博　士　論　文

Growth rate of height functions associated with ample divisors and its applications

（豊富な因子に付随する高さ関数の増大度とその応用）

佐野　薫

京都大学大学院理学研究科
数学・数理解析専攻（数学系）
２０１９年3月

Growth rate of height functions associated with ample divisors and its applications

A thesis presented
by

Kaoru Sano

to
The Division of Mathematics and Mathematical Sciences
Graduate School of Science
for the degree of Doctor of Science

Kyoto University
March 2019

To my family.

Abstract

For a projective variety defined over a global field, the ample height, i.e., the Weil height function associated with an ample divisor, measures the arithmetic complexity of a rational point. In this paper, we investigate the growth rate of the ample height under the iteration of a surjective self-morphism on a smooth, or more generally normal, projective variety. More precisely, we mainly deal with Kawaguchi-Silverman conjecture, which asserts that the arithmetic degree, the arithmetic complexity of a forward orbit, at a point whose forward orbit is Zariski dense is equal to the dynamical degree, the geometric complexity of the self-morphism.

As a preparation, we define the dynamical degree and the arithmetic degree and prove their properties. Then we prove Kawaguchi-Silverman conjecture for any surjective self-morphisms on a smooth projective surface and on a semi-abelian variety.

As related topics, we introduce the dynamical canonical height for Jordan blocks of the action of the pull-back of the divisors by a surjective self-morphism. Then we give an equivalent condition of the preperiodicity of a point under some assumptions. We also give an explicit asymptotic formula of the growth rate of the height functions associated with ample divisors. Moreover, the explicit formula gives the proof of a variant of the dynamical Mordell-Lang conjecture, which asserts that the sequence of the pair of the number of the iteration of two surjective self-morphisms under which the accident, hitting two orbits, occur forms an arithmetic progression.

Acknowledgements

The author is grateful for the Top Global University project for Kyoto University (abbrev. KTGU project). With the support of the KTGU project, the author had a chance to visit Professor Joseph H. Silverman during February-April, 2017 and October-November 2017. The author would appreciate his hospitality when the author was staying at Brown University. These were great opportunities to discuss the research and to consider the future works. The author also thanks Tetsuhi Ito, the author's supervisor, for the polite and detailed education on the research. The author is happy for meeting with Yohsuke Matsuzawa and Takahiro Shibata, who did collaborated research with the author. The author appreciates the family and his girlfriend for multilateral support.

Contents

CHAPTER 1

Introduction

1.1. Preparation

For a number field K, let M_K be the set of the all absolute values on K such that its restriction on $\mathbb{Q}$ is the standard Archimedean absolute value or the p-adic absolute value for some prime p. For each $v \in M_K$, let K_v and $\mathbb{Q}_v$ be the completion of K and $\mathbb{Q}$, respectively, respect to the distance

$$d_v(x, y) = |x - y|_v.$$

Let n_v be the extension degree $[K_v, \mathbb{Q}_v]$. Then we define the naive height h_{nv} of a point $P = [x_0 : x_1 : \cdots : x_N] \in \mathbb{P}^N(K)$ by

$$h_{\mathrm{nv}}(P) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \max_{0 \leq i \leq N} |x_i|_v^{n_v}.$$

The naive height function $h_{\mathrm{nv}} \colon \mathbb{P}^N(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$ is independent of the choice of the number field K such that $P \in \mathbb{P}^N(K)$ and the choice of the homogeneous coordinates $x_0, x_1, \ldots, x_N$.

For a projective variety X defined over $\overline{\mathbb{Q}}$ and a very ample Cartier divisor $H \in \mathrm{Pic}(X)$, fix a basis $\{f_0, f_1, \ldots, f_N\}$ of the linear system $|H|$. Then we get the closed embedding $\phi_{|H|} \colon X \hookrightarrow \mathbb{P}^N$ over $\overline{\mathbb{Q}}$, and we define the Weil height function $h_H \colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$ associated with H by

$$h_H := h_{\mathrm{nv}} \circ \phi_{|H|}.$$

Of course, the Weil height h_H depends on the choice of the basis of $|H|$. But changing the basis only makes the bounded difference of h_H. Hence the Weil height function h_H is well-defined as a function up to bounded functions. See [**HS**, Theorem B. 3.2] and [**Lan**, Chapter 4, Theorem 5.1] for the definitions and basic properties of Weil height functions.

The Weil height function h_H associated with a very ample divisor H measures the arithmetic complexity of rational points. For a dominant rational self-map $f \colon X \dashrightarrow X$, let $X_f(\overline{\mathbb{Q}}) = \left\{P \in X(\overline{\mathbb{Q}}) \mid f^n(P) \neq I_f\right\}$, where I_f is the indeterminacy locus of f. For a rational point $P \in X_f(\overline{\mathbb{Q}})$, the following problem is considered.

Problem 1.1.1. How does the sequence $(h_H(f^n(P)))_{n \in \mathbb{Z}_{\geq 0}}$ grow?

On this problem, the arithmetic degree

$$\alpha_f(P) = \lim_{n \to \infty} \max\{h_H(f^n(P)), 1\}^{1/n}$$

of f at P is introduced in [**Sil1**]. The arithmetic degree measures the arithmetic complexity of the orbit.

On the other hand, the (first) dynamical degree δ_f, which measures the geometric complexity of f, is defined by

$$\lim_{n \to \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n},$$

where the pull-back $f^*\colon \mathrm{N}^1(X) \longrightarrow \mathrm{N}^1(X)$ is defined as follows: take a resolution $\pi\colon \tilde{X} \longrightarrow X$ of the indeterminacy of f with $\tilde{X}$ smooth projective and let $\tilde{f}\colon \tilde{X} \longrightarrow X$ be the induced morphism, then we define $f^* = \pi_* \tilde{f}^*$.

On Problem 1.1.1, Kawaguchi and Silverman conjectured that the growth rate of $h_H(f^n(P))$ is controlled by the dynamical degree as follows:

Conjecture 1.1.2 ([KS3, Conjecture 6]). *Let X be a smooth projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$. Let $P \in X_f(\overline{\mathbb{Q}})$. The limit defining $\alpha_f(P)$ exists and if the forward f-orbit $O_f(P) = \{f^n(P) \mid n \in \mathbb{Z}_{\geq 0}\}$ is Zariski dense in X, then*

$$\alpha_f(P) = \delta_f.$$

holds.

See Chapter 2 for the details and the properties of the arithmetic and the dynamical degrees. For example generically finite invariance of the arithmetic degree is proved in Section 2.2. In Chapter 3, we introduce the known results on 1.1.2, and prove Conjecture 1.1.2 for certain cases as the by-product of the birational invariance of the arithmetic degrees. Moreover, we prove the existence of a point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) = \delta_f$ for any surjective self-morphisms $f\colon X \longrightarrow X$ in Chapter 3.

1.2. Main results

In Chapter 4 and Chapter 5, we prove the following theorem:

Theorem 1.2.1. *Conjecture 3.1.1 is true for any surjective self-morphisms on any smooth projective surfaces, and any semi-abelian varieties, respectively.*

Moreover, for the case of semi-abelian varieties, we give the precise description of the set of the arithmetic degrees of f.

As a related topics, when $f\colon X \longrightarrow X$ is a surjective self-morphism on a normal projective variety X, we introduce the dynamical canonical height for Jordan blocks of $f^*\colon \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Then we give an equivalent condition of the preperiodicity of a point under some assumptions.

Theorem 1.2.2. *Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective self-morphism on X over $\overline{\mathbb{Q}}$. Assume that 1 does not appear as the complex absolute value of an eigenvalue of the linear self-map*

$$f^*\colon \overline{V_H} \longrightarrow \overline{V_H},$$

where $\overline{V_H}$ is the subspace of $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by the set $\{(f^n)^*H \mid n \geq 0\}$. Then for every point $P \in X(\overline{\mathbb{Q}})$, the following conditions are equivalent.

- *P is preperiodic under f.*
- *$\alpha_f(P) = 1$.*

This theorem gives a proof of Conjecture 3.1.1 for any automorphisms on a smooth projective variety of Picard number 2, see Theorem 6.5.2 for detail.

Using the dynamical canonical height for Jordan blocks, we also give an explicit asymptotic formula of the growth rate of the height functions associated with ample divisors as follows:

Theorem 1.2.3 ([San3, Theorem 1.1]). *Let X be a normal projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \longrightarrow X$ a surjective self-morphism of X over $\overline{\mathbb{Q}}$. Let H be an ample divisor on X over $\overline{\mathbb{Q}}$. Then for any point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) > 1$, there is a non-negative integer $t_f(P) \in \mathbb{Z}_{\geq 0}$, positive real numbers $C_0, C_1 > 0$, and an integer N_0 such that the inequalities*

$$C_0 n^{t_f(P)} \alpha_f(P)^n < h_H(f^n(P)) < C_1 n^{t_f(P)} \alpha_f(P)^n$$

hold for all $n \geq N_0$.

Moreover, the explicit formula gives a variant of the dynamical Mordell-Lang conjecture as follows:

Theorem 1.2.4 ([San3, Theorem 1.2]). *Let X be a normal projective variety over $\overline{\mathbb{Q}}$, and $f, g\colon X \longrightarrow X$ étale self-morphisms of X over $\overline{\mathbb{Q}}$. Let $P, Q \in X(\overline{\mathbb{Q}})$ be points satisfying the following two conditions:*

- *$\alpha_f(P)^p = \alpha_g(Q)^q > 1$ for some $p, q \in \mathbb{Z}_{\geq 1}$, and*
- *$t_f(P) = t_g(Q)$, where $t_f(P)$ and $t_g(Q)$ are as in Theorem 1.2.3.*

Then the set

$$S_{f,g}(P,Q) := \{(m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid f^m(P) = g^n(Q)\}$$

is a finite union of the sets of the form

$$\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$.

1.2.1. Notation. In this paper, the ground field k is $\overline{\mathbb{Q}}$ if there are no attention. A *variety* is a separated irreducible reduced scheme of finite type over the ground field k. A *point* of a variety always means a closed point (or equivalently, a k-valued point). Let X be a variety over k and $f\colon X \dashrightarrow X$ a rational map. We use the following notation:

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the set of rational numbers, real numbers, and complex numbers, respectively.
- A *divisor* on a variety X means a Cartier divisor on X defined over $\overline{\mathbb{Q}}$.
- The group of codimension one cycles modulo rational equivalence is denoted by $\mathrm{CH}^1(X)$.
- $\mathrm{Div}(X)$ is the group of divisors on X defined over $\overline{\mathbb{Q}}$.
- $\mathrm{Pic}(X)$ is the Picard group of X.
- $\mathrm{N}^1(X)$ is the quotient group of $\mathrm{Pic}(X)$ by the numerical equivalence.
- $\mathrm{NS}(X)$ is the Néron–Severi group of X. It is well-known that $\mathrm{NS}(X)$ is a finitely generated abelian group. We put $\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.
- The indeterminacy locus of f is denoted by I_f.
- We write $X_f(k) = \{P \in X(k) \mid f^n(x) \notin I_f \text{ for every } n \geq 0\}$.
- The forward f-orbit $\{f^n(P) \mid n \geq 0\}$ of $P \in X_f$ is denoted by $O_f(P)$.
- A point $P \in X_f(\overline{k})$ is called *preperiodic* if the forward f-orbit $O_f(P)$ of P is a finite set. This is equivalent to the condition that $f^n(P) = f^m(P)$ for some $n, m \geq 0$ with $n \neq m$.
- For a projective variety defined over $\overline{\mathbb{Q}}$ and a Cartier divisor $D \in \mathrm{Pic}(X)$, fix a Weil height function

$$h_D\colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R};$$

see [**HS**, Theorem B. 3.2] and [**Lan**, Chapter 4, Theorem 5.1] for the definitions and basic properties of Weil height functions. For a $\mathbb{C}$-divisor $D = \sum_{i=1}^r a_i D_i \in$

$\mathrm{Div}(X)_{\mathbb{C}}$ ($a_i \in \mathbb{C}$), the Weil height function h_D associated with D is defined by

$$h_D := \sum_{i=1}^{r} a_i h_{D_i} \colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{C}.$$

- For a linear self-map $\varphi\colon V \longrightarrow V$ of a finite dimensional real vector space V, the set of the eigenvalues of φ is denoted by $\mathrm{EV}(\varphi; V)$. The maximum among the absolute values of the eigenvalues of φ is called the spectral radius of T and denoted by $\rho(\varphi)$.
- For a polynomial $F \in \mathbb{C}[t]$, the maximum among the absolute values of the roots of F is called the spectral radius of F and denoted by $\rho(F)$.
- For a $\mathbb{Z}$-module M and a field K, we write $M_K = M \otimes_{\mathbb{Z}} K$.
- Let X be a commutative algebraic group and $a \in X(k)$ a point. The translation by a is denoted by T_a.
- An *self-morphism* on a variety X means a morphism from X to itself defined over k. A *non-invertible self-morphism* is a surjective self-morphism which is not an automorphism.
- A *curve* (resp. *surface*) simply means a smooth projective variety of dimension 1 (resp. dimension 2) unless otherwise stated.
- For any curve C, the genus of C is denoted by $g(C)$.
- The symbols $\equiv$, $\sim$, $\sim_{\mathbb{Q}}$ and $\sim_{\mathbb{R}}$ mean algebraic equivalence, linear equivalence, $\mathbb{Q}$-linear equivalence, and $\mathbb{R}$-linear equivalence, respectively.
- For a surjective morphism $f\colon X \longrightarrow X$, we denote the degree $[\overline{\mathbb{Q}}(X) : f^*\overline{\mathbb{Q}}(X)]$ of extension of function fields by $\deg f$.
- Let f, g and h be real-valued functions on a domain S. The equality $f = g + O(h)$ means that there is a positive constant C such that $|f(x) - g(x)| \leq C|h(x)|$ for every $x \in S$. The equality $f = g + O(1)$ means that there is a positive constant C' such that $|f(x) - g(x)| \leq C'$ for every $x \in S$.

CHAPTER 2

Arithmetic degrees and dynamical degrees

2.1. The dynamical degrees

In this section, let $f\colon X \dashrightarrow X$ be a dominant rational map on a smooth projective variety both defined over an algebraically closed field k of characteristic 0 and let H be an ample divisor on X defined over k.

We define the pull-back $f^*\colon \mathrm{NS}(X) \longrightarrow \mathrm{NS}(X)$ as follows. Take a resolution $\pi\colon \tilde{X} \longrightarrow X$ of indeterminacy of f with $\tilde{X}$ smooth projective and let $\tilde{f}\colon \tilde{X} \longrightarrow X$ be the induced morphism. Then we define $f^* = \pi_* \tilde{f}^*$. This is independent of the choice of resolution.

Definition 2.1.1. The (first) *dynamical degree* δ_f of f is

$$\delta_f := \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim(X)-1})^{1/n}.$$

Remark 2.1.2.

(1) We introduce another definitions of the dynamical degree. Fix a norm $\|\cdot\|$ on $\mathrm{Hom}(\mathrm{N}^1(X)_{\mathbb{R}}, \mathrm{N}^1(X)_{\mathbb{R}})$. Then $\delta_f = \lim_{n\to\infty} \|(f^n)^*\|^{1/n}$. When f is a morphism, δ_f is the spectral radius of $f^*\colon \mathrm{N}^1(X)_{\mathbb{R}} \longrightarrow \mathrm{N}^1(X)_{\mathbb{R}}$. If the ground field is $\mathbb{C}$, this is equal to the spectral radius of $f^*\colon H^{1,1}(X) \longrightarrow H^{1,1}(X)$ (cf. [**DS3**, §4]).

(2) Dynamical degree is a birational invariant. That is, if $\pi\colon X \dashrightarrow X'$ is a birational map and $f\colon X \dashrightarrow X$ and $f'\colon X' \dashrightarrow X'$ are conjugate by π, then $\delta_f = \delta_{f'}$.

(3) Let X, Y be smooth projective varieties and $f\colon X \dashrightarrow X$, $g\colon Y \dashrightarrow Y$ dominant rational maps. Then, by definition, it is easy to see that $\delta_{f\times g} = \max\{\delta_f, \delta_g\}$.

Remark 2.1.3. The limit defining δ_f exists, and δ_f does not depend on the choice of H (see [**DS2**, Corollary 7], [**Gue**, Proposition 1.2]). Note that if f is a self-morphism, we have $(f^n)^* = (f^*)^n$ as a linear self-map on $\mathrm{NS}(X)$. But if f is merely a rational self-map, we have $(f^n)^* \neq (f^*)^n$ in general.

Remark 2.1.4. The first dynamical degree of a dominant rational self-map on a smooth complex projective variety was first defined by Dinh and Sibony in [**DS1, DS2**]. Dang and Truong gave an algebraic definition of the dynamical degrees in [**Dan**] and [**Tru**], respectively.

Remark 2.1.5 ([**Gue**, Proposition 1.2 (iii)], [**KS3**, Remark 7]**).** Let $\rho(f^*)$ be the spectral radius of the linear self-map $f^*\colon \mathrm{NS}(X) \longrightarrow \mathrm{NS}(X)$. The dynamical degree δ_f is equal to the limit $\lim_{n\to\infty}(\rho((f^n)^*))^{1/n}$. Thus we have $\delta_{f^n} = \delta_f^n$ for every $n \geq 1$.

Remark 2.1.6. Let X be a smooth projective complex variety and $f\colon X \dashrightarrow X$ a dominant rational map. Fix an closed immersion $\iota\colon X \longrightarrow \mathbb{P}^N$ and a hyperplane H in $\mathbb{P}^N$.

Put $H_X := \iota^* H$ and $\omega_X := \omega_{FS}$, where ω_{FS} is the Fubini-Study form on $\mathbb{P}^N$. Then we have

$$\begin{aligned}\delta_f &= \lim_{n\to\infty}((f^n)^* H_X \cdot H_X^{\dim X-1})^{1/n}\\ &= \lim_{n\to\infty}\Big(\int (f^n)^*\omega_X \wedge \omega_X^{\dim X-1}\Big)^{1/n}\\ &= \lim_{n\to\infty}\|(f^n)^*\|_{1,1}^{1/n},\end{aligned}$$

where $\|(f^n)^*\|_{1,1}$ is the operator norm of $f^*\colon H^{1,1}(X,\mathbb{C}) \longrightarrow H^{1,1}(X,\mathbb{C})$. The second equality follows from the property [**Ful**, Corollary 19.2 (b)] of the cycle map $\mathrm{cl}\colon \mathrm{NS}(X) \longrightarrow H^2(X,\mathbb{C})$ and the comparison theorem $H^2_{dR}(X) \cong H^2(X,\mathbb{C})$. The third equality follows from [**DS2**, Corollary 7] or [**Gue**, Proposition 1.2 (iii)].

