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A COMMENT ON BAGARIA-SHELAH'S FRAGMENT OF MARTIN'S AXIOM

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INTRODUCTION

Harrington-Shelah showed that, for $x \in \mathbb{R}$, if ${\aleph_1}^{L[x]} = \aleph_1$ and Martin's axiom MA holds, then there exists a $\Delta^1_3(x)$-set of reals without Baire property [4]. Bagaria extended their result as follow: For $x \in \mathbb{R}$, if ${\aleph_1}^{L[x]} = \aleph_1$, $\mathsf{MA}(\sigma\text{-centered})$ holds and every Aronszajn tree is special, then there exists a $\Delta^1_3(x)$-set of reals without Baire property [1]. Harrington-Shelah also showed that, if $\aleph_1$ is inaccessible to the reals and MA holds, then $\aleph_1$ is weakly compact in L, and that the existence of a weakly compact cardinal is equiconsistent to the assertion that MA holds and every projective set of reals has the Baire property [4]. So in [1], Bagaria asked if $\mathsf{MA}(\sigma\text{-centered})$ plus the assertion that every Aronszajn tree is special implies $\mathsf{MA}(\sigma\text{-linked})$. In [2], Bagaria-Shelah proved the consistency of the assertion (*) that $\mathsf{MA}(\sigma\text{-centered})$ holds, every Aronszajn tree is special, and $\mathsf{MA}(\sigma\text{-linked})$ fails.

In this paper, the consistency of the assertion (*) is proved by use of the idea due to Bagaria-Shelah in [2] combining with the rectangle refining property due to Larson-Todorčević [6]. §1 provides Bagaria-Shelah's work in [2] and introduces their fragment of Martin's axiom. §2 provides some remarks of the rectangle refining property and a proof of the theorem in this paper. In §3, some previous works on fragments of Martin's axiom are mentioned and are compared to the theorem in this paper.

1. BAGARIA-SHELAH'S FRAGMENT OF MARTIN'S AXIOM

In [2], Bagaria-Shelah introduced the following property of forcing notions.

Definition 1.1 ([2, DEFINITION 1]). For an integer $k \geq 2$, a forcing notion $\mathbb{P}$ satisfies the property Pr_k if, for any $\{p_\alpha : \alpha \in \omega_1\} \in [\mathbb{P}]^{\aleph_1}$, there exists a pairwise disjoint uncountable family $\{u_\xi : \xi \in \omega\}$ of non-empty finite subsets of ω_1 such that, for each $\{\xi_i : i \in k\} \in [\omega_1]^k$, there exists $\langle \alpha_i : i \in k\rangle \in \prod_{i\in k} u_{\xi_i}$ such that $\{p_{\alpha_i} : i \in k\}$ has a common extension in $\mathbb{P}$.

$\mathsf{MA}(\mathsf{Pr}_k)$ denotes the forcing axiom for forcing notions with the property Pr_k.

The property Pr_k is stronger than the countable chain condition. A σ-centered forcing satisfies the property Pr_k for every integer $k \geq 2$. Bagaria-Shelah proved that a specialization of an Aronszajn tree by finite approximations also satisfies the property Pr_k for every integer $k \geq 2$ [2, LEMMA 2]. So, for every integer $k \geq 2$, $\mathsf{MA}(\mathsf{Pr}_k)$ implies $\mathsf{MA}(\sigma\text{-centered})$ and the assertion that every Aronszajn tree is special. They also showed that, for any integer $k \geq 2$, the property Pr_k is preserved under finite support iterations [2, LEMMA 3].

In [2], Bagaria-Shelah introduced the following forcing notion that plays a role of the failure of MA(σ-linked) in the extension with finite support iterations of forcing notions with the property Pr_k.

