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Validity of bilateral classical logic and its application

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1 Mathematical Preliminary

In this section, we introduce Knaster-Tarski Theorem [7].

Theorem 1. *Let $\mathcal{L}$ be a complete lattice, $f\colon \mathcal{L} \to \mathcal{L}$ be an increasing function and F be the set of all fixed points of f. Then, F forms a complete lattice. In particular, F is not empty.*

Proof. We only prove existence of greatest and least fixed points. Let

$$u := \bigvee \{x \in \mathcal{L} \mid x \leq f(x)\}. \tag{1}$$

Let $x \leq f(x)$. Then, $f(x) \leq f(u)$ by the definition of u and f being increasing. Because $x \leq f(x)$, $x \leq f(u)$. Thus, $u \leq f(u)$. Therefore, $f(u) \leq f(f(u))$. This means $f(u) \in \{x \in \mathcal{L} \mid x \leq f(x)\}$. We can conclude $f(u) \leq u$ and therefore, $f(u) = u$. Thus, u is the greatest fixed point.

Applying a similar construction to the dual lattice $\mathcal{L}^{\mathrm{op}}$, we obtain the least fixed point

$$l := \bigwedge \{x \in \mathcal{L} \mid f(x) \leq x\}. \tag{2}$$

□

This proof, although simple and short, uses an impredicative definition, because we assume u is already contained in the set $\{x \in \mathcal{L} \mid x \leq f(x)\}$. We discuss the philosophical issues which arise by using an impredicative definition to define validity in Section 5.

We have induction principles on the least and greatest fixed points of f.

Lemma 1. *Let $P \subseteq \mathcal{L}$. Assume that $x \in P$ implies $f(x) \in P$. Further, if $(x_i)_{i \in I}$ are elements of P, $\bigwedge_{i \in I} x_i \in P$. Then, $u \in P$. Similarly, assume $\bigvee_{i \in I} x_i \in P$ if $(x_i)_{i \in I}$ are elements of P. Then, $l \in P$.*

Proof. Let $L := \{y \geq u \mid y \in P\}$. By assumption, $u_P := \bigwedge L \in P$. Because $f(u_P)$ satisfies P and $u \leq f(u_P)$, $f(u_P) \in L$. Thus $u_P \leq f(u_P)$. Therefore, $u_P \leq u$. By definition. $u \leq u_P$. Therefore, $u = u_P \in P$. The case for the least fixed point is proved similarly. □

2 Bilateral classical logic

Bilateral classical logic is a formal system for classical logic based on the idea that in classical logic, statements can have two linguistic forces, not only affirmation but in addition, denial. The system is most famously proposed by Rumfitt [5], but similar ideas appear in the other literature [3, 6]. In this paper, we only consider the implicational fragment of propositional logic for simplicity.

Definition 1 (Proposition, Statement). *Atomic propositions are denoted by symbols* $a, b, a_1, \ldots$. *Propositions* $A, B, A_1, \ldots$ *are defined by*

$$A := a \mid A \to A. \tag{3}$$

Statements $\alpha, \beta, \alpha_1, \ldots$ *are defined by*

$$\alpha := +A \mid -A. \tag{4}$$

In addition, a special symbol $\bot$ *appears in derivations.* $\bot$ *should be understood as a punctuation symbol, not a statement.*

Definition 2. $+a$ *and* $-a$ *for an atomic proposition* a *are called* atomic statements. *For a statement* α*, its* conjugate α^* *is defined as*

$$(+A)^* \equiv -A \qquad\qquad (-A)^* \equiv +A \tag{5}$$

Definition 3 (Logical rules).

$$\frac{\begin{matrix}[+A]\\ \vdots \\ +B\end{matrix}}{+A \to B} + \to I \qquad \frac{+A \to B \quad +A}{+B} + \to E \tag{6}$$

$$\frac{+A \quad -B}{-A \to B} - \to I \qquad \frac{-A \to B}{+A} - \to E1 \qquad \frac{-A \to B}{-B} - \to E1 \tag{7}$$

Definition 4 (Coordination rules).

$$\frac{+A \quad -A}{\bot} \bot \qquad\qquad \frac{\begin{matrix}[\alpha]\\ \vdots \\ \bot\end{matrix}}{\alpha^*} \text{RAA} \tag{8}$$

Definition 5. *For a set S of atomic statements, the derivations of the system* BCL(S) *are derivations starting from atomic statements $\alpha \in S$ and assumptions $[\beta]$ using the logical rules and coordination rules. In the rules* $+\to I$ *and* RAA, *the assumption α is* discharged *and no longer* open. *If a derivation π does not contain an open assumption, π is called* closed. *Otherwise, π is called* open.

