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A local analysis of the radial configuration for the two-phase torsion problem in the ball

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Abstract

This is a resume of [4]. We consider the configuration given by two concentric balls, made of different materials. We get precise information on the local optimality of this radially symmetric configuration for the two-phase torsional rigidity functional under the effect of perturbations that act exclusively on the inner ball while satisfying the volume (or surface area) preserving constraint. We perform shape derivatives up to the second order and make use of spherical harmonics to aid our calculations. Depending on the difference of the two conductivities, a symmetry breaking phenomenon occurs.

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Contents

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be the unit open ball centred at the origin. Moreover, let $\omega \subset\subset \Omega$ be a sufficiently regular open set. Fix two positive constants σ_-, σ_+ and consider the following *distribution of conductivities*:

$$\sigma := \sigma_\omega := \begin{cases} \sigma_- & \text{in } \omega, \\ \sigma_+ & \text{in } \Omega \setminus \omega. \end{cases}$$

We consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}(\sigma_\omega \nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

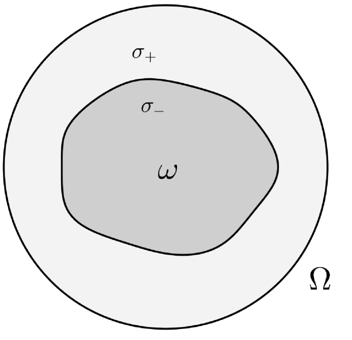

Figure 1: Our problem setting.

By solution of problem (1.1) we mean a function $u \in H_0^1$ that satisfies the following weak formulation:

$$\int_\Omega \sigma_\omega \nabla u \cdot \nabla \varphi = \int_\Omega \varphi \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{1.2}$$

Moreover, since σ_ω is piecewise constant, the following alternative formulation of (1.1) is also known:

$$\begin{cases} -\sigma_\omega \Delta u = 1 & \text{in } \omega \cup \left(\Omega \setminus \overline{\omega}\right), \\ \sigma_- \partial_n u_- = \sigma_+ \partial_n u_+ & \text{on } \partial\omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here, by n we mean the outward unit normal to $\partial\omega$ or $\partial\Omega$ and $\partial_n := \frac{\partial}{\partial n}$ denotes the usual normal derivative. Throughout the paper we will use the $+$ and $-$ subscripts to denote quantities in the two different phases. The second equality of (1.3), known in the literature as *transmission condition*, has to be intended in the sense of traces. In the sequel, the notation $[f] := f_+ - f_-$ will be used to denote the jump of a function f through the interface $\partial\omega$ (for example, the transmission condition can be written as "$[\sigma_\omega \partial_n u] = 0$ on $\partial\omega$").

We aim to study the following *torsional rigidity functional*:

$$E(\omega) := \int_\Omega \sigma_\omega |\nabla u_\omega|^2 = \sigma_- \int_\omega |\nabla u_\omega|^2 + \sigma_+ \int_{\Omega \setminus \overline{\omega}} |\nabla u_\omega|^2, \tag{1.4}$$

where u_ω is the unique solution of (1.1).

Physically speaking, the value $E(\omega)$ represents the torsional rigidity of an infinitely long composite beam whose cross section is depicted in Figure 1. The values σ_-, σ_+, then, represent the hardness of the material of each phase.

The study of similar energy functionals is not new. The one-phase version of this problem was first studied by Pólya in [17] by means of symmetric rearrangement inequalities.

Pólya's result tells us that homogeneous beams with a spherical section are the "most resistant" (precisely speaking, the ball maximises the one-phase torsional rigidity functional among all Lipschitz domains of a fixed volume). Unfortunately, the technique employed by Pólya cannot be applied directly to a two-phase setting because of the discontinuity of the coefficients. Inspired by the result of Pólya, we perform a local analysis of the configuration given by ω and Ω being concentric balls. In [4], we study what happens to the torsional rigidity after applying a small perturbation to the inner ball. We make use of the shape derivative machinery that has been used by Conca and Mahadevan in [2], and Dambrine and Kateb in [6] for the minimisation of the first Dirichlet eigenvalue in a similar two-phase setting (Ω being a ball).

In [5] we deal with more general perturbations: namely we allow pertubations that act on both $\partial\omega$ and $\partial\Omega$ simultaneously. This might give rise to some *resonance effect* that does not appear when $\partial\omega$ or $\partial\Omega$ are perturbed in isolation.