Definition 1.2 ([2, LEMMA 4]). For an integer $k \geq 2$, the forcing notion $\mathbb{P}^k_*$ consists of triples $p = \langle u_p, A_p, h_p \rangle$ such that

- u_p is a finite subset of ω_1,
- A_p is a subset of $[u_p]^{k+1}$ (which is a finite set),
- h_p is a function from the set $\{v \subseteq u_p : [v]^{k+1} \cap A_p = \emptyset\}$ into ω such that, for every $l \in \mathrm{ran}(h_p)$ and $\rho \in \left[{h_p}^{-1}[\{l\}]\right]^k$, $\bigcup \rho$ belongs to $\mathrm{dom}(h_p)$,

ordered by: $q \leq_{\mathbb{P}^k_*} p$ iff $u_q \supseteq u_p$, $A_p = A_q \cap [u_p]^{k+1}$ and $h_q \supseteq h_p$.

Note that $\mathbb{P}^k_*$ is of size $\aleph_1$. Bagaria-Shelah proved that $\mathbb{P}^k_*$ has precaliber $\aleph_1$ [2, LEMMA 4 (1)]. They also define the following $\mathbb{P}^k_*$-names.

Definition 1.3 ([2, LEMMA 4]). Let k be an integer not smaller than 2. Define the $\mathbb{P}^k_*$-name $\dot{A}^*_k$ by

$$\dot{A}^k_* := \left\{(\check{v}, p) : p \in \mathbb{P}^k_*, v \in A_p\right\},$$

and define the $\mathbb{P}^k_*$-name $\dot{\mathbb{Q}}^k_*$ by

$$\dot{\mathbb{Q}}^k_* := \left\{(\check{v}, p) : p \in \mathbb{P}^k_*, v \in \mathrm{dom}(h_p)\right\}.$$

We notice that

$$\Vdash_{\mathbb{P}^k_*} \text{“ } \dot{A}^k_* = \bigcup_{p \in \dot{G}_{\mathbb{P}^k_*}} A_p \text{ and } \dot{\mathbb{Q}}^k_* = \left\{v \in [\omega_1]^{<\aleph_0} : [v]^{k+1} \cap \dot{A}^k_* = \emptyset\right\} \text{”}.$$

By considering $\dot{\mathbb{Q}}^k_*$ as a $\mathbb{P}^k_*$-name for a forcing notion, ordered by $\supseteq$, Bagaria-Shelah proved that

[2, LEMMA 4 (3)]: $\Vdash_{\mathbb{P}^k_*}$ “ $\dot{\mathbb{Q}}^k_*$ is σ-k-linked ”, and

[2, LEMMA 4 (5)]: $\Vdash_{\mathbb{P}^k_*}$ “ for any $\{v_\alpha : \alpha \in \omega_1\} \in [\dot{\mathbb{Q}}^k_*]^{\aleph_1}$ with $v_\alpha \not\subseteq \alpha$, and any pairwise disjoint uncountable family $\{u_\xi : \xi \in \omega\}$ of non-empty finite subsets of ω_1, there exists $\{\xi_i : i \in k+1\} \in [\omega_1]^{k+1}$ such that, for every $\langle \alpha_i : i \in k+1 \rangle \in \prod_{i \in k+1} u_{\xi_i}$, $\bigcup_{i \in k+1} v_{\alpha_i}$ does not belong to $\dot{\mathbb{Q}}^k_*$ ” .

The last assertion implies that

$$\Vdash_{\mathbb{P}^k_*} \text{“ } \dot{\mathbb{Q}}^k_* \text{ is not } \sigma\text{-}(k+1)\text{-linked ”}.$$

Definition 1.4 ([2, LEMMA 4]). For an integer $k \geq 2$ and $\alpha \in \omega_1$, define the $\mathbb{P}^k_*$-name $\dot{I}_\alpha$ such that

$$\Vdash_{\mathbb{P}^k_*} \text{“ } \dot{I}_\alpha := \left\{v \in \dot{\mathbb{Q}}^k_* : v \not\subseteq \alpha\right\} \text{”}.$$

Note that

$$\Vdash_{\mathbb{P}^k_*} \text{“ } \dot{I}_\alpha \text{ is dense in } \dot{\mathbb{Q}}^k_* \text{”}$$

[2, LEMMA 4 (4)]. The following is a key point of the proof of the failure of MA(σ-linked) in the extension with finite support iterations of forcing notions with the property Pr_k.