Remark 2.1.7. Let X be a smooth projective variety over $\mathbb{C}$, and $f\colon X \longrightarrow X$ a surjective self-morphism. Let $\omega \in \mathrm{Pic}(X)$ be an ample divisor class. Let $\xi \in H^1(X,\mathcal{O}_X) = H^{0,1}(X)$ be the eigenvector of $f^*\colon H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X)$ associated with the eigenvalue whose complex absolute value is the spectral radius λ of f^* on $H^{0,1}(X)$. Then we have

$$(\xi \cdot \overline{\xi} \cdot \omega^{2\dim X-2}) > 0$$

by Hodge theory. In particular $\xi \cdot \overline{\xi} \in H^{1,1}(X)$ is non-zero. Hence the class $\xi \cdot \overline{\xi}$ is an eigenvector of $f^*\colon H^{1,1}(X) \longrightarrow H^{1,1}(X)$ associated with the eigenvalue $|\lambda|^2$. Consequently, the spectral radius of f^* on $H^{1,1}(X)$ is greater than or equal to the square of the spectral radius of f^* on $H^{0,1}(X)$.

When f is a self-morphism, the dynamical degree is actually the spectral radius of the linear self-map f^* on a smaller space than $\mathrm{NS}(X)_\mathbb{R}$ as Theorem 2.1.11.

Definition 2.1.8. When $f\colon X \longrightarrow X$ is a surjective self-morphism, for a divisor D on X, let V_D be the $\mathbb{Q}$-vector subspace of $\mathrm{Pic}(X) \otimes_\mathbb{Z} \mathbb{Q}$ generated by the set

$$\{(f^n)^* D \mid n \geq 0\}.$$

Let $\overline{V_D}$ be the image of V_D in $\mathrm{NS}(X)$.

Lemma 2.1.9. *Let X be a normal projective variety, and $f\colon X \longrightarrow X$ a surjective self-morphism on X both defined over $\overline{\mathbb{Q}}$. Then there is a monic integral polynomial $P_f(t) \in \mathbb{Z}[t]$ such that $P_f(f^*)$ annihilates* $\mathrm{Pic}(X)$.

Proof. See [**KS2**, Lemma 19]. Note that when X is projective geometrically integral and geometrically normal, $\mathrm{Pic}^0(X)$ is projective (see [**FAG**, Theorem 9.5.4]). Moreover $\mathrm{NS}(X)$ is finitely generated. Hence the proof of [**KS2**, Lemma 19] works. $\square$

We also recall an important lemma about the finiteness of the dimension of the $\mathbb{Q}$-vector space V_H.

Lemma 2.1.10. *The $\mathbb{Q}$-vector subspace V_H of* $\mathrm{Pic}(X)_\mathbb{Q}$ *is finite dimensional.*

Proof. Let $P_f(t) \in \mathbb{Z}[t]$ be the monic polynomial as in Lemma 2.1.9. Consider the $\mathbb{Q}$-subspace V'_H of $\mathrm{Pic}(X)_\mathbb{Q}$ spanned by the set

$$\{(f^n)^* H \mid 0 \leq n \leq \deg P_f(t) - 1\}.$$

Since $P_f(f^*)$ annihilates H, the space V'_H satisfies $f^*(V'_H) \subset V'_H$. Hence, we have $V_H = V'_H$, and so V_H is finite dimensional. See also the top of the proof of [**KS2**, Theorem 3]. $\square$

Theorem 2.1.11 ([San2, Remark 5.10]). *When $f\colon X \longrightarrow X$ is a surjective self-morphism on a normal projective variety X over $\overline{\mathbb{Q}}$, we have*

$$\delta_f = \max_{\lambda \in \mathrm{EV}(f^*;\ \overline{V_H})} |\lambda|.$$

Proof. For $D \in \mathrm{NS}(X)_{\mathbb{R}}$, we put

$$\|D\| := \inf\left\{ (D_1 \cdot H^{\dim X - 1}) + (D_2 \cdot H^{\dim X-1}) \;\middle|\; \substack{D = D_1 - D_2, \\ D_1, D_2 \in \mathrm{NS}(X) \text{ are effective}} \right\}.$$

Then since $\|\cdot\|$ is a non-trivial norm on $\mathrm{NS}(X)_{\mathbb{R}}$, the quantity

$$\|(f^n)^*\| := \sup_{D \in \mathrm{NS}(X)_{\mathbb{R}}} \left(\frac{\|(f^n)^* D\|}{\|D\|} \right)$$

is the operator norm of $(f^n)^*\colon\ \mathrm{NS}(X)_{\mathbb{R}} \longrightarrow \mathrm{NS}(X)_{\mathbb{R}}$. Thus we see that

$$\begin{aligned}
\lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n} &= \lim_{n\to\infty} \left(\frac{((f^n)^* H \cdot H^{\dim X-1})}{\|H\|} \right)^{1/n} \\
&\leq \lim_{n\to\infty} \left(\frac{\|(f^n)^* H\|}{\|H\|} \right)^{1/n} \\
&\leq \lim_{n\to\infty} \sup_{D \in \mathrm{NS}(X)_{\mathbb{R}}} \left(\frac{\|(f^n)^* D\|}{\|D\|} \right)^{1/n} \\
&= \limsup_{n\to\infty} \|(f^n)^*\|^{1/n} \\
&= \max_{\lambda \in \mathrm{EV}(f^*;\ \mathrm{NS}(X)_{\mathbb{R}})} |\lambda|.
\end{aligned}$$

Hence we obtain the following inequalities

$$\begin{aligned}
\delta_f &= \max_{\lambda \in \mathrm{EV}(f^*;\ \mathrm{NS}(X)_{\mathbb{R}})} |\lambda| \\
&\geq \max_{\lambda \in \mathrm{EV}(f^*;\ (\overline{V_H})_{\mathbb{R}})} |\lambda| \\
&\geq \lim_{n\to\infty} ((f^n)^* H \cdot H^{\dim X-1})^{1/n} \\
&= \delta_f.
\end{aligned}$$

Consequently δ_f is equal to the spectral radius of $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$. □

Remark 2.1.12. It is a natural problem to understand the difference of $\mathrm{EV}(f^*; V_H)$ and $\mathrm{EV}(f^*; \overline{V_H})$, and the difference of the size of the Jordan blocks of the linear maps $f^*|_{V_H}$ and $f^*|_{\overline{V_H}}$.

Let L be the $\mathbb{Z}$-submodule of $\mathrm{Pic}(X)$ generated by the set $\{(f^n)^* H \mid n \geq 0\}$. Let $\overline{L}$ be the image of L in $N^1(X) = \mathrm{NS}(X)/(\text{torsion})$. By [**KS2**, Lemma 19], L and $\overline{L}$ are finitely generated abelian groups. If it is necessary, by replacing H by a multiple of H, we may assume L and $\overline{L}$ do not have torsion elements. Let $\pi\colon L \longrightarrow \overline{L}$ be the canonical surjection, and put $M := \ker \pi$. We also put $W\colon = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Then M is a subgroup of $\mathrm{Pic}^0(X)$. Let

$$\varphi\colon\ \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X)$$

be the morphism induced by f. Let

$$\varphi'\colon H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X)$$

be the $\mathbb{C}$-linear map induced by f. Let $F(t)$ be the characteristic polynomial of φ'. Since $H^1(X, \mathcal{O}_X)$ is isomorphic to the Lie algebra of $\mathrm{Pic}^0(X)$, we have $F(\varphi) = 0$ on $\mathrm{Pic}^0(X)$. Hence in particular, we get $F(\varphi|_M) = 0$. Since W is generated by M, we also get $F(\varphi|_M)$=0. Consequently, the spectral radius of $f^*\colon W \longrightarrow W$ is less than or equal to the spectral radius of φ'. Combining Remark 2.1.6, Remark 2.1.7, and Theorem 2.1.11, we obtain that the spectral radius of $f^*\colon W \longrightarrow W$ is less than or equal to $\sqrt{\delta_f}$.

Therefore, for $\lambda \in \mathbb{C}$ with $|\lambda| > \sqrt{\delta_f}$, the Jordan normal form of $f^*\colon V_H \longrightarrow V_H$ associated with the eigenvalue λ is identified with the Jordan normal form of $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$ associated with λ by the canonical projection $\mathrm{pr}\colon V_H \longrightarrow \overline{V_H}$.

2.2. The arithmetic degrees

The *arithmetic degree* of f at a point $P \in X_f(\overline{\mathbb{Q}})$ is defined as follows. Fix the (absolute logarithmic) Weil height function

$$h_H\colon X(\overline{\mathbb{Q}}) \longrightarrow [0, \infty)$$

associated with H (see [**HS**, Theorem B3.2]). We put

$$h_H^+(P) := \max\{h_H(P), 1\}.$$

We call

$$\overline{\alpha}_f(P) := \limsup_{n\to\infty} h_H^+(f^n(P))^{1/n} \text{ and}$$
$$\underline{\alpha}_f(P) := \liminf_{n\to\infty} h_H^+(f^n(P))^{1/n}$$

the upper arithmetic degree and *the lower arithmetic degree* of f at P, respectively. It is known that $\overline{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ do not depend on the choice of H and h_H (see [**KS3**, Proposition 12]). If $\overline{\alpha}_f(P) = \underline{\alpha}_f(P)$, the limit

$$\alpha_f(P) := \lim_{n\to\infty} h_H^+(f^n(P))^{1/n}$$

is called *the arithmetic degree of f at P*.

Remark 2.2.1. Let D be a $\mathbb{C}$-divisor on X and f a dominant rational self-map on X. Take $P \in X_f(\overline{\mathbb{Q}})$. Then we can easily check that

$$\overline{\alpha}_f(P) \geq \limsup_{n\to\infty} \max\{|h_D(f^n(P))|, 1\}^{1/n}, \text{ and}$$
$$\underline{\alpha}_f(P) \geq \liminf_{n\to\infty} \max\{|h_D(f^n(P))|, 1\}^{1/n}.$$

So when these limits exist, we have

$$\alpha_f(P) \geq \lim_{n\to\infty} \max\{|h_D(f^n(P))|, 1\}^{1/n}.$$

Remark 2.2.2. When f is a self-morphism, the existence of the limit defining the arithmetic degree $\alpha_f(P)$ was proved by Kawaguchi and Silverman (see [**KS2**, Theorem 3] or Chapter 7). But it is not known in general.

Lemma 2.2.3 (Generically finite invariance of the arithmetic degree, cf [**MSS1**, Lemma3.3, Theorem 3.4] **and** [**MS**, Lemma 2.7]**).** *Let $f\colon X \dashrightarrow X$ and $g\colon Y \dashrightarrow Y$ be* *dominant rational self-maps on smooth projective varieties. Let μ is a generically finite map such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\mu"'] & X \arrow[d, dashed, "\mu"] \\ Y \arrow[r, dashed, "g"'] & Y. \end{tikzcd}$$

Suppose that there is a non-empty Zariski open subset $U \subset X$ and $V \subset Y$ such that $\mu|_U \colon U \longrightarrow V$ is a finite morphism. Then equalities $\overline{\alpha}_f(P) = \overline{\alpha}_g(\mu(P))$ and $\underline{\alpha}_f(P) = \underline{\alpha}_g(\mu(P))$ hold for all $P \in X_f(\overline{\mathbb{Q}})$ such that $O_f(x) \subset U(\overline{\mathbb{Q}})$. Hence for such P, $\alpha_f(P)$ exists if and only if $\alpha_g(\mu(P))$. Moreover, if they exist, they are equal to each other.

Proof. Taking the resolution of indeterminacy of μ, we may assume that μ is a morphism.

At first, we prove the assertion in the situation that μ is birational, so $\mu|_U \colon U \longrightarrow V$ is isomorphism.

Let H_Y be an ample divisor on Y. Then there is a μ-exceptional effective divisor E such that $H_X := \mu^* H_Y - E$ is an ample divisor on X. Since U does not intersect with the support of E, we have

$$\begin{aligned} h_{H_Y} \circ \mu &= h_{H_X} + h_E + O(1) \\ &\geq h_{H_X} + O(1) \end{aligned}$$

on $U(\overline{\mathbb{Q}})$. On the other hand, since there is a constant $C > 0$ such that $CH_X - \mu^* H_Y$ is ample, we have

$$Ch_X \geq h_{H_Y} \circ \mu + O(1)$$

on $X(\overline{\mathbb{Q}})$. Hence, there are positive constants C_1 and C_2 such that for a point $P \in U \cap X_f(\overline{\mathbb{Q}}) \cap \mu^{-1}(Y_g(\overline{\mathbb{Q}}))$, the following inequalities hold:

$$Ch_{H_X}(f^n(x)) + C_1 \geq h_{H_Y}(g^n(\mu(P))) + C_2 \geq h_{H_X}(f^n(P)).$$

Taking $\limsup_{n\to\infty} \max\{1, -\}^{1/n}$ (resp. $\liminf_{n\to\infty} \max\{1, -\}^{1/n}$), we consequently get

$$\overline{\alpha}_f(P) = \overline{\alpha}_g(\mu(P))$$

(resp. $\underline{\alpha}_f(P) = \underline{\alpha}_g(\mu(P))$.

In general situation, let $X \xrightarrow{p} Z \xrightarrow{q} Y$ be the Stein factorization of μ. Let $\beta \colon \tilde{Z} \longrightarrow Z$ be the resolution of singularities that is a composite of blow-up centers do not intersect with $W := q^{-1}(V)$. Let $\mu \colon \widetilde{X} \longrightarrow X$ be a resolution of indeterminacy of the induced rational map $X \dashrightarrow \widetilde{Z}$ that is a composite of blow-ups along smooth centers outside W. The situation is summarized in the following diagram:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\mu"'] \arrow[dr, "\widetilde{\beta}"] & \\ X \arrow[dr, "p"] \arrow[dd, "\pi"'] & & \widetilde{Z} \arrow[dl, "\beta"] \\ & Z \arrow[dl, "q"] & \\ Y & & . \end{tikzcd}$$

Then since β is a sequence of blow-ups, there exists an effective β-exceptional $\mathbb{Q}$-divisor E_β on $\tilde{Z}$ such that

$$\beta^* q^* H_Y - E_\beta$$

is ample (cf [**Har**, II Proposition 7.10]). Since $\tilde{Z}$ is smooth, there exists an effective $\tilde{p}$-exceptional $\mathbb{Q}$-divisor $E_{\tilde{p}}$ such that

$$\tilde{p}^*(\beta^* q^* H_Y - E_\beta) - E_{\tilde{p}}$$

is ample. (To see this, use [**KM**, Lemma 2.62] for example.) Set $H_{\tilde{X}} = \tilde{p}^*(\beta^* q^* H_Y - E_\beta) - E_{\tilde{p}}$ and $E = \tilde{p}^* E_\beta + E_{\tilde{p}}$. Then $H_{\tilde{X}}$ is ample, E is effective, $\mathrm{Supp}(E) \cap \tilde{\beta}^*(U) = \emptyset$, and $\tilde{\beta}^* \mu^* H_Y = H_{\tilde{X}} + E$.

Let $\tilde{P} := \tilde{\beta}^{-1}(P)$. Since $f(P) \in U$ and $\tilde{\beta}$ is isomorphic on U, the rational map $\tilde{f} := \tilde{\beta}^{-1} \circ g \circ \tilde{\beta}$ induced by g is well-defined. Since $\tilde{\beta}$ is birational and the assertion is proved in the case of birational map, now it is enough to show the assertion for $\tilde{X} \longrightarrow Y$.

Choose height function $h_{H_{\tilde{X}}+E}$ so that $h_{H_{\tilde{X}}+E} = h_{H_Y} \circ \mu \circ \tilde{\beta}$. Then since $O_{\tilde{f}}(\tilde{x})$ does not intersect with $\mathrm{Supp}(E)$, we have the following inequalities:

$$\begin{aligned} h_{H_{\tilde{X}}}(\tilde{f}^n(\tilde{P})) + O(1) &\leq h_{H_{\tilde{X}}+E}(\tilde{f}^n(\tilde{P})) \\ &= h_{H_Y} \circ \mu \circ \tilde{\beta} \circ \tilde{f}^n(\tilde{P}) \\ &= h_{H_Y}(g^n(\mu(P))). \end{aligned} \tag{2.2.1}$$

On the other hand, let $C > 0$ be a positive constant such that $CH_{\tilde{X}} - (H_{\tilde{X}} + E)$ is ample. Then we have

$$h_{H_{\tilde{X}}+E}(\tilde{f}^n(\tilde{P})) \leq C h_{H_{\tilde{X}}}(\tilde{f}^n(\tilde{P})). \tag{2.2.2}$$

By the same argument of the situation that μ is birational, the assertion follows from the inequalities (2.2.1) and (2.2.2). $\square$

Remark 2.2.4. The assertion of Lemma 2.2.3 contains the invariance of the arithmetic degree under birational or finite morphisms which commutes with the dynamics. Hence we can use Lemma 2.2.3 to reduce the proof of Conjecture 3.1.1 for surjective self-morphisms on any projective surfaces to those for surjective self-morphisms on minimal models. Moreover, the independence of the arithmetic degree from the choice of an embedding of the algebraic torus to the projective space is not proved in [**Sil1**]. By Lemma 2.2.3, the upper or lower arithmetic degree of a self-morphisms on a smooth quasi-projective variety (containing an algebraic torus or more generally a semi-abelian variety) at a point is well-defined.

Remark 2.2.5. By Theorem 2.2.3, the arithmetic degree of a rational self-map on a quasi-projective variety at a point does not depend on the choice of an open immersion of the quasi-projective variety to a projective variety. Furthermore, by the birational invariance of dynamical degrees, we can state Conjecture 3.1.1 for rational self-maps on quasi-projective varieties, such as semi-abelian varieties.

