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Exact blow-up profile for a heat equation with a nonlinear boundary condition

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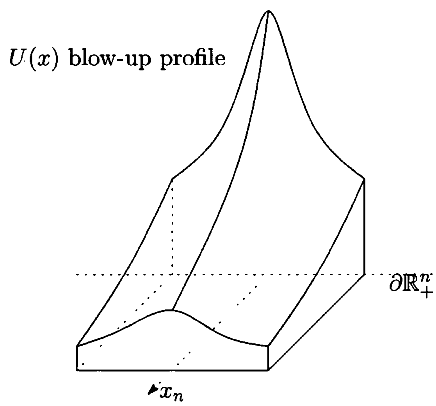
We study the asymptotic behavior of positive blow-up solutions for the heat equation.

$$\begin{cases} u_t = \Delta u, & x \in \mathbb{R}^n_+,\ t \in (0,T), \\ \partial_\nu u = u^q, & x \in \partial\mathbb{R}^n_+,\ t \in (0,T), \\ u = u_0, & x \in \mathbb{R}^n_+,\ t = 0, \end{cases} \tag{P}$$

where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}; x_n > 0\}$, $\partial_\nu = -\partial/\partial x_n$ and $1 < q < n/(n-2)$. A function $u(x,t)$ is said to be x_n-axial symmetric, if $u(x,t)$ depends only on $|x'|$ and x_n for any $t \in [0,T)$. We focus on x_n-axial symmetric functions throughout this report. Let $u(x,t)$ be a blow-up solution of (P) and T be its blow-up time. If a limit

$$U(x) = \lim_{t \to T} u(x,t) \in [0, \infty]$$

exists for any $x \in \mathbb{R}^n_+ \cup \partial\mathbb{R}^n_+$, we call $U(x)$ a blow-up profile of $u(x,t)$.



A goal of this report is to determine the singularity of blow-up profiles. The blow-up profile for (P) was first studied in [4, 5] for a one dimensional case. Indeed they constructed blow-up solutions satisfying

$$U(x) = Ax^{\frac{-1}{q-1}} \quad \text{for } x \in \mathbb{R}_+. \tag{1}$$

The author [6] extended their results to a multidimensional case and proved that if $u(x,t)$ is a positive x_n-axial symmetric blow-up solution, its blow-up profile is given by

$$U(x) = A(1 + o(1))(\cos\theta)^{\frac{-1}{q-1}} |x|^{\frac{-1}{q-1}} \quad \text{along } x_n = |x| \cos\theta \tag{2}$$

for any fixed $\theta \in [0, \pi/2)$. This profile function coincides with (1) if $\theta = 0$. Unfortunately (2) does not hold on the boundary, since $\theta = \pi/2$ on the boundary. On the other hand, the boundary singularity was studied in [7]. He obtained the following inequalities under some monotonicity conditions.

$$\mathsf{k}_1 \left(|\log|x|| / |x|^2 \right)^{\frac{1}{2(q-1)}} \le U(x) \le \mathsf{k}_2 \left(|\log|x|| / |x|^2 \right)^{\frac{1}{2(q-1)}}, \quad x \in \partial\mathbb{R}^n_+,\ |x| < 1.$$

In this report, we improve this bound and consider more general situations.

Let $u(x,t)$ be a positive x_n-axial symmetric solution which blows up at the origin. We introduce self-similar variables.

$$w(y,s)=(T-t)^{\frac{1}{2(q-1)}}u(\sqrt{T-t}\cdot y,t),\qquad T-t=e^{-s}.$$

The rescaled function $w(y,s)$ satisfies

$$\begin{cases} w_s=\Delta w-\frac{y}{2}\cdot\nabla w-\frac{w}{2(q-1)}, & y\in\mathbb{R}^n_+,\ s>-\log T,\\ \partial_\nu w=w^q, & y\in\partial\mathbb{R}^n_+,\ s>-\log T.\end{cases}$$

It is known that $w(y,s)$ converges to the $\kappa(y)$ as $s\to\infty$, where $\kappa(y)=\kappa(y_n)$ is the unique positive bounded solution of

$$\frac{d^2\kappa}{dy_n^2}-\frac{y_n}{2}\frac{d\kappa}{dy_n}-\frac{\kappa}{2(q-1)}=0 \tag{3}$$