A direct calculation shows that the function u, solution to (1.3) where $\omega = B_R$, has the following expression:

$$u(x) = \begin{cases} \frac{1-R^2}{2N\sigma_+} + \frac{R^2-|x|^2}{2N\sigma_-} & \text{for } |x| \in [0,R], \\ \frac{1-|x|^2}{2N\sigma_+} & \text{for } |x| \in [R,1]. \end{cases} \tag{1.5}$$

In this paper we will use the following notation for Jacobian and Hessian matrix respectively.

$$(Dv)_{ij} := \frac{\partial v_i}{\partial x_j}, \quad (D^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

for all smooth real valued function f and vector field $v = (v_1, \ldots, v_N)$ defined on Ω. We will introduce some differential operators from tangential calculus that will be used in the sequel. For smooth f and v defined on $\partial\omega$ we set

$$\begin{aligned} \nabla_\tau f &:= \nabla \widetilde{f} - (\nabla \widetilde{f} \cdot n) n \qquad &\text{(tangential gradient)}, \\ \operatorname{div}_\tau v &:= \operatorname{div} \widetilde{v} - n \cdot (D\widetilde{v} n) \qquad &\text{(tangential divergence)}, \end{aligned} \tag{1.6}$$

where $\widetilde{f}$ and $\widetilde{v}$ are some smooth extensions on the whole Ω of f and v respectively. It is known that the differential operators defined in (1.6) do not depend on the choice of the extensions. Moreover we let $D_\tau v$ denote the matrix whose i-th row is given by $\nabla_\tau v_i$. We define the (additive) mean curvature of $\partial\omega$ as $H := \operatorname{div}_\tau n$ (cf. [8,12]). According to this definition, the mean curvature H of ∂B_R is given by $(N-1)/R$.

A first key result of this paper is the following.

Theorem 1.1. *For all suitable perturbations that fix the volume, the first order shape derivative of E at B_R vanishes.*

Actually, Theorem 1.1 holds true under the weaker assumption that our perturbation satisfies the first order volume preserving condition (2.11). An improvement of Theorem 1.1 is given by the following precise result (obtained by studying second order shape derivatives).

Theorem 1.2. *Let $\sigma_-, \sigma_+ > 0$ and $R \in (0,1)$. If $\sigma_- > \sigma_+$ then B_R is a local maximiser for the functional E under the fixed volume constraint.*

On the other hand, if $\sigma_- < \sigma_+$ then B_R is a saddle shape for the functional E under the fixed volume constraint.

This work is organised as follows: in section 2 the concept of shape derivative is introduced and results concerning the first order shape derivative of the functional E are presented. In section 3 we deal with the second order shape derivative of the functional E and the study of its sign by means of a spherical harmonic expansion. In section 4 we examine the differences that arise when we replace the volume constraint with a surface area one.

2 Computation of the first order shape derivative: Proof of Theorem 1.1

We consider the following class of perturbations that act on B_R without altering $\partial\Omega$:

$$\mathcal{A} := \left\{ \Phi \in \mathcal{C}^\infty\big([0,1) \times \mathbb{R}^N, \mathbb{R}^N\big) \;\middle|\; \begin{array}{l} \Phi(0,\cdot) = \mathrm{Id},\ \exists R_0 \in (R,1) \text{ such that} \\ \Phi(t,x) = x \text{ for } t \in [0,1),\, |x| \geq R_0 \end{array} \right\}. \tag{2.7}$$

For $\Phi \in \mathcal{A}$ we will write $\Phi(t)$ to denote $\Phi(t,\cdot)$ and, for all domain D in $\mathbb{R}^N$, $\Phi(t)(D)$ will denote the set of all $\Phi(t,x)$ for $x \in D$. In the sequel the following notation for the first order approximation (in the "time" variable) of Φ will be used:

$$\Phi(t) = \mathrm{Id} + th + o(t) \quad \text{as } t \to 0, \qquad \text{for some smooth } h : \mathbb{R}^N \to \mathbb{R}^N. \tag{2.8}$$

In particular it will be useful to separate the normal and tangential component of h: we write $h_n := h \cdot n$ and $h_\tau := h - h_n n$ on ∂B_R. We define the shape derivative of a shape functional J with respect to a deformation field Φ in $\mathcal{A}$ as follows:

$$\frac{d}{dt} J\big(\Phi(t)(D)\big)\bigg|_{t=0} = \lim_{t \to 0} \frac{J\big(\Phi(t)(D)\big) - J(D)}{t}.$$

This subject is very deep. Many different formulations of shape derivatives associated to various kinds of deformation fields have been proposed over the years. We refer to [8] for a detailed analysis on the equivalence between the various methods. For the study of second (or even higher) order shape derivatives and their computation we refer to [8, 13, 16, 18].