CHAPTER 3

The Kawaguchi-Silverman Conjecture

3.1. The Kawaguchi-Silverman Conjecture

We have quantities of two types. The first is the arithmetic degree which is number theoretic. The second is the dynamical degree which is topological. In this chapter, we introduce the conjecture on the connection of the arithmetic degrees at a Zariski-dense orbit and the dynamical degree, which is stated by Kawaguchi and Silverman in [**KS3**]. And we prove that there are (possibly Zariski non-dense) orbits at which the arithmetic degree is equal to the dynamical degree for any surjective self-morphisms. Finally, as applications of birational invariance of the arithmetic and dynamical degrees, we give examples for which Kawaguchi-Silverman conjecture is true.

Conjecture 3.1.1. *Let X be a smooth quasi-projective variety and $f\colon X \dashrightarrow X$ a dominant rational map, both defined over $\overline{\mathbb{Q}}$. Let $P \in X_f(\overline{\mathbb{Q}})$.*

(1) *The limit defining $\alpha_f(P)$ exists.*
(2) *The arithmetic degree $\alpha_f(P)$ is an algebraic integer.*
(3) *If the orbit $O_f(P) = \{f^n(P) \mid n = 0, 1, 2, \dots\}$ is Zariski dense in X, then*

$$\alpha_f(P) = \delta_f.$$

This is the Kawaguchi-Silverman conjecture, and we abbreviate it as KSC.

Remark 3.1.2. In the original statement in [**KS3**], the finiteness of the collections of the arithmetic degrees $\{\alpha_f(P) \mid x \in X_f(\overline{\mathbb{Q}})\}$ is conjectured, but a counterexample is given in [**LS2**, Theorem 2].

Remark 3.1.3. In [**KS3**], the conjecture is formulated for smooth projective varieties. Of course, quasi–projective version of the conjecture is equivalent to the projective version.

Remark 3.1.4. Kawaguchi, Lesieutre, Matsuzawa, Satriano, Shibata, Silverman and the author proved Conjecture 3.1.1 in the following cases (for details, see [**KS1**], [**KS2**], [**LS1**], [**MSS1**], [**MS**] [**San1**], [**Sil1**], [**Sil2**],).

(1) ([**KS1**, Theorem 2 (a)]) f is a self-morphism and the Néron-Severi group of X has rank one.
(2) ([**KS1**, Theorem 2 (b)]) f is the extension to $\mathbb{P}^N$ of a regular affine automorphism on $\mathbb{A}^N$.
(3) ([**Kaw**, Assertion A], [**KS1**, Theorem 2 (c)]) X is a smooth projective surface and f is an automorphism on X.
(4) ([**Sil1**, Proposition 19]) f is the extension to $\mathbb{P}^N$ of a monomial map on $\mathbb{G}_m^N$ and $P \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$.
(5) ([**KS2**, Corollary 32], [**Sil2**, Theorem 2]) X is an abelian variety. Note that any rational map between abelian varieties is automatically a morphism.
(6) ([**MSS1**, Theorem 1.3]) f is a non-invertible self-morphism of a smooth projective surface.

(7) ([**MS**, Theorem 1.1]) f is a self-morphism of a semi-abelian varieties.
(8) ([**LS1**, Theorem 1.2]) f is a surjective self-morphism of hyper Kahler varieties.
(9) ([**LS1**, Proposition 1.6]) f is a non-invertible surjective self-morphism of smooth projective threefold of Kodaira dimension 0.
(10) ([**San1**, Theorem 1.2]) f is a self-morphism and X is the product $\prod_{i=1}^n X_i$ of smooth projective varieties, with the assumption that each variety X_i satisfies one of the following conditions:
   - the first Betti number of $(X_i)_{\mathbb{C}}$ is zero and the Néron–Severi group of X_i has rank one,
   - X_i is an abelian variety,
   - X_i is an Enriques surface, or
   - X_i is a $K3$ surface.
(11) ([**San1**, Theorem 1.3]) f is a self-morphism and X is the product $X_1 \times X_2$ of positive dimensional varieties such that one of X_1 or X_2 is of general type. (In fact, there do not exist Zariski dense forward f-orbits on such $X_1 \times X_2$.)

Moreover, the arithmetic degrees over function fields are studied in [**MSS2**].

Theorem 3.1.5 ([Matz, Theorem 1.4] and [KS3, Theorem 4]). *Let $f\colon X \dashrightarrow X$ be a dominant rational self-map on a smooth projective variety X over $\overline{\mathbb{Q}}$. Then the inequality*

$$\overline{\alpha}_f(x) \leq \delta_f$$

holds for all $x \in X_f(\overline{\mathbb{Q}})$.

Remark 3.1.6. Let X be a complex smooth projective variety with $\kappa(X) > 0$, $\Phi : X \dashrightarrow W$ the Iitaka fibration of X, and $f\colon X \dashrightarrow X$ a dominant rational self-map on X. Nakayama and Zhang proved that there exists an automorphism $g\colon W \longrightarrow W$ of finite order such that $\Phi \circ f = g \circ \Phi$ (see [**NZ**, Theorem A]). This implies that any dominant rational self-map on a smooth projective variety of positive Kodaira dimension does not have a Zariski dense orbit. So the latter half of Conjecture 3.1.1 is meaningful only for smooth projective varieties of non-positive Kodaira dimension.

3.2. Reductions

We recall some lemmata which are useful to reduce the proof of some cases of Conjecture 3.1.1 to easier cases.

Lemma 3.2.1. *Let X be a smooth projective variety and $f\colon X \longrightarrow X$ a surjective self-morphism. Then Conjecture 3.1.1 holds for f if and only if Conjecture 3.1.1 holds for f^t for some $t \geq 1$.*

Proof. See [**San1**, Lemma 3.3]. □

Lemma 3.2.2 ([Sil2, Lemma 6]). *Let $\psi\colon X \longrightarrow Y$ be a finite morphism between smooth projective varieties. Let $f_X\colon X \longrightarrow X$ and $f_Y\colon Y \longrightarrow Y$ be surjective self-morphisms on X and Y, respectively. Assume that $\psi \circ f_X = f_Y \circ \psi$.*
(i) *For any $P \in X(\overline{\mathbb{Q}})$, we have $\alpha_{f_X}(P) = \alpha_{f_Y}(\psi(P))$.*
(ii) *Assume that ψ is surjective. Then Conjecture 3.1.1 holds for f_X if and only if Conjecture 3.1.1 holds for f_Y.*

Proof. (i) This is a special case of Lemma 2.2.3.

(ii) For a point $P \in X(\overline{\mathbb{Q}})$, the forward f_X-orbit $O_{f_X}(P)$ is Zariski dense in X if and only if the forward f_Y-orbit $O_{f_Y}(\psi(P))$ is Zariski dense in Y since ψ is a finite surjective morphism. Moreover we have $\dim X = \dim Y$. So we obtain

$$\begin{aligned}\delta_{f_X} &= \lim_{n\to\infty}((f_X^n)^*\psi^*H \cdot (\psi^*H)^{\dim X-1})^{1/n}\\ &= \lim_{n\to\infty}(\psi^*(f_Y^n)^*H \cdot (\psi^*H)^{\dim Y-1})^{1/n}\\ &= \lim_{n\to\infty}(\deg(\psi)((f_Y^n)^*H \cdot H^{\dim Y-1}))^{1/n}\\ &= \delta_{f_Y}.\end{aligned}$$

Therefore the assertion follows. □

3.3. Existence of a rational point with the full arithmetic degree

As we see in Remark 3.1.6 and in the proof of Theorem 4.1.1, there does not always exist a Zariski dense orbit for a given self-map. For instance, a self-map cannot have a Zariski dense orbit if it is a self-map on a variety of positive Kodaira dimension. So it is also important to consider whether a self-map has a point whose orbit has full arithmetic complexity, that is, whose arithmetic degree coincides with the dynamical degree. We prove that such a point always exists for any surjective self-morphism on any smooth projective variety.

Theorem 3.3.1. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective self-morphism on X. Then there exists a $\overline{\mathbb{Q}}$-rational point $P \in X(\overline{\mathbb{Q}})$ such that $\alpha_f(P) = \delta_f$.*

If f is an automorphism, we can construct a "large" collection of points whose orbits have full arithmetic complexity.

Theorem 3.3.2. *Let k be a number field, X a smooth projective variety over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ an automorphism. Then there exists a subset $S \subset X(\overline{\mathbb{Q}})$ which satisfies all of the following conditions.*

1. *For every $P \in S$, $\alpha_f(P) = \delta_f$.*
2. *For $P, Q \in S$ with $P \neq Q$, $O_f(P) \cap O_f(Q) = \emptyset$.*
3. *S is Zariski dense in X.*

In this section, we prove Theorem 3.3.1 and Theorem 3.3.2. Theorem 3.3.1 follows from the following lemma. A subset $\Sigma \subset V(\overline{\mathbb{Q}})$ is called a *set of bounded height* if for some (or, equivalently, any) ample divisor A on V, the height function h_A associated with A is a bounded function on Σ.

Lemma 3.3.3. *Let X be a smooth projective variety and $f\colon X \longrightarrow X$ a surjective self-morphism with $\delta_f > 1$. Let $D \not\equiv 0$ be a nef $\mathbb{R}$-divisor such that $f^*D \equiv \delta_f D$. Let $V \subset X$ be a closed subvariety of positive dimension such that $(D^{\dim V} \cdot V) > 0$. Then there exists a non-empty open subset $U \subset V$ and a set $\Sigma \subset U(\overline{\mathbb{Q}})$ of bounded height such that for every $P \in U(\overline{\mathbb{Q}}) \setminus \Sigma$ we have $\alpha_f(P) = \delta_f$.*

Remark 3.3.4. By Perron-Frobenius-type result of [**Bir**, Theorem], there is a nef $\mathbb{R}$-divisor $D \not\equiv 0$ satisfying the condition $f^*D \equiv \delta_f D$ since f^* preserves the nef cone.

Proof. Fix a height function h_D associated with D. For every $P \in X(\overline{\mathbb{Q}})$, the following limit exists (cf. [**KS3**, Theorem 5]).

$$\widehat{h}_D(P) = \lim_{n\to\infty} \frac{h_D(f^n(P))}{\delta_f^n}$$

The function $\widehat{h}_D$ has the following properties (cf. [**KS3**, Theorem 5]).

(i) $\widehat{h}_D = h_D + O(\sqrt{h_H})$ where H is any ample divisor on X and $h_H \geq 1$ is a height function associated with H.

(ii) If $\widehat{h}_D(P) \neq 0$, then $\alpha_f(P) = \delta_f$.

Since $(D^{\dim V} \cdot V) > 0$, we have $(D|_V{}^{\dim V}) > 0$ and $D|_V$ is big. Thus we can write $D|_V \sim_{\mathbb{R}} A + E$ with an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor E on V. Therefore we have

$$\widehat{h}|_{V(\overline{\mathbb{Q}})} = h_A + h_E + O(\sqrt{h_A})$$

where h_A, h_E are height functions associated with A, E and h_A is taken to be $h_A \geq 1$. In particular, there exists a positive real number $B > 0$ such that $h_A + h_E - \widehat{h}|_{V(\overline{\mathbb{Q}})} \leq B\sqrt{h_A}$. Then we have the following inclusions.

$$\begin{aligned}\{P \in V(\overline{\mathbb{Q}}) \mid \widehat{h}(P) \leq 0\} &\subset \{P \in V(\overline{\mathbb{Q}}) \mid h_A(P) + h_E(P) \leq B\sqrt{h_A(P)}\}\\ &\subset \operatorname{Supp} E \cup \{P \in V(\overline{\mathbb{Q}}) \mid h_A(P) \leq B\sqrt{h_A(P)}\}\\ &= \operatorname{Supp} E \cup \{P \in V(\overline{\mathbb{Q}}) \mid h_A(P) \leq B^2\}.\end{aligned}$$

Hence we can take $U = V \setminus \operatorname{Supp} E$ and $\Sigma = \{P \in U(\overline{\mathbb{Q}}) \mid \widehat{h}_D(P) \leq 0\}$. □

Corollary 3.3.5. Let X be a smooth projective variety of dimension N and $f\colon X \longrightarrow X$ a surjective self-morphism. Let C be a irreducible curve which is a complete intersection of ample effective divisors $H_1, \ldots, H_{N-1}$ on X. Then for infinitely many points P on C, we have $\alpha_f(P) = \delta_f$.

Proof. We may assume $\delta_f > 1$. Let D be as in Lemma 3.3.3. Then $(D \cdot C) = (D \cdot H_1 \cdots H_{N-1}) > 0$ (cf. [**KS3**, Lemma 18]). Since $C(\overline{\mathbb{Q}})$ is not a set of bounded height, the assertion follows from Lemma 3.3.3. □

To prove Theorem 3.3.2, we need the following theorem which is a corollary of the dynamical Mordell–Lang conjecture for étale finite morphisms.

Theorem 3.3.6 (Bell–Ghioca–Tucker [BGT1, Corollary 1.4]). *Let $f\colon X \longrightarrow X$ be an étale finite morphism of smooth quasi-projective variety X. Let $P \in X(\overline{\mathbb{Q}})$. If the orbit $O_f(P)$ is Zariski dense in X, then any proper closed subvariety of X intersects $O_f(P)$ in at most finitely many points.*

Proof of Theorem 3.3.2. We may assume $\dim X \geq 2$. Since we are working over $\overline{\mathbb{Q}}$, we can write the set of all proper subvarieties of X as

$$\{V_i \subsetneq X \mid i = 0, 1, 2, \ldots\}.$$

By Corollary 3.3.5, we can take a point $P_0 \in X \setminus V_0$ such that $\alpha_f(P) = \delta_f$. Assume we can construct $P_0, \ldots, P_n$ satisfying the following conditions.

(1) $\alpha_f(P_i) = \delta_f$ for $i = 0, \ldots, n$.

(2) $O_f(P_i) \cap O_f(P_j) = \emptyset$ for $i \neq j$.

(3) $P_i \notin V_i$ for $i = 0, \ldots, n$.

Now, take a complete intersection curve $C \subset X$ satisfying the following conditions.

- For $i = 0, \dots, n$, $C \not\subset O_f(P_i)$ if $\overline{O_f(P_i)} \neq X$.
- For $i = 0, \dots, n$, $C \not\subset O_{f^{-1}}(P_i)$ if $\overline{O_{f^{-1}}(P_i)} \neq X$.
- $C \not\subset V_{n+1}$.

By Theorem 3.3.6, if $\mathcal{O}_{f^{\pm}}(P_i)$ is Zariski dense in X, then $O_{f^{\pm}}(P_i) \cap C$ is a finite set. By Corollary 3.3.5, there exists a point

$$P_{n+1} \in C \setminus \left(\bigcup_{0 \leq i \leq n} O_f(P_i) \cup \bigcup_{0 \leq i \leq n} O_{f^{-1}}(P_i) \cup V_{n+1} \right)$$

such that $\alpha_f(P_{n+1}) = \delta_f$. Then $P_0, \dots, P_{n+1}$ satisfy the same conditions. Therefore we get a subset $S = \{P_i \mid i = 0, 1, 2, \dots\}$ of X which satisfies the desired conditions. $\square$

3.3.1. Applications of the birational invariance. As by-products of birational invariance of the arithmetic degree, we obtain the following two cases for which Conjecture 3.1.1 holds:

Theorem 3.3.7. *Let k be a number field, X a smooth projective irrational surface over $\overline{\mathbb{Q}}$, and $f \colon X \dashrightarrow X$ a birational automorphism on X. Then Conjecture 3.1.1 holds for f.*

Theorem 3.3.8. *Let k be a number field, X a smooth projective toric variety over $\overline{\mathbb{Q}}$, and $f \colon X \longrightarrow X$ a toric surjective self-morphism on X. Then Conjecture 3.1.1 holds for f.*

Lin gives a precise description of the arithmetic degrees of toric self-maps on toric varieties in [**Lin**].

Proof of Theorem 3.3.7. Take a point $P \in X_f(\overline{\mathbb{Q}})$. If $\mathcal{O}_f(P)$ is finite, the limit $\alpha_f(P)$ exists and is equal to 1. Next, assume that the closure $\overline{\mathcal{O}_f(P)}$ of $\mathcal{O}_f(P)$ has dimension 1. Let Z be the normalization of $\overline{\mathcal{O}_f(P)}$ and $\nu \colon Z \longrightarrow X$ the induced morphism. Then an self-morphism $g \colon Z \longrightarrow Z$ satisfying $\nu \circ g = f \circ \nu$ is induced. Take a point $P' \in Z$ such that $\nu(P') = P$. Then $\alpha_g(P') = \alpha_f(P)$ since ν is finite by Lemma 3.2.2 (i). It follows from [**KS2**, Theorem 3] that $\alpha_g(P')$ exists (note that [**KS2**, Theorem 3] holds for possibly non-surjective self-morphisms on possibly reducible normal varieties). Therefore $\alpha_f(P)$ exists.

