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Alternation hierarchies and fragments of modal μ-calculus

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Abstract

Modal μ-calculus, the logic obtained by adding (non-first-order) least and greatest fixpoint operators to the modal logic, has attracted great interests from computer science and mathematical logic. It is natural to classify the formulas of modal μ-calculus by the number of alternating blocks of fixpoint operators, which is called the alternation hierarchy. A fundamental issue is the strictness of such alternation hierarchies. We review the historical and latest studies on the (semantical) strictness of alternation hierarchy with respect to various transition systems. We also introduce the variable hierarchy and observe that, the simple alternation hierarchy of the one-variable fragment of modal μ-calculus is strict over finitely branching transition systems.

1 Introduction

Modal μ-calculus, introduced by Kozen [19], is an extension of modal logic by adding greatest and least fixpoint operators. Such a logic is capable of capturing the greatest and least solutions of the equation $X = \Gamma(X)$, where Γ is a monotone function with a set variable X. By modal logic, we mean the propositional logic with modalities $\Box$ (universal modality, which is interpreted as necessity) and $\Diamond$ (existential modality, which is interpreted as possibility). Other kinds of μ-calculus can be found in [3].

Modal μ-calculus is closely related with tree automata and parity games. Niwinski (1988) showed that Rabin tree automata can be translated into μ-calculus. For another direction, he (1986) proved that a restricted fragment of μ-calculus without conjunctions can be translated into Rabin tree automata. The equivalence between modal μ-calculus and tree automata over binary trees is established by Emerson and Jutla [15]. Study along this line is motivated by Rabin's study on the decidability of monadic second order logic with two successors, and also highly concerned with the positional determinacy of parity games [15, 16].

A fundamental issue on modal μ-calculus is the strictness of *alternation hierarchy*. The alternation hierarchy classifies formulas by their *alternation depth*, namely, the number of alternating blocks of greatest and least fixpoint operators. Note that the alternation depth, in a game-theoretic view, is related with the number of priorities in parity games, and from an automata-theoretic perspective, it concerns with the Rabin index of Rabin tree automata.

The strictness of alternation hierarchy of modal μ-calculus was established by Bradfield [11, 10, 9], and at the same time by Lenzi [21]. In sequel, Arnold [4] and Bradfield [8] showed that the alternation hierarchy of modal μ-calculus is strict over infinite binary trees. Alberucci and Facchini [1] proved that the alternation hierarchy is strict over reflexive transition systems. D'Agostino and Lenzi [12] further showed the hierarchy of the modal μ-calculus over reflexive and symmetric graphs is infinite.

On the other hand, the hierarchy collapses with respect to some classes of transition systems. Note that the alternation hierarchy of modal μ-calculus collapses to the *alternation-free fragment of modal μ-calculus* (AFMC, namely, the fragment consists of formulas with no nested alternation of least and greatest fixpints) over finite directed acyclic transition systems. This means that, for any modal μ-calculus formula ϕ, there is an alternation-free formula φ such that for any finite directed acyclic transition system $\mathcal{M}$, $\mathcal{M} \models \phi$ iff $\mathcal{M} \models \varphi$. Such a collapse consists in the fact that the semantics of the least and greatest fixpoint operators make difference when the transition system contains infinite paths. The finite directed acyclic graphs only have finite paths. The hierarchy collapses to AFMC over transitive and directed transition systems, while it even collapses to the *modal fragment* (namely, modal logic, denoted as ML) with respect to transitive undirected transition systems [1]. Gutierrez *et al.* [17] strengthened the collapse results and proved that it collapses to AFMC over the finite directed transition systems with "feedback vertex sets" of a bounded size. Remark that, if the vertices in a feedback vertex set are removed, then the transition system is decomposed into several finite directed acyclic graphs, which in fact makes the transition system contains only finite paths.

When the transition systems are restricted to linear ones (not necessarily finite), the alternation hierarchy may also collapse. For instance, over ω-languages, the alternation hierarchy of modal μ-calculus collapses to the AFMC [24]. When the transition systems are restricted to visibly pushdown ω-languages, Arenas *et al.* [2] showed the hierarchy collapses to L_μ^1, where L_μ^k denotes the fragment of modal μ-calculus that consists of formulas of alternation depth at most k (e.g., the AFMC is L_μ^0). This result was improved by Gutierrez *et al.* [17], who proved that the hierarchy collapses to AFMC for visibly pushdown ω-languages.