(see [3], [2]). To investigate the asymptotic behavior of solutions, we consider the linearization around $\kappa(y)$.

$$\Delta\phi-\frac{y}{2}\cdot\nabla\phi-\frac{\phi}{2(q-1)}=-\lambda\phi\ \text{ in }\mathbb{R}^n_+,\qquad \partial_\nu\phi=q\kappa^{q-1}\phi\ \text{ on }\partial\mathbb{R}^n_+. \tag{4}$$

Under x_n-axial symmetric case setting, the eigenfunction is written as

$$\phi_{k,l}(y)=h_k(|y'|)I_l(y_n),\qquad k=0,1,2,\cdots,\quad l\in\mathbb{N}.$$

Let $\lambda_k^{\mathbb{R}^{n-1}}$ be the kth eigenvalue corresponding $h_k(|y'|)$ and $\lambda_l^{\mathbb{R}_+}$ be the lth eigenvalue corresponding $I_l(y_n)$. The eigenvalue of (4) is given by $\lambda_{k,l}=\lambda_k^{\mathbb{R}^{n-1}}+\lambda_l^{\mathbb{R}_+}+\frac{1}{2(q-1)}$. It is known that

$$\lambda_{0,1}<0,\qquad \lambda_{1,1}=0,\qquad \lambda_{k,l}>0\quad(k\geq 2\text{ or }l\geq 2).$$

Let $L^2_\rho(\mathbb{R}^n_+)$ be a weighted Lebesgue space defined by $L^2_\rho(\mathbb{R}^n_+)=\{w\in L^1_{\text{loc}}(\mathbb{R}^n_+);\|w\|_{L^2_\rho(\mathbb{R}^n_+)}=(\int_{\mathbb{R}^n_+}w(y)^2\,e^{-|y|^2/4}dy)^{1/2}<\infty\}$. We recall our previous result.

Theorem 1 (Theorem 1.1 [7]). *Let $u(x,t)$ be a positive x_n-axial symmetric solution of* (P) *which blows up at the origin. Then one of the following two cases occurs.*

(c1) $\|w(s)-(\kappa+\mathsf{c}_1 s^{-1}\phi_{1,1})\|_{L^2_\rho(\mathbb{R}^n_+)}=o(s^{-1})$, *where $\mathsf{c}_1<0$ is a constant depending only on n,q or*

(c2) $w(s)-\kappa$ decays exponentially in $L^2_\rho(\mathbb{R}^n_+)$.

We now state our main result in this report.

Theorem 2. *Let $u(x,t)$ be a positive x_n-axial symmetric solution of* (P) *which blows up at the origin.*

(i) *If $w(y,s)$ behaves as* (c1) *in* Theorem 1, *then the blow-up profile $U(x)\in C(\mathbb{R}^n_+\cup\partial\mathbb{R}^n_+\setminus\{0\})$ exists and satisfies*

$$U(x)=\mathsf{k}\left(|\log|x||/|x|^2\right)^{\frac{1}{2(q-1)}}(1+o(1)),\qquad x\in\partial\mathbb{R}^n_+,\ |x|<1,$$

where k is a positive constant depending only on n and q.

(ii) *If $w(y,s)$ behaves as* (c2) *in* Theorem 1, *then*

$$U(x)=\mathsf{k}|x|^{-\frac{k}{(q-1)}}(1+o(1)),\qquad x\in\partial\mathbb{R}^n_+,\ |x|<1$$

for some $k\geq 2$.

Generally a blow-up solution behaves like

$$w(s)=\kappa+\mathsf{c}_1e^{-\lambda_{k,l}s}\phi_{k,l}+o(e^{-\lambda_{k,l}s})\quad\text{in }L^2_\rho(\mathbb{R}^n_+)$$

for some $k = 0, 1, 2, \cdots$ and $l \in \mathbb{N}$. Unfortunately we have no idea about the case $l \geq 2$.

