NOTES ON PRODUCTS OF LINDELÖF SPACES WITH POINTS $G_{\delta}$ (Iterated Forcing Theory and Cardinal Invariants)

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NOTES ON PRODUCTS OF LINDELOF SPACES WITH POINTS $G_{\delta}$

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ABSTRACT. In this note, under some extra assumptions, we study some constructions of regular $T_1$ Lindelöf spaces with points $G_{\delta}$ whose product have a large extent.

1. INTRODUCTION

For a topological space $X$, the Lindelöf degree of $X$, $L(X)$, is the minimal cardinal $\kappa$ such that every open cover of $X$ has a subcover of size $\leq \kappa$. A space $X$ is Lindelöf if $L(X) = \omega$, that is, every open cover of $X$ has a countable subcover. The extent of $X$, $e(X)$, is $\sup\{|C| \mid C \subseteq X \text{ is closed and discrete}\}$. It is clear that $|X| \geq L(X) \geq e(X)$.

It is well-known that the product of compact spaces is compact. In contrast with compact spaces, it is also known that the product of Lindelöf spaces needs not to be Lindelöf; If $S$ is the Sorgenfrey line, $S$ is Lindelöf but $e(S \times S) = 2^\omega \leq L(S \times S)$.

This fact suggests the following natural question:

**Question 1.1.** Are there Lindelöf spaces $X$ and $Y$ with $e(X \times Y) > 2^\omega$?

For this question, Shelah [3] and Gorelic [1] proved the following consistency results:

**Fact 1.2.** (1) Under $V=L$, there are regular $T_1$ Lindelöf spaces $X$ and $Y$ with points $G_{\delta}$ such that $e(X \times Y) = (2^\omega)^+$. (2) Suppose CH. Then there is a $\sigma$-closed, $\omega_2$-c.c. forcing notion which forces the following statement: There are regular $T_1$ Lindelöf spaces $X$ and $Y$ with points $G_{\delta}$ such that $e(X \times Y) = 2^{\omega_1}$ and $2^{\omega_1}$ is arbitrary large.

However it is still open whether the existence of such spaces is provable from ZFC. In this note, we will give relatively simple proofs of Shelah and Gorelic's results. First, we show that the Cohen forcing creates such spaces:

**Theorem 1.3.** The Cohen forcing forces the following: There are regular $T_1$ Lindelöf spaces $X$ and $Y$ with points $G_{\delta}$ such that $e(X \times Y) = (2^{\omega_1})^V$.

We also prove the following theorem. A space $X$ is a P-space if every $G_{\delta}$-subset of $X$ is open in $X$. 
Theorem 1.4. Suppose there is a regular $T_1$ Lindelöf $P$-space $X$ with character $\leq \omega_1$. Then there are regular Lindelöf spaces $X_0$ and $X_1$ with points $G_\delta$ such that $e(X_0 \times X_1) = |X|$.

It is known that under $V = L$, there is a regular $T_1$ Lindelöf $P$-space of weight $\omega_1$ and size $(2^\omega)^+ = \omega_2$ (Juhász-Weiss [2]).

2. In the Cohen forcing extension

For Theorem 1.3, we prove the following which would be interesting in its own right.

Proposition 2.1. Let $X$ be a zero-dimensional $T_1$ Lindelöf space $X$ with points $G_\delta$. Then the Cohen forcing forces the following statement: There are zero-dimensional $T_1$ Lindelöf spaces $X_0$ and $X_1$ with points $G_\delta$ such that $e(X_0 \times X_1) = |X|$.

We can obtain Theorem 1.3 by the combination of the proposition with the following fact:

Fact 2.2 (Usuba [4]). The Cohen forcing forces the following: There exists a zero-dimensional $T_1$ Lindelöf space $X$ with points $G_\delta$ such that $|X| = (2^{\omega_1})^V$.