At the same time, the number of variables contained in a formula also serves as an important measure of complexity for formulas of modal μ-calculus. Berwanger [5] proved that the two-variable fragment of modal μ-calculus is enough to express properties in arbitrary level of alternation hierarchy of modal μ-calculus. Then it is natural to ask whether $L_\mu[2] = L_\mu$ and whether the distinct bounded variables are redundant. This problem was subsequently negatively answered by Berwanger, Grädel and Lenzi in [6, 7] by showing the strictness of variable hierarchy.

The remainder of this paper is organized as follows. We start with the syntax and semantics of modal μ-calculus (Section 2), and also two kinds of alternation hierarchy for modal μ-calculus, namely, simple and Niwiński alternation hierarchy (Section 3). In Section 4, the studies on semantical strictness of such hierarchies are reviewed. Section 5 is concerned with the variable hierarchy and fragments of modal μ-calculus. We observe the strictness of simple alternation hierarchy of one-variable fragment of modal μ-calculus, by checking the correspondence to the counterparts of weak alternating tree automata and weak games.

2 Syntax and semantics

2.1 Syntax

The formulas of modal μ-calculus are built by using conjunction ($\vee$), disjunction ($\wedge$), existential and universal modalities ($\Diamond$ and $\Box$) and the least and a greatest fixpoint operators (μ and ν). Given a language consisting of

- *Prop*: a set of atomic propositions, p, q,...
- $\mathfrak{X}$: a countable set of second-order variables, X, Z, ...
- L: a set of labels, a, b, c, ...

the set L_μ of formulas of modal μ-calculus, is defined inductively as follows:

(1) for all atomic propositions $p \in Prop$, p, $\neg p \in L_\mu$,
(2) for $p \in Prop$, $\mathbf{tt}(:= p \vee \neg p) \in L_\mu$ and $\mathbf{ff}(:= p \wedge \neg p) \in L_\mu$,
(3) for all variables $X \in \mathfrak{X}$, $X \in L_\mu$,
(4) if φ, $\psi \in L_\mu$, then $\varphi \vee \psi$, $\varphi \wedge \psi \in L_\mu$,
(5) if $\varphi \in L_\mu$, then $\Diamond_a\varphi$, $\Box_a\varphi \in L_\mu$ for $a \in L$.
(6) if $X \in \mathfrak{X}$, $\varphi \in L_\mu$ and X occurs positively in φ, then $\mu X.\varphi$, $\nu X.\varphi \in L_\mu$.

Remark 1 (Negation). *The negation can be defined in the languages inductively as follows:*

$$\neg(\neg p) = p, \quad \neg(\neg X) = X, \quad \neg(\psi \vee \varphi) = \neg\psi \wedge \neg\varphi, \quad \neg(\psi \wedge \varphi) = \neg\psi \vee \neg\varphi$$
$$\neg\Box_a\varphi = \Diamond_a\neg\varphi, \quad \neg\Diamond_a\varphi = \Box_a\neg\varphi, \quad \neg\mu X.\varphi(X) = \nu X.\neg\varphi(\neg X), \quad \neg\nu X.\varphi(X) = \mu X.\neg\varphi(\neg X)$$

Remark 2. *For $\mu X.\varphi$ ($\nu X.\varphi$), it only allows positive occurrence of X in φ, namely, an even number of negations. Such a condition ensures the function denoted by $\varphi(X)$ containing X is a monotone function in X. Thus the least and greatest fixpoints of the function exist.*

The set of free variables of a formula $\varphi \in L_\mu$, denoted by $\mathsf{free}(\varphi)$, is defined inductively as follows:

1. $\mathsf{free}(\varphi \vee \phi) = \mathsf{free}(\varphi \wedge \phi) = \mathsf{free}(\varphi) \cup \mathsf{free}(\phi)$,
2. $\mathsf{free}(\Diamond\varphi) = \mathsf{free}(\Box\varphi) = \mathsf{free}(\varphi)$,
3. $\mathsf{free}(\mu X.\varphi) = \mathsf{free}(\nu X.\varphi) = \mathsf{free}(\varphi) - \{X\}$.