The structure theorem for first and second order shape derivatives (cf. [12, Theorem 5.9.2, page 220] and the subsequent corollaries) yields the following expansion. For every shape functional J, domain D and pertubation field Φ in $\mathcal{A}$, under suitable smoothness assumptions the following holds:

$$J\big(\Phi(t)(D)\big) = J(D) + t\, l_1^J(D)(h_n) + \frac{t^2}{2}\Big(l_2^J(D)(h_n, h_n) + l_1^J(D)(Z)\Big) + o(t^2) \text{ as } t \to 0, \tag{2.9}$$

for some linear $l_1^J(D) : \mathcal{C}^\infty(\partial D) \to \mathbb{R}$ and bilinear form $l_2^J(D) : \mathcal{C}^\infty(\partial D) \times \mathcal{C}^\infty(\partial D) \to \mathbb{R}$ to be determined eventually. Moreover for the ease of notation we have set

$$Z := \big(V' + Dh\, h\big) \cdot n + ((D_\tau n) h_\tau) \cdot h_\tau - 2\nabla_\tau h_n \cdot h_\tau,$$

where $V(t,\Phi(t)) := \partial_t\Phi(t)$ and $V' := \partial_t V(t,\cdot)$. Perturbations of the form $\Phi = \mathrm{Id} + t\,h_n n$ on the boundary of D are usually called *Hadamard perturbation.* As (2.9) shows, using only Hadamard perturbations is enough to compute the first order shape derivative of a functional (and also the bilinear part of its second order shape derivative l_2^J for that matter). On the other hand, second order derivatives contain an extra term $l_1^J(D)(Z)$ that depends on higher terms of the expansion of Φ . It is woth noticing that, (see [12, Corollary 5.9.4, page 221]) Z vanishes in the special case when Φ is a Hadamard perturbation (this is a key observation, crucial to the computation of the bilinear form l_2^J in [4, Theorem 3.1]).

We introduce the class of perturbations in $\mathcal{A}$ that fix the volume of B_R:

$$\mathcal{B} := \left\{\Phi \in \mathcal{A} \mid \mathrm{Vol}\big(\Phi(t)(B_R)\big) = \mathrm{Vol}(B_R) \text{ for all } t \in [0,1)\right\}.$$

The following expansion is also well known. For all $\Phi \in \mathcal{A}$ we have

$$\mathrm{Vol}\big(\Phi_t(B_R)\big) = \mathrm{Vol}(B_R) + t\int_{\partial B_R} h_n + \frac{t^2}{2}\left(\int_{\partial B_R} Hh_n^2 + \int_{\partial B_R} Z\right) + o(t^2) \text{ as } t \to 0. \quad (2.10)$$

This yields the following two volume preserving conditions:

$$\int_{\partial B_R} h_n = 0, \qquad (1^{\text{st}} \text{ order volume preserving}) \qquad (2.11)$$

$$\int_{\partial B_R} Hh_n^2 + \int_{\partial B_R} Z = 0. \qquad (2^{\text{nd}} \text{ order volume preserving}) \qquad (2.12)$$

Notice that condition (2.12) implies that Hadamard perturbations cannot be volume preserving (that is one of the reasons why whe had to include more general perturbations in the definition of $\mathcal{A}$, (2.7)).

Usually, shape functionals can also depend on the domain indirectly, by means of some functions defined on it, those are called *state functions* in literature. In our case, the function u_ω, solution of problem (1.1) is the only state function for the functional E defined in (1.4). We will now introduce the concepts of "shape" and "material" derivative of a state function defined on Ω. Fix an admissible perturbation field $\Phi \in \mathcal{A}$ and let $u = u(t,x)$ be defined on $[0,1) \times \Omega$. Computing the partial derivative with respect to t at a fixed point $x \in \Omega$ is usually called *shape derivative* of u; we will write:

$$u'(t_0,x) := \frac{\partial u}{\partial t}(t_0,x), \quad \text{for } x \in \Omega, t_0 \in [0,1).$$

On the other hand differentiating along the trajectories gives rise to the *material derivative*:

$$\dot{u}(t_0,x) := \frac{\partial v}{\partial t}(t_0,x), \; x \in \Omega, t_0 \in [0,1);$$

where $v(t,x) := u(t,\Phi(t,x))$. From now on for the sake of brevity we will omit the dependency on the "time" variable and write $u(x)$, $u'(x)$ and $\dot{u}(x)$ for $u(0,x)$, $u'(0,x)$ and $\dot{u}(0,x)$. The following relationship between shape and material derivatives hold true:

$$u' = \dot{u} - \nabla u \cdot h \qquad (2.13)$$

In the case where $u(t,\cdot) := u_{\Phi(t)(B_R)}$ (i.e. it is the solution to problem (1.1) when $\omega = \Phi(t)(B_R)$), by symmetry, we have:

$$u' = \dot{u} - (\partial_n u) h_n \quad \text{on } \partial B_R. \tag{2.14}$$

In accordance with the classical theory (see for example [16]), for $k \in \mathbb{N}$, the k-th order shape derivative of an integral functional depends on the $(k-1)$-st shape derivative of its state functions. Therefore we need only compute u'. We give the following characterisation (see [4, Proposition 2.3]):