Finally, assume that $\mathcal{O}_f(P)$ is Zariski dense. If $\delta_f = 1$, then $1 \leq \underline{\alpha}_f(P) \leq \overline{\alpha}_f(P) \leq \delta_f = 1$ by Theorem 3.1.5, so $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f = 1$. So we may assume that $\delta_f > 1$. Since X is irrational and $\delta_f > 1$, $\kappa(X)$ must be non-negative (cf. [**DF**, Theorem 0.4, Proposition 7.1 and Theorem 7.2]). Take a birational morphism $\mu \colon X \longrightarrow Y$ to the minimal model Y of X and let $g \colon Y \dashrightarrow Y$ be the birational automorphism on Y defined as $g = \mu \circ f \circ \mu^{-1}$. Then g is in fact an automorphism since, if g has indeterminacy, Y must have a K_Y-negative curve. It is obvious that $\mathcal{O}_g(\mu(P))$ is also Zariski dense in Y. Since $\mu(\mathrm{Exc}(\mu))$ is a finite set, there is a positive integer n_0 such that $\mu(f^n(P)) = g^n(\mu(P)) \notin \mu(\mathrm{Exc}(\mu))$ for $n \geq n_0$. So we have $f^n(P) \notin \mathrm{Exc}(\mu)$ for $n \geq n_0$. Replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset X \setminus \mathrm{Exc}(\mu)$. Applying Lemma 2.2.3 to P, it follows that $\alpha_f(P) = \alpha_g(\mu(P))$. We know that $\alpha_g(\mu(P))$ exists since g is a morphism. So $\alpha_f(P)$ also exists. The equality $\alpha_g(\mu(P)) = \delta_g$ holds as a consequence of Conjecture 3.1.1 for automorphisms on surfaces (cf. Remark 3.1.4 (3)). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. $\square$

Proof of Theorem 3.3.8. Let $\mathbb{G}_m^d \subset X$ be the torus embedded as an open dense subset in X. Then $f|_{\mathbb{G}_m^d} : \mathbb{G}_m^d \longrightarrow \mathbb{G}_m^d$ is a homomorphism of algebraic groups by assumtion. Let $\mathbb{G}_m^d \subset \mathbb{P}^d$ be the natural embedding of $\mathbb{G}_m^d$ to the projective space $\mathbb{P}^d$ and $g \colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be the induced rational self-map. Then g is a monomial map.

Take $P \in X(\overline{\mathbb{Q}})$ such that $\mathcal{O}_f(P)$ is Zariski dense. Note that $\alpha_f(P)$ exists since f is a morphism. Since $\mathcal{O}_f(P)$ is Zariski dense and $f(\mathbb{G}_m^d) \subset \mathbb{G}_m^d$, there is a positive integer n_0 such that $f^n(P) \in \mathbb{G}_m^d$ for $n \geq n_0$. By replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset \mathbb{G}_m^d$. Applying Lemma 2.2.3 to P, it follows that $\alpha_f(P) = \alpha_g(P)$.

The equality $\alpha_g(P) = \delta_g$ holds as a consequence of Conjecture 3.1.1 for monomial maps (cf. Remark 3.1.4 (4)). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. $\square$

CHAPTER 4

Self-morphisms on surfaces

4.1. Outline of this chapter

In this chapter we prove Conjecture 3.1.1 for surjective self-morphisms on smooth projective surfaces.

Theorem 4.1.1. *Let X be a smooth projective surface over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective self-morphism on X. Then Conjecture 3.1.1 holds for f.*

The main idea of the proof of Theorem 4.1.1 is reducing the cases to the lower dimensional cases along minimal model program and [**San1**, Lemma 5.2]. The proof of [**LS1**, Proposition 1.6] is also by this idea.

In Section 4.2, by using the Enriques classification of smooth projective surfaces, we reduce Theorem 4.1.1 to three cases, i.e. the case of $\mathbb{P}^1$-bundles, hyperelliptic surfaces, and surfaces of Kodaira dimension one. In Section 4.3 we recall fundamental properties of $\mathbb{P}^1$-bundles over curves. In Section 4.4, Section 4.5, and Section 4.6, we prove Theorem 4.1.1 in each case explained in Section 4.2.

4.2. Self-morphisms on surfaces

We start to prove Theorem 4.1.1. Since Conjecture 3.1.1 for automorphisms on surfaces is already proved by Kawaguchi (see Remark 3.1.4 (3)), it is sufficient to prove Theorem 4.1.1 for *non-invertible* self-morphisms, that is, surjective self-morphisms which are not automorphisms.

Let $f\colon X \longrightarrow X$ be a non-invertible self-morphism on a surface. We divide the proof of Theorem 4.1.1 according to the Kodaira dimension of X.

(I) $\kappa(X) = -\infty$; we need the following result due to Nakayama.

Lemma 4.2.1 (cf. [Nak, Lemma 10]). *Let $f\colon X \longrightarrow X$ be a non-invertible self-morphism on a surface X with $\kappa(X) = -\infty$. Then there is a positive integer m such that $f^m(E) = E$ for any irreducible curve E on X with negative self-intersection.*

Proof. See [**Nak**, Lemma 10]. □

Let $\mu\colon X \longrightarrow X'$ be the contraction of a (-1)-curve E on X. By Lemma 4.2.1, there is a positive integer m such that $f^m(E) = E$. Then f^m induces an self-morphism $f'\colon X' \longrightarrow X'$ such that $\mu \circ f^m = f' \circ \mu$. Using Lemma 3.2.1 and Lemma 2.2.3, the assertion of Theorem 4.1.1 for f follows from that for f'. Continuing this process, we may assume that X is relatively minimal.

When X is irrational and relatively minimal, X is a $\mathbb{P}^1$-bundle over a curve C with $g(C) \geq 1$.

When X is rational and relatively minimal, X is isomorphic to $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$ with $n \neq 1$. Note that Conjecture 3.1.1 holds for surjective self-morphisms on projective spaces (see Remark 3.1.4 (1)).

(II) $\kappa(X) = 0$; for surfaces with non-negative Kodaira dimension, we use the following result due to Fujimoto.

Lemma 4.2.2 (cf. [Fuj, Lemma 2.3 and Proposition 3.1]). *Let $f\colon X \longrightarrow X$ be a non-invertible self-morphism on a surface X with $\kappa(X) \geq 0$. Then X is minimal and f is étale.*

PROOF. See [**Fuj**, Lemma 2.3 and Proposition 3.1] $\square$

So X is either an abelian surface, a hyperelliptic surface, a K3 surface, or an Enriques surface. Since f is étale, we have $\chi(X, \mathcal{O}_X) = \deg(f)\chi(X, \mathcal{O}_X)$. Now $\deg(f) \geq 2$ by assumption, so $\chi(X, \mathcal{O}_X) = 0$ (cf. [**Fuj**, Corollary 2.4]). Hence X must be either an abelian surface or a hyperelliptic surface because K3 surfaces and Enriques surfaces have non-zero Euler characteristics. Note that Conjecture 3.1.1 is valid for self-morphisms on abelian varieties (see Remark 3.1.4 (5)).

(III) $\kappa(X) = 1$; this case will be treated in Section 4.6.

(IV) $\kappa(X) = 2$; the following fact is well-known.

Lemma 4.2.3. *Let X be a smooth projective variety of general type. Then any surjective self-morphism on X is an automorphism. Furthermore, the group of automorphisms* $\mathrm{Aut}(X)$ *on X has finite order.*

PROOF. See [**Fuj**, Proposition 2.6], [**Iit**, Theorem 11.12], or [**Matm**, Corollary 2]. $\square$

So there is no non-invertible self-morphism on X. As a summary, the remaining cases for the proof of Theorem 4.1.1 are the following:

- Non-invertible self-morphisms on $\mathbb{P}^1$-bundles over a curve.
- Non-invertible self-morphisms on hyperelliptic surfaces.
- Non-invertible self-morphisms on surfaces of Kodaira dimension 1.

These three cases are studied in Sections 4.3-4.6 below.

Remark 4.2.4. Fujimoto and Nakayama gave a complete classification of surfaces which admit non-invertible self-morphisms (cf. [**FN2**, Theorem 1.1], [**Fuj**, Proposition 3.3], [**Nak**, Theorem 3], and [**FN1**, Appendix to Section 4]).

4.3. Some properties of $\mathbb{P}^1$-bundles over curves

In this section, we recall and prove some properties of $\mathbb{P}^1$-bundles (see [**Har**, Chapter V.2], [**Hom1**], [**Hom2**] for details). In this section, let X be a $\mathbb{P}^1$-bundle over a curve C. Let $\pi\colon X \longrightarrow C$ be the projection.

Proposition 4.3.1. *We can represent X as $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 on C such that $H^0(\mathcal{E}) \neq 0$ but $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for all invertible sheaves $\mathcal{L}$ on C with* $\deg \mathcal{L} < 0$. *The integer $e := -\deg \mathcal{E}$ does not depend on the choice of such $\mathcal{E}$. Furthermore, there is a section $\sigma\colon C \longrightarrow X$ with image C_0 such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$.*

PROOF. See [**Har**, V Proposition 2.8]. $\square$

Lemma 4.3.2. *The Picard group and the Néron–Severi group of X have the following structure:*

$$\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \pi^* \mathrm{Pic}(C),$$
$$\mathrm{NS}(X) \cong \mathbb{Z} \oplus \pi^* \mathrm{NS}(C) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Furthermore, the image C_0 of the section $\sigma\colon C \longrightarrow X$ in Proposition 4.3.1 generates the first direct factor of $\mathrm{Pic}(X)$ *and* $\mathrm{NS}(X)$.

Proof. See [**Har**, V, Proposition 2.3]. □

Lemma 4.3.3. *Let $F \in \mathrm{NS}(X)$ be a fiber $\pi^{-1}(p) = \pi^*p$ over a point $p \in C(\overline{\mathbb{Q}})$, and e the integer defined in Proposition 4.3.1. Then the intersection numbers of generators of* $\mathrm{NS}(X)$ *are as follows.*

$$\begin{aligned} F \cdot F &= 0, \\ F \cdot C_0 &= 1, \\ C_0 \cdot C_0 &= -e. \end{aligned}$$

Proof. It is easy to see that the equalities $F \cdot F = 0$ and $F \cdot C_0 = 1$ hold. For the last equality, see [**Har**, V, Proposition 2.9]. □

We say that f *preserves fibers* if there is an self-morphism f_C on C such that $\pi \circ f = f_C \circ \pi$. In our situation, since there is a section $\sigma\colon C \longrightarrow X$, f preserves fibers if and only if, for any point $p \in C$, there is a point $q \in C$ such that $f(\pi^{-1}(p)) \subset \pi^{-1}(q)$.

The following lemma appears in [**Ame**, p. 18] in more general form. But we need it only in the case of $\mathbb{P}^1$-bundles on a curve, and the proof in general case is similar to our case. So we deal only with the case of $\mathbb{P}^1$-bundle on a curve.

Lemma 4.3.4. *For any surjective self-morphism f on X, the iterate f^2 preserves fibers.*

Proof. By the projection formula, the fibers of $\pi\colon X \longrightarrow C$ can be characterized as connected curves having intersection number zero with any fiber $F_p = \pi^*p$, $p \in C$. Hence, to check that the iterate f^2 sends fibers to fibers, it suffices to show that $(f^2)^*(\pi^*\mathrm{NS}(C)_{\mathbb{R}}) = \pi^*\mathrm{NS}(C)_{\mathbb{R}}$. Now $\dim \mathrm{NS}(X)_{\mathbb{R}} = 2$ and the set of the numerical classes in X with self-intersection zero forms two lines, one of which is $\pi^*\mathrm{NS}(C)_{\mathbb{R}}$, and f^* fixes or interchanges them. So $(f^2)^*$ fixes $\pi^*\mathrm{NS}(C)_{\mathbb{R}}$. □

The following might be well-known, but we give a proof for the reader's convenience.

Lemma 4.3.5. *A surjective self-morphism f preserves fibers if and only if there exists a non-zero integer a such that $f^*F \equiv aF$. Here, F is the numerical class of a fiber.*

Proof. Assume $f^*F \equiv aF$. For any point $p \in C$, we set $F_p := \pi^{-1}(p) = \pi^*p$. If f does not preserve fibers, there is a point $p \in C$ such that $f(F_p) \cdot F > 0$. Now we can calculate the intersection number as follows:

$$\begin{aligned} 0 = F \cdot aF = F \cdot (f^*F) &= F_p \cdot (f^*F) \\ &= (f_*F_p) \cdot F = \deg(f|_{F_p}) \cdot (f(F_p) \cdot F) > 0. \end{aligned}$$

This is a contradiction. Hence f preserves fibers.

Next, assume that f preserves fibers. Write $f^*F = aF + bC_0$. Then we can also calculate the intersection number as follows:

$$\begin{aligned} b = F \cdot (aF + bC_0) &= F \cdot f^*F = (f_*F) \cdot F \\ &= \deg(f|_F) \cdot (F \cdot F) = 0. \end{aligned}$$

Further, by the injectivity of f^*, we have $a \neq 0$. The proof is complete. □

Lemma 4.3.6. *If $\mathcal{E}$ splits, i.e., if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$, the invariant e of $X = \mathbb{P}(\mathcal{E})$ is non-negative.*

Proof. See [**Har**, V, Example 2.11.3]. ☐

Lemma 4.3.7. *Assume that $e \geq 0$. Then for a divisor $D = aF + bC_0 \in \mathrm{NS}(X)$, the following properties are equivalent.*

- *D is ample.*
- *$a > be$ and $b > 0$.*

In other words, the nef cone of X is generated by F and $eF + C_0$.

Proof. See [**Har**, V, Proposition 2.20]. ☐

We can prove a result stronger than Lemma 4.3.4 as follows.

Lemma 4.3.8. *Assume that $e > 0$. Then any surjective self-morphism $f\colon X \longrightarrow X$ preserves fibers.*

Proof. By Lemma 4.3.5, it is enough to prove $f^*F \equiv aF$ for some integer $a > 0$. We can write $f^*F \equiv aF + bC_0$ for some integers $a, b \geq 0$.

Since we have

$$aF + bC_0 = (a - be)F + b(eF + C_0)$$

and f preserves the nef cone and the ample cone, either of the equalities $a - be = 0$ or $b = 0$ holds.

We have

$$\begin{aligned} 0 &= \deg(f)(F \cdot F) = (f_*f^*F) \cdot F \\ &= (f^*F) \cdot (f^*F) = (aF + bC_0) \cdot (aF + bC_0) \\ &= 2ab - b^2e = b(2a - be). \end{aligned}$$

So either of the equalities $b = 0$ or $2a - be = 0$ holds.

If we have $b \neq 0$, we have $a - be = 0$ and $2a - be = 0$. So we get $a = 0$. But since $e \neq 0$, we obtain $b = 0$. This is a contradiction. Consequently, we get $b = 0$ and $f^*F \equiv aF$. ☐

Lemma 4.3.9. *Fix a fiber $F = F_p$ for a point $p \in C(\overline{\mathbb{Q}})$. Let f be a surjective self-morphism on X preserving fibers, f_C the self-morphism on C satisfying $\pi \circ f = f_C \circ \pi$, $f_F := f|_F\colon F \longrightarrow f(F)$ the restriction of f to the fiber F. Set $f^*F \equiv aF$ and $f^*C_0 \equiv cF + dC_0$. Then we have $a = \deg(f_C)$, $d = \deg(f_F)$, $\deg(f) = ad$, and $\delta_f = \max\{a, d\}$.*

Proof. Our assertions follow from the following equalities of divisor classes in $\mathrm{NS}(X)$ and of intersection numbers:

$$\begin{aligned} aF &= f^*F = f^*\pi^*p \\ &= \pi^*f_C^*p = \pi^*(\deg(f_C)p) \\ &= \deg(f_C)\pi^*p = \deg(f_C)F, \\ \deg(f)F &= f_*f^*F = f_*f^*\pi^*p \\ &= f_*\pi^*f_C^*p = f_*\pi^*(\deg(f_C)p) \\ &= \deg(f_C)f_*F = \deg(f_C)\deg(f_F)f(F) \\ &= \deg(f_C)\deg(f_F)F \\ \deg(f) &= \deg(f)C_0 \cdot F = (f_*f^*C_0) \cdot F \\ &= (f^*C_0) \cdot (f^*F) = (cF + dC_0) \cdot aF = ad. \end{aligned}$$

The last assertion $\delta_f = \max\{a, d\}$ follows from the functoriality of f^* and the equality $\delta_f = \lim_{n\to\infty} \rho((f^n)^*)^{1/n} = \rho(f^*)$ (cf. Remark 2.1.5). ☐

Lemma 4.3.10. *Let Notation be as in Lemma 4.3.9. Assume that $e \geq 0$. Then both F and C_0 are eigenvectors of $f^*\colon \mathrm{NS}(X)_{\mathbb{R}} \longrightarrow \mathrm{NS}(X)_{\mathbb{R}}$. Further, if e is positive, then we have $\deg(f_C) = \deg(f_F)$.*

Proof. Set $f^*F = aF$ and $f^*C_0 = cF + dC_0$ in $\mathrm{NS}(X)$. Then we have

$$\begin{aligned} -ead = -e \deg f &= (f_* f^* C_0) \cdot C_0 \\ &= (f^*C_0)^2 = (cF + dC_0)^2 = 2cd - ed^2. \end{aligned}$$

Hence, we get $c = e(d - a)/2$. We have the following equalities in $\mathrm{NS}(X)$:

$$f^*(eF + C_0) = aeF + (cF + dC_0) = (ae + c)F + dC_0.$$

By the fact that f^*D is ample if and only if D is ample, it follows that $eF + C_0$ is an eigenvector of f^*. Thus, we have

$$de = ae + c = ae + e(d - a)/2 = e(d + a)/2.$$

Therefore, the equality $e(d - a) = 0$ holds. So $c = e(d - a)/2 = 0$ holds.

Further, we assume that $e > 0$. Then it follows that $d - a = 0$. So we have $\deg(f_C) = a = d = \deg(f_F)$. □

The following lemma is used in Subsection 4.4.2.

Lemma 4.3.11. *Let $\mathcal{L}$ be a non-trivial invertible sheaf of degree 0 on a curve C with $g(C) \geq 1$, $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, and $X = \mathbb{P}(\mathcal{E})$. Let C_0, C_1 be sections corresponding to the projections $\mathcal{E} \longrightarrow \mathcal{L}$ and $\mathcal{E} \longrightarrow \mathcal{O}_C$. If $\sigma\colon C \longrightarrow X$ is a section such that $(\sigma(C))^2 = 0$, then $\sigma(C)$ is equal to C_0 or C_1.*

Proof. Note that $e = 0$ in this case and thus $(C_0^2) = 0$. Moreover, $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}^{-1}$. Set $\sigma(C) \equiv aC_0 + bF$. Then $a = (\sigma(C) \cdot F) = 1$ and $2ab = (\sigma(C)^2) = 0$. Thus $\sigma(C) \equiv C_0$. Therefore, $\mathcal{O}_X(\sigma(C)) \cong \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{N}$ for some invertible sheaf $\mathcal{N}$ of degree 0 on C. Then

$$\begin{aligned} 0 \neq H^0(X, \mathcal{O}_X(\sigma(C))) &= H^0(C, \pi_*\mathcal{O}_X(C_0) \otimes \mathcal{N}) \\ &= H^0(C, (\mathcal{L} \oplus \mathcal{O}_C) \otimes \mathcal{N}) \end{aligned}$$

and this implies $\mathcal{N} \cong \mathcal{O}_C$ or $\mathcal{N} \cong \mathcal{L}^{-1}$. Hence $\mathcal{O}_X(\sigma(C))$ is isomorphic to $\mathcal{O}_X(C_0)$ or $\mathcal{O}_X(C_0) \otimes \pi^*\mathcal{L}^{-1} = \mathcal{O}_X(C_1)$. Since $\mathcal{L}$ is non-trivial, we have $H^0(\mathcal{O}_X(C_0)) = H^0(\mathcal{O}_X(C_1)) = \overline{\mathbb{Q}}$ and we get $\sigma(C) = C_0$ or C_1. □

4.4. $\mathbb{P}^1$-bundles over curves

In this section, we prove Conjecture 3.1.1 for non-invertible self-morphisms on $\mathbb{P}^1$-bundles over a curve. We divide the proof according to the genus of the base curve.