A formula is said to be in *positive normal form*, if it is in positive form and in addition all bounded variables are distinct. Any formula can be converted in to positive normal form by using de Morgan laws and *α-conversion* (cf. Section 2.4.2 of [3]).

2.2 Kripke semantics

Formulas of the modal μ-calculus can be evaluated via labelled transition systems $\mathcal{M} = (S, [\![\ ]\!]^{\mathcal{M}})$ (a Kripke model) at a particular vertex in S, in the sense that

$$\mathcal{M}, v \models \phi \overset{def}{\iff} v \in [\![\phi]\!]^{\mathcal{M}}.$$

More formally, a *Kripke model* is a tuple $\mathcal{M} = (S, (E_a)_{a \in L}, \mathcal{I})$, where

- S is a set of vertices,
- $E_a \subseteq S \times S$ is a transition relation, for $a \in L$ and L a set of labels, and
- $\mathcal{I}: \ Prop \to \mathcal{P}(S)$.

A Kripke model is a directed *labelled transition system*, in which the edge is labelled by elements of L and the vertices are labelled by a subset of $Prop$, that is, a vertex $v \in S$ is labelled by $\{p \in Prop : v \in \mathcal{I}(p)\} \subseteq Prop$.

Given a Kripke model $\mathcal{M}$ and a valuation function $\mathcal{V} : \mathfrak{X} \to \mathcal{P}(S)$, the denotation $[\![\varphi]\!]^{\mathcal{M}}_{\mathcal{V}} \subseteq S$ is defined as follows:

$$\begin{aligned}
[\![\mathbf{tt}]\!]^{\mathcal{M}}_{\mathcal{V}} &:= S \text{ and } [\![\mathbf{ff}]\!]^{\mathcal{M}}_{\mathcal{V}} := \emptyset, \\
[\![X]\!]^{\mathcal{M}}_{\mathcal{V}} &:= \mathcal{V}(X) \quad \text{for all } X \in \mathfrak{X}, \\
[\![p]\!]^{\mathcal{M}}_{\mathcal{V}} &:= \mathcal{I}(p) \text{ and } [\![\neg p]\!] := S - \mathcal{I}(p) \quad \text{for all } p \in Prop, \\
[\![\varphi \vee \psi]\!]^{\mathcal{M}}_{\mathcal{V}} &:= [\![\varphi]\!]^{\mathcal{M}}_{\mathcal{V}} \cup [\![\psi]\!]^{\mathcal{M}}_{\mathcal{V}}, \\
[\![\varphi \wedge \psi]\!]^{\mathcal{M}}_{\mathcal{V}} &:= [\![\varphi]\!]^{\mathcal{M}}_{\mathcal{V}} \cap [\![\psi]\!]^{\mathcal{M}}_{\mathcal{V}}, \\
[\![\Diamond_a \varphi]\!]^{\mathcal{M}}_{\mathcal{V}} &:= \{v \in S : \exists w. (v, w) \in E_a \wedge w \in [\![\varphi]\!]^{\mathcal{M}}_{\mathcal{V}}\}, \\
[\![\Box_a \varphi]\!]^{\mathcal{M}}_{\mathcal{V}} &:= \{v \in S : \forall w. (v, w) \in E_a \to w \in [\![\varphi]\!]^{\mathcal{M}}_{\mathcal{V}}\}.
\end{aligned}$$

We may omit the symbols $\mathcal{M}$ and $\mathcal{V}$ in the notation $[\![\varphi]\!]_{\mathcal{V}}^{\mathcal{M}}$ if they are clear from the context. It remains to define $[\![\theta X.\varphi]\!]$ for $\theta = \mu, \nu$. To define the semantics of fixpoints formulas of modal μ-calculus, we recall the following well-known result.