Lemma 1.5 ([2, LEMMA 6]). *For any integer $k \geq 2$ and any $\mathbb{P}^k_*$-name $\dot{\mathbb{Q}}$ for a forcing notion with the property Pr_{k+1},*

$\Vdash_{\mathbb{P}^k_* * \dot{\mathbb{Q}}}$ " *there are no directed subset G of $\dot{\mathbb{Q}}^k_*$ such that $\dot{I}_\alpha \cap G \neq \emptyset$ for all $\alpha \in \omega_1$* ".

This implies that, for any integer $k \geq 2$ and any $\mathbb{P}^k_*$-name $\dot{\mathbb{Q}}$ for a forcing notion with the property Pr_{k+1},

$$\Vdash_{\mathbb{P}^k_* * \dot{\mathbb{Q}}} \text{“}\mathsf{MA}_{\aleph_1}(\sigma\text{-}k\text{-linked}) \text{ fails”}.$$

Therefore, the following has been concluded.

Theorem 1.6 ([2, LEMMA 6]). *It is consistent that $\mathsf{MA}(\mathsf{Pr}_{k+1})$ holds, and $\mathsf{MA}_{\aleph_1}(\sigma$-$k$-linked) fails.*

2. THE RECTANGLE REFINING PROPERTY AND THE MAIN RESULT

Larson-Todorčević introduced a property of ccc partitions on $[\omega_1]^2$, called the rectangle refining property, and obtained a consistency of the affirmative of Katětov's problem [6]. The following is a version of the rectangle refining property for forcing notions. A similar definition of the following is appeared in [9, 11]. The following notation is inspired by [3, Theorem 3.1].

Definition 2.1. A forcing notion $\mathbb{P}$ satisfies the rectangle refining property if there exists a function w from $\mathbb{P}$ into $[\omega_1]^{<\aleph_0}$ such that

- **(•)** for any pair of compatible conditions p and q in $\mathbb{P}$, there exists a common extension r of p and q in $\mathbb{P}$ such that $w(r) = w(p) \cup w(q)$, and
- **(rec)** for any uncountable subsets I and J of $\mathbb{P}$, if the set $\{w(p); p \in I \cup J\}$ forms a Δ-system, then there are uncountable subsets I' and J' of I and J respectively such that each element of I' is compatible with any element of J' in $\mathbb{P}$.

The rectangle refining property is a stronger property than the countable chain condition. Like the property Pr_k, typical examples of forcing notions with the rectangle refining property are a σ-centered forcing notion and a specialization of an Aronszajn tree by finite approximations. For other examples, see [9, 10, 11]. Note that forcing notions with the rectangle refining property satisfies Chodounský-Zapletal's Y-cc [3]. Let $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \right\rangle$ be a finite support iteration of forcing notions with the rectangle refining property, and, for each $\beta < \lambda$, let $\dot{w}_\beta$ be a $\mathbb{P}_\beta$-name for a function that witnesses the rectangle refining property of $\dot{\mathbb{Q}}_\beta$. By induction on $\alpha \leq \lambda$, it can be prove that, for every $p \in \mathbb{P}_\alpha$, there is an extension q of p in $\mathbb{P}_\alpha$ such that, for each $\xi \in \operatorname{supp}(q)$, there exists $w^q_\xi \in [\omega_1]^{<\aleph_0}$ such that

$$q \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“ } \dot{w}_\xi(q(\xi)) = w^q_\xi \text{”}.$$