4.4.1. $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.

Theorem 4.4.1. *Let $\pi\colon X \longrightarrow \mathbb{P}^1$ be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and $f\colon X \to X$ be a non-invertible self-morphism. Then Conjecture 3.1.1 holds for f.*

Proof. Take a locally free sheaf $\mathcal{E}$ of rank 2 on $\mathbb{P}^1$ such that $X \cong \mathbb{P}(\mathcal{E})$ and $\deg \mathcal{E} = -e$ (cf. Proposition 4.3.1). Then $\mathcal{E}$ splits (see [**Har**, V. Corollary 2.14]). When X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the case of $e = 0$, the assertion holds by [**San1**, Theorem 1.3]. When X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the case of $e > 0$, the self-morphism f preserves fibers and induces an self-morphism $f_{\mathbb{P}^1}$ on the base curve $\mathbb{P}^1$. By Lemma 4.3.10, we have $\delta_f = \delta_{f_{\mathbb{P}^1}}$.

Fix a point $p \in \mathbb{P}^1$ and set $F = \pi^* p$. Let $P \in X(\overline{\mathbb{Q}})$ be a point whose forward f-orbit is Zariski dense in X. Then the forward $f_{\mathbb{P}^1}$-orbit of $\pi(P)$ is also Zariski dense in $\mathbb{P}^1$. Now the assertion follows from the following computation.

$$\begin{aligned}\alpha_f(P) &\geq \lim_{n\to\infty} h_F(f^n(P))^{1/n} = \lim_{n\to\infty} h_{\pi^*p}(f^n(P))^{1/n}\\ &= \lim_{n\to\infty} h_p(\pi\circ f^n(P))^{1/n} = \lim_{n\to\infty} h_p(f^n_{\mathbb{P}^1}\circ\pi(P))^{1/n} = \delta_{f_{\mathbb{P}^1}} = \delta_f.\end{aligned}$$

□

4.4.2. $\mathbb{P}^1$-bundles over genus one curves. In this subsection, we prove Conjecture 3.1.1 for any self-morphisms on a $\mathbb{P}^1$-bundle on a curve C of genus one.

The following result is due to Amerik. Note that Amerik in fact proved it for $\mathbb{P}^1$-bundles over varieties of arbitrary dimension (cf. [**Ame**]).

Lemma 4.4.2. *Let $X = \mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^1$-bundle over a curve C. If X has a fiber-preserving surjective self-morphism whose restriction to a general fiber has degree greater than 1, then $\mathcal{E}$ splits into a direct sum of two line bundles after a finite base change. Furthermore, if $\mathcal{E}$ is semistable, then $\mathcal{E}$ splits into a direct sum of two line bundles after an étale base change.*

Proof. See [**Ame**, Theorem 2 and Proposition 2.4]. □

Lemma 4.4.3. *Let E be a curve of genus one with an self-morphism $f: E \longrightarrow E$. If $g: E' \longrightarrow E$ is a finite étale covering of E, there exists a finite étale covering $h: E'' \longrightarrow E'$ and an self-morphism $f': E'' \longrightarrow E''$ such that $f\circ g\circ h = g\circ h\circ f'$. Furthermore, we can take h as satisfying $E'' = E$.*

Proof. At first, since E' is an étale covering of genus one curve E, E' is also a genus one curve. By fixing a rational point $p \in E'(\overline{\mathbb{Q}})$ and $g(p) \in E(\overline{\mathbb{Q}})$, these curves E and E' are regarded as elliptic curves, and g can be regarded as an isogeny between elliptic curves. Let $h := \hat{g}: E \longrightarrow E'$ be the dual isogeny of g. The morphism f is decomposed as $f = \tau_c \circ \psi$ for a homomorphism ψ and a translation map τ_c by $c \in E(\overline{\mathbb{Q}})$. Fix a rational point $c' \in E(\overline{\mathbb{Q}})$ such that $[\deg(g)](c') = c$ and consider the translation map $\tau_{c'}$, where $[\deg(g)]$ is the multiplication by $\deg(g)$. We set $f' = \tau_{c'} \circ \psi$. Then we have the following equalities.

$$\begin{aligned}&f\circ g\circ h = \tau_c\circ\psi\circ g\circ\hat{g}\\ &= \tau_c\circ\psi\circ[\deg(g)] = \tau_c\circ[\deg(g)]\circ\psi\\ &= [\deg(g)]\circ\tau_{c'}\circ\psi = g\circ h\circ f'.\end{aligned}$$

This is what we want. □

Proposition 4.4.4. *Let $\mathcal{E}$ be a locally free sheaf of rank 2 on a genus one curve C and $X = \mathbb{P}(\mathcal{E})$. Suppose Conjecture 3.1.1 holds for any non-invertible self-morphism on X with $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle of degree zero on C. Then Conjecture 3.1.1 holds for any non-invertible self-morphism on $X = \mathbb{P}(\mathcal{E})$ for any $\mathcal{E}$.*

Proof. By Lemma 4.3.4 and Lemma 3.2.1, we may assume that f preserves fibers. We can prove Conjecture 3.1.1 in the case of $\deg(f|_F) = 1$ in the same way as in the case of $g(C) = 0$ since $\deg(f|_F) = 1 \leq \deg(f_C)$. Since we are considering the case of $g(C) = 1$, if $\mathcal{E}$ is indecomposable, then $\mathcal{E}$ is semistable (see [**Muk**, 10.2 (c), Proposition 10.49] or [**Har**, V. Exercise 2.8 (c)]). By Lemma 4.4.2, if $\deg(f|_F) > 1$ and $\mathcal{E}$ is indecomposable, there is a finite étale covering $g: E \longrightarrow C$ satisfying that $E \times_C X \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ over E. Furthermore, by Lemma 4.4.3, we can take E equal to C and there is an self-morphism $f'_C \colon C \longrightarrow C$ satisfying $f_C \circ g = g \circ f'_C$. Then by the universality of cartesian product $X \times_{C,g} C$, we have an induced self-morphism $f' \colon X \times_{C,g} C \longrightarrow X \times_{C,g} C$. By Lemma 3.2.2, it is enough to prove Conjecture 3.1.1 for the self-morphism f'. Thus, we may assume that $\mathcal{E}$ is decomposable, i.e., $X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. Then the invariant e is non-negative by Lemma 4.3.6. When e is positive, by the same method as the proof of Theorem 4.1.1 in the case of $g(C) = 0$, the proof is complete. When $e = 0$, we have $\deg \mathcal{L} = 0$ and the assertion holds by the assumption. □

In the rest of this subsection, we keep the following notation. Let C be a genus one curve and $\mathcal{L}$ an invertible sheaf on C with degree 0. Let $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{O}_C \oplus \mathcal{L}))$ and $\pi \colon X \longrightarrow C$ the projection. When $\mathcal{L}$ is trivial, we have $X \cong C \times \mathbb{P}^1$, and by [**San1**, Theorem1.3], Conjecture 3.1.1 is true for X. Thus we may assume $\mathcal{L}$ is non-trivial. In this case, we have two sections of $\pi \colon X \longrightarrow C$ corresponding to the projections $\mathcal{O}_C \oplus \mathcal{L} \longrightarrow \mathcal{L}$ and $\mathcal{O}_C \oplus \mathcal{L} \longrightarrow \mathcal{O}_C$. Let C_0 and C_1 denote the images of these sections. Then we have $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) = \mathcal{O}_X(1) \otimes \pi^*\mathcal{L}^{-1}$. Since $\mathcal{L}$ is non-trivial, we have $C_0 \neq C_1$. But since $\deg \mathcal{L} = 0$, C_0 and C_1 are numerically equivalent. Thus $(C_0 \cdot C_1) = (C_0^2) = 0$ and therefore $C_0 \cap C_1 = \emptyset$.

Let f be a non-invertible self-morphism on X such that there is a surjective self-morphism $f_C \colon C \longrightarrow C$ with $\pi \circ f = f_C \circ \pi$.

Lemma 4.4.5. *When $\mathcal{L}$ is a torsion element of* $\mathrm{Pic}\, C$*, Conjecture 3.1.1 holds for f.*

Proof. We fix an algebraic group structure on C. Since $\mathcal{L}$ is torsion, there exists a positive integer $n > 0$ such that $[n]^*\mathcal{L} \cong \mathcal{O}_C$. Then the base change of $\pi \colon X \longrightarrow C$ by $[n] \colon C \longrightarrow C$ is the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times C \longrightarrow C$. Applying Lemma 4.4.3 to $g = [n]$, we get a finite morphism $h \colon C \longrightarrow C$ such that the base change of $\pi \colon X \longrightarrow C$ by $h \colon C \longrightarrow C$ is $\mathbb{P}^1 \times C \longrightarrow C$ and there exists a finite morphism $f'_C \colon C \longrightarrow C$ with $f_C \circ h = h \circ f'_C$. Then f induces a non-invertible self-morphism $f' \colon \mathbb{P}^1 \times C \longrightarrow \mathbb{P}^1 \times C$. By [**San1**, Theorem 1.3], Conjecture 3.1.1 holds for f'. By Lemma 3.2.2, Conjecture 3.1.1 holds also for f. □

Now, let F be the numerical class of a fiber of π. By Lemma 4.3.10, we have

$$f^*F \equiv aF,$$
$$f^*C_0 \equiv bC_0$$

for some integers $a, b \geq 1$. Note that $a = \deg f_C$, $b = \deg f|_F$ and $ab = \deg f$ (cf. Lemma 4.3.9).

Lemma 4.4.6.

(1) *One of the equalities $f(C_0) = C_0$, $f(C_0) = C_1$ and $f(C_0) \cap C_0 = f(C_0) \cap C_1 = \emptyset$ holds. The same is true for $f(C_1)$.*
(2) *If $f(C_0) \cap C_i = \emptyset$ for $i = 0, 1$, then the base change of $\pi \colon X \longrightarrow C$ by $f_C \colon C \longrightarrow C$ is isomorphic to $\mathbb{P}^1 \times C$. In particular, $f_C^*\mathcal{L} \cong \mathcal{O}_C$ and $\mathcal{L}$ is a torsion element of* $\mathrm{Pic}\, C$*. The same conclusion holds under the assumption that $f(C_1) \cap C_i = \emptyset$ for $i = 0, 1$.*

Proof. (1) Since $f^*C_i \equiv bC_i$, $C_0 \equiv C_1$ and $(C_0^2) = 0$, we have $(f_*C_i \cdot C_j) = 0$ for every i and j. Thus the assertion follows.

(2) Assume $f(C_0) \cap C_i = \emptyset$ for $i = 0, 1$. Consider the following Cartesian diagram.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ C \arrow[r, "f_C"] & C \end{tikzcd}$$

Then Y is a $\mathbb{P}^1$-bundle over C associated with the vector bundle $\mathcal{O}_C \oplus f_C^*\mathcal{L}$. The pull-backs $C_i = g^{-1}(C_i), i = 0, 1$ are sections of π'. By the projection formula, we have $(C_i'^2) = 0$. Let $\sigma\colon C \longrightarrow X$ be the section with $\sigma(C) = C_0$. Since $\pi \circ f \circ \sigma = f_C$, we get a section $s\colon C \longrightarrow Y$ of π'.

$$\begin{tikzcd} & C \arrow[d, "\sigma"] \arrow[dd, bend right=60, "s"'] \arrow[ddd, bend right=90, "\mathrm{id}"'] \\ & X \arrow[d, "f"] \\ Y \arrow[r, "g"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ C \arrow[r, "f_C"'] & C \end{tikzcd}$$

Note that $g(s(C)) = f(C_0) \neq C_0, C_1$. Thus $s(C), C_0', C_1'$ are distinct sections of π'. Moreover, by the projection formula, we have $(s(C) \cdot C_0') = 0$. Thus we have three sections which are numerically equivalent to each other. Then Lemma 4.3.11 implies $f_C^*\mathcal{L} \cong \mathcal{O}_C$ and $Y \cong \mathbb{P}^1 \times C$. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is an isogeny, the kernel of f_C^* is finite and thus $\mathcal{L}$ is a torsion element of $\operatorname{Pic} C$. $\square$

Lemma 4.4.7.

(1) *Suppose that*
- $\mathcal{L}$ *is non-torsion in* $\operatorname{Pic} C$,
- $f(C_0) = C_0$ *or* C_1, *and*
- $f(C_1) = C_0$ *or* C_1.

Then $f(C_0) = C_0$ *and* $f(C_1) = C_1$, *or* $f(C_0) = C_1$ *and* $f(C_1) = C_0$.

(2) *If the equalities* $f(C_0) = C_0$ *and* $f(C_1) = C_1$ *hold, then* $f^*C_i \sim_{\mathbb{Q}} bC_i$ *for* $i = 0$ *and* 1.

Proof. (1) Assume that $f(C_0) = C_0$ and $f(C_1) = C_0$. Then $f_*C_0 = aC_0$ and $f_*C_1 = aC_0$ as cycles. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is surjective, there exists a degree zero divisor M on C such that $f_C^*\mathcal{O}_C(M) \cong \mathcal{L}$. Then $C_1 \sim C_0 - \pi^* f_C^* M$. Hence

$$aC_0 = f_*C_1 \sim (f_*C_0 - f_*\pi^* f_C^* M) = (aC_0 - f_*\pi^* f_C^* M)$$

and

$$0 \sim f_*\pi^* f_C^* M \sim f_* f^* \pi^* M \sim (\deg f)\pi^* M.$$

Thus $\pi^* M$ is torsion and so is M. This implies that $\mathcal{L}$ is torsion, which contradicts the assumption.

The same argument shows that the case when $f(C_0) = C_1$ and $f(C_1) = C_1$ does not occur.

(2) In this case, we have $f_*C_0 \sim aC_0$. We can write $f^*C_0 \sim bC_0 + \pi^* D$ for some degree zero divisor D on C. Thus

$$(\deg f)C_0 \sim f_* f^* C_0 \sim abC_0 + f_*\pi^* D = (\deg f)C_0 + f_*\pi^* D$$

and $f_*\pi^*D \sim 0$. Since $f_C^*\colon \operatorname{Pic}^0 C \longrightarrow \operatorname{Pic}^0 C$ is surjective, there exists a degree zero divisor D' on C such that $f_C^* D' \sim D$. Then

$$0 \sim f_*\pi^*D \sim f_*\pi^* f_C^* D' \sim f_* f^* \pi^* D' \sim (\deg f)\pi^* D'.$$

Hence $\pi^*D' \sim_{\mathbb{Q}} 0$ and $D' \sim_{\mathbb{Q}} 0$. Therefore $D \sim_{\mathbb{Q}} 0$ and $f^*C_0 \sim_{\mathbb{Q}} bC_0$.

Similarly, we have $f^*C_1 \sim_{\mathbb{Q}} bC_1$. $\square$

Lemma 4.4.8. *Suppose $a < b$. If $f^*C_i \sim_{\mathbb{Q}} bC_i$ for $i = 0, 1$, the line bundle $\mathcal{L}$ is a torsion element of* $\operatorname{Pic} C$.

Proof. Let L be a divisor on C such that $\mathcal{O}_C(L) \cong \mathcal{L}$. Note that $C_1 \sim C_0 - \pi^*L$. Thus

$$f^*\pi^*L \sim f^*(C_0 - C_1) \sim_{\mathbb{Q}} bC_0 - bC_1 \sim b\pi^*L$$

and $f_C^*L \sim_{\mathbb{Q}} bL$ hold.

Thus, from the following lemma, $\mathcal{L}$ is a torsion element. $\square$

Lemma 4.4.9. *Let a, b be integers such that $1 \le a < b$. Let C be a curve of genus one defined over an algebraically closed field k. Let $f_C\colon C \longrightarrow C$ be an self-morphism of* $\deg f_C = a$. *If L is a divisor on C of degree* 0 *satisfying*

$$f_C^* L \sim_{\mathbb{Q}} bL,$$

the divisor L is a torsion element of $\operatorname{Pic}^0(C)$.

Proof. By the definition of $\mathbb{Q}$-linear equivalence, we have $f_C^* rL \sim brL$ for some positive integer r. Since the curve C is of genus one, the group $\operatorname{Pic}^0(C)$ is an elliptic curve. Assume the (group) endomorphism

$$f_C^* - [b]\colon \operatorname{Pic}^0(C) \longrightarrow \operatorname{Pic}^0(C)$$

is the 0 map. Then we have the equalities $a = \deg f_C = \deg f_C^* = \deg[b] = b^2$. But this contradicts to the inequality $1 \le a < b$. Hence the map $f_C^* - [b]$ is an isogeny, and $\operatorname{Ker}(f_C^* - [b]) \subset \operatorname{Pic}^0(C)$ is a finite group scheme. In particular, the order of $rL \in \operatorname{Ker}(f_C^* - [b])(k)$ is finite. Thus, L is a torsion element. $\square$

Remark 4.4.10. We can actually prove the following. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f\colon X \longrightarrow X$ be a surjective morphism over $\overline{\mathbb{Q}}$ with first dynamical degree δ. If an $\mathbb{R}$-divisor D on X satisfies

$$f^*D \sim_{\mathbb{R}} \lambda D$$

for some $\lambda > \delta$, then one has $D \sim_{\mathbb{R}} 0$.