We start the proof of the proposition. Fix a space $X$ as in the assumption of the proposition. For each $x \in X$, take open sets $G_n^x (n < \omega)$ such that $\bigcap_{n<\omega} G_n^x = \{x\}$. By the assumption for $X$, we may assume that each $G_n^x$ is clopen and $G_0^x \supseteq G_1^x \supseteq \ldots$. Let $H_n^x = G_n^x \setminus G_{n+1}^x$. Note that the following:

1. $H_n^x$ is clopen.
2. $H_n^x \cap H_m^x = \emptyset$ for every $n < m < \omega$.
3. $x \notin H_n^x$.
4. $G_m^x = \{x\} \cup \bigcup_{m \leq n < \omega} H_n^x$.

Let $C$ be the Cohen forcing notion $2^{<\omega}$. Take a $(V,C)$-generic $G$, and we work in $V[G]$. Let $a = \{n < \omega \mid \bigcup G(n) = 0\}$ and $b = \{n < \omega \mid \bigcup G(n) = 1\}$. We define the space $X_a$ as the following manner. For $x \in X$, let $W_a^x = \bigcup\{H_n^x \mid n \in a\} \cup \{x\}$. $W_a^x$ is a closed subset of $X$. Then the topology of $X_a$ is generated by the family $\{O \subseteq X \mid O$ is open in $X\} \cup \{W_a^x \mid x \in X\}$ as a subbase. One can check that $X_a$ is a zero-dimensional $T_1$-space with points $G_\delta$. We define $X_b$ by the same way but replacing $a$ by $b$. $X_a$ and $X_b$ are finer spaces than $X$. We shall show that $X_a$ and $X_b$ are required spaces.

Lemma 2.3. $e(X_a \times X_b) = |X|$. Namely, the diagonal $\Delta = \{\langle x, x \rangle \mid x \in X\}$ is a closed discrete subset of $X_a \times X_b$.

Proof. Since $X_a$ and $X_b$ are Hausdorff, it is clear that $\Delta$ is closed. To see that $\Delta$ is discrete, take $\langle x, x \rangle \in \Delta$. Consider $W_a^x \times W_b^x$. It is obvious that $W_a^x \times W_b^x$ is an open neighborhood of $\langle x, x \rangle$ in $X_a \times X_b$. We check that $\Delta \cap (W_a^x \times W_b^x) = \{\langle x, x \rangle\}$. Take
\( \langle y, y \rangle \in \Delta \cap (W_a^x \times W_b^x) \) and suppose \( y \neq x \). Since \( W_a^x = \bigcup \{ H_n^x \mid n \in a \} \cup \{ x \} \) and \( W_b^x = \bigcup \{ H_n^x \mid n \in b \} \cup \{ x \} \), there are \( n_a \in a \) and \( n_b \in b \) such that \( y \in H_{n_a}^x \cap H_{n_b}^x \). \( n_a \neq n_b \) because \( a \cap b = \emptyset \). However then \( H_{n_a}^x \) is disjoint from \( H_{n_b}^x \), this is a contradiction. \( \square \)

**Lemma 2.4.** \( X_a \) and \( X_b \) are Lindelöf.

**Proof.** We prove it only for \( X_a \). \( X_b \) can be checked by the same argument. Our argument which will be used in this proof came from Usuba [5].

Let \( \mathcal{U} \) be an open cover of \( X_a \). We may assume that every element of \( \mathcal{U} \) is of the form \( O \cap \bigcap W_{a}^{x_0} \cap \cdots \cap W_{a}^{x_n} \) for some open set \( O \) in \( X \) and \( x_0, \ldots, x_n \in X \). Let \( W_{a}^{x_0,\ldots,x_n} = W_{a}^{x_0} \cap \cdots \cap W_{a}^{x_n} \). Take a \( \mathbb{C} \)-name \( \dot{\mathcal{U}} \) for \( \mathcal{U} \), and let \( \dot{a} \) be a name for \( a \).

Return to \( \mathcal{V} \). Let \( p \in \mathbb{C} \) be such that \( p \models \dot{\mathcal{U}} \) is an open cover of \( X_a \). Take a sufficiently large regular cardinal \( \theta \), and a countable \( M < H_\theta \) containing all relevant objects. We see that \( p \models \dot{\mathcal{U}} \{ O \cap W_{a}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}} \mid O, x_0, \ldots, x_n \in M \} \) is a cover of \( X_a \). Since \( M \) is countable, we have that \( p \models \dot{\mathcal{U}} \) has a countable subcover as required.