Such a q is here called a nice condition of $\mathbb{P}_\alpha$. Note that the set of nice conditions of $\mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$. We say that an uncountable set $\{p_\zeta : \zeta \in \omega_1\}$ of nice conditions of $\mathbb{P}_\alpha$ forms a Δ-system as a set of conditions of the iteration if $\{\operatorname{supp}(p_\zeta) : \zeta \in \omega_1\}$ forms a Δ-system with root $\triangle$ and, for each $\xi \in \triangle$, $\left\{ w^{p_\zeta}_\xi : \zeta \in \omega_1 \right\}$ also forms a Δ-system. Note that every uncountable set of nice conditions of $\mathbb{P}_\alpha$ has an uncountable subset that forms a Δ-system as a set of conditions of the iteration. The rectangle refining property is preserved under finite support iterations in the following sense.

Lemma 2.2. *Suppose that $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \right\rangle$ is a finite support iteration of forcing notions with the rectangle refining property, and let $\alpha \leq \lambda$. Then, for every uncountable set $\{p_\zeta : \zeta \in \omega_1\}$ of nice conditions of $\mathbb{P}_\alpha$, and every pair of uncountable subsets I and J of ω_1, if $\{p_\zeta : \zeta \in \omega_1\}$ forms a Δ-system as a set of conditions of the iteration, then there are $I' \in [I]^{\aleph_1}$ and $J' \in [J]^{\aleph_1}$ such that, for each $\zeta \in I$ and $\eta \in J'$, p_ζ and p_η are compatible in $\mathbb{P}_\alpha$.*

Proof. This is proved by induction on $\alpha \leq \lambda$. In the case that α is a limit ordinal, this is proved by inductive hypothesis. Suppose that $\mathbb{P}_\alpha$ satisfies the conclusion of the lemma, that $\{p_\zeta : \zeta \in \omega_1\}$ is an uncountable set of nice conditions of $\mathbb{P}_{\alpha+1}$ such that $\{\mathrm{supp}(p_\zeta) : \zeta \in \omega_1\}$ forms a Δ-system with root $\triangle$ with $\alpha \in \triangle$, and, for each $\xi \in \triangle$, $\left\{ w_\xi^{p_\zeta(\xi)} : \zeta \in \omega_1 \right\}$ also forms a Δ-system, and that I and J are uncountable subsets of ω_1.

By inductive hypothesis, there are $I^{(1)} \in [I]^{\aleph_1}$ and $J^{(1)} \in [J]^{\aleph_1}$ such that, for each $\zeta \in I^{(1)}$ and $\eta \in J^{(1)}$, $p_\zeta \restriction \alpha$ and $p_\eta \restriction \alpha$ are compatible in $\mathbb{P}_\alpha$. By refining $I^{(1)}$ and $J^{(1)}$ if necessary, we may assume that $I^{(1)}$ is disjoint from $J^{(1)}$. Define $\mathbb{P}_\alpha$-names $\dot{I}^{(2)}$ and $\dot{J}^{(2)}$ such that

$$\Vdash_{\mathbb{P}_\alpha} \text{“ } \dot{I}^{(2)} := \left\{ \zeta \in I^{(1)} : p_\zeta \restriction \alpha \in \dot{G}_{\mathbb{P}_\alpha} \right\} \text{ and } \dot{J}^{(2)} := \left\{ \zeta \in J^{(1)} : p_\zeta \restriction \alpha \in \dot{G}_{\mathbb{P}_\alpha} \right\} \text{”}.$$

Claim that there exists $q \in \mathbb{P}_\alpha$ such that

$$q \Vdash_{\mathbb{P}_\alpha} \text{“ both } \dot{I}^{(2)} \text{ and } \dot{J}^{(2)} \text{ are uncountable”}.$$

To see this, assume not. Then, since $\mathbb{P}_\alpha$ is ccc, there is $\delta \in \omega_1$ such that

$$\Vdash_{\mathbb{P}_\alpha} \text{“ } \dot{I}^{(2)}, \dot{J}^{(2)} \subseteq \delta \text{”}.$$

Take $\zeta \in I^{(1)} \setminus \delta$ and $\eta \in J^{(1)} \setminus \delta$, and take a common extension r of $p_\zeta \restriction \alpha$ and $p_\eta \restriction \alpha$ in $\mathbb{P}_\alpha$. Then

$$r \Vdash_{\mathbb{P}_\alpha} \text{“ } \zeta \in \dot{I}^{(2)} \setminus \delta \text{ and } \eta \in \dot{J}^{(2)} \setminus \delta \text{”},$$

which is a contradiction.