3 Normalization

The normalization procedure of BCL(S) is defined as follows. The idea is to reduce every introduction-elimination pair in a main branch of an inference, counting introduction rules for logical symbols and RAA as introduction rules and elimination rules for logical symbols and the law of contradiction as elimination rules. In addition, contradiction/RAA-pair is reduced for simpler normal forms.

$$\dfrac{\dfrac{\begin{matrix}[+A]\\ \vdots\\ +B\end{matrix}}{+A\to B}\quad \begin{matrix}\vdots\\ +A\end{matrix}}{+B} \quad\Longrightarrow\quad \begin{matrix}\vdots\\ +A\\ \vdots\\ +B\end{matrix} \tag{9}$$

$$\dfrac{\dfrac{+A\quad -B}{-A\to B}}{+A} \quad\Longrightarrow\quad +A \tag{10}$$

$$\dfrac{\dfrac{+A\quad -B}{-A\to B}}{-B} \quad\Longrightarrow\quad -B \tag{11}$$

$$\dfrac{\dfrac{\begin{matrix}[+A]\\ \vdots\\ +B\end{matrix}}{+A\to B}\quad \dfrac{\begin{matrix}\vdots\\ +A\end{matrix}\quad -B}{-A\to B}}{\bot} \quad\Longrightarrow\quad \dfrac{\begin{matrix}\vdots\\ +A\\ \vdots\\ +B\end{matrix}\quad -B}{\bot} \tag{12}$$

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \vdots\\ \bot\end{matrix}}{\alpha^*}\quad \begin{matrix}\vdots\\ \alpha\end{matrix}}{\bot} \quad\Longrightarrow\quad \begin{matrix}\vdots\\ \alpha\\ \vdots\\ \bot\end{matrix} \tag{13}$$

$$\dfrac{\dfrac{\begin{matrix}[-A\to B]\\ \vdots\\ \bot\end{matrix}}{+A\to B}\quad \begin{matrix}\vdots\\ +A\end{matrix}}{+B} \quad\Longrightarrow\quad \dfrac{\begin{matrix}\dfrac{\begin{matrix}\vdots\\ +A\end{matrix}\quad [-B]}{-A\to B}\\ \vdots\\ \bot\end{matrix}}{+B} \tag{14}$$

$$\dfrac{\dfrac{\begin{array}{c}[+A\to B]\\ \vdots\\ \bot\end{array}}{-A\to B}}{+A} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[+A]\quad[-A]}{\bot}}{+B}}{+A\to B}\\ \vdots\\ \bot\end{array}}{+A} \tag{15}$$

$$\dfrac{\dfrac{\begin{array}{c}[+A\to B]\\ \vdots\\ \bot\end{array}}{-A\to B}}{-B} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\dfrac{[+B]}{+A\to B}\\ \vdots\\ \bot\end{array}}{-B} \tag{16}$$

4 Validity and its application

Next, we define a notion of *validity* following Prawitz [4]. However, slight modification is made. We introduce the distinction of *evidences* and *valid derivations*, in which the former represent *direct* verification while the latter represent *indirect* verification's which represent *constructions* of direct verification's.

The set $[\![\alpha]\!](S)$ of *evidences* of a statement α in BCL(S) is defined by induction on α, using fixed point construction. The construction is inspired by construction of reducibility candidates in Yamagata [8, 9]. Use of fixed point construction to define reducibility is started by Barbanera and Berardi [1, 2].

The set $[\![\bot]\!](S)$ is defined as the set of derivations

$$\dfrac{\alpha \quad \alpha^*}{\bot} \tag{17}$$

where $\alpha, \alpha^* \in S$.

