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On the infinite Ramsey property for random graph

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Abstract

The aim of this article is to give a brief sketch of an example of an edge coloring on K_n-free random graph (Rado graph) which has no monochromatic K_n-free random subgraph.

1 Introduction

Let L be a finite relational language and let K be a class of (isomorphism types of) finite L-structure. We say K has (structural) Ramsey property if for any $A, B \in K$ there is $C \in K$ such that $C \to (B)^A_k$ for every $k \in \omega$. The Ramsey property of K is just the classical (finite) Ramsey theorem where L is the empty:

Fact 1 (Ramsey theorem)**.**

1. Let $k, n, m \in \omega$. There is $l \in \omega$ such that $l \to (m)^n_k$.
2. Let $k, n \in \omega$. Then $\omega \to (\omega)^n_k$.

A famous nontrivial example of structural Ramsey property is the class of totally ordered (K_n-free) finite graphs, which is proved by Neštřil and Rödl[3]. If we consider about edge-coloring, then the order is not needed. Hence, as a corollary, we have

Fact 2. Let B be any (K_n-free) finite graph. Then there is a (K_n-free) finite graph C such that for every edge-coloring $f : E(C) \to k$ there is an subgraph $B' \subset C$ which is isomorphic to B such that $f|E(B')$ is constant.

(In this article, subgraph always means induced subgraph.) Now we can ask that if there is any infinite Ramsey property with respect to graphs like classical Ramsey theorem. A natural infinite graph containing every K_n-free finite graph is a K_n-free random graph (Rado graph), countable homogeneous graph containing all K_n-free finite graphs. The Ramsey property of Random graph is investigated by, for example, Erdös, Hajnal, Póza, Komjáth, Pouzet and Sauer[1][2][4].

Erdös, Hajnal and Póza[1] realized that the following:

Fact 3. There is an edge-coloring $f : E(G) \to 2$ such that for every random subgraph $G' \subset G$, the number $|f(E(G'))| = 2$.

This seems that we may not expect random graph has Ramsey property. However, in Pouzet and Sauer's paper [4], the following is proved using a dense linear order on random graph:

Theorem 4. Let G be a random graph. Let $f : E(G) \to k$ be an edge-coloring with $k \in \omega$. Then there is a random subgraph $G' \subset G$ such that $|f(E(G'))| \leq 2$.

Therefore, we can say random graph has a kind of infinite Ramsey property.

In this article, we show Fact 3 for K_n-free random graph. The idea of the coloring is essentially same to the one discussed in Pouzet and Sauer's paper. However, we will see the coloring can be applied for K_n-free graphs.

2 A coloring on K_n-free random graph.

Let L be a finite relational language.

Definition 5. A countable L-structure M is said to be ultrahomogeneous if every isomorphism between finite substructures of M can be extended to an automorphism in $Aut(M)$.

Let $H = (V(H), E(H))$ be an infinite graph such that

- $V(H) = \{h_i : i \in \omega\}$,
- $\{h_0, h_i\} \in E(H)$ if and only if i is odd,
- $\{h_i, h_{i+1}\} \in E(H)$ for every $i \in \omega$.

Note that we do not require that $\{h_i, h_j\} \in E$ or not, for $0 < i < i+1 < j$. Let G be any K_n-free random graph. Note that G contains H.

Lemma 6. Let $G = \{g_i : i \in \omega\}$ be any enumerations of G. Then there is an embedding $\sigma : H \to G$ preserving the enumeration, i.e. $i < j$ implies $k < l$ where $\sigma(h_i) = g_k$ and $\sigma(h_j) = g_l$.

Proof. Since $\mathrm{Th}(G)$ is ω-categorical and admits quantifier elimination, for any finite $aA \subset G$, there are infinitely many realization of $\mathrm{tp}(a/A)$ in G. Hence we can embed H into G step by step, preserving the enumerations. □

In this section, we prove the following:

Theorem 7. There is an edge coloring $f : E(G) \to 2$ such that for every copy $G' \subset G$ of G, $|f(E(G'))| = 2$.

In what follows, we assume $V(H) \subset V(G) = \omega$ (hence $h_i \in \omega$) and $h_i < h_j \leftrightarrow i < j$. We define $f : E(G) \to 2$ as follows.

Definition 8.
1. For given $i < j \in G$, let $t(i,j)$ be the minimum $t \in \omega$ such that $E(t,i) \not\leftrightarrow E(t,j)$.
2. Let $\{i < j\} \in E(G)$. Define $f(\{i,j\}) = 0$ if and only if $t(i,j) < i$ and $\{t(i,j), i\} \in E(G)$.

Now fix a copy $G' \subset G$ of G and let $\sigma : H \to G'$ be an embedding such that $\sigma(h_i) < \sigma(h_j) \leftrightarrow i < j$. For the simplicity, let $n_i = \sigma(h_i)$ for each $i \in \omega$.

Proof of Theorem 7. Without loss of generality, assume that $f|E(G') = \{0\}$. Since $n_0 < n_i < n_{i+1}$ and $E(n_0, n_i) \not\leftrightarrow E(n_0, n_{i+1})$, we know that $t(n_i, n_{i+1}) \leq n_0$ for every $i \in \omega$. For each $i \in \omega$, let *code*(i) be the $\{0,1\}$-sequence $s^i_0 s^i_1 \ldots s^i_{n_0}$ such that $s^i_k = 1$ if and only if $\{k, n_i\} \in E(G)$. (Hence there is at least one 0 in *code*(i) and $s^i_k = s^{i+1}_k$ for every $k < t(n_i, n_{i+1})$.) We will consider *code*(i) as a binary number and discuss the natural order (lexicographic order) on them.

Claim A. *code*$(i) >$ *code*$(i+1)$ for every $i \in \omega$.

Fix i and put $t = t(n_i, n_{i+1})$. It is implied that $\{t, n_i\} \in E(G)$ and $\{t, n_{i+1}\} \notin E(G)$ from $f(n_i, n_{i+1}) = 0$, so that $s^i_t = 1$ and $s^{i+1}_t = 0$. Since $s^i_k = s^{i+1}_k$ for every $k < t(n_i, n_{i+1})$, *code*(i) must be greater that *code*$(i+1)$. (End of proof of the claim.)

The claim implies a contradiction because $\{$*code*$(i) : i \in \omega\}$ is finite. □

References

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