Let $q \in \mathbb{P}_\alpha$ be a condition of $\mathbb{P}_\alpha$ that forces $\dot{I}^{(2)}$ and $\dot{J}^{(2)}$ to be uncountable. Since $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a forcing notion with the rectangle refining property, there are $\mathbb{P}_\alpha$-names $\dot{I}^{(3)}$ and $\dot{J}^{(3)}$ such that

> $q \Vdash_{\mathbb{P}_\alpha}$ “ $\dot{I}^{(3)}$ and $\dot{J}^{(3)}$ are uncountable subsets of $\dot{I}^{(2)}$ and $\dot{J}^{(2)}$ respectively, and, for each $\zeta \in \dot{I}^{(3)}$ and $\eta \in \dot{J}^{(3)}$, $p_\zeta(\alpha)$ and $p_\eta(\alpha)$ are compatible in $\dot{\mathbb{Q}}_\alpha$ ”.

For each $i \in \omega_1$, take an extension q_i of q in $\mathbb{P}_\alpha$ and $\zeta_i, \eta_i \in \omega_1$ such that, for each $i \in \omega_1$, q_i forms a nice condition and

$$q_i \Vdash_{\mathbb{P}_\alpha} \text{“ } \zeta_i \in \dot{I}^{(3)} \text{ and } \eta_i \in \dot{J}^{(3)} \text{”},$$

and, for each $i, j \in \omega_1$ with $i < j$,

$$\max\{\zeta_i, \eta_i\} < \min\{\zeta_j, \eta_j\}.$$

Take an uncountable subset K of ω_1 such that $\{q_i : i \in K\}$ forms a Δ-system as a set of conditions of the iteration. By inductive hypothesis, take uncountable disjoint subsets K_0 and K_1 of K such that, for each $i \in K_0$ and $j \in K_1$, q_i and q_j are compatible in $\mathbb{P}_\alpha$. Define $I' := \{\zeta_i : i \in K_0\}$ and $J' := \{\eta_j : j \in K_1\}$. Since

$$q_i \Vdash_{\mathbb{P}_\alpha} \text{“ } \zeta_i \in \dot{I}^{(3)} \subseteq \dot{I}^{(2)} \subseteq I^{(1)} \subseteq I \text{”}$$

for each $i \in K_0$, I' is an uncountable subset of I. Similarly, $J' \subseteq J$. The pair I' and J' is what we want. $\square$

Lemma 2.3. *Suppose that k is an integer not smaller than 2, and $\dot{\mathbb{Q}}$ is a $\mathbb{P}^k_*$-name for a forcing notion with the rectangle refining property. Then*

$$\Vdash_{\mathbb{P}^k_* * \dot{\mathbb{Q}}} \text{“ there are no } K \in [\omega_1]^{\aleph_1} \text{ such that } [K]^{k+1} \subseteq \dot{\mathbb{Q}}^k_* \text{ ”}.$$

This lemma implies that $\mathbb{P}^k_* * \dot{\mathbb{Q}}$ forces the failure of $\mathsf{MA}_{\aleph_1}(\sigma\text{-}k\text{-linked})$ for $\dot{\mathbb{Q}}^k_*$ in some strong sense. See in the next section.