In order to show it, fix \( x^* \in X \) and \( p' \leq p \). We will find \( r \leq p' \) and \( O, x_0, \ldots, x_n \in M \) with \( r \models \dot{\mathcal{U}} \). For a condition \( q \leq p' \) and \( x \in X \), let \( a_q = \{ n \in \text{dom}(q) \mid q(n) = 0 \} \) and \( W_q^x = \bigcup \{ H_n^x \mid n \in a_q \} \cup G_{\text{dom}(q)}^x \). Then for \( x_0, \ldots, x_n \in X \), let \( W_{q}^{x_0,\ldots,x_n} = W_{q}^{x_0} \cap \cdots \cap W_{q}^{x_n} \). Note that \( W_{q}^{x_0,\ldots,x_n} \) is open in \( X \).

Now let \( \mathcal{V} \) be the set of all \( O \cap W_{q}^{x_0,\ldots,x_n} \) such that \( q \leq p' \) and \( q \models \dot{\mathcal{U}} \). We claim that \( \mathcal{V} \) is an open cover of \( X \). Take \( y \in X \). Then there are \( q \leq p \), open \( O \subseteq X \), and \( x_0, \ldots, x_n \in X \) such that \( q \models \dot{\mathcal{U}} \). Clearly \( y \in O \), and \( y \in G_q^x \) for every \( i \leq n \). We see \( y \in W_{q}^{x_i} \). Fix \( i \leq n \).

Case 1: \( y = x_i \). Then trivially \( y \in W_{q}^{x_i} \).

Case 2: \( y \neq x_i \). Then there is a unique \( m < \omega \) with \( y \in H_{m}^{x_i} \). If \( m \geq \text{dom}(q) \) or \( m \in \text{dom}(q) \) but \( q(m) = 1 \), we can take a condition \( q' \leq q \) with \( m \in \text{dom}(q') \) and \( q'(m) = 1 \). \( q' \models \dot{\mathcal{U}} \). By the same argument we can find \( w \in W_{q'}^{x_i} \). Since \( W_{q'}^{x_i} \) is open, we have \( y \in W_{q'}^{x_i} \).

In either cases, we have \( y \in W_{q}^{x_i} \), hence \( y \in \bigcap_{i \leq n} W_{q}^{x_i} = W_{q}^{x_0,\ldots,x_n} \).

By the elementarity of \( M \), we have \( \mathcal{V} \in M \). Since \( X \) is Lindelöf, \( \mathcal{V} \) has a countable subcover \( \mathcal{V}' \). We may assume that \( \mathcal{V}' \in M \). \( \mathcal{V}' \subseteq M \) because \( \mathcal{V}' \) is countable. Now, we can take \( O \cap W_{q}^{x_0,\ldots,x_n} \in \mathcal{V}' \). Clearly \( O \subseteq X \), and \( x_0, \ldots, x_n \in M \). We find \( r \leq q \) with \( r \models \dot{\mathcal{U}} \). Finally we have to find \( r \models \dot{\mathcal{U}} \). Since \( x^* \in W_{q}^{x_0,\ldots,x_n} \), for each \( i \leq n \), if \( x^* \neq x_i \) then \( x^* \in H_{k_i}^x \) for some \( k < \omega \). Hence there is a large \( m < \omega \) such that \( m > \text{dom}(q) \) and \( x^* \neq x_i \) then \( x^* \in H_{k_i}^x \) for some \( k < m \). Define \( r \leq q \) by \( \text{dom}(r) = m \) and \( r(k) = 0 \) for every \( \text{dom}(q) \leq k < m \). We have \( r \models \dot{\mathcal{U}} \). It is clear \( r \models \dot{\mathcal{U}} \). Finally we have to find \( r \models \dot{\mathcal{U}} \). Suppose \( x^* \neq x_i \). We can find \( k < m \) such that \( x^* \in H_{k_i}^x \). By the choice of \( r \), we have \( r \models \dot{\mathcal{U}} \). \( \square \)
Remark 2.5. As in [5], for each positive $n < \omega$ we can prove the following: In the Cohen forcing extension, there is a regular Lindelöf space $X$ with points $G_\delta$ such that $X^n$ is Lindelöf but $e(X^{n+1}) = (2^{\omega_1})^V$.