Theorem 1 (Knaster-Tarski). *Given a set S and a function monotone with respect to the set inclusion $f : \mathcal{P}(S) \to \mathcal{P}(S)$, then f has a least fixpoint* $\mathbf{lfp}f = \bigcap\{S' \subseteq S : f(S') \subseteq S'\}$ *and greatest fixpoint* $\mathbf{gfp}f = \bigcup\{S' \subseteq S : f(S') \supseteq S'\}$.

Given $\mathcal{M}$, the μ or ν formula φ is associated with an operator $\varphi^{\mathcal{M}} : \mathcal{P}(S) \to \mathcal{P}(S)$. Since $\varphi^{\mathcal{M}}$ is monotone by Remark 2, it follows from Theorem 1 that $\varphi^{\mathcal{M}}$ has least and greatest fixed points: $\mathbf{lfp}(\varphi^{\mathcal{M}}) = \bigcap\{X : \varphi^{\mathcal{M}}(X) \subseteq X\}$ and $\mathbf{gfp}(\varphi^{\mathcal{M}}) = \bigcup\{X : \varphi^{\mathcal{M}}(X) \supseteq X\}$.

Thus we can define that $[\![\mu X.\varphi]\!] := \mathbf{lfp}(\varphi^{\mathcal{M}})$ and $[\![\nu X.\varphi]\!] := \mathbf{gfp}(\varphi^{\mathcal{M}})$.

Then, together with Remark 1, we have

Fact 1. *For any sentence φ, any Kripke model $\mathcal{M} = (S, (E_a)_{a\in L}, \mathcal{I})$ with valuation function $\mathcal{V}$, it holds that $[\![\neg\varphi]\!]_{\mathcal{V}}^{\mathcal{M}} = S - [\![\varphi]\!]_{\mathcal{V}}^{\mathcal{M}}$.*

3 Alternation hierarchy of modal μ-calculus

We start with the so-called simple (or syntactic) alternation hierarchy by counting simply syntactic alternation of μ and ν as follows, where the superscript S means simple or syntactic.

Definition 1. *The simple alternation hierarchy of modal μ-calculus is defined as follows.*

- $\Sigma_0^{S\mu}, \Pi_0^{S\mu}$: *the class of formulas with no fixpoint operators*
- $\Sigma_{n+1}^{S\mu}$: *containing $\Sigma_n^{S\mu} \cup \Pi_n^{S\mu}$ and closed under the following operations*

  (i) if $\varphi_1, \varphi_2 \in \Sigma_{n+1}^{S\mu}$, then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\Box_R\varphi_1$, $\Diamond_R\varphi_1 \in \Sigma_{n+1}^{S\mu}$,

  (ii) if $\varphi \in \Sigma_{n+1}^{S\mu}$, then $\mu X.\varphi \in \Sigma_{n+1}^{S\mu}$
- *dually for* $\Pi_{n+1}^{S\mu}$
- $\Delta_n^{S\mu} := \Sigma_n^{S\mu} \bigcap \Pi_n^{S\mu}$

A formula is strict $\Sigma_n^{S\mu}$ if it is in $\Sigma_n^{S\mu} - \Pi_n^{S\mu}$.

Notice that the above notion of simple alternation does not capture the complexity of dependence of fixpoints. For instance, it does not distinguish the following two formulas:

- $\Phi_1 = \nu X.\mu Z.\Diamond_a X \vee (p \wedge \Diamond_b Z)$
- $\Phi_2 = \nu X.\Diamond_a X \wedge (\mu Z.p \vee \Diamond_b Z)$

Both Φ_1 and Φ_2 are strict $\Pi_2^{S\mu}$. But the former is more complex, since its inner fixpoint depends on the outer one.

A stronger notion of alternation hierarchy is introduced by Niwiński.

