On the number of independent orders (Model theoretic aspects of the notion of independence and dimension)

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On the number of independent orders

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Abstract

In this note, we present and prove some lemmas that are useful when studying the number of independent orders. We can show \( \kappa_{sr}(T) = \infty \Rightarrow \kappa_{sr}(T) = \infty \), using these lemmas. Its proof will be given in a forthcoming paper. (The details are not given in this note.)

We fix a complete theory \( T \), and we work in a very saturated model of \( T \). Letters \( x, y, \ldots \) are used to denote finite tuples of variables. \( X \) is a set of \( x \)-tuples and \( Y \) is a set of \( y \)-tuples. In many cases, they have the form

\[
X = (x_{\eta})_{\eta \in \omega^n} \quad \text{and} \quad Y = (y_{\nu})_{\nu \in n \times \omega},
\]

where \( n \in \omega \). For sets \( Z, W \) of finite tuples of variables and a set \( \Gamma(Z, W) \) of formulas, the set of all formulas \( \exists z_0 \ldots \exists z_{m-1}(\gamma_0(z, w) \land \ldots \gamma_{m-1}(z, w)) \), where \( m \in \omega, \gamma_i(z_i, w_i) \in \Gamma, z_i \subset Z, w_i \subset W \), is denoted by \( \exists Z \Gamma(Z, W) \).

Definition 1. Let \( n \in \omega \).

1. Let \( X = (x_{\eta} : \eta \in \omega^n) \) be a set of variables. Let \( \Delta(X) \) be a set of formulas whose free variables are in \( X \). We say that \( \Delta \) has the subarray property if there is a set \( A = (a_{i_0, \ldots, i_{n-1}} : (i_0, \ldots, i_{n-1}) \in \omega^n) \) such that for any strictly increasing functions \( f_i : \omega \to \omega \) (\( i < n \)), \( A_{f_0, \ldots, f_{n-1}} = (a_{f_0(i_0), \ldots, f_{n-1}(i_{n-1})} : (i_0, \ldots, i_{n-1}) \in \omega^n) \) realizes \( \Delta \).

2. Let \( Y = (y_{\nu})_{\nu \in n \times \omega} \). Let \( \mathcal{E}(Y) \) be a set of formulas whose free variables are in \( Y \). We say that \( \mathcal{E} \) has the \( (n \text{-dimensional}) \) subsequence property if there is a set \( B = (b_{i,j})_{(i,j) \in n \times \omega} \) such that for any strictly increasing functions \( f_i : \omega \to \omega \) (\( i < n \)), \( B_{f_0, \ldots, f_{n-1}} = (b_{f_i(i), \ldots, f_{i}(j)} : (i, j) \in n \times \omega) \) realizes \( \mathcal{E}(Y) \).
Lemma 2. Suppose that $\Delta(X)$, where $X = (x_{\eta} : \eta \in \omega^n)$, has the sub-array property. Then a realization $A = (a_{\eta} : \eta \in \omega^n)$ of $\Delta$ can be chosen as an indiscernible array in the following sense:

(*) For finite subsets $F, F'$ of $\omega^n$, if $F$ and $F'$ are isomorphic as $\{\leq_0, \ldots, \leq_{n-1}\}$-structures then $a_F$ and $a_{F'}$ have the same $L$-type.

Proof. For simplicity, we assume $n = 2$. We write $X$ as $X = (X_0, X_1, \ldots)$, where $X_i = (x_{ij})_{j \in \omega}$. For each $i$, let $X_i = (x_{ij})_{j \in \omega}$ be the $i$-th row vector of $X$. Then

$$\Delta = \Delta((X_i)_{i \in \omega}) = \Delta(X_0, X_1, \ldots)$$

has the subsequence property. So, for $A = (A_i)_{i \in \omega}$ realizing $\Delta$, we can assume the $A_i$'s form an indiscernible sequence. Similarly, we can also assume $(A_j')_{j \in \omega}$, where $A_j' = (a_{i,j})_{i \in \omega}$, is an indiscernible sequence. So $A$ is an indiscernible array.

For $A = (a_{\eta})_{\eta \in \omega^n}$ and a subset $F$ of $\omega^2$, $a_F$ will denote the set $(a_{\eta})_{\eta \in F}$.

Lemma 3. Suppose that $\Delta(X)$ is realized by an indiscernible array $A = (a_{\eta} : \eta \in \omega^n)$. Let $X^* = (x_{\eta})_{\eta \in I^n}$, where $I$ is an arbitrary ordered set. We define $\Delta^*(X^*)$ by: For all $\varphi$ and $F^* \subset \text{fin} I^n$:

$$\varphi(x_{F^*}) \in \Delta^* \iff \varphi(x_F) \in \Delta, \text{ for some } F \subset \omega^n \text{ with } F \equiv_{\leq_0, \ldots, \leq_{n-1}} F^*.$$ 

Then $\Delta^*$ is consistent and is realized by an indiscernible array.

Proof. It is sufficient to show the consistency, since the indiscernibility condition can be added to $\Delta^*$. Let $\varphi_i(x_{F^*_i}) \in \Delta^* (i < m)$. Choose $F_i \subset \omega^n (i < m)$ witnessing the definition of $\Delta^*$. Then $\varphi_i(a_{F_i})$ holds for all $i < m$. We can also choose $F'_i \subset \omega^n$ such that $F'_0 \ldots F'_{n-1} \equiv F'_0 \ldots F'_{n-1}$. By the indiscernibility, $\varphi_i(a_{F'_i})$ holds for all $i < m$. This shows that $\bigwedge \varphi_i(x_{F^*_i})$ is satisfiable.