Sketch of the proof. Consider the canonical height

$$\widehat{h}_D(P) = \lim_{n\to\infty} h_D(f^n(P))/\lambda^n$$

where h_D is a height associated with D (cf. [**CS**]). If $\widehat{h}_D(P) \ne 0$ for some P, then we can prove $\overline{\alpha}_f(P) \ge \lambda$. This contradicts the fact $\delta \ge \overline{\alpha}_f(P)$ and the assumption $\lambda > \delta$. Thus one has $\widehat{h}_D = 0$ and therefore $h_D = \widehat{h}_D + O(1) = O(1)$. By a theorem of Serre, we get $D \sim_{\mathbb{R}} 0$ (see [**Ser**, 2.9. Theorem]). $\square$

Proposition 4.4.11. *Let $\mathcal{L}$ be an invertible sheaf of degree zero on a genus one curve C and $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. For any non-invertible self-morphism $f\colon X \longrightarrow X$, Conjecture 3.1.1 holds.*

Proof. By Lemma 4.4.5 and Proposition 4.4.9 we may assume $a \geq b$. In this case, $\delta_f = a$ and Conjecture 3.1.1 can be proved as in the proof of Proposition 4.4.1. $\square$

Proof of Theorem 4.1.1 for $\mathbb{P}^1$-bundles over genus one curves. As we argued at the first of Section 4.2, we may assume that the self-morphism $f\colon X \longrightarrow X$ is not an automorphism. Then the assertion follows from Proposition 4.4.4 and Proposition 4.4.11. $\square$

Remark 4.4.12. In the above setting, the line bundle $\mathcal{L}$ is actually an eigenvector for f_C^* up to linear equivalence. More precisely, for a $\mathbb{P}^1$-bundle $\pi\colon X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ over a curve C with $\deg \mathcal{L} = 0$ and an self-morphism $f\colon X \longrightarrow X$ that induces an self-morphism $f_C\colon C \longrightarrow C$, there exists an integer t such that $f_C^*\mathcal{L} \cong \mathcal{L}^t$. Indeed, let C_0 and C_1 be the sections defined above. Since $(f^*(C_0) \cdot C_0) = 0$, we can write $\mathcal{O}_X(f^{-1}(C_0)) \cong \mathcal{O}_X(mC_0) \otimes \pi^*\mathcal{N}$ for some integer m and degree zero line bundle $\mathcal{N}$ on C. Since

$$\begin{aligned} 0 \neq H^0(\mathcal{O}_X(f^{-1}(C_0))) &= H^0(\mathcal{O}_X(mC_0) \otimes \pi^*\mathcal{N}) \\ &= H^0(\mathrm{Sym}^m(\mathcal{O}_C \oplus \mathcal{L}) \otimes \mathcal{N}) = \bigoplus_{i=0}^{m} H^0(\mathcal{L}^i \otimes \mathcal{N}), \end{aligned}$$

we have $\mathcal{N} \cong \mathcal{L}^r$ for some $-m \leq r \leq 0$. Thus $f^*\mathcal{O}_X(C_0) \cong \mathcal{O}_X(mC_0) \otimes \pi^*\mathcal{L}^r$. The key is the calculation of global sections using projection formula. Since $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(C_0) \otimes \pi^*\mathcal{L}^{-1}$, we have $\pi_*\mathcal{O}_X(mC_1) \cong \pi_*\mathcal{O}_X(mC_0) \otimes \mathcal{L}^{-m}$. Moreover, since C_0 and C_1 are numerically equivalent, we can similarly get $f^*\mathcal{O}_X(C_1) \cong \mathcal{O}_X(mC_0) \otimes \pi^*\mathcal{L}^s$ for some integer s. Thus, $f^*\pi^*\mathcal{L} \cong \pi^*\mathcal{L}^{r-s}$. Therefore, $\pi^* f_C^*\mathcal{L} \cong \pi^*\mathcal{L}^{r-s}$. Since $\pi^*\colon \operatorname{Pic} C \longrightarrow \operatorname{Pic} X$ is injective, we get $f_C^*\mathcal{L} \cong \mathcal{L}^{r-s}$.

4.4.3. $\mathbb{P}^1$-bundles over curves of genus ≥ 2. By the following proposition, Conjecture 3.1.1 trivially holds in this case.

Proposition 4.4.13. *Let C be a curve with $g(C) \geq 2$ and $\pi\colon X \longrightarrow C$ be a $\mathbb{P}^1$-bundle over C. Let $f\colon X \longrightarrow X$ be a surjective self-morphism. Then there exists an integer $t > 0$ such that f^t is a morphism over C, that is, f^t satisfies $\pi \circ f^t = \pi$. In particular, f admits no Zariski dense orbit.*

Proof. By Lemma 4.3.4, we may assume that f induces a surjective self-morphism $f_C\colon C \longrightarrow C$ with $\pi \circ f = f_C \circ \pi$. Since C is of general type, f_C is an automorphism of finite order and the assertion follows. $\square$

Remark 4.4.14. One can also show that any surjective self-morphism over a curve of genus at least two admits no dense orbit by using the Mordell conjecture (Faltings's theorem).

4.5. Hyperelliptic surfaces

Theorem 4.5.1. *Let X be a hyperelliptic surface and $f\colon X \longrightarrow X$ a non-invertible self-morphism on X. Then Conjecture 3.1.1 holds for f.*

Proof. Let $\pi\colon X \longrightarrow E$ be the Albanese map of X. By the universality of π, there is a morphism $g\colon E \longrightarrow E$ satisfying $\pi \circ f = g \circ \pi$. It is well-known that E is a genus one curve, π is a surjective morphism with connected fibers, and there is an étale cover $\phi\colon E' \longrightarrow E$ such that $X' = X \times_E E' \cong F \times E'$, where F is a genus one curve (cf. [**Bad**, Chapter 10]). In particular, X' is an abelian surface. By Lemma 4.4.3, taking a further étale base change, we may assume that there is an self-morphism $h\colon E' \longrightarrow E'$ such that

$\phi \circ h = g \circ \phi$. Let $\pi' \colon X' \longrightarrow E'$ and $\psi \colon X' \longrightarrow X$ be the induced morphisms. Then, by the universality of fiber products, there is a morphism $f' \colon X' \longrightarrow X'$ satisfying $\pi' \circ f' = \pi' \circ h$ and $\psi \circ f' = f \circ \psi$. Applying Lemma 3.2.2, it is enough to prove Conjecture 3.1.1 for the self-morphism f'. Since X' is an abelian variety, this holds by [**KS2**, Corollary 32] and [**Sil2**, Theorem 2]. □

4.6. Surfaces with $\kappa(X) = 1$

Let $f \colon X \longrightarrow X$ be a non-invertible self-morphism on a surface X with $\kappa(X) = 1$. In this section we shall prove that f does not admit any Zariski dense forward f-orbit. Although this result is a special case of [**NZ**, Theorem A] (see Remark 3.1.6), we will give a simpler proof of it.

By Lemma 4.2.2, X is minimal and f is étale. Since $\deg(f) \geq 2$, we have $\chi(X, \mathcal{O}_X) = 0$.

Let $\phi = \phi_{|mK_X|} \colon X \longrightarrow \mathbb{P}^N = \mathbb{P}H^0(X, mK_X)$ be the Iitaka fibration of X and set $C_0 = \phi(X)$. Since f is étale, it induces an automorphism $g \colon \mathbb{P}^N \longrightarrow \mathbb{P}^N$ such that $\phi \circ f = g \circ \phi$ (cf. [**FN2**, Lemma 3.1]). The restriction of g to C_0 gives an automorphism $f_{C_0} \colon C_0 \longrightarrow C_0$ such that $\phi \circ f = f_{C_0} \circ \phi$. Take the normalization $\nu \colon C \longrightarrow C_0$ of C_0. Then ϕ factors as $X \xrightarrow{\pi} C \xrightarrow{\nu} C_0$ and π is an elliptic fibration. Moreover, f_{C_0} lifts to an automorphism $f_C \colon C \longrightarrow C$ such that $\pi \circ f = f_C \circ \pi$.

So we obtain an elliptic fibration $\pi \colon X \longrightarrow C$ and an automorphism f_C on C such that $\pi \circ f = f_C \circ \pi$ In this situation, the following holds.

Theorem 4.6.1. *Let X be a surface with $\kappa(X) = 1$, $\pi \colon X \longrightarrow C$ an elliptic fibration, $f \colon X \longrightarrow X$ a non-invertible self-morphism, and $f_C \colon C \longrightarrow C$ an automorphism such that $\pi \circ f = f_C \circ \pi$. Then $f_C^t = \mathrm{id}_C$ for a positive integer t.*

Proof. Let $\{P_1, \ldots, P_r\}$ be the points over which the fibers of π are multiple fibers (possibly $r = 0$, i.e. π does not have any multiple fibers). We denote by m_i denotes the multiplicity of the fiber $\pi^* P_i$ for every i. Then we have the canonical bundle formula:

$$K_X = \pi^*(K_C + L) + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \pi^* P_i,$$

where L is a divisor on C such that $\deg(L) = \chi(X, \mathcal{O}_X)$. Then $\deg(L) = 0$ because f is étale and $\deg(f) \geq 2$ (cf. Lemma 4.2.2). Since $\kappa(X) = 1$, the divisor $K_C + L + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} P_i$ must have positive degree. So we have

$$2(g(C) - 1) + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} > 0.$$

For any i, set $Q_i = f_C^{-1}(P_i)$. Then $\pi^* Q_i = \pi^* f_C^* P_i = f^* \pi^* P_i$ is a multiple fiber. So $(f_C)|_{\{P_1, \ldots, P_r\}}$ is a permutation of $\{P_1, \ldots, P_r\}$ since f_C is an automorphism.

We divide the proof into three cases according to the genus $g(C)$ of C:

(1) $g(C) \geq 2$; then the automorphism group of C is finite. So $f_C^t = \mathrm{id}_C$ for a positive integer t.

(2) $g(C) = 1$; by $(*)$, it follows that $r \geq 1$. For a suitable t, all P_i are fixed points of f_C^t. We put the algebraic group structure on C such that P_1 is the identity element. Then f_C^t is a group automorphism on C. So $f_C^{ts} = \mathrm{id}_C$ for a suitable s since the group of group automorphisms on C is finite.

(3) $g(C) = 0$; again by $(*)$, it follows that $r \geq 3$. For a suitable t, all P_i are fixed points of f_C^t. Then f_C^t fixes at least three points, which implies that f_C^t is in fact the identity map. $\square$

Immediately we obtain the following corollary.

Corollary 4.6.2. Let $f\colon X \longrightarrow X$ be a non-invertible self-morphism on a surface X with $\kappa(X) = 1$. Then there does not exist any Zariski dense f-orbit.

Therefore Conjecture 3.1.1 trivially holds for non-invertible self-morphisms on surfaces of Kodaira dimension 1.

CHAPTER 5

Self-morphisms on semi-abelian varieties

5.1. Outline of this chapter

Let X be a smooth projective variety and $f\colon X \dashrightarrow X$ a rational self-map, both defined over $\overline{\mathbb{Q}}$.

Let $A(f)$ be the set of the arithmetic degrees of f, i.e.

$$A(f) = \{\alpha_f(x) \mid x \in X(\overline{\mathbb{Q}})\}.$$

Determining the set $A(f)$ for a given f is an interesting problem. As we noted in Theorem 3.3.1, for any surjective morphism f, there exists a point $x \in X$ such that $\alpha_f(x) = \delta_f$. When X is a toric variety and f is a self-rational map on X that is induced by a group homomorphism of the algebraic torus, the set $A(f)$ is completely determined in [**Sil2, Lin**].

When X is quasi-projective, the arithmetic degrees and dynamical degrees can be defined by taking a smooth compactification of X. Since as we noted in Chapter 2, the arithmetic degrees and the dynamical degrees are birational invariant, they are well-defined for self-morphisms of semi-abelian varieties. In this chapter, we prove Conjecture 3.1.1 for self-morphisms of semi-abelian varieties and determine the set $A(f)$.

Theorem 5.1.1. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ a self-morphism (not necessarily surjective), both defined over $\overline{\mathbb{Q}}$.*

(1) *Suppose f is surjective. Then for any point $x \in X(\overline{\mathbb{Q}})$ with Zariski dense f-orbit, we have $\alpha_f(x) = \delta_f$.*

(2) *For every $x \in X(\overline{\mathbb{Q}})$, the arithmetic degree $\alpha_f(x)$ exists. Moreover, write $f = T_a \circ g$ where T_a is the translation by a point $a \in X(\overline{\mathbb{Q}})$ and g is a group homomorphism. Then $A(f) = A(g)$.*

(3) *Suppose f is a group homomorphism. Let $F(t)$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ and*

$$F(t) = t^{e_0} F_1(t)^{e_1} \cdots F_r(t)^{e_r}$$

the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0 \geq 0$ and $e_i > 0$ for $i = 1, \dots, r$. Let $\rho(F_i)$ be the maximum among the absolute values of the roots of F_i. Then we have

$$A(f) \subset \{1, \rho(F_1), \rho(F_1)^2, \dots, \rho(F_r), \rho(F_r)^2\}.$$

More precisely, set

$$X_i = f^{e_0} F_1(f)^{e_1} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X).$$

Define

$$A_i = \begin{cases} \{\rho(F_i)\} & \text{if } X_i \text{ is an algebraic torus,} \\ \{\rho(F_i)^2\} & \text{if } X_i \text{ is an abelian variety,} \\ \{\rho(F_i), \rho(F_i)^2\} & \text{otherwise.} \end{cases}$$

Then we have

$$A(f) = \{1\} \cup A_1 \cup \cdots \cup A_r.$$

Remark 5.1.2. Actually, in the situation of Theorem 5.1.1 (3), f is conjugate by an isogeny to a homomorphism of the form

$$f_0 \times \cdots \times f_r \colon X_0 \times \cdots \times X_r \longrightarrow X_0 \times \cdots \times X_r$$

where $A(f_0) = \{1\}$ and $A(f_i) = \{1\} \cup A_i$ for $i = 1, \dots, r$.

We can characterize the set of points whose arithmetic degrees are equal to 1 as follows (cf. [**San2**] for related results).

Theorem 5.1.3. *Let X be a semi-abelian variety and $f \colon X \longrightarrow X$ a surjective morphism both defined over $\overline{\mathbb{Q}}$. Write $f = T_a \circ g$ where T_a is the translation by $a \in X(\overline{\mathbb{Q}})$ and g is an isogeny. Suppose that the minimal polynomial of g has no irreducible factor that is a cyclotomic polynomial. Then there exists a point $b \in X(\overline{\mathbb{Q}})$ such that, for any $x \in X(\overline{\mathbb{Q}})$, the following are equivalent:*

(1) $\alpha_f(x) = 1$;
(2) $\# O_f(x) < \infty$;
(3) $x \in b + X(\overline{\mathbb{Q}})_{\mathrm{tors}}$.

Here $X(\overline{\mathbb{Q}})_{\mathrm{tors}}$ is the set of torsion points.

Remark 5.1.4. When f is an isogeny, we can take $b = 0$.

Remark 5.1.5. If the minimal polynomial of g has irreducible factor that is a cyclotomic polynomial, then one of f_i in Remark 5.1.2 (applied to $f = g$) has dynamical degree 1.

To prove the above theorems, we calculate the dynamical degrees of self-morphisms of semi-abelian varieties.

Theorem 5.1.6. *Let X be a semi-abelian variety over an algebraically closed field of characteristic zero.*

(1) *Let $f \colon X \longrightarrow X$ be a surjective group homomorphism. Let*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence with T a torus and A an abelian variety. Then f induces surjective group homomorphisms

$$\begin{aligned} f_T := f|_T \colon T &\longrightarrow T \\ g \colon A &\longrightarrow A \end{aligned}$$

with $g \circ \pi = \pi \circ f$. Then we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\}.$$

Moreover, let P_T and P_A be the monic minimal polynomials of f_T and g as elements of $\mathrm{End}(T)_{\mathbb{Q}}$ and $\mathrm{End}(A)_{\mathbb{Q}}$ respectively. Then, $\delta_{f_T} = \rho(P_T)$ and $\delta_g = \rho(P_A)^2$.

(2) *Let $f \colon X \longrightarrow X$ be a surjective homomorphism and $a \in X$ a closed point. Then $\delta_{T_a \circ f} = \delta_f$.*

Remark 5.1.7. The description of δ_{f_T} in Theorem 5.1.6(1) is well-known (see for example [**Sil1**]).

The outline of this chapter is as follows. In §5.2,we summarize the basic properties of the arithmetic degrees. In §5.3, we prove a lemma that says every homomorphism of a semi-abelian variety "splits into rather simple ones". In §5.4, we prove our main theorems for isogenies of abelian varieties. We use these to prove the main theorems. In §5.5.1, we calculate the first dynamical degrees of self-morphisms of semi-abelian varieties and prove Theorem 5.1.6. In §5.5.2, we prove Theorem 5.1.1 and 5.1.3.

5.2. Preliminaries

In this section, the ground field is $\overline{\mathbb{Q}}$.

Definition 5.2.1. Let $f\colon X \dashrightarrow X$ be a rational self-map of a smooth quasi-projective variety. If $\alpha_f(x)$ exists for every $x \in X_f(\overline{\mathbb{Q}})$, we write $A(f) = \{\alpha_f(x) \mid x \in X_f(\overline{\mathbb{Q}})\}$.

Remark 5.2.2. By definition, $1 \leq \underline{\alpha}_f(x) \leq \overline{\alpha}_f(x)$. When x is f-preperiodic, $\alpha_f(x) = 1$.