If α is an atomic statement and $\alpha \in S$, $\mathrm{Ax}(\alpha)(S)$ consists of the derivation

$$\dfrac{\alpha \in S}{\alpha} \tag{18}$$

Then, $[\![+a]\!]$ and $[\![-a]\!]$ are defined by equations

$$[\![+a]\!](S) := \mathrm{Ax}(+a)(S) \cup [\![-a]\!]^*(S) \tag{19}$$

$$[\![-a]\!](S) := \mathrm{Ax}(-a)(S) \cup [\![+a]\!]^*(S). \tag{20}$$

$[\![+A \to B]\!]$ and $[\![-A \to B]\!]$ are defined simultaneously by fixed point construction. Let the semantic space $\mathcal{M}(\alpha)$ be the set of all maps $m \in \mathcal{M}(\alpha)$ sending a set of atomic sentences S to the set $m(S)$ of closed derivations of α.

$$[\![+A \to B]\!](S) := \to ([\![+A]\!], [\![+B]\!])(S) \cup [\![-A \to B]\!]^*(S) \tag{21}$$

$$[\![-A \to B]\!](S) := \bullet([\![+A]\!], [\![-B]\!])(S) \cup [\![+A \to B]\!]^*(S) \tag{22}$$

Here, $\to, \bullet$ and * are defined as follows. $\bullet$ is simplest to define. Let $\mathcal{S}_A \in \mathcal{M}(+A)$ and $\mathcal{S}_B \in \mathcal{M}(+B)$. Then, $\pi \in \bullet(\mathcal{S}_A, \mathcal{S}_B)(S)$ if π has a form

$$\dfrac{\begin{array}{c} \vdots\, \sigma_A \\ +A \end{array} \quad \begin{array}{c} \vdots\, \sigma_B \\ -B \end{array}}{-A \to B} \tag{23}$$

and all normalization sequences of σ_A lead to elements of $\mathcal{S}_A(S)$ and all normalization sequences of σ_B lead to elements of $\mathcal{S}_B(S)$. Similarly, $\pi \in \to (\mathcal{S}_A, \mathcal{S}_B)(S)$ if π has a form

$$\dfrac{\begin{array}{c} [+A] \\ \vdots\, \pi' \\ +B \end{array}}{+A \to B} \tag{24}$$

and for any set $S' \supseteq S$ of atomic statements and $\sigma \in S_A(S')$,

$$\begin{array}{c} \vdots\, \sigma \\ +A \\ \vdots\, \pi' \\ +B \end{array} \tag{25}$$

always reduces to an element of $S_B(S')$. Let $\mathcal{S}_\alpha \in \mathcal{M}(\alpha)$. Then, $\mathcal{S}^*_\alpha(S) \in \mathcal{M}(\alpha^*)$ is defined by the set of closed derivations of α^* such that $\pi \in \mathcal{S}^*_\alpha(S)$ if π has a form

$$\dfrac{\begin{array}{c} [\alpha] \\ \vdots\, \pi' \\ \bot \end{array}}{\alpha^*} \tag{26}$$

and for any set $S' \supseteq S$ of atomic statements and any $\sigma \in \mathcal{S}_\alpha(S')$,

$$\begin{array}{c} \vdots\, \sigma \\ \alpha \\ \vdots\, \pi' \\ \bot \end{array} \tag{27}$$

always reduces an element of $[\![\bot]\!](S')$.

(19), (20), (21) and (22) depend each other, thus they might not appear well-defined. However, this circularity is not vicious. To see this, expand the definition (21).

$$[\![+A \to B]\!](S) = \to ([\![+A]\!], [\![+B]\!])(S) \cup [\![-A \to B]\!]^*(S) \tag{28}$$

$$= \to ([\![+A]\!], [\![+B]\!])(S) \cup (\bullet([\![+A]\!], [\![-B]\!])(S) \cup [\![+A \to B]\!]^*(S))^*(S) \tag{29}$$

Let an operator $F\colon \mathcal{M}(+A \to B) \to \mathcal{M}(+A \to B)$ as follows.

$$F(m)(S) = \to ([\![+A]\!], [\![+B]\!])(S) \cup (\bullet([\![+A]\!], [\![-B]\!])(S) \cup m^*(S))^*(S) \quad (30)$$

$\mathcal{M}(+A \to B)$ is a complete Boolean algebra by the point-wise ordering. Because * is a contra-variant operator, F is an increasing operator. Thus, Knaster and Tarski theorem allows the construction of the least fixed point. We choose the least fixed point as a solution of (28). The definition of the solution of (19) and (20) are similar.

Definition 6. *A closed derivation π of α is* valid *in* BCL(S) *if all normalization sequences of π lead to elements of $[\![\alpha]\!](S)$. For a open derivation π with assumptions $\alpha_1, \ldots, \alpha_n$, π is valid if, for any $S' \supseteq S$ and closed valid derivations $\sigma_1, \ldots, \sigma_n$ of* BCL(S), *when substituting $\sigma_1, \ldots, \sigma_n$ to $\alpha_1, \ldots, \alpha_n$, π is valid in* BCL(S').