Definition 2. *The Niwiński alternation hierarchy of modal μ-calculus is defined as follows.*

- $\Sigma_0^{N\mu}, \Pi_0^{N\mu}$: *the class of formulas with no fixpoint operators*
- $\Sigma_{n+1}^{N\mu}$: *containing $\Sigma_n^{N\mu} \cup \Pi_n^{N\mu}$ and closed under the following operations*
  - *(i) if $\varphi_1, \varphi_2 \in \Sigma_{n+1}^{N\mu}$, then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\Box_R\varphi_1$, $\Diamond_R\varphi_1 \in \Sigma_{n+1}^{N\mu}$,*
  - *(ii) if $\varphi \in \Sigma_{n+1}^{N\mu}$, then $\mu Z.\varphi \in \Sigma_{n+1}^{N\mu}$, and*
  - *(iii) if $\varphi(X)$, $\psi \in \Sigma_{n+1}^{N\mu}$ and no free variable of ψ is captured by φ, then $\varphi(X\backslash\psi) \in \Sigma_{n+1}^{N\mu}$.*
- *dually for $\Pi_{n+1}^{N\mu}$*
- $\Delta_n^{N\mu} := \Sigma_n^{N\mu} \bigcap \Pi_n^{N\mu}$

The Niwiński alternation depth of a formula ϕ is the least n such that $\phi \in \Delta_{n+1}^{N\mu}$.

Condition (iii) means that if one substitute a free variable X of $\varphi \in \Sigma_{n+1}^{N\mu}$ by $\psi \in \Sigma_{n+1}^{N\mu}$ satisfying that X is not bounded in ψ, then resulted formula $\varphi(X\backslash\psi)$ is still $\Sigma_{n+1}^{N\mu}$.

Fact 2. $\Sigma_n^{S\mu} \subsetneq \Sigma_n^{N\mu}$.

Note that the above two kinds of alternation hierarchy are defined in a syntactic way. The syntactic alternation depth of a formula is an upper bound on the descriptive complexity of its model-checking parity games.

Let $\mathbb{T}$ denotes the sets of all directed transition systems. The semantical simple alternation hierarchy of modal μ-calculus is defined as follows:

$$\begin{aligned}
\Sigma_n^{S\mu}[\mathbb{T}] &= \{[\![\varphi]\!]^{\mathcal{M}} : \varphi \in \Sigma_n^{S\mu} \text{ and } \mathcal{M} \in \mathbb{T}\},\\
\Pi_n^{S\mu}[\mathbb{T}] &= \{[\![\varphi]\!]^{\mathcal{M}} : \varphi \in \Pi_n^{S\mu} \text{ and } \mathcal{M} \in \mathbb{T}\}.\\
\Delta_n^{S\mu}[\mathbb{T}] &= \Sigma_n^{S\mu}[\mathbb{T}] \bigcap \Pi_n^{S\mu}[\mathbb{T}].
\end{aligned}$$

The semantical Niwiński alternation hierarchy of modal μ-calculus can be defined in the same manner. In the following, by the alternation hierarchy of modal μ-calculus, we always mean the semantical alternation hierarchy of modal μ-calculus, unless otherwise stated.

If we restrict the transition systems to a special class with a certain property P, which is denoted as $\mathbb{T}^{\mathsf{P}}$, the semantical alternation hierarchy of modal μ-calculus over $\mathbb{T}^{\mathsf{P}}$ can be defined similarly.

4 Strictness of alternation hierarchy for modal μ-calculus

The strictness of alternation hierarchy of modal μ-calculus was first established by Bradfield [11, 10, 9], and at the same time by Lenzi [21].

In [11. 10], Bradfield obtained the strictness results by analyzing the effective descriptive complexity of properties in the modal μ-calculus. He transferred Lubarsky's results on strictness of hierarchy for arithmetic μ-calculus [22] to that of modal μ-calculus via *recursively presented Kripke models.* Note that in a recursive presented Kripke model, the set of vertices is recursively codable as a recursive set of integers, set of labels likewise and the transition relation is also recursive, with recursive valuations for the free variables.

Recall that we have introduced two kinds of alternation hierarchy of modal μ-calculus, namely, simple $(\Sigma_n^{S_\mu})$ and Niwiński $(\Sigma_n^{N_\mu})$. In the following, we may just write Σ_n^μ when the distinction does not make sense or the arguments can be applied to these two kinds of definitions.

Theorem 2. *For any n, there is a formula $\psi \in \Sigma_n^\mu$, but no Π_n^μ formula equivalent to ψ.*

Simultaneously, Lenzi [21] also proved a slight weaker version of Theorem 2 for *Emerson-Lei alternation hierarchy* over n-ary trees.