Proof. Throughout the proof, we work in the extension with $\mathbb{P}^k_*$. Suppose that $\mathbb{Q}$ is a forcing notion with the rectangle refining property (in the extension with $\mathbb{P}^k_*$), $q \in \mathbb{Q}$ and $\dot{K}$ is a $\mathbb{Q}$-name for an uncountable subset of ω_1 such that

$$q \Vdash_{\mathbb{Q}} \text{“ } \left[\dot{K}\right]^{k+1} \subseteq \mathbb{Q}^k_* \text{ ”}.$$

For each $\alpha \in \omega_1$, take an extension q_α of q in $\mathbb{Q}$ and $\delta_\alpha \in \omega_1$ such that, for each $\alpha \in \omega_1$, q_α forms a nice condition and

$$q_\alpha \Vdash_{\mathbb{P}_\alpha} \text{“ } \delta_\alpha \in \dot{K} \text{ ”},$$

and, for each $\alpha, \alpha' \in \omega_1$ with $\alpha < \alpha'$, $\delta_\alpha < \delta_{\alpha'}$. Note that each set $\{\delta_\alpha\}$ is a condition of $\mathbb{Q}^k_*$. Since $\mathbb{Q}$ satisfies the rectangle refining property, we can take uncountable subsets I_l, $l \leq k$, of ω_1 such that, for each $\langle \alpha_l : l \leq k \rangle \in \prod_{l \leq k} I_l$, $\{q_{\alpha_l} : l \leq k\}$ has a common extension in $\mathbb{Q}$. We build a pairwise disjoint uncountable family $\{u_\xi : \xi \in \omega_1\}$ of finite subsets of ω_1 such that each u_ξ contains some member of I_l for all $l \leq k$. Applying $\{\{\delta_\alpha\} : \alpha \in \omega_1\}$ and $\{u_\xi : \xi \in \omega_1\}$ to [2, Lemma 4 (5)] (which is a property of $\mathbb{Q}^k_*$, mentioned above), we can find $\langle \alpha_l : l \leq k \rangle \in \prod_{l \leq k} I_l$ such that $\{\delta_{\alpha_l} : l \leq k\} \notin \mathbb{Q}^k_*$. However, $\{q_{\alpha_l} : l \leq k\}$ has a common extension r in $\mathbb{Q}$, and then,

$$r \Vdash_{\mathbb{Q}} \text{“ } \{\delta_{\alpha_l} : l \leq k\} \in \left[\dot{K}\right]^{k+1} \subseteq \mathbb{Q}^k_* \text{ ”},$$

which is a contradiction. $\square$

Therefore, we obtain the following theorem

Theorem 2.4. *For each integer $k \geq 2$, it is consistent that* MA(rec) *holds, and* $\mathsf{MA}_{\aleph_1}$*(σ-k-linked) fails.*

3. Concluding remark

Connections between several fragments of Martin's axiom have been studied. For example, Bagaria proved that it is consistent that MA(σ-centered) holds and MA(σ-linked) fails [1, 3.6], and that it is consistent that MA(productive ccc) holds, every Aronszajn tree is special and MA fails. Chodounský-Zapletal introduced the property of forcing notions, called Y-cc, which is a stronger property than the countable chain condition, and showed that it is consistent that MA(Y-cc) holds and $\mathsf{cov}(\mathcal{N}) = \aleph_1$ [3]. Note that MA(Y-cc) implies MA(σ-centered) and the assertion that every Aronszajn tree is special*1. Hence, this Chodounský-Zapletal's result also implies the consistency of the assertion (*)*2.

In 1980s, Todorčević investigated Martin's Axiom from the view point of Ramsey theory, and introduced the following fragments of Martin's axiom: $\mathcal{K}_{<\omega}$ denotes the assertion that every ccc forcing notion has precaliber $\aleph_1$ (that is, every uncountable subset has a centered subset I, which means that any finite subset of I has a common extension); $\mathcal{K}_n$ denotes the assertion that every ccc forcing notion has the property K_n (that is, every uncountable subset of a ccc forcing notion has an uncountable n-linked subset I, which means that any n-many elements of I has a common extension)*3; $\mathcal{K}'_{<\omega}$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^{<\aleph_0}$ has an uncountable K_0-homogeneous set, for each $n \in \omega$; $\mathcal{K}'_n$ denotes the assertion that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable K_0-homogeneous set.*4 The following diagram is a summary of implications of these fragments of $\mathsf{MA}_{\aleph_1}$. The triangle on the left side of the diagram is the Todorčević-Veličković theorem [8].