Proposition 2.1. *For any given admissible $\Phi \in \mathcal{A}$, the corresponding u' can be characterised as the (unique) solution to the following problem in the class of functions that are smooth in the open set $B_R \cup (\Omega \setminus \overline{B_R})$:*

$$\begin{cases} \Delta u' = 0 & \text{in } B_R \cup (\Omega \setminus \overline{B_R}), \\ [\sigma \partial_n u'] = 0 & \text{on } \partial B_R, \\ [u'] = -\left[\partial_n u\right] h_n & \text{on } \partial B_R, \\ u' = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

As usual when dealing with perturbations that act on the interface of discontinuity of the coefficients, the Hadamard's formulas cannot be applied directly (we refer to [12, formula (5.17) page 176 and formulas (5.110)-(5.111) page 227] for a proof of those useful formulas in the smooth one-phase setting). Instead, we have to split the integral as done in (1.4) and apply the Hadamard's formula to each integral (this is a standard procedure, followed for example by [2,6], among many others). This gives rise to surface integrals on the interface as shown in the following theorem. We refer to [4, Theorem 2.4] for a proof.

Theorem 2.2. *For all $\Phi \in \mathcal{A}$ we have*

$$l_1^E(B_R)(h_n) = -\int_{\partial B_R} \left[\sigma |\nabla u|^2\right] h_n.$$

In particular, by symmetry, for all Φ satisfying the first order volume preserving condition (2.11) *(and thus for all $\Phi \in \mathcal{B}$) we get $l_1^E(B_R)(h_n) = 0$.*

3 Computation of the second order shape derivative: Proof of Theorem 1.2

The result of the previous chapter tells us that the configuration corresponding to B_R is a critical shape for the functional E under the fixed volume constraint. In order to obtain more precise information, we will need an explicit formula for the second order shape derivative of E. The first step consists of the computation of the bilinear form $l_2^E(B_R)(h_n, h_n)$ (we refer to [4, Theorem 3.1] for the proof).

Theorem 3.1. *For all $\Phi \in \mathcal{A}$ we have*

$$l_2^E(B_R)(h_n h_n) = -2\int_{\partial B_R} \sigma_- \partial_n u_- \left[\partial_n u'\right] h_n - 2\int_{\partial B_R} \sigma_- \partial_n u_- \left[\partial_{nn}^2 u\right] h_n^2 - \int_{\partial B_R} \sigma_- \partial_n u_- \left[\partial_n u\right] H h_n^2.$$

By Theorem(2.2), for all $\Phi \in \mathcal{B}$ and $t > 0$ small, the expansion (2.9) corresponding to the functional E reads

$$E\left(\Phi(t)(B_R)\right) = E(B_R) + \frac{t^2}{2}\left(l_2^E(B_R)(h_n h_n) - \int_{\partial B_R} \left[\sigma |\nabla u|^2\right] Z\right) + o(t^2) \text{ as } t \to 0. \quad (3.16)$$

Employing the use of the second order volume preserving condition (2.12) and the fact that, by symmetry, the quantity $\left[\sigma|\nabla u|^2\right]$ is constant on the interface ∂B_R we have

$$-\int_{\partial B_R} \left[\sigma|\nabla u|^2\right] Z = \int_{\partial B_R} \left[\sigma|\nabla u|^2\right] H h_n^2.$$

Combining this with the result of Theorem 3.1 yields

$$E\left(\Phi(t)(B_R)\right) = E(B_R) + t^2\left\{-\int_{\partial B_R} \sigma_- \partial_n u_- \left[\partial_n u'\right] h_n - \int_{\partial B_R} \sigma_- \partial_n u_- \left[\partial_{nn}^2 u\right] h_n^2\right\} + o(t^2).$$

We will denote the expression between braces in the above by $Q(h_n)$. Since u' depends linearly on h_n (see (2.15)), it follows immediately that $Q(h_n)$ is a quadratic form in h_n. Some elementary calculation involving (1.5) and (2.15) yields

$$Q(h_n) = \frac{R}{N}\left(\frac{1}{\sigma_-} - \frac{1}{\sigma_+}\right)\left(-\int_{\partial B_R} \sigma_- \partial_n u'_- h_n + \frac{1}{N}\int_{\partial B_R} h_n^2\right) \quad (3.17)$$