Bradfield [9] gave explicit examples for such hard formula ψ. Note that $\mu\!\!\nu$ denotes μ if n is odd, otherwise ν.

Theorem 3. *The L_μ formula $\mu X_n.\nu X_{n-1}\ldots \mu\!\!\nu X_1.\Box_c X_1 \vee \bigvee_{i=1..n} \Diamond_{R_i} X_i$ is Σ_n^μ but not Π_n^μ.*

He [9] also indicated that formulas expressing winning positions in parity games with n priorities, which are introduced by Emerson and Jutla [15] and Walukiewicz [25], are strict formulas of level n of the alternation hierarchy of modal μ-calculus. Consider a parity game $\mathcal{G} = (V_I, V_{II}, \Omega)$ with n priorities, where player I (player II) takes charge the set V_I (V_{II}) and the priority function is $\Omega : V_I \bigcup V_{II} \to \{1, 2, \ldots, n\}$. Then the winning region of player I in the parity game $\mathcal{G}$ can be expressed by the following L_μ formula

$$W^n := \mu X_n.\nu X_{n-1}\ldots \mu\!\!\nu X_1.\left(V_I \to \Diamond \bigwedge_{i=1}^{n}(\Omega_i \to X_i)\right) \wedge \left(V_{II} \to \Box \bigwedge_{i=1}^{n}(\Omega_i \to X_i)\right).$$

Bradfield claimed that

Theorem 4 ([9]). *W^n is a strict Σ_n^μ formula of alternation hierarchy of modal μ-calculus.*

Thereafter, Arnold [4] provided a more direct proof for Theorem 4. The main ideas of his proof are as follows.

- Reduce any Σ_n^μ formula ψ to W^n via some mapping F^ψ, namely, $\mathcal{M} \models \psi \Leftrightarrow F^\psi(\mathcal{M}) \models W^n$.

• Since F^{ψ} is a contracting mapping on the compact metric space of binary trees, by Banach fixed-point theorem, F^{ψ} has a fixpoint $\mathcal{M}^{\psi}$. Then, if $\neg W^n$ is equivalent to a Σ_n^{μ} formula ψ, it is a contradiction that $\mathcal{M}^{\psi} \models \psi \Leftrightarrow \mathcal{M}^{\psi} \models \neg\psi$.

Alberucci and Facchini proved that

Theorem 5 ([1]). *The Niwiński alternation hierarchy of modal μ-calculus is strict over reflexive transition systems.*

D'Agostino and Lenzi further showed that

Theorem 6 ([12]). *The alternation hierarchy of the modal μ-calculus over reflexive and symmetric transition systems is infinite.*

As a summary, studies on the strictness of the alternation hierarchy of modal μ-calculus with respect to several classes of transition systems are listed as follows:

Table 1: Summary of results on the alternation hierarchy of modal μ-calculus

Class of transition systems	Alternation hierarchy of modal μ-calculus	References
$\mathbb{T}^{rp}$	strict	[11, 10]
$\mathbb{T}^{n\text{-}tree}$	strict	[21]
$\mathbb{T}^{2\text{-}tree}$	strict	[4, 8]
$\mathbb{T}^{R}$	strict	[1]
$\mathbb{T}^{RS}$	strict	[12]
$\mathbb{T}^{fda}$	collapse to AFMC	[23]
$\mathbb{T}^{t}$	collapse to AFMC	[1, 13, 14]
$\mathbb{T}^{t'}$	collapse to AFMC	[17]
$\mathbb{T}^{tud}$	collapse to ML	[1, 14]
$\mathbb{T}^{\mathbf{REG}_\omega}$	collapse to AFMC	[24]
$\mathbb{T}^{\mathbf{VPL}_\omega}$	collapse to AFMC	[17]

$\mathbb{T}^{rp}$: the class of recursive presentive transition systems

$\mathbb{T}^{n\text{-}tree}$: the class of n-ary trees

$\mathbb{T}^{2\text{-}tree}$: the class of binary trees

$\mathbb{T}^{R}$: the class of reflexive transition sytsems

$\mathbb{T}^{RS}$: the class of reflexive and symmetric transition systems

$\mathbb{T}^{fda}$: the class of finite directed acyclic transition sytsems

$\mathbb{T}^t$: the class of transitive transition sytsems

$\mathbb{T}^{t'}$: the class of transitive transition sytsems with feedback vertex sets of a bounded size

$\mathbb{T}^{tud}$: the class of transitive and undirected graphs

$\mathbb{T}^{\mathbf{REG}_\omega}$: the class of ω-regular languages, and

$\mathbb{T}^{\mathbf{VPL}_\omega}$: the class of visibly pushdown ω-languages.