$$
\begin{array}{ccccccccccccc}
 & & \mathcal{K}_{<\omega} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}_{n+1} & \longrightarrow & \mathcal{K}_n & \longrightarrow \cdots \longrightarrow & \mathcal{K}_3 & \longrightarrow & \mathcal{K}_2 \\
 & \nearrow\!\!\!\swarrow & \updownarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathsf{MA}_{\aleph_1} & & & & & & & & & & & & \\
 & \searrow\!\!\!\nwarrow & & & & & & & & & & & \\
 & & \mathcal{K}'_{<\omega} & \longrightarrow & \cdots & \longrightarrow & \mathcal{K}'_{n+1} & \longrightarrow & \mathcal{K}'_n & \longrightarrow \cdots \longrightarrow & \mathcal{K}'_3 & \longrightarrow & \mathcal{K}'_2
\end{array}
$$

It is not known whether any other implications in this diagram hold under ZFC. Bagaria-Shelah's lemma [2, LEMMA 6] can be modified the lemma for the failure of $\mathcal{K}'_{k+1}$ for $\dot{\mathbb{Q}}^k_*$ in the extension with $\mathbb{P}^k_* * \dot{\mathbb{Q}}$. So it is proved that, for each integer $k \geq 2$, it is consistent that $\mathsf{MA}(\mathsf{Pr}_{k+1})$ holds and $\mathcal{K}'_{k+1}$ fails.

Larson-Todorčević showed that a Suslin tree forces that there exists a ladder system coloring which cannot be uniformized [5, THEOREM 6.2], and that, for each non-principal ultrafilter U in the ground model, $(2^{\omega_1}, <_{\text{lex}})$ cannot be embedded into ω^ω/U [5, THEOREM 6.3]. It is proved that $\mathcal{K}'_4$ implies that every ladder system coloring can be uniformized [8, §2], and that $\mathcal{K}'_3$ implies that, for every non-principal ultrafilter U in the ground model, $(2^{\omega_1}, <_{\text{lex}})$ can be embedded into ω^ω/U [7, 7.7. THEOREM]. Larson-Todorčević proved that it is consistent that a Suslin tree can force $\mathcal{K}'_2(\text{rec})$ [6]. In [11], the author develops their result to $\mathcal{K}_{<\omega}(\text{rec})$ in some sense. Therefore, it is proved that it is consistent that $\mathcal{K}_{<\omega}(\text{rec})$ holds in some sense and $\mathcal{K}'_3$ fails, by use of forcing with a Suslin tree. **Lemma 2.3** says that $\mathcal{K}'_{k+1}$ for $\dot{\mathbb{Q}}^k_*$ fails in the extension with $\mathbb{P}^k_* * \dot{\mathbb{Q}}$. So consequently, it is proved that it is consistent that $\mathsf{MA}(\text{rec})$ holds, and both $\mathcal{K}'_3$ and $\mathsf{MA}_{\aleph_1}(\sigma\text{-linked})$ fail. This cannot be concluded by use of a forcing extension with a Suslin tree.

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*1Because both a σ-centered forcing and a specialization of an Aronszajn tree by finite approximations satisfy Y-cc.

*2Notice that Random forcing is σ-linked.

*3A forcing notion with the property K_n satisfies the property Pr_n.

*4They are defined by Todorčević in several papers. In [5, Definition 4.9] and [8, §2], $\mathcal{K}_n$'s are defined as assertions for ccc forcing notions, however in [6, §4] and [7, §7], $\mathcal{K}_n$'s are defined as assertions for ccc partitions. To separate them, we use the notations as above. These notations are same to ones in [10].

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