Lemma 2. *If all* one-step reducta *of π are valid, π is valid*

Lemma 3. *σ is evidence, its one-step reducta are also evidences*

Corollary 1. *Evidences are valid*

Theorem 2. *All derivations of* BCL(S) *are valid.*

Proof. By induction on the construction of the derivations π. For the sake of simplicity, we assume that π has no open assumption. If π has open assumptions, the proof is only different in the respect that we has to consider $\pi[\sigma_1/\beta_1, \ldots, \sigma_n/\beta_n]$ (the derivation which obtained by substituting $\sigma_1, \ldots, \sigma_n$ to $\beta_1, \ldots, \beta_n$ respectively) instead of π itself.

$$\frac{\alpha \in S}{\alpha} \quad (31)$$

Because no further normalization is possible, and the derivation is contained in $[\![\alpha]\!](S)$, the theorem holds for this case.

$$\frac{[\alpha]}{\alpha} \quad (32)$$

For this case, the theorem is proved because $[\![\alpha]\!]$ is already valid.

$$\frac{\begin{array}{c}[\alpha] \\ \vdots\ \pi_1 \\ \bot\end{array}}{\alpha^*} \quad (33)$$

By induction hypothesis, π_1 is valid. Thus, $\pi_1 \in [\![\alpha]\!]^*(S)$. By the definition of evidences, π is already an evidence of α^* in BCL(S).

$$\dfrac{\begin{matrix}\vdots\,\pi_1 & \vdots\,\pi_2\\ \alpha & \alpha^*\end{matrix}}{\bot} \tag{34}$$

It suffices to show that all one-step normalization lead to $[\![\bot]\!](S)$. We only consider the case in which π has a form

$$\dfrac{\dfrac{\begin{matrix}[\alpha^*]\\ \vdots\,\pi_1'\\ \bot\end{matrix}}{\alpha} \quad \begin{matrix}\vdots\,\pi_2\\ \alpha^*\end{matrix}}{\bot} \tag{35}$$

and reduces

$$\begin{matrix}\vdots\,\pi_2\\ \alpha^*\\ \vdots\,\pi_1'\\ \bot.\end{matrix} \tag{36}$$

By induction hypothesis, π_2' is valid. Thus, by induction hypothesis, substituting π_2' to α^* yields a valid derivation. Therefore, π reduces an evidence.

If the last rule of π is a introduction rule, the proof is easy consequence from the construction of the evidences.

For elimination rules, we only consider $E+\rightarrow$-rule.

$$\dfrac{\begin{matrix}\vdots\,\pi_1 & \vdots\,\pi_2\\ +A\rightarrow B & +A\end{matrix}}{+B} \tag{37}$$

If all derivations obtained by one-step normalization are valid, this derivation is also valid. The only non-trivial case is that (14) is applied. Therefore, π has a form

$$\dfrac{\dfrac{\begin{matrix}[-A\rightarrow B]\\ \vdots\,\pi_1'\\ \bot\end{matrix}}{+A\rightarrow B} \quad \begin{matrix}\vdots\,\pi_2\\ +A\end{matrix}}{+B} \tag{38}$$

By induction hypothesis, for any $\sigma \in [\![-B]\!](S'), S' \supset S$,

$$\dfrac{\begin{matrix}\vdots\,\pi_2 & \vdots\,\sigma\\ +A & -B\end{matrix}}{-A\rightarrow B} \tag{39}$$

is an element of $[\![-A \to B]\!](S')$. Thus, by induction hypothesis,

$$\begin{array}{c} \dfrac{\begin{array}{cc} \vdots\, \pi_2 & \vdots\, \sigma \\ +A & -B \end{array}}{-A \to B} \\ \vdots\, \pi_1' \\ \bot \end{array} \qquad (40)$$

always reduces an element of $[\![\bot]\!](S')$. Therefore,

$$\begin{array}{c} \dfrac{\begin{array}{cc} \vdots\, \pi_2 & \\ +A & [-B] \end{array}}{-A \to B} \\ \vdots\, \pi_1' \\ \dfrac{\bot}{+B} \end{array} \qquad (41)$$

is an element of $[\![-B]\!]^*(S)$, thus an evidence of $+B$ in BCL(S). □

Lemma 4. *If S is inconsistent, for any statement α, $[\![\alpha]\!](S)$ is non-empty and its elements are strongly normalizable.*