5 Variable hierarchy and fragments of modal μ-calculus

5.1 Variable hierarchy of modal μ-calculus

The alternation depth offers a complexity measure of L_μ by counting the alternation of least and greatest fixpoints. It is well-known that in descriptive set theory, the numbers of variables contained in a formula also serves as an important measure of complexity [18]. In this section, we will review an analogue measure for modal μ-calculus, the number of distinct variables bounded by the least and greatest fixpoints, which induces the variable hierarchy of modal μ-calculus. For any n, we denote $L_\mu[n]$ the set of L_μ formulas that has at most n distinct variables bounded by μ or ν, and likewise for $\Sigma_i^\mu[n]$, $\Pi_i^\mu[n]$, $\Delta_i^\mu[n]$ for all level i of L_μ. For instance, we can define the simple alternation hierarchy of $L_\mu[1]$ by modifying the definition of simple alternation hierarchy for L_μ, via level-by-level restricting the formulas with only one fixpoint variable in Definition 1, e.g., $\Sigma_n^{S\mu}[1] = \Sigma_n^{S\mu} \bigcap L_\mu[1]$.

Theorem 4 tells us that the formulas expressing winning regions of parity games exhaust the finite levels of alternation hierarchy of L_μ. However, Berwanger [5] showed that when we consider variable hierarchy, all such formulas can be reduced to $L_\mu[2]$.

Theorem 7 ([5]). *The alternation hierarchy of $L_\mu[2]$ is strict and not contained in any finite level of the L_μ.*

That is, formulas in $L_\mu[2]$ can express properties in arbitrary level of alternation hierarchy of L_μ. Then it is natural to ask whether $L_\mu[2] = L_\mu$ and whether the distinct bounded variables are redundant. This problem was negatively answered by Berwanger, Grädel and Lenzi by showing the strictness of variable hierarchy as follows:

Theorem 8 ([6, 7]). *For any n, there exists a formula $\phi \in L_\mu[n]$ which is not equivalent to any formula in $L_\mu[n-1]$.*

Thus $L_\mu[1] \subsetneq L_\mu[2] \subsetneq L_\mu[3] \subsetneq \cdots \subsetneq L_\mu$. Recall that $L_\mu[2]$ is enough to express properties in arbitrary level of (simple and Niwiński) alternation hierarchy of L_μ. Combining these results, we have

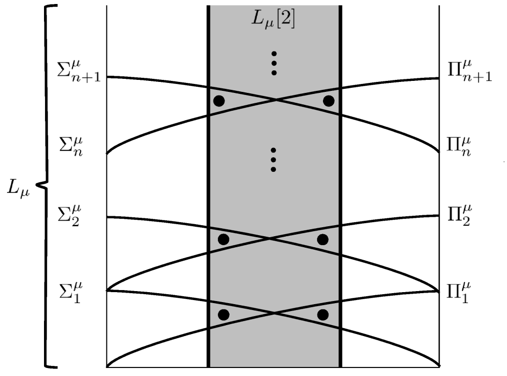

Figure 1: Alternation hierarchy of L_μ and $L_\mu[2]$

In Figure 1, the grey-covered area is $L_\mu[2]$, which is properly contained in the full modal μ-calculus L_μ. The symbols • at each level of the hierarchy denote the strict formulas, such as the two-variable formulas of [5], which witness the strictness of (simple and Niwiński) alternation hierarchy of both L_μ and $L_\mu[2]$.