In the following we will try to find an explicit expression for u'. To this end we will perform the spherical harmonic expansion of the function $h_n : \partial B_R \to \mathbb{R}$. We set

$$h_n(R\theta) = \sum_{k=1}^{\infty}\sum_{i=1}^{d_k} \alpha_{k,i} Y_{k,i}(\theta) \quad \text{for all } \theta \in \partial B_1. \quad (3.18)$$

The functions $Y_{k,i}$ are called *spherical harmonics* in the literature. They form a complete orthonormal system of $L^2(\partial B_1)$ and are defined as the solutions of the following eigenvalue problem:

$$-\Delta_\tau Y_{k,i} = \lambda_k Y_{k,i} \quad \text{on } \partial B_1,$$

where $\Delta_\tau := \mathrm{div}_\tau \nabla_\tau$ is the Laplace-Beltrami operator on the unit sphere. We impose the following normalisation condition

$$\int_{\partial B_1} Y_{k,i}^2 = R^{1-N}. \quad (3.19)$$

The following expressions for the eigenvalues λ_k and the corresponding multiplicities d_k are also known (for some reason, the expression for d_k appearing in [4, (4.26)] is wrong):

$$\lambda_k = k(k+N-2), \qquad d_k = \frac{(2k+N-2)(k+N-3)!}{k!(N-2)!}. \quad (3.20)$$

Notice that the value $k = 0$ has to be excluded from the summation in (3.18) because we require h_n to verify the first order volume preserving condition (2.11) (also look at Figure 2 to get the gist of how perturbations related to different values of k work).

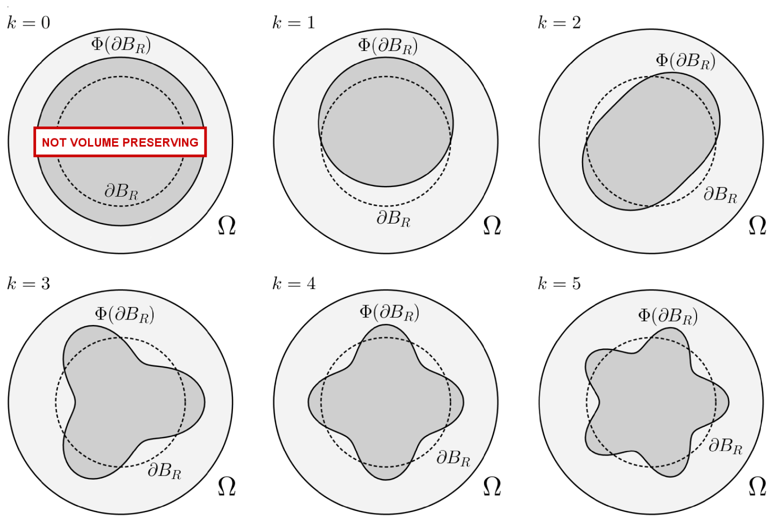

Figure 2: How $\Phi(t)(B_R)$ looks like for small t when $h_n(R\cdot) = Y_{k,i}$, in 2 dimensions.

Let us pick an arbitrary $k \in \{1, 2, \dots\}$ and $i \in \{1, \dots, d_k\}$. We will use the method of separation of variables to find the solution of problem (2.15) in the particular case when $h_n(R\theta) = Y_{k,i}(\theta)$, for all $\theta \in \partial B_1$ and then the general case will be recovered by linearity.

We will be searching for solutions to (2.15) of the form $u' = u'(r, \theta) = f(r)g(\theta)$ (where $r := |x|$ and $\theta := x/|x|$ for $x \neq 0$). Using the well known decomposition formula for the Laplacian into its radial and angular components, the equation $\Delta u' = 0$ in $B_R \cup (\Omega \setminus \overline{B_R})$ can be rewritten as

$$0 = \Delta u'(x) = f_{rr}(r)g(\theta) + \frac{N-1}{r} f_r(r)g(\theta) + \frac{1}{r^2} f(r)\Delta_\tau g(\theta) \text{ for } r \in (0, R) \cup (R, 1),\ \theta \in \partial B_1.$$

Take $g = Y_{k,i}$. Under this assumption, we get the following equation for f:

$$f_{rr} + \frac{N-1}{r} f_r - \frac{\lambda_k}{r^2} f = 0 \quad \text{in } (0, R) \cup (R, 1). \tag{3.21}$$

It can be easily checked that, on each interval $(0, R)$ and $(R, 1)$, any solution to the above consists of a linear combination of the following two independent solutions:

$$f_{sing}(r) := r^{2-N-k} \qquad \text{and} \qquad f_{reg}(r) := r^k. \tag{3.22}$$