Our proof is a slight modification of arguments in a series of papers [8, 9, 10, 11, 12]. We here focus on the asymptotic behavior of $w(y,s)$ for large $|y|$. We introduce another rescale.

$$W(z,\zeta,s) = w(\mathsf{R}(s)z,\zeta,s) \quad \text{for } z \in \mathbb{R}^{n-1},\ \zeta > 0.$$

The function $\mathsf{R}(s)$ ($\lim_{s\to\infty}\mathsf{R}(s) = \infty$) is determined by the long time behavior of $w(y,s)$. A goal is to determine the asymptotic behavior of $W(z,\zeta,s)$ for $|z| + |\zeta| < R$ as $s \to \infty$. We see that $W(z,\zeta,s)$ satisfies

$$\begin{cases} W_s = \mathsf{R}^{-2}\Delta_z W - (\frac{1}{2} - \frac{\mathsf{R}_s}{\mathsf{R}})z \cdot \nabla_z W + W_{\zeta\zeta} - \frac{\zeta}{2}W_\zeta - mW, & z \in \mathbb{R}^{n-1},\ \zeta > 0,\ s > -\log T, \\ \partial_\nu W = W^q, & z \in \mathbb{R}^{n-1},\ \zeta = 0,\ s > -\log T. \end{cases}$$

To investigate the long time behavior of $W(z,\zeta,s)$, we study a limiting problem of this equation. For the case (i), since $w(y,s) = \kappa(y) + \mathsf{c}_1 s^{-1}\phi_{1,1}(y) + o(s^{-1})$ in $L^2_\rho(\mathbb{R}^n_+)$ (see Theorem 1 (c1)) and $\phi_{1,1}(y) = c(|y'|^2 - 2(n-1))I_1(y_n)$, we take

$$\mathsf{R}(s) = \sqrt{s}.$$

A limiting equation is given by

$$\begin{cases} -\frac{1}{2}z \cdot \nabla_z W + W_{\zeta\zeta} - \frac{\zeta}{2}W_\zeta - mW = 0 & \text{for } z \in \mathbb{R}^{n-1},\ \zeta > 0, \\ \partial_\nu W = W^2 & \text{on } z \in \mathbb{R}^{n-1},\ \zeta = 0. \end{cases} \tag{5}$$

We will see that

$$W(z,\zeta) = (1 + \mathsf{b}|z|^2)^{-m}\kappa(\tfrac{\zeta}{\sqrt{1+\mathsf{b}|z|^2}}) \tag{6}$$

gives a solution of (5) (see below), where $\mathsf{b} > 0$ is a free parameter. We choose b to match (c1) in Theorem 1. Then $W(z,\zeta,s)$ gives a better approximation of $w(y,s)$ than $\kappa + \mathsf{c}_1 s^{-1}\phi_{1,1}$.

From now, we derive (6) in a formal approach. Since $z \cdot \nabla_z |z|^{2j} = 2k|z|^{2j}$, we look for solutions in the form of

$$W(z,\zeta) = \sum_{j=0}^{\infty} a_j(\zeta)|z|^{2j}.$$

Substituting this into (5), we have

$$a_j'' - \tfrac{\zeta}{2}a_j' - (m+j)a_j = 0, \qquad \zeta > 0.$$

Let $A_j(\eta) = a_j(\zeta)$ with $\eta = \frac{\zeta^2}{4}$ for $\eta > 0$. Then A_j satisfies

$$\eta A_j'' + (\tfrac{1}{2} - \eta)A_j' - (m+j)A_j = 0, \qquad \eta > 0.$$

This is the confluent hypergeometric equation (see (8.1.1) p. 201 [1]). One solution of this equation is give by the Kummer function $M(m+j, \frac{1}{2}; \eta)$, another is given by

$$U(m+j, \tfrac{1}{2}; \eta) = \tfrac{1}{\Gamma(m+j)} \int_0^\infty e^{-\eta t} t^{m+j-1}(1+t)^{\frac{1}{2}-(m+j)-1} dt$$