3. USING P-SPACES

In this section, we prove Theorem 1.3. Fix a regular Lindelöf $T_1$ P-space $X$ with character $\leq \omega_1$. Let $Y = \{x \in X \mid \chi(x, X) = \omega_1\}$. Note that every $x \in X \setminus Y$ is an isolated point.

Let $S$ be the Sorgenfrey line, namely, the underlying set of $S$ is the real line $\mathbb{R}$, and the topology of $S$ is generated by the family $\{[r, s) \mid r, s \in \mathbb{R}\}$ as an open base. $S$ is a first countable regular $T_1$ Lindelöf space.

For a subset $A \subseteq X$, let $A^z = \bigcup\{\{x\} \times \mathbb{R} \mid x \in A \cap Y\} \cup (A \setminus Y)$. By the assumption, for each $x \in Y$, there is a sequence $\langle G_\alpha^x \mid \alpha < \omega_1 \rangle$ such that:

1. $G_\alpha^x$ is clopen in $X$.
2. $\{G_\alpha^x \mid \alpha < \omega_1\}$ is $\subseteq$-decreasing, and $G_\alpha^x = \bigcap_{\beta < \alpha} G_\beta^x$ if $\alpha$ is limit.
3. $\bigcap_{\alpha < \omega_1} G_\alpha^x = \{x\}$.

Fix an injection $\sigma : \omega_1 \to \mathbb{R}$. For $x \in Y$, $\alpha < \omega_1$, and open $O \subseteq S$, let $W(x, \alpha, O) = \bigcup\{[G_\beta^x \setminus G_{\beta+1}^x] \mid \beta < \alpha, \sigma(\beta) \in O\} \cup (\{x\} \times O)$.

We define the space $X_0$ as in [4] using $X$ and $S$. The underlying set of $X_0$ is $[X]$. The topology of $X_0$ is generated by the family $\{\{W\} \mid W \subseteq X$ is open$\} \cup \{W(x, \alpha, O) \mid x \in Y, \alpha < \omega_1, O \subseteq S$ is open$\}$ as an open base. We know that $X_0$ is a regular $T_1$ Lindelöf space with points $G_\delta$ (see [4]).

For $X_1$, let $S'$ be the space $\mathbb{R}$ equipped with the reverse Sorgenfrey topology, namely, it is generated by the family $\{[r, s) \mid r, s \in \mathbb{R}\}$ as an open base. $S'$ is also a first countable regular $T_1$ Lindelöf space. We define $X_1$ by the same way to $X_0$ but replacing $S$ with $S'$. Again, $X_1$ is a regular Lindelöf space with points $G_\delta$.

We show that $e(X_0 \times X_1) = |X|$. Let $\Delta = \{(x, x) \mid x \in X \setminus Y\} \cup \{(\langle x, r \rangle, \langle x, r \rangle) \mid x \in Y, r \in \mathbb{R}\}$. One can check that $\Delta$ is closed in $X_0 \times X_1$. We see that $\Delta$ is discrete. If $x \in X \setminus Y$, then $x$ is isolated in $X$, hence $\langle x, x \rangle$ is also isolated in $X_0 \times X_1$. Let $x \in Y$ and $r \in \mathbb{R}$. Let $O_0 = [r, r+1)$ and $O_1 = (r-1, r]$. $O_0$ is open in $S$ with $r \in O_0$, and $O_1$ is open in $S'$ with $r \in O_1$. Moreover we have $O_0 \cap O_1 = \{r\}$. Consider open sets $W(x, 0, O_0) \subseteq X_0$ and $W(x, 0, O_1) \subseteq X_1$. By the choice of $O_0$ and $O_1$, we have $W(x, 0, O_0) \cap W(x, 0, O_1) = \{\langle x, r \rangle\}$. Then $\Delta \cap (W(x, 0, O_0) \times W(x, 0, O_1)) = \langle\langle x, r \rangle, \langle x, r \rangle\rangle$, as required.

Remark 3.1. The existence of a regular $T_1$ Lindelöf P-space with character $\leq \omega_1$ and size $> \omega_1$ is independent from ZFC.

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