Since equation (3.21) is defined for $r \in (0, R) \cup (R, 1)$, we have that the following holds for some real constants A_k, B_k, C_k and D_k;

$$f(r) = \begin{cases} A_k r^{2-N-k} + B_k r^k & \text{for } r \in (0, R), \\ C_k r^{2-N-k} + D_k r^k & \text{for } r \in (R, 1). \end{cases}$$

Moreover, since $2 - N - k$ is negative, A_k must vanish, otherwise a singularity would occur at $r = 0$. The other three constants can be obtained by the interface and boundary conditions of problem (2.15) bearing in mind that $u'(r,\theta) = f(r)Y_{k,i}(\theta) = f(r)h_n(R\theta)$. We get the following system:

$$\begin{cases} C_k R^{2-N-k} + D_k R^k - B_k R^k = -\frac{R}{N\sigma_-} + \frac{R}{N\sigma_+}, \\ \sigma_- k B_k R^{k-1} = \sigma_+(2-N-k)C_k R^{2-N-k} + \sigma_+ k D_k R^{k-1}, \\ C_k + D_k = 0. \end{cases}$$

We solve the system above for B_k:

$$B_k = \frac{R^{1-k}}{N\sigma_-} \cdot \frac{k(\sigma_- - \sigma_+) - (2-N-k)(\sigma_- - \sigma_+)R^{2-N-2k}}{k(\sigma_- - \sigma_+) + ((2-N-k)\sigma_+ - k\sigma_-)R^{2-N-2k}}. \tag{3.23}$$

Therefore, in the particular case when $h_n(R\,\cdot) = Y_{k,i}$ we obtain

$$u'_- = u'_-(r,\theta) = B_k r^k Y_{k,i}(\theta), \quad r \in [0,R),\ \theta \in \partial B_1.$$

By linearity, we recover the expansion of u'_- in the general case (i.e. when (3.18) holds):

$$\begin{aligned} u'_-(r,\theta) &= \sum_{k=1}^{\infty}\sum_{i=1}^{d_k} \alpha_k B_k r^k Y_{k,i}(\theta), \quad r \in [0,R),\ \theta \in \partial B_1, \quad \text{and therefore} \\ \partial_n u'_-(R,\theta) &= \sum_{k=1}^{\infty}\sum_{i=1}^{d_k} \alpha_k B_k k R^{k-1} Y_{k,i}(\theta), \qquad \theta \in \partial B_1. \end{aligned} \tag{3.24}$$

We can now diagonalise the quadratic form Q in (3.17). In other words we can consider only the case $h_n(R\,\cdot) = Y_{k,i}$ for all possible pairs (k,i). Actually, the dependence on the parameter i can be removed: without loss of generality we can consider Q as a function of k as follows:

$$\begin{aligned} Q(h_n) = Q(k,i) = Q(k) &= \frac{R}{N}\left(\frac{\sigma_+ - \sigma_-}{\sigma_+\sigma_-}\right)\left(-\sigma_- B_k k R^{k-1} + \frac{1}{N}\right) = \\ \frac{R}{N^2}\left(\frac{\sigma_+ - \sigma_-}{\sigma_+\sigma_-}\right)&\left\{1 - k\frac{k(\sigma_- - \sigma_+) + (N-2+k)(\sigma_- - \sigma_+)R^{2-N-2k}}{\sigma_- k(1 - R^{2-N-2k}) + \sigma_+(k + (2-N-k)R^{2-N-2k})}\right\}. \end{aligned} \tag{3.25}$$

Notice that the denominator in the expression in braces above is always negative for $N \geq 2$, $k \geq 1$ and $R \in (0,1)$: as a consequence, $Q(k)$ is well defined.

Remark 3.2. *The fact that, as shown in* (3.25), $Q(k,i) = Q(k)$ *holds true for all* $k \geq 1$ *and* $i \in \{1,\ldots,d_k\}$ *(i.e. the "orientation" of the perturbation is not relevant) is a consequence of the radial symmetry of the configuration that we are studying and of the rotational invariance of the functional* E*. On the other hand, when both* ∂B_R *and* $\partial\Omega$ *are perturbed simultaneously, as done in* [5]*, the rotational symmetry is lost and the parameter* i *must also be taken into account.*

We are now ready to prove the main result of the paper.

Theorem 3.3. *Let $\sigma_-, \sigma_+ > 0$ and $R \in (0,1)$. If $\sigma_- > \sigma_+$ then*

$$\frac{d^2}{dt^2} E\big(\Phi(t)(B_R)\big)\Big|_{t=0} < 0 \quad \textit{for all } \Phi \in \mathcal{B}.$$

Hence, B_R is a local maximiser for the functional E under the fixed volume constraint. On the other hand, if $\sigma_- < \sigma_+$, then there exist some Φ_1 and Φ_2 in $\mathcal{B}$, such that

$$\frac{d^2}{dt^2} E\big(\Phi_1(t)(B_R)\big)\Big|_{t=0} < 0, \quad \frac{d^2}{dt^2} E\big(\Phi_2(t)(B_R)\big)\Big|_{t=0} > 0.$$

In other words, B_R is a saddle shape for the functional E under the fixed volume constraint.