Lemma 4. Suppose that $\mathcal{E}(Y)$, where $Y = (y_{(i,j)} : \langle i, j \rangle \in n \times \omega)$, has the $n$-dimensional subsequence property. Then $\mathcal{E}(Y)$ is realized by $B = (b_{(i,j)} : \langle i, j \rangle \in n \times \omega)$ with the following property:

(**) By letting $B_i = (b_{i,j})_{j \in \omega} (i < n)$, $B_i$ is an indiscernible sequence over $\bigcup_{k \neq i} B_k$.

Proof. Easy.
Example 5. Let $\varphi(x, y)$ be a formula. We say that $T$ has $n$ independent orders uniformly defined by $\varphi$ if there are $A = (a_\eta : \eta \in \omega^n)$ and $B = (b_{i,j})_{(i,j) \in n \times \omega}$ such that, for all $\eta \in \omega^n$ and $(i, j) \in n \times \omega$,

$$\varphi(a_\eta, b_{ij}) \text{ holds iff } j \geq \eta(i).$$

Let

$$\Gamma(X, Y) := \{ \varphi(x_\eta, y_{i,j})^{\text{if } j \geq \eta(i)} : \eta \in \omega^n, (i, j) \in n \times \omega \}.$$

Then $T$ has $n$ independent orders iff $\Gamma(X, Y)$ is consistent (with $T$). The set $\Delta(X) := \exists Y \Gamma(X, Y)$ has the subarray property and $\mathcal{E}(Y) := \exists X \Gamma(X, Y)$ has the $n$-dimensional subsequence property. (Notice that $\Delta$ and $\mathcal{E}$ are sets of first-order formulas.)

![Diagram](image)

2-dimensional case

From now on, $\Gamma_{\varphi, n, \omega}(X, Y)$ denotes the set described by the above example. By Lemma 3 (or by a direct argument), $\Gamma_{\varphi, n, \mathbb{Q}}$ is naturally defined. In particular, if $T$ has $n$ independent orders defined by $\varphi$, then $\Gamma_{\varphi, n, \mathbb{Q}}(X, Y)$ is consistent, and $\Delta(X) := \exists Y \Gamma_{n, \varphi, \mathbb{Q}}(X, Y)$ has the subarray property. We simply write $\Gamma_{\varphi, n}$ if we are not interested in the ordered set ($\omega$ or $\mathbb{Q}$).

Definition 6 (The Number of Independent Orders). Let $m, n \in \omega$. We write
1. $\kappa_{ird}^m(T) \geq n$ if $\Gamma_{\varphi(x,y),n}$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa_{ird}^m(T) = n$ if $\kappa_{ird}^m(T) \geq n$ and $\kappa_{ird}^m(T) \not\geq n+1$.
3. $\kappa_{ird}^m(T) = \infty$ if $\kappa_{ird}^m(T) \geq n$ ($\forall n$).

**Definition 7** (The Number of Independent Strict Orders). Let $\Gamma_{\varphi(x,y),n}^s(X,Y)$ be the set:

$$\Gamma_{\varphi(x,y),n}(X,Y) \cup \bigcup_{j<n} \{ \forall x (\varphi(x, y_{i,j}) \rightarrow \varphi(x, y_{i+1,j})) : i \in \omega \}.$$

We write

1. $\kappa_{srd}^m(T) \geq n$ if $\Gamma_{\varphi(x,y),n}^s$ is consistent for some $\varphi(x,y)$ with $|x| = m$.
2. $\kappa_{srd}^m(T) = n$ if $\kappa_{srd}^m(T) \geq n$ and $\kappa_{srd}^m(T) \not\geq n+1$.
3. $\kappa_{srd}^m(T) = \infty$ if $\kappa_{srd}^m(T) \geq n$ ($\forall n$).

The definition of above invariants are due to Shelah, but with a slight modification.

**Remark 8.** 1. Suppose that $T$ has the independence property. Then $\kappa_{ird}^1(T) = \infty$: Since $T$ has the independence property, there is a formula $\varphi(x,y)$ with $|x| = 1$ and $I = (b_i)_{i \in \omega}$ such that $\{ \varphi(x, b_i) : i \in F \}$ is consistent for any $F \subseteq \omega$. Choose an indiscernible sequence $I^* = (b_i)_{i \in \omega^2}$ extending $I$. Then $I^*$ realizes $\exists X \Delta_{\varphi,\omega}(X, Y)$. By compactness, this shows $\kappa_{ird}^1(T) = \infty$.

2. Let $T_{rg}$ be the theory of random graphs. Then $\kappa_{ird}^1(T_{rg}) = \infty$ and $\kappa_{srd}^m(T_{rg}) = 1$.

3. If $T$ has the order property, then $\kappa_{ird}^m(T) \geq m+1$. If $T$ has the strict order property, then $\kappa_{srd}^m(T) \geq m+1$: Both can be proven similarly. For the case of strict order property, choose $\psi(x,y)$ with $|x| = 1$ and $I = (b_i)$ witnessing the property. For $u = u_0, \ldots, u_{m-1}$, let $\varphi_i(u,y) := \psi(u_i,y)$ ($i < m$). Then $\{ \varphi_i(u, b_j) : i < m, j \in \omega \}$ is consistent, for any $\eta \in \omega^m$. This shows $\kappa_{srd}^m(T) \geq m + 1$, since there is a formula with additional variables such that each $\varphi_i$ is a specialization of the formula.
References