Lemma 5.2.3. *Let X, Y be smooth quasi-projective varieties and $f\colon X \dashrightarrow X$, $g\colon Y \dashrightarrow Y$ rational maps. Let $x \in X_f(\overline{\mathbb{Q}})$ and $y \in Y_g(\overline{\mathbb{Q}})$. If $\alpha_f(x)$ and $\alpha_g(y)$ exist, then $\alpha_{f\times g}(x,y)$ also exists and*

$$\alpha_{f\times g}(x,y) = \max\{\alpha_f(x), \alpha_g(y)\}.$$

Proof. It is enough to prove when X, Y are projective. Take ample divisors H_X, H_Y on X, Y respectively. Fix associated height functions h_{H_X}, h_{H_Y} so that $h_{H_X} \geq 1$ and $h_{H_Y} \geq 1$. Then $h := h_{H_X} \circ \mathrm{pr}_1 + h_{H_Y} \circ \mathrm{pr}_2$ is an ample height function on $X \times Y$. Then

$$\begin{aligned}
&\lim_{n\to\infty} h((f\times g)^n(x,y))^{1/n} = \lim_{n\to\infty} (h_{H_X}(f^n(x)) + h_{H_Y}(g^n(y)))^{1/n} \\
&= \max\{\lim_{n\to\infty} h_{H_X}(f^n(x))^{1/n}, \lim_{n\to\infty} h_{H_Y}(g^n(y))^{1/n}\} = \max\{\alpha_f(x), \alpha_g(y)\}.
\end{aligned}$$

□

Lemma 5.2.4. *Consider the following commutative diagram*

$$\begin{array}{ccc}
Y & \overset{g}{\dashrightarrow} & Y \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
X & \underset{f}{\dashrightarrow} & X
\end{array}$$

where X, Y are smooth quasi-projective varieties, f, g rational maps and π a surjective morphism. Let $y \in Y_g(\overline{\mathbb{Q}})$ be a point such that $\pi(y) \in X_f(\overline{\mathbb{Q}})$. Then

$$\overline{\alpha}_g(y) \geq \overline{\alpha}_f(x)$$
$$\underline{\alpha}_g(y) \geq \underline{\alpha}_f(x).$$

Proof. We may assume X, Y are projective. Take an ample divisor H_X on X and fix an associated height function h_{H_X} with $h_{H_X} \geq 1$. Take an ample divisor H_Y on Y so that $H_Y - \pi^*H_X$ is ample. Then we can take a height function h_{H_Y} associated with H_Y so that $h_{H_Y} \geq h_{H_X} \circ \pi$. Then

$$\begin{aligned}
\overline{\alpha}_f(x) &= \limsup_{n\to\infty} h_{H_X}(f^n(x))^{1/n} = \limsup_{n\to\infty} h_{H_X}(\pi(g^n(y)))^{1/n} \\
&\leq \limsup_{n\to\infty} h_{H_Y}(g^n(y))^{1/n} = \overline{\alpha}_g(y).
\end{aligned}$$

The second inequality can be proved similarly. □

The following results are used later.

Theorem 5.2.5. [**KS2**, Theorem 4],[**Sil1**, Theorem 4, Corollary 32],[**Sil2**, Theorem 2]

(1) *For any self-morphisms of abelian varieties, Conjecture 3.1.1 is true.*
(2) *Let X be an algebraic torus and $f\colon X \longrightarrow X$ be a homomorphism. Then Conjecture 3.1.1 is true for f. Moreover, let $F(t)$ be the minimal monic polynomial of f as an element of $\mathrm{End}(X)_{\mathbb{Q}}$ and $F(t) = t^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$ the irreducible decomposition. Then $A(f) = \{1, \rho(F_1), \dots, \rho(F_r)\}$.*

5.3. Splitting lemma

In this section, the ground field is an algebraically closed field of characteristic zero. Let X be a a semi-abelian variety, i.e. a commutative algebraic group that is an extension of an abelian variety by an algebraic torus. Note that X is divisible i.e. the morphism $X \longrightarrow X; x \mapsto nx$ is surjective for every $n > 0$.

Lemma 5.3.1. *Let $f\colon X \longrightarrow X$ be a homomorphism. Let $F(t) \in \mathbb{Z}[t]$ be a polynomial such that $F(f) = 0$ in $\mathrm{End}(X)$. Suppose $F(t) = F_1(t)F_2(t)$ where $F_1, F_1 \in \mathbb{Z}[t]$ are coprime in $\mathbb{Q}[t]$. Set $X_1 = F_2(f)(X)$ and $X_2 = F_1(f)(X)$. Then $X = X_1 + X_2$ and $X_1 \cap X_2$ is finite. In other words, the morphism $X_1 \times X_2 \longrightarrow X; (x_1, x_2) \mapsto x_1 + x_2$ is an isogeny.*

Proof. See [**Sil2**, Lemma 5]. □

In the situation of Lemma 5.3.1, write $f_i = f|_{X_i}$. Then $F_i(f_i) = 0$ and we have the following commutative diagram:

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "f_1\times f_2"] \arrow[d, "\pi"'] & X_1 \times X_2 \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X. \end{tikzcd}$$

Here π is the isogeny defined by $\pi(x_1, x_2) = x_1 + x_2$.

Since X is divisible, we have $\mathrm{End}(X) \subset \mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. Let $f \in \mathrm{End}(X)$ and $F(t) \in \mathbb{Z}[t]$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$. (The monic minimal polynomial has integer coefficients because those of (group) endomorphisms of a torus and an abelian variety have integer coefficients.) Let

$$F(t) = F_0(t)^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$$

be the decomposition into irreducible factors where $F_0(t) = t$, $e_0 \geq 0$, $e_i > 0, i = 1, \dots, r$ and $F_i(t)$ are distinct monic irreducible polynomials. Note that r is possibly zero. Set

$$X_i = F_0(f)^{e_0}\cdots F_{i-1}(f)^{e_{i-1}}F_{i+1}(f)^{e_{i+1}}\cdots F_r(f)^{e_r}(X)$$

and $f_i = f|_{X_i}$. Here, X_i are also (semi-)abelian varieties since they are images of a (semi-)abelian variety. Then we get the commutative diagram

$$\begin{tikzcd} X_0 \times \cdots \times X_r \arrow[r, "f_0\times\cdots\times f_r"] \arrow[d, "\pi"'] & X_0 \times \cdots \times X_r \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

where $\pi(x_0,\dots,x_r)=x_0+\cdots+x_r$. Note that the monic minimal polynomial of f_i as an element of $\mathrm{End}(X_i)\otimes_{\mathbb{Z}}\mathbb{Q}$ is $F_i(t)^{e_i}$. Note that f is surjective if and only if $e_0=0$ and if this is the case, we have $\delta_f=\delta_{f_0\times\cdots\times f_r}=\max\{\delta_{f_1},\dots,\delta_{f_r}\}$ (cf. Remark 2.1.2).

5.4. Arithmetic and dynamical degrees of isogenies of abelian varieties

Theorem 5.4.1 (Theorem 5.1.1(3) for abelian varieties). *Let X be an abelian variety and $f\colon X\longrightarrow X$ be a homomorphism, both defined over $\overline{\mathbb{Q}}$. Let $F(t)$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X)_{\mathbb{Q}}$ and*

$$F(t)=t^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$$

the irreducible decomposition in $\mathbb{Q}[t]$ where $e_0\geq 0$ and $e_i>0$ for $i=1,\dots,r$. Then we have

$$A(f)=\{1,\rho(F_1)^2,\dots,\rho(F_r)^2\}$$

Theorem 5.4.2 (Theorem 5.1.3 for isogenies of abelian varieties). *Let X be an abelian variety and $f\colon X\longrightarrow X$ an isogeny, both defined over $\overline{\mathbb{Q}}$. Suppose that the minimal polynomial of f has no irreducible factor that is a cyclotomic polynomial. Then for any $x\in X(\overline{\mathbb{Q}})$,*

$$\alpha_f(x)=1\iff \#\,O_f(x)<\infty\iff x\in X(\overline{\mathbb{Q}})_{\mathrm{tors}}$$

where $X(\overline{\mathbb{Q}})_{\mathrm{tors}}$ is the set of torsion points.

Lemma 5.4.3. *Let X be an abelian variety of dimension g over an algebraically closed field of characteristic zero and $f\colon X\longrightarrow X$ an isogeny. Let $P(t)$ be the monic minimal polynomial of f as an element of $\mathrm{End}(X)_{\mathbb{Q}}$, which has integer coefficient, and ρ the maximum among the absolute values of the roots of $P(t)$. Then we have $\delta_f=\rho^2$.*

Remark 5.4.4. The minimal polynomial of f as an element of $\mathrm{End}(X)_{\mathbb{Q}}$ is equal to the minimal polynomial of $T_l(f)$ for every prime number l. If the ground field is $\mathbb{C}$, these are also equal to the minimal polynomial of the analytic representation of f.

Proof. By the Lefschetz principle, we may assume that the ground field is $\mathbb{C}$. Let $X=\mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$. Let $f_r\colon\Lambda\longrightarrow\Lambda$ be the rational representation and $f_a\colon\mathbb{C}^g\longrightarrow\mathbb{C}^g$ the analytic representation of f.

We have a natural isomorphism $H^r(X;\mathbb{Z})\simeq\mathrm{Hom}_{\mathbb{Z}}(\bigwedge^r\Lambda,\mathbb{Z})$ (cf. [**Mum**, §1 (4)]). If we identify $H^r(X;\mathbb{Z})$ with $\mathrm{Hom}_{\mathbb{Z}}(\bigwedge^r\Lambda,\mathbb{Z})$ by this isomorphism, then $f^*\colon H^r(X;\mathbb{Z})\longrightarrow H^r(X;\mathbb{Z})$ is $(\bigwedge^r f_r)^*$. Therefore, the eigenvalues of f^* are products of r eigenvalues of f_r. Since $f_a|_\Lambda=f_r$, the characteristic polynomial of f_r as an $\mathbb{R}$-linear map is $Q(t)\overline{Q(t)}$ where $Q(t)$ is the characteristic polynomial of f_a as a $\mathbb{C}$-linear map. (Take a basis $e_1,\dots,e_g$ of $\mathbb{C}^g$ so that f_a is represented by an upper triangular matrix. Then compute the characteristic polynomial of f_a,f_r using bases $\{e_1,\dots,e_g\},\{e_1,ie_1,\dots,e_g,ie_g\}$ respectively.) Note that the set of roots of $P(t)$ and $Q(t)$ are the same. Therefore, the spectral radius of $f^*\colon H^2(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow H^2(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}$ is equal to the square of spectral radius of f_r. Note that the spectral radius of $f^*\curvearrowright H^2(X;\mathbb{Z})$ is equal to the spectral radius of $f^*\curvearrowright H^{1,1}(X)$ (cf. the inequality above Proposition 4.4 in [**DS3**]), this proves the theorem. ☐

Now, let X be an abelian variety and $f\colon X\longrightarrow X$ a homomorphism, both defined over $\overline{\mathbb{Q}}$. Let $F(t)$ be the monic minimal polynomial of f and

$$F(t)=t^{e_0}F_1(t)^{e_1}\cdots F_r(t)^{e_r}$$

the decomposition into irreducible factors in $\mathbb{Q}[t]$. Here F_i are distinct monic irreducible polynomial in $\mathbb{Z}[t]$ with $F_i(0) \neq 0$. Write $F_0(t) = t$. Set

$$X_i = F_0(f)^{e_0} \cdots F_{i-1}(f)^{e_{i-1}} F_{i+1}(f)^{e_{i+1}} \cdots F_r(f)^{e_r}(X).$$

Then by §3, we have the following commutative diagram:

$$\begin{tikzcd} X_0 \times \cdots \times X_r \arrow[r, "f_0 \times \cdots \times f_r"] \arrow[d, "\pi"'] & X_0 \times \cdots \times X_r \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X. \end{tikzcd}$$

Here, the vertical arrows are isogenies. Note that the minimal polynomial of f_i is $F_i(t)^{e_i}$.

Lemma 5.4.5. *Let $f\colon X \longrightarrow X$ be an isogeny over $\overline{\mathbb{Q}}$ such that the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible monic polynomial in $\mathbb{Z}[t]$. For any $x \in X(\overline{\mathbb{Q}})$, if $\alpha_f(x) < \delta_f$, then x is a torsion point. In particular, x is a f-preperiodic point and $\alpha_f(x) = 1$.*

Remark 5.4.6. Note that $\alpha_f(x) < \delta_f$ happens only if $\delta_f > 1$. In the above situation, $\delta_f = 1$ if and only if $F(t)$ is a cyclotomic polynomial. This follows from Lemma 5.4.3 and the fact that if the absolute value of every root of an irreducible monic polynomial with integer coefficients is less than or equal to one, then the polynomial is cyclotomic.

Proof. We prove the claim by induction on $\dim X$. If $\dim X = 0$, there is nothing to prove. Suppose $\dim X = d > 0$ and the claim holds for dimension $< d$. Take a nef $\mathbb{R}$-divisor D such that $f^*D \equiv \delta_f D$. Let q be the quadratic part of the canonical height of D, i.e. $q(x) = \lim_{n\to\infty} h_D(nx)/n^2$. By [**KS2**, Theorem 29, Lemma 31], there exists a f-invariant subabelian variety $Y \subsetneq X$ such that

$$\{x \in X(\overline{\mathbb{Q}}) \mid q(x) = 0\} = Y(\overline{\mathbb{Q}}) + X(\overline{\mathbb{Q}})_{\mathrm{tors}}.$$

Assume $\alpha_f(x) < \delta_f$. Then $x = y + z$ for some $y \in Y(\overline{\mathbb{Q}})$ and some torsion point z. It is enough to show that y is a torsion point. If Y is a point, we are done. Suppose $\dim Y > 0$. Since Y is f-invariant, the minimal polynomial of $f|_Y$ divides $F(t)^e$ and is not equal to 1. Thus $\delta_{f|_Y} = \delta_f > \alpha_f(x) = \alpha_f(y) = \alpha_{f|_Y}(y)$. Here, we use the fact that $\alpha_f(x) = \alpha_f(y + z) = \alpha_f(y)$. This follows from the definition of arithmetic degree and the fact that the Neron-Tate height associated with a symmetric ample divisor is invariant under torsion translate. By the induction hypothesis, y is a torsion point. □

Proof of Theorem 5.4.1. We use the notation of §3. Set $f_i = f|_{X_i}$. By [**Sil2**, Lemma 6], $A(f) = A(f_0 \times \cdots \times f_r)$. Since $\alpha_{f_i}(0) = 1$ and $\alpha_{f_0\times\cdots\times f_r}(x_0, \ldots, x_r) = \max\{\alpha_{f_0}(x_0), \ldots, \alpha_{f_r}(x_r)\}$ (see Lemma 5.2.3), we have $A(f_0\times\cdots\times f_r) = A(f_0)\cup\cdots\cup A(f_r)$. Note that $A(f_0) = \{1\}$ since $f_0^{e_0} = 0$. By Lemma 5.4.5 and the fact that there always exists a point whose arithmetic degree equals the dynamical degree (cf. [**KS2**, Corollary 32] or [**MSS1**, Theorem 1.6]), we have $A(f_i) = \{1, \delta_{f_i}\}$ for $i = 1, \ldots, r$. Thus $A(f) = \{1, \delta_{f_1}, \ldots, \delta_{f_r}\}$. By Lemma 5.4.3, δ_{f_i} is equal to $\rho(F_i)^2$. □

Proof of Theorem 5.4.2. By §3, we may assume the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible polynomial that is not cyclotomic. Then $\rho(F)$ is greater than one. Thus $\delta_f > 1$. By Lemma 5.4.5, if $\alpha_f(x) = 1$ then x is a torsion point. □

5.5. Arithmetic and dynamical degrees of self-morphisms of semi-abelian varieties

5.5.1. Dynamical degrees. In this subsection, the ground field is an algebraically closed field of characteristic zero.

Proposition 5.5.1. *Let X be a semi-abelian variety. Let $f\colon X \longrightarrow X$ be a surjective group homomorphism. Let*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be an exact sequence with T a torus and A an abelian variety. Then f induces surjective group homomorphisms

$$f_T := f|_T\colon T \longrightarrow T$$
$$g\colon A \longrightarrow A$$

with $g \circ \pi = \pi \circ f$. Then we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\}$$

Proof. This follows from the product formula of dynamical degrees ([**DN**, Theorem 1.1]) and [**DN**, Remark 3.4]. To apply [**DN**, Remark 3.4], take the standard compactification of X as in [**Voj**, §2 (2.3)]. □

Lemma 5.5.2. *Let $f\colon X \longrightarrow X$ be a surjective homomorphism of a semi-abelian variety X and $a \in X$ a closed point. Then $\delta_{T_a \circ f} = \delta_f$.*

Proof. Let $\overline{X}$ be the standard compactification of X as in [**Voj**, §2 (2.3)]. Then T_a extends to an automorphism of $\overline{X}$, which we also denote by T_a, and the pull-back $T_a^*\colon \mathrm{N}^1(\overline{X}) \longrightarrow \mathrm{N}^1(\overline{X})$ is the identity. (We can deduce these facts from the description of the group law in terms of the compactification, cf. [**Voj**, the proof of Proposition 2.6].) Thus, as an endomorphisms of $\mathrm{N}^{1^1}(\overline{X})$, we have

$$((T_a \circ f)^n)^* = (T_b \circ f^n)^* = (f^n)^* \circ T_b^* = (f^n)^*$$

where $b = a + f(a) + \cdots + f^{n-1}(a)$. Therefore,

$$\delta_{T_a \circ f} = \lim_{n\to\infty} \|((T_a \circ f)^n)^*\|^{1/n} = \lim_{n\to\infty} \|(f^n)^*\|^{1/n} = \delta_f$$

where $\|\cdot\|$ is a norm on $\mathrm{End}_{\mathbb{R}}(\mathrm{N}^1(\overline{X})_{\mathbb{R}})$. □

Proof of Theorem 5.1.6. (2) is Lemma 5.5.2. (1) follows from Proposition 5.5.1, Lemma 5.4.3 and Remark 5.1.7. □

5.5.2. Kawaguchi-Silverman conjecture and arithmetic degrees. In this subsection, the ground field is $\overline{\mathbb{Q}}$.