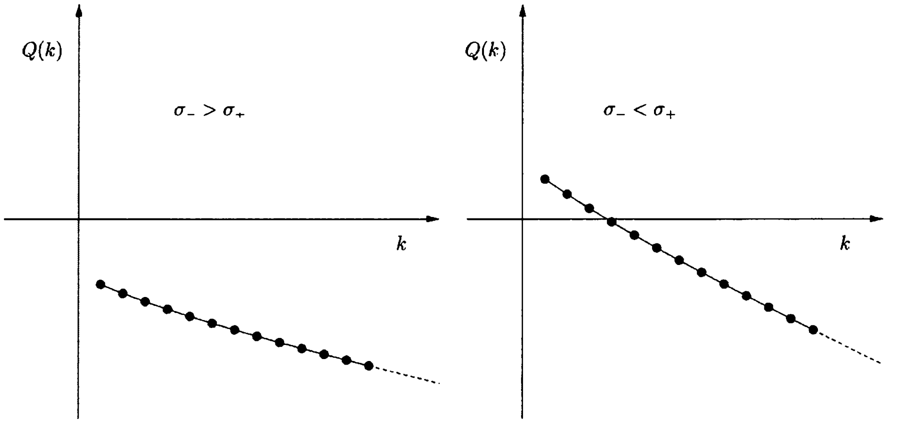

Figure 3: The graph of $Q(k)$ when $\sigma_- > \sigma_+$ (left) and when $\sigma_- < \sigma_+$ (right).

Proof. We will rewrite (3.25) more compactly as follows:

$$Q(k) = \frac{R}{N^2}\left(\frac{\sigma_+ - \sigma_-}{\sigma_+\sigma_-}\right)\left(1 - k\frac{\mathcal{N}}{\mathcal{D}}\right).$$

First, suppose that $\sigma_- > \sigma_+$. As remarked after (3.25), the denominator $\mathcal{D}$ is negative; moreover, since by hypothesis $\sigma_- > \sigma_+$, we also have $\mathcal{N} > 0$. This implies that $Q(k)$ is negative for all $k \geq 1$, and hence

$$\frac{d^2}{dt^2} E\big(\Phi(t)(B_R)\big)\Big|_{t=0} < 0 \quad \text{for all } \Phi \in \mathcal{B},$$

as claimed.

Now suppose that $\sigma_- < \sigma_+$. We have

$$Q(1) = \frac{R}{N}\underbrace{\left(\frac{\sigma_+ - \sigma_-}{\sigma_+}\right)}_{>0}\underbrace{\frac{-R^{-N}}{\sigma_-(1 - R^{-N}) + \sigma_+(-1 + (1-N)R^{-N})}}_{>0} > 0. \tag{3.26}$$

On the other hand, an elementary calculation shows that (actually for all $\sigma_-, \sigma_+ > 0$)

$$\lim_{k\to\infty} Q(k) = -\infty. \tag{3.27}$$

Combining (3.26) and (3.27) we get that, when $\sigma_- < \sigma_+$, B_R is a saddle shape for the functional E under the fixed volume constraint. □

As Figure 3 suggests, the function $k \mapsto Q(k)$ is actually strictly decreasing. This is proven in [4, Lemma 4.1] by treating k as a real variable and studying the sign of the derivative $\frac{d}{dk}Q(k)$.

4 Some comments on the surface area preserving case

The method employed in this paper can be applied to other geometrical constraints as well. In particular we studied what happens when volume preserving perturbations are replaced by surface area preserving ones.

We need to replace (2.11)-(2.12) by the following well known first and second order surface area preserving conditions (see for instance [12, page 225]):

$$\int_{\partial B_R} H h_n = 0, \quad \int_{\partial B_R} |\nabla_\tau h_n|^2 + \int_{\partial B_R} \Big(H^2 - \operatorname{tr}\big((D_\tau n)^T D_\tau n\big)\Big) h_n^2 + \int_{\partial B_R} HZ = 0. \tag{4.28}$$

Notice that, as H is constant on B_R, the first equality in (4.28) is equivalent to (2.11) and hence, the result of Theorem 2.2 holds true for surface area preserving perturbations as well.