5.2 One-variable fragment of modal μ-calculus

In the following, we would like to analyze the expressive power of $L_\mu[1]$, namely, the one-variable fragment of modal μ-calculus. We first note that one-variable fragment of modal μ-calculus is contained in the whole *weak alternation hierarchy* (cf. [20]). By definition, it is obvious that the following relation holds over finitely branching transition systems:

$$\underbrace{\bigcup_{n<\omega} \Sigma_n^{S\mu}[1]}_{\text{Simple altern. hierar. of } L_\mu[1]} \subseteq \Delta_2^{N\mu}[2] \subseteq \Delta_2^{N\mu} = comp(\Sigma_1^{N\mu}, \Pi_1^{N\mu}) = \underbrace{\bigcup_{n<\omega} \Sigma_n^{W\mu}}_{\text{Weak altern. hierar.}}.$$

From an automata-theoretic point of view, the weak alternation hierarchy is defined as the tree languages that are accepted by weak alternating tree automata. An alternating tree automaton $\mathcal{A} = (Q, X, \delta, n, \Omega)$ is said to be *weak* if δ satisfies that for all $q \in Q$ and $a \in X$, $q' \in \delta(p, a)$ implies $\Omega(q') \leq \Omega(q)$.

The strictness of weak alternation hierarchy of weak alternating tree automata is first proved by Mostowski, and the proof was simplified by using the reduction and diagonal arguments due to Arnold [4].

Theorem 9 (Mostowski). *The weak alternation hierarchy of weak tree automata is strict.*

Analogue to the relation between parity alternating tree automata and parity game, Arnold [4] also proved the following correspondence between weak alternating automata and weak games.

Theorem 10 ([4]). *The acceptance of a binary tree t by a weak alternating tree automaton $\mathcal{A}$ with n priorities can be expressed as the existence of a winning strategy in a weak game $\mathbb{G}_n$ associated with $\mathcal{A}$ and t, and vice versa.*

By the diagonal argument from [4], as well as Theorem 9, we can see that the formulas expressing the winning regions in weak games G_n for $n < \omega$ witness the strictness of the weak alternation hierarchy.

Next, we concentrate on the connection between weak alternation hierarchy and one-variable fragment of modal μ-calculus. From now on, we relax the condition that "all the fixpoint variables should be distinct". In such a relaxed context, a set variable can be bounded by μ and/or ν more than once.

In the following we observe that one variable is enough to express the winning region of weak games. A *weak game* can be given as a rooted structure $\mathcal{G}$, v_0 with $\mathcal{G} = (V, V_\Diamond, V_\Box, E, \Omega, n)$. The rules of such games from a certain vertex v_0 is as follows: if $v_0 \in V_\Diamond$, player $\Diamond$ chooses a vertex v_1 such that $(v_0, v_1) \in E$, if $v_0 \in V_\Box$, it is player $\Box$'s turn to choose a vertex v_1 such that $(v_0, v_1) \in E$, and so on. Player $\Diamond$ wins with a play x if the priority sequence of x is nonincreasing and the eventual priority of x is even. Note that we assume $\mathcal{G}$ is finitely branching and each branch is infinite.

Given n, we consider the following formulas for $i = 1, \dots, n$,

$$\begin{cases} \varphi_i := \nu X.\Big(\varphi_{i-1} \vee (\Omega_i \wedge \rhd X)\Big), & \text{if } i \text{ is odd} \\ \varphi_i := \mu X.\Big(\varphi_{i-1} \vee (\Omega_i \wedge \rhd X)\Big), & \text{if } i \text{ is even} \end{cases}$$

where $\rhd X := (V_\Diamond \wedge \Diamond X) \vee (V_\Box \wedge \Box X)$ and $\varphi_0 = \emptyset$. The formula φ_n describes that player $\Box$ has a winning strategy in a weak game with priority n.

Inductively, we can show that

Theorem 11. *Winning regions in weak games can be expressed by formulas of one-variable fragment of modal μ-calculus.*

Thus, by Theorem 9 and Theorem 10, as well as Theorem 11, we have

Theorem 12. *The simple alternation hierarchy of $L_\mu[1]$ is strict over finitely branching transition systems. Moreover, the simple alternation hierarchy of $L_\mu[1]$ exhausts the weak alternation hierarchy.*

That is, $L_\mu[1]$ formulas are enough to express properties at any level of the weak alternation hierarchy.

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