(see (8.2.1) p. 204 [1]). Since $M(m+j, \frac{1}{2}; \eta)$ behaves like $\eta^{m+j-\frac{1}{2}}e^\eta$ as $\eta \to \infty$ (see (8.1.8) p. 203 [1]), we take

$$A_j(\eta) = \alpha_j \pi^{-\frac{1}{2}}\Gamma(m+j+\tfrac{1}{2})U(m+j, \tfrac{1}{2}, \eta),$$

where α_j is a constant. Since $U(a,c;0) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)}$ if $1 - c > 0$ (see (8.2.5) p. 205 [1]), it is clear that

$$A_j(0) = \alpha_j. \tag{7}$$

Form now, we determine α_j. We substitute

$$W(z,\zeta)=\sum_{j=0}^{\infty}A_j(\tfrac{\zeta^2}{4})|z|^{2j}$$

into a boundary condition in (5). Since $U'(a,c;x)=-aU(a+1,c+1,x)$ (see (8.5.8) p. 212 [1]) and $U(a,c;x)=\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}(1+o(1))$ as $x\to 0+$ if $1-c<0$ (see p. 205 [1]), we get

$$\begin{aligned}\tfrac{d}{d\zeta}U(m+j,\tfrac12,\tfrac{\zeta^2}{4})|_{\zeta=0} &= \tfrac{\zeta}{2}U'(m+j,\tfrac12,\tfrac{\zeta^2}{4})|_{\zeta=0}\\ &= -\tfrac{(m+j)\zeta}{2}U(m+j+1,\tfrac32,\tfrac{\zeta^2}{4})|_{\zeta=0}\\ &= -\tfrac{(m+j)\zeta}{2}\tfrac{\Gamma(\frac12)}{\Gamma(m+j+1)}(\tfrac{\zeta^2}{4})^{-\frac12}(1+o(1))|_{\zeta=0}\\ &= -(m+j)\tfrac{\Gamma(\frac12)}{\Gamma(m+j+1)}=-\tfrac{\Gamma(\frac12)}{\Gamma(m+j)}.\end{aligned}$$

Therefore it holds that

$$\tfrac{d}{d\zeta}A_j(\tfrac{\zeta^2}{4})|_{\zeta=0}=-\alpha_j\pi^{-\frac12}\Gamma(m+j+\tfrac12)\tfrac{\Gamma(\frac12)}{\Gamma(m+j)}=-\alpha_j\tfrac{\Gamma(m+j+\frac12)}{\Gamma(m+j)}.$$

From now we consider the case $q=2$. Since $m=\frac{1}{2(q-1)}=\frac12$, it follows that

$$\partial_\nu W=-\partial_\zeta W|_{\zeta=0}=\sum_{j=0}^{\infty}\alpha_j\tfrac{\Gamma(m+j+\frac12)}{\Gamma(m+j)}|z|^{2j}=\sum_{j=0}^{\infty}\alpha_j\tfrac{\Gamma(1+j)}{\Gamma(\frac12+j)}|z|^{2j}. \tag{8}$$

On the other hand, it holds from (7) that

$$\begin{aligned}W|_{\zeta=0}^2 &= (\alpha_0+\alpha_1|z|^2+\alpha_2|z|^4+\alpha_3|z|^6+\cdots)(\alpha_0+\alpha_1|z|^2+\alpha_2|z|^4+\alpha_3|z|^6+\cdots)\\ &= \alpha_0^2+2\alpha_0\alpha_1|z|^2+(2\alpha_0\alpha_2+\alpha_1^2)|z|^4+(2\alpha_0\alpha_3+2\alpha_1\alpha_2)|z|^6\\ &\quad+(2\alpha_0\alpha_4+2\alpha_1\alpha_3+\alpha_2^2)|z|^8+(2\alpha_0\alpha_5+2\alpha_1\alpha_4+2\alpha_2\alpha_3)|z|^{10}+\cdots.\end{aligned} \tag{9}$$

Therefore from (8) and (9), we obtain

$$\alpha_0\tfrac{\Gamma(1)}{\Gamma(\frac12)}=\alpha_0^2,\qquad \alpha_1\tfrac{\Gamma(2)}{\Gamma(\frac32)}=2\alpha_0\alpha_1,\qquad \alpha_2\tfrac{\Gamma(3)}{\Gamma(\frac52)}=2\alpha_0\alpha_2+\alpha_1^2,$$

$$\alpha_3\tfrac{\Gamma(4)}{\Gamma(\frac72)}=2\alpha_0\alpha_3+2\alpha_1\alpha_2,\qquad \alpha_4\tfrac{\Gamma(5)}{\Gamma(\frac92)}=2\alpha_0\alpha_4+2\alpha_1\alpha_3+\alpha_2^2,$$

$$\alpha_5\tfrac{\Gamma(5)}{\Gamma(\frac92)}=2\alpha_0\alpha_5+2\alpha_1\alpha_4+2\alpha_2\alpha_3.$$