The study of the second order shape derivative of E under this constraint is done as before by replacing (2.12) by the second equation in (4.28). We are able to write the shape Hessian of E as a quadratic form in h_n. It can be then diagonalised by considering $h_n(R\cdot) = Y_{k,i}$ for all possible pairs (k,i), under the normalisation (3.19). Under this assumption, by (3.20) we get

$$\int_{\partial B_R} |\nabla_\tau h_n|^2 = \frac{\lambda_k}{R^2} = \frac{k(k+N-2)}{R^2}.$$

We obtain

$$E\big(\Phi(t)(B_R)\big) = E(B_R) + t^2\,\widetilde{Q}(k) + o(t^2) \text{ as } t \to 0,$$

where $\widetilde{Q}(k)$ is given by the following:

$$\frac{R}{N^2}\left(\frac{\sigma_+ - \sigma_-}{\sigma_+\sigma_-}\right)\left(\frac{3}{2} - \frac{k(k+N-2)}{2(N-1)} - k\,\frac{k(\sigma_- - \sigma_+) - (2-N-k)(\sigma_- - \sigma_+)R^{2-N-2k}}{k(\sigma_- - \sigma_+) + ((2-N-k)\sigma_+ - k\sigma_-)R^{2-N-2k}}\right).$$

It is immediate to check that $\widetilde{Q}(1) = Q(1)$ and therefore, $\widetilde{Q}(1)$ is negative for $\sigma_- > \sigma_+$ and positive otherwise. On the other hand, $\lim_{k\to\infty}\widetilde{Q}(k) = \infty$ for $\sigma_- > \sigma_+$ and $\lim_{k\to\infty}\widetilde{Q}(k) = -\infty$ for $\sigma_- < \sigma_+$. In other words, under the surface area preserving constraint, B_R is always a saddle shape, independently of the relation between σ_- and

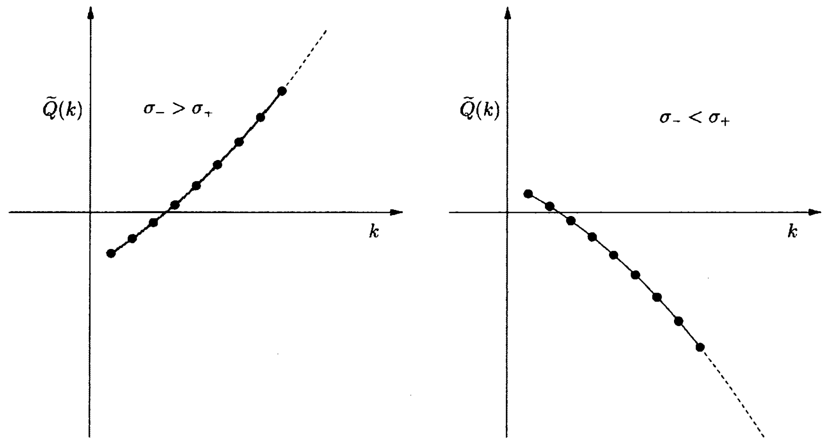

Figure 4: The graph of $\widetilde{Q}(k)$ when $\sigma_- > \sigma_+$ (left) and when $\sigma_- < \sigma_+$ (right).

σ_+. This result has the following intuitive geometric interpretation. Since the case $k = 1$ corresponds to deformations that coincide with translations at first order, it is natural to expect a similar behaviour under both volume and surface area preserving constraint. On the other hand, high frequency perturbations (i.e. those corresponding to a very large eigenvalue) lead to the formation of indentations in the surface of B_R as shown in Figure 2. Hence, in order to prevent the surface area of B_R from expanding, its volume must inevitably shrink. This behaviour is shown in Figure 5. Together with the number of "indentations", this shrinking effect must become stronger the larger k is, this suggests that the behaviour of $E\big(\Phi(t)(B_R)\big)$ for large k might be approximated by that of the extreme case $\omega = \emptyset$.

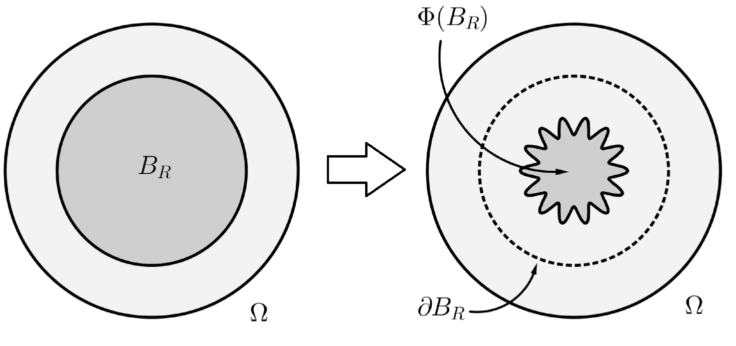

Figure 5: Under the surface area preserving constraint, $\Phi(B_R)$ progressively "shrinks" as k gets larger and larger (here $k = 14$).

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