Proof. By induction on α, using Lemma 1. □

Corollary 2. BCL($\emptyset$) *is strongly normalizable.*

Proof. By Theorem 2, every derivation π of α in BCL($\emptyset$) is valid. Let $[\beta_1], \ldots, [\beta_n]$ be open assumptions in π. Let $\sigma_1 \in [\![\beta_1]\!], \ldots, \sigma_n \in [\![\beta_n]\!]$. We choose an inconsistent S. Then, there are always such $\sigma_1, \ldots, \sigma_n$. The derivation π' which is obtained by substituting $\sigma_1, \ldots, \sigma_n$ to open assumptions of π is valid. Thus, all normalization sequences of π' lead to evidences of α. By Lemma 4, evidences are strongly normalizable. Therefore, π' is strongly normalizable. Because any normalization sequence of π' terminates, any normalization sequence of π also terminates. □

5 Relation to Dummett's verificationist semantics

A natural question is how the notion of evidence and validity are related to Dummett's verificationist semantics. In Dummett's program, the meaning of a statement is given by its direct verification, and the whole practice of inferences is justified because inferences somehow *produce* direct evidences. Our notion of an evidence and validity fits in this picture.

However, Dummett requires "decidability" to the notion of a direct evidence, otherwise, he argues, such notion is impossible to *manifest* on a competent speaker of the language.

Our notion of an evidence appears hopelessly undecidable, due to use of fixed point construction in the definition. Despite this appearance, we argue that the situation is not so simple.

Here, we do *not* try to show that our notion of an evidence is decidable. However, we *do* try to show that "decidability" is a delicate notion which requires further elucidation.

A possible interpretation of "decidability" is a realist one. In this view, a property is decidable or not, independent of our knowledge. The decidability of a property is shown by, for example, giving a procedure to decide its membership and to show termination of the procedure for every possible input. To show termination, we may just appeal experience ("The procedure terminates for all inputs so far") or use a mathematical theory of which correctness is justified by, again, experience. From a realist view, proving decidability can also be non-constructive. The proof of decidability then does not need to give a concrete decision procedure.

Clearly, the realist view is not Dummett's view when he is talking about "decidability". Still, this view has a merit to consider, because we are not necessarily seeking to *total* anti-realism, but content with anti-realism on a particular domain. We may just want anti-realism on, say, set theory, past events or ethics, while maintain realism on "concrete" objects such as decision procedures, proof trees or natural numbers.

Another view to decidability is that, the notion of decidability depends on an underlining (possibly informal) theory T. This view is better fitted to Dummett's anti-realist view. However, it is not clear what theory T Dummett has in his mind when he is talking about verificationist semantics. In particular, he does not offer an argument against a theory T with impredicative definitions.

If we accept Knaster and Tarski theorem and Lemma 1, we can prove that the set $[\![\alpha]\!](S)$ of evidences is decidable, even though its definition appears highly non-constructive. It is clear that the set of evidences is decidable if α is atomic, so we concentrate the case $\alpha \equiv +A \to B$ and $\alpha \equiv -A \to B$.

Proposition 1. *$\pi \in [\![+A \to B]\!](S)$ if and only if π has a form either*

$$\dfrac{\begin{array}{c}[+A]\\ \vdots\ \pi_1\\ +B\end{array}}{+A \to B} \tag{42}$$

where π_1 is a valid derivation, or

$$\dfrac{\begin{array}{c}[-A \to B]\\ \vdots\ \pi_1\\ \bot\end{array}}{+A \to B} \tag{43}$$

where π_1 *is a valid derivation. Similarly,* $\pi \in [\![-A \to B]\!](S)$ *if and only if* π *has a form either*

$$\dfrac{\begin{array}{cc}\vdots\ \pi_1 & \vdots\ \pi_2\\ +A & -B\end{array}}{+A \to B} \tag{44}$$

where π_1, π_2 *are valid derivations, or*

$$\dfrac{\begin{array}{c}[+A \to B]\\ \vdots\ \pi_1\\ \bot\end{array}}{-A \to B} \tag{45}$$

where π_1 *is a valid derivation.*

Proof. We only prove the case for $+A \to B$. The *if* part is clear from the definition, so we prove the only if part. However, all derivations are valid by Theorem 2. Therefore, the only if part is trivial. □

Corollary 3. *A derivation of which the last inference is an introduction of logical connectives or RAA is an evidence. Therefore, the set of evidences are decidable.*

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