amalgamation property ?*

By the definition of construction (extension), we must take maximal disjoint paths at each stage. I will consider classes without this condition.

3. More restricted classes

In Diestel's paper [1], they construct a universal graph for the class $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$. On the other hand, the complete graph K_n belongs to $\mathbf{K_2}$ for $n < \aleph_0$. We can exclude them from $\mathbf{K_2}$ by some properties of labels.

Definition 18 Let $P \in \mathcal{P}_i$ be a path. We say that the labelling $T(P)$ is *compatible* if there is no single $P' \in \mathcal{P}_j$ such that $T(P) \subset P'$ for some $j < i$ (that is, there are independent paths $P_k \in \mathcal{P}_{j_k}$ for $k < 2$ and $j_k < i$ such that $T(P)$ and P_k are not edge-disjoint for $k < 2$).

Now we determine some restricted class $\mathbf{K_2}'$ of $\mathbf{K_2}$.

Definition 19 Let $\mathbf{K_2}'$ be the class of finite graphs G satisfying that ;
1) G is constructible with respect to labels with $ln(G) = 2$, and
2) G has no edges contained in different compatible labels (at the same stage in the construction).

Remark 20 $\mathbf{K_2}'$ *contains all finite members of* $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ *with length* 2. *And the free amalgam* $B \otimes_A C$ *of Example 8 is in* $\mathbf{K_2}'$.

Problem 21 *Let* $\mathbf{K_2}'$ *be the class of finite graphs satisfying the conditions as above.*
Then does $\mathbf{K_2}'$ *have (free) amalgamation property ?*

4. Further problems

Problem 22 *Are there other classes of finite graphs which have amalgamation property ?*
Can we extend $\mathbf{K_2}$ *to the class of finite graphs which is constructible w.r.t labels with the length n naturally ?*

Problem 23 *Can we characterize decomposable generic graphs by the notion of predimension or dimension ?*
More generally, can we classify decomposable graphs by stability theoretic notions ?

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