It is clear that $\alpha_0=\frac{\Gamma(1)}{\Gamma(\frac12)}=\pi^{-\frac12}$. For simplicity, we put $\alpha_j=\pi^{-\frac12}\beta_j$. Then we get

$$(\tfrac{2!}{\frac32\cdot\frac12}-2)\beta_2=\beta_1^2,\qquad (\tfrac{3!}{\frac52\cdot\frac32\cdot\frac12}-2)\beta_3=2\beta_1\beta_2,\qquad (\tfrac{4!}{\frac72\cdot\frac52\cdot\frac32\cdot\frac12}-2)\beta_4=2\beta_1\beta_3+\beta_2^2,$$

$$(\tfrac{5!}{\frac92\cdot\frac72\cdot\frac52\cdot\frac32\cdot\frac12}-2)\beta_5=2\beta_1\beta_4+2\beta_2\beta_3.$$

Furthermore we put $\beta_j=(2\beta_1)^j\gamma_j$. We easily see that

$$\tfrac83\gamma_2=1,\qquad \tfrac65\gamma_3=\gamma_2,\qquad \tfrac{58}{35}\gamma_4=\gamma_3+\gamma_2^2,\qquad \tfrac{130}{63}\gamma_5=\gamma_4+2\gamma_2\gamma_3.$$

This implies

$$\gamma_2=\tfrac38=\tfrac{\frac12\cdot\frac32}{2!},\qquad \gamma_3=\tfrac{5}{16}=\tfrac{\frac52\cdot\frac32\cdot\frac12}{3!},\qquad \gamma_4=\tfrac{35}{128}=\tfrac{\frac72\cdot\frac52\cdot\frac32\cdot\frac12}{4!},\qquad \gamma_5=\tfrac{63}{256}=\tfrac{\frac92\cdot\frac72\cdot\frac52\cdot\frac32\cdot\frac12}{5!}.$$

From this relation, we assume

$$\gamma_j = \frac{\frac{2j-1}{2}.\frac{2j-3}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}}{j!} = \pi^{-\frac{1}{2}}\frac{\Gamma(j+\frac{1}{2})}{j!}.$$

Since $\alpha_j = \pi^{-1}(2\beta_1)^j\gamma_j$, $W(z,\zeta)$ is written by

$$\begin{aligned} W(z,\zeta) &= \frac{\alpha_0}{\sqrt{\pi}}\Gamma(1)U(\tfrac{1}{2},\tfrac{1}{2};\eta) + \frac{\alpha_1}{\sqrt{\pi}}\Gamma(2)U(\tfrac{3}{2},\tfrac{1}{2};\eta)|z|^2 + \frac{1}{\sqrt{\pi}}\sum_{j=2}^{\infty}\alpha_j\Gamma(1+j) \\ &\quad \times U(\tfrac{1}{2}+j,\tfrac{1}{2};\eta)|z|^{2j} \\ &= \frac{1}{\pi}U(\tfrac{1}{2},\tfrac{1}{2};\eta) + \frac{\beta_1}{\pi}U(\tfrac{3}{2},\tfrac{1}{2};\eta)|z|^2 + \frac{1}{\pi^{\frac{3}{2}}}\sum_{j=2}^{\infty}\Gamma(j+\tfrac{1}{2})U(\tfrac{1}{2}+j,\tfrac{1}{2};\eta)(2\beta_1)^j|z|^{2j} \\ &= \frac{1}{\pi^{\frac{3}{2}}}\sum_{j=0}^{\infty}\Gamma(j+\tfrac{1}{2})U(\tfrac{1}{2}+j,\tfrac{1}{2};\eta)(2\beta_1)^j|z|^{2j}. \end{aligned} \tag{10}$$