Lemma 5.5.3. *Let $f\colon X \longrightarrow X$ be a surjective group homomorphism of a semi-abelian variety. Fix an exact sequence*

$$0 \longrightarrow T \longrightarrow X \xrightarrow{p} A \longrightarrow 0.$$

The morphisms induced by f is denoted by

$$f_T\colon T \longrightarrow T$$
$$g\colon A \longrightarrow A.$$

Suppose the minimal polynomial of f as an element of $\mathrm{End}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ *is the form of* $F(t)^e$ *where* $F(t)$ *is a monic irreducible polynomial that is not cyclotomic and* $e > 0$. *(Note that the minimal polynomial is automatically monic with integer coefficient because it is the case for* f_T *and* g.*) Then, for* $x \in X(\overline{\mathbb{Q}})$, *either*

(1) $O_g(p(x))$ *is infinite and* $\alpha_f(x) = \delta_f$ *or,*

(2) $p(x)$ *is a torsion point and* $\alpha_f(x) = 1$ *or* δ_{f_T}.

Moreover,

$$A(f) = \{1, \delta_{f_T}, \delta_g\} = \begin{cases} \{1, \rho(F)\} & \text{if } X = T, \\ \{1, \rho(F)^2\} & \text{if } X = A, \\ \{1, \rho(F), \rho(F)^2\} & \text{otherwise.} \end{cases}$$

Lemma 5.5.4. *Let X be a semi-abelian variety and $f\colon X \longrightarrow X$ be a surjective group homomorphism. Let $x \in X(\overline{\mathbb{Q}})$ be a point and $n > 0$ a positive integer. If $\alpha_f(nx)$ exists, then $\alpha_f(x)$ also exists and $\alpha_f(nx) = \alpha_f(x)$.*

Proof. Apply Lemma 2.2.3
Taking m-th root and limit as $m \to \infty$, we get the claim. □

Proof of Lemma 5.5.3. First of all, we have

$$\delta_f = \max\{\delta_g, \delta_{f_T}\} = \max\{\rho(F)^2, \rho(F)\} = \rho(F)^2 = \delta_g$$

(see Theorem 5.2.5(2), Proposition 5.5.1, Lemma 5.4.3). By Lemma 5.4.5, we have

$$\alpha_g(p(x)) = \begin{cases} 1 & \text{if } p(x) \text{ is torsion} \\ \delta_g = \delta_f & \text{otherwise.} \end{cases}$$

Note that, by Lemma 5.2.4 and Remark 2.1.2 (3), we have $\alpha_g(p(x)) \leq \underline{\alpha}_f(x) \leq \delta_f$. Thus, if $\alpha_g(p(x)) = \delta_f$, $\alpha_f(x)$ exists and is equal to δ_f.

Now, suppose $p(x)$ is a torsion point. Take a positive integer n such that $np(x) = 0$. Then $nx \in T$ and therefore $\alpha_f(nx) = \alpha_{f_T}(nx)$ exists and is equal to 1 or $\rho(F) = \delta_{f_T}$ (Theorem 5.2.5(2)). By Lemma 5.5.4, $\alpha_f(x)$ exists and is equal to 1 or δ_{f_T}.

The claim $A(f) = \{1, \delta_{f_T}, \delta_f\}$ follows from the facts that $A(f_T) = \{1, \delta_{f_T}\}$ (Theorem 5.2.5(2)), $A(g) = \{1, \delta_g\}$ (Lemma 5.4.5) and $\alpha_f(x) \geq \alpha_g(p(x))$ (Lemma 5.2.4). □

Lemma 5.5.5. *Let $f\colon X \longrightarrow X$ be a homomorphism of a semi-abelian variety. Let $F(t)$ be the minimal monic polynomial of f. Assume $F(1) \neq 1$. Let $a \in X(\overline{\mathbb{Q}})$ be any point. Then there exists a point $b \in X(\overline{\mathbb{Q}})$ such that $h := T_b \circ (T_a \circ f) \circ T_{-b}$ is a homomorphism. For every such b, the minimal polynomial of h is also $F(t)$.*

Proof. Since $F(1) \neq 1$, $f - \mathrm{id}$ is surjective. For any $b \in X(\overline{\mathbb{Q}})$ with $f(b) - b = a$, the morphism $T_b \circ (T_a \circ f) \circ T_{-b}$ is a group homomorphism.

Now we prove the second part. By symmetry, it is enough to prove $F(h) = 0$. We have

$$h^n = T_b \circ (T_a \circ f)^n \circ T_{-b} = T_b \circ T_{a+f(a)+\cdots+f^{n-1}(a)} \circ f^n \circ T_{-b}.$$

Note that since h is a homomorphism, we have $h(0) = 0$, in other words, $a = (f - \mathrm{id})(b)$. Thus

$$h^n = T_b \circ T_{f^n(b)-b} \circ f^n \circ T_{-b} = T_{f^n(b)} \circ f^n \circ T_{-b}.$$

Therefore, for any $x \in X(\overline{\mathbb{Q}})$

$$F(h)(x) = F(f)(b) + F(f)(x - b) = 0.$$

□

Proof of Theorem 5.1.1. Let X be a semi-abelian variety and first assume $f\colon X \longrightarrow X$ is a homomorphism. We use the notation of §5.3. Apply Lemma 2.2.3 for a suitable smooth compactification of

$$\begin{tikzcd} X_0 \times \cdots \times X_r \arrow[r, "f_0\times\cdots\times f_r"] \arrow[d, "\pi"'] & X_0 \times \cdots \times X_r \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X. \end{tikzcd}$$

By Lemma 5.5.3, $\alpha_{f_i}(x)$ exists for every i and every point $x \in X_i(\overline{\mathbb{Q}})$. Therefore, by Lemma 5.2.3 and again Lemma 2.2.3, $A(f) = A(f_0 \times \cdots \times f_r) = A(f_0) \cup \cdots \cup A(f_r)$. Since f_0 is nilpotent, $A(f_0) = \{1\}$. If F_i is a cyclotomic polynomial, then $\delta_{f_i} = 1$ and $A(f_i) = \{1\}$. Therefore by Lemma 5.5.3, we have

$$\begin{aligned} A(f) &= A(f_1) \cup \cdots \cup A(f_r) \\ &= \{1\} \cup A_1 \cup \cdots \cup A_r. \end{aligned}$$

Now, consider any self-morphism of X. For any self-morphism $f\colon X \longrightarrow X$ satisfying $f(0) = 0$, applying [**Bri**, Lemma 5.4.8] for $\varphi\colon X \times X \longrightarrow$ given by $\varphi(x, y) = f(x + y) - f(x) - f(y)$, one can see that f is a group homomorphism. Hence any self-morphism on X is the form of $T_a \circ f$ where T_a is the translation by $a \in X(\overline{\mathbb{Q}})$ and f is a homomorphism. There exist points $a_i \in X_i(\overline{\mathbb{Q}})$ such that $\pi(a_0, \ldots, a_r) = a_0 + \cdots + a_r = a$. Then we have the following commutative diagram:

$$\begin{tikzcd} X_0 \times \cdots \times X_r \arrow[r, "f_0\times\cdots\times f_r"] \arrow[d, "\pi"'] & X_0 \times \cdots \times X_r \arrow[r, "T_{a_0}\times\cdots\times T_{a_r}"] \arrow[d, "\pi"] & X_0 \times \cdots \times X_r \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X \arrow[r, "T_a"'] & X. \end{tikzcd}$$

As above, we have $A(T_a \circ f) = A\big((T_{a_0} \circ f_0) \times \cdots \times (T_{a_r} \circ f_r)\big)$. Since f_0 is nilpotent, every orbit of $T_{a_0} \circ f_0$ is finite and therefore $A(T_{a_0} \circ f_0) = \{1\} = A(f_0)$. If $F_i(t)$ is a cyclotomic polynomial, by Lemma 5.5.2 we have $\delta_{T_{a_i} \circ f_i} = \delta_{f_i} = 1$ and therefore $A(T_{a_i} \circ f_i) = \{1\} = A(f_i)$. If $F_i(t), i \geq 1$ is not a cyclotomic polynomial, by Lemma 5.5.5, $T_{a_i} \circ f_i$ is conjugate by a translation to a group homomorphism h_i with minimal polynomial $F_i^{e_i}$. In particular, $A(T_{a_i} \circ f_i) = A(h_i) = A(f_i)$. Therefore

$$\begin{aligned} A\big((T_{a_0} \circ f_0) \times \cdots \times (T_{a_r} \circ f_r)\big) &= A(T_{a_0} \circ f_0) \cup \cdots \cup A(T_{a_r} \circ f_r) \\ &= A(f_0) \cup \cdots \cup A(f_r) = A(f). \end{aligned}$$

If the $T_a \circ f$-orbit of a point $x \in X(\overline{\mathbb{Q}})$ is Zariski dense, then by Lemma 5.2.3 and Lemma 5.5.3, we have

$$\alpha_f(x) = \max\{\delta_{h_i} = \delta_{f_i} \mid F_i \text{ is not a cyclotomic polynomial}\} = \delta_f.$$

$\square$

Proof of Theorem 5.1.3. Since $F(1) \neq 1$, by Lemma 5.5.5, there exists a point $b \in X(\overline{\mathbb{Q}})$ such that $T_{-b} \circ f \circ T_b$ is a homomorphism. Thus it is enough to prove the equivalence of (1), (2) and (3) for every homomorphism f and $b = 0$. (3) $\Rightarrow$ (2). This follows from the fact that the set of n-torsion points of X is finite for each $n > 0$ and that the image of an n-torsion point by a homomorphism is also an n-torsion point. (2) $\Rightarrow$ (1) is trivial. To prove (1) $\Rightarrow$ (3), let $x \in X(\overline{\mathbb{Q}})$ be a point with $\alpha_f(x) = 1$. By §3, we may assume that the minimal polynomial of f is the form of $F(t)^e$ where F is an irreducible monic polynomial that is not cyclotomic. We use the notation of Lemma 5.5.3. By Theorem 5.4.2 and the inequality $\alpha_f(x) \geq \alpha_g(p(x))$, $p(x)$ is a torsion point. Take $n > 0$ so that $np(x) = 0$. Then $nx \in T$. By Lemma 5.5.4, $\alpha_{f_T}(nx) = \alpha_f(nx) = \alpha_f(x) = 1$. Since the minimal polynomial of f_T divides $F(t)^e$, we can use [**Sil2**, Proposition 21(d)] and have $nx \in T(\overline{\mathbb{Q}})_{\mathrm{tors}}$. Hence $x \in X(\overline{\mathbb{Q}})_{\mathrm{tors}}$. $\square$

CHAPTER 6

The canonical heights for Jordan blocks

6.1. Outline of this chapter

Let X be a normal projective variety and $f\colon X \longrightarrow X$ a surjective self-morphism on X both defined over $\overline{\mathbb{Q}}$.

We shall introduce the notion of canonical heights for Jordan blocks of small eigenvalues, which are defined on the set

$$\varprojlim_{f} X(\overline{\mathbb{Q}}) := \left\{\text{sequence } (P_n)_{n\in\mathbb{Z}} \in X(\overline{\mathbb{Q}})^{\mathbb{Z}} \,\middle|\, f(P_n) = P_{n+1} \text{ for all } n \in \mathbb{Z}\right\}.$$

For each $m \in \mathbb{Z}$, we set

$$\begin{aligned} \mathrm{pr}_m\colon \varprojlim_{f} X(\overline{\mathbb{Q}}) &\longrightarrow X(\overline{\mathbb{Q}}) \\ (P_n)_{n\in\mathbb{Z}} &\mapsto P_m. \end{aligned}$$

Note that if f is an automorphism, each element of the set $\varprojlim_{f} X(\overline{\mathbb{Q}})$ is identified with the set $\{f^n(P) \mid n \in \mathbb{Z}\}$, which is the union of the forward f-orbit of P and the forward f^{-1}-orbit of P.

The canonical heights for Jordan blocks associated with an eigenvalue $\lambda \in \mathbb{C}$ satisfying $|\lambda| > 1$ were introduced by Kawaguchi and Silverman in [**KS2**, Theorem 13]. We generalize their results to eigenvalues whose complex absolute values are different from 0 and 1 using the set $\varprojlim_{f} X(\overline{\mathbb{Q}})$.

We shall define the dynamical canonical heights for Jordan blocks which is used to prove the boundedness of some heights in Section 6.3, and to study the growth rate of ample heights in Chapter 7.

Theorem 6.1.1 (The canonical heights for Jordan blocks). *Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective self-morphism over $\overline{\mathbb{Q}}$. Let $\lambda \in \mathbb{C}$ be a complex number with $|\lambda| \neq 0, 1$. Let*

$$D_j \in \mathrm{Div}(X)_{\mathbb{C}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{C} \quad (0 \leq j \leq l)$$

be $\mathbb{C}$-divisors satisfying the following linear equivalences

$$f^*D_j \sim D_{j-1} + \lambda D_j \quad (0 \leq j \leq l),$$

where we set $D_{-1} := 0$. For each D_j, fix a Weil height function h_{D_j}. Then there are unique functions

$$\widehat{h}_{D_j}\colon \varprojlim_{f} X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{C} \quad (0 \leq j \leq l)$$

satisfying the normalization condition

$$\widehat{h}_{D_j} = h_{D_j} \circ \mathrm{pr}_0 + O(1)$$

and the functional equations

$$\widehat{h}_{D_j} \circ f = \widehat{h}_{D_{j-1}} + \lambda \widehat{h}_{D_j} \quad (0 \leq j \leq l),$$

where we set $\widehat{h}_{D_{-1}} := 0$.

Remark 6.1.2. Kawaguchi and Silverman proved the following results in [**KS1**]. Assume that $f\colon X \longrightarrow X$ is a polarized surjective self-morphism (i.e., there is an ample $\mathbb{R}$-divisor H satisfying the numerical equivalence $f^*H \equiv \delta_f H$ for some $\delta_f \in \mathbb{R}_{>1}$). Then we have $\alpha_f(P) \in \{1, \delta_f\}$ for any point $P \in X(\overline{\mathbb{Q}})$. Furthermore, we have $\alpha_f(P) = 1$ if and only if P is preperiodic under f; see [**KS1**, Proposition 7] for details. So Theorem 6.3.1 can be regarded as a generalization of this fact.

We now briefly sketch the plan of this chapter. In Section 6.2, we introduce the notion of the canonical heights for Jordan blocks associated with eigenvalues whose complex absolute values are different from 0 and 1. It simplifies the proof of Theorem 6.3.1. In Section 6.3, we prove Theorem 6.3.1. Since Theorem 6.3.1 is a refinement of the results proved by Kawaguchi and Silverman in [**KS2**], we often refer to their paper [**KS2**]. In Section 6.5, we provide an application to the relation between the dynamical degree and the arithmetic degree; see Proposition 6.5.2. We prove a conjecture proposed by Kawaguchi and Silverman for certain self-morphisms on smooth projective varieties. Finally, we give some remarks on the dynamical degrees and the proof of Theorem 6.3.1.

Remark 6.1.3. Kawaguchi and Silverman proved the existence of the arithmetic degree $\alpha_f(P)$; see [**KS2**, Theorem 3]. It is also known that $\alpha_f(P)$ does not depend on the choice of H and of h_H; see [**KS3**, Proposition 12].

Definition 6.1.4. For a linear self-map $\varphi\colon V \longrightarrow V$ on a finite dimensional vector space V over a subfield K of $\mathbb{C}$, the set of all the eigenvalues of φ on $V \otimes_K \mathbb{C}$ is denoted by $\mathrm{EV}(\varphi; V)$. For a real number $B \in \mathbb{R}$, we define

$$\mathrm{EV}(\varphi, B; V) := \{\lambda \in \mathrm{EV}(\varphi; V) \mid |\lambda| \geq B\}.$$

Definition 6.1.5. We say a sequence $(P_n)_{n\in\mathbb{Z}} \in X(\overline{\mathbb{Q}})^{\mathbb{Z}}$ is an *f-orbit* if it satisfies $f(P_n) = P_{n+1}$ for all $n \in \mathbb{Z}$. An *f-orbit* of $P \in X(\overline{\mathbb{Q}})$ is an f-orbit satisfying $P_0 = P$. Let $\varprojlim_f X(\overline{\mathbb{Q}})$ be the set of all f-orbits. For each integer $m \in \mathbb{Z}$, let

$$\mathrm{pr}_m\colon \varprojlim_f X(\overline{\mathbb{Q}}) \longrightarrow X(\overline{\mathbb{Q}}), \quad (P_n)_{n\in\mathbb{Z}} \mapsto P_m$$

be the projection to the m-th component. For an f-orbit $(P_n)_{n\in\mathbb{Z}}$, let $f((P_n)_{n\in\mathbb{Z}})$ be the f-orbit whose m-th component is P_{m+1} for all $m \in \mathbb{Z}$. Then a self-map

$$f\colon \varprojlim_f X(\overline{\mathbb{Q}}) \longrightarrow \varprojlim_f X(\overline{\mathbb{Q}})$$

is defined and we regard it as the left shift operator. We also define the right shift operator

$$R\colon \varprojlim_f X(\overline{\mathbb{Q}}) \longrightarrow \varprojlim_f X(\overline{\mathbb{Q}})$$

such that $\mathrm{pr}_m \circ R((P_n)) = P_{m-1}$ for all $m \in \mathbb{Z}$.

Definition 6.1.6. For a non-negative integer $l \geq 0$ and a complex number $\lambda \in \mathbb{C}$, let

$$\Lambda := \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & O & \\ & 1 & \ddots & & \\ O & & \ddots & \lambda & \\ & & & 1 & \lambda \end{pmatrix}$$

be the Jordan block matrix of the size $(l+1) \times (l+1)$. We put $N := \Lambda - \lambda I$.

Definition 6.1.7. The symbol $\| \cdot \|$ denotes the sup norm of a (column) vector or a matrix of complex numbers, i.e., for vectors $v = {}^t(a_0, \ldots, a_l) \in \mathbb{C}^{l+1}$ and matrices $A = (a_{i,j})_{0 \leq i,j \leq l}$ with complex coordinates, we set

$$\|v\| := \max_{0 \leq i \leq l} |a_i| \quad \text{and} \quad \|A\| := \max_{0 \leq i,j \leq l} |a_{i,j}|.$$

Remark 6.1.8. For a (column) vector $v \in \mathbb{C}^{l+1}$ and a square matrix A of size $(l+1) \times (l+1)$, we have

$$\|Av\| \leq (l+1) \cdot \|A\| \cdot \|v\|.$$

We frequently use this inequality.

6.2. The canonical heights for