We recall the following formula (see p. 219 [1]).

$$U(a,\tfrac{1}{2},\tfrac{\zeta^2}{4}) = \frac{\sqrt{\pi}}{\Gamma(a+\frac{1}{2})\Gamma(a)}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(a+\frac{l}{2})}{l!}\zeta^l. \tag{11}$$

We apply this formula in (10).

$$\begin{aligned} W(z,\zeta) &= \frac{1}{\pi^{\frac{3}{2}}}\sum_{j=0}^{\infty}\Gamma(j+\tfrac{1}{2})U(\tfrac{1}{2}+j,\tfrac{1}{2};\eta)(2\beta_1)^j|z|^{2j} \\ &= \frac{1}{\pi}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(\frac{1}{2}+\frac{l}{2}+j)}{j!\cdot l!}(2\beta_1)^j|z|^{2j}\zeta^l. \end{aligned}$$

Since

$$(1-x)^{-p} = 1 + \frac{\Gamma(p+1)}{\Gamma(p)}x + \frac{\Gamma(p+2)}{2!\cdot\Gamma(p)}x^2 + \cdots \frac{\Gamma(p+n)}{n!\cdot\Gamma(p)}x^n + \cdots,$$

we obtain

$$\begin{aligned} W(z,\zeta) &= \frac{1}{\pi}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{l}{2}+j)}{j!\cdot l!}(2\beta_1)^j|z|^{2j}\zeta^l \\ &= \frac{1}{\pi}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(\frac{1}{2}+\frac{l}{2})}{l!}\zeta^l\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{l}{2}+j)}{j!\cdot\Gamma(\frac{1}{2}+\frac{l}{2})}(2\beta_1|z|^2)^j \\ &= \frac{1}{\pi}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(\frac{1}{2}+\frac{l}{2})}{l!}\zeta^l(1-2\beta_1|z|^2)^{-(\frac{1}{2}+\frac{l}{2})} \\ &= \frac{1}{\pi}(1-2\beta_1|z|^2)^{-\frac{1}{2}}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(\frac{1}{2}+\frac{l}{2})}{l!}\Big(\frac{\zeta}{\sqrt{1-2\beta_1|z|^2}}\Big)^l. \end{aligned}$$

We set $\eta_1 = \frac{\zeta^2}{4(1-2\beta_1|z|^2)}$. We again use (11) to get

$$\begin{aligned} W(z,\zeta) &= \frac{1}{\pi}(1-2\beta_1|z|^2)^{-\frac{1}{2}}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(\frac{1}{2}+\frac{l}{2})}{l!}\Big(\frac{\zeta}{\sqrt{1-2\beta_1|z|^2}}\Big)^l \\ &= \frac{1}{\pi}(1-2\beta_1|z|^2)^{-\frac{1}{2}}U(\tfrac{1}{2},\tfrac{1}{2};\eta_1). \end{aligned}$$

We put $-2\beta_1 = \mathsf{b}$. We finally obtain

$$W(z,\zeta) = \frac{1}{\pi}(1+\mathsf{b}|z|^2)^{-\frac{1}{2}}U(\tfrac{1}{2},\tfrac{1}{2};\tilde{\eta}_1), \qquad \tilde{\eta}_1 = \frac{\zeta^2}{4(1+\mathsf{b}|z|^2)}.$$

Let $\kappa(\zeta)$ be given in (3). Since $\kappa(\zeta) = \frac{1}{\pi}U(\frac{1}{2},\frac{1}{2},\frac{\zeta^2}{4})$ (see (3.6) in [5]), we obtain

$$W(z,\zeta) = (1 + \mathsf{b}|z|^2)^{-\frac{1}{2}}\kappa(\tfrac{\zeta}{\sqrt{1+\mathsf{b}|z|^2}}).$$

By a direct computation, we can check this actually satisfies (5) with $q = 2$. By the same way, we find that

$$W(z,\zeta) = (1 + \mathsf{b}|z|^2)^{-m}\kappa(\tfrac{\zeta}{\sqrt{1+\mathsf{b}|z|^2}}).$$

gives a solution of (5) for any $q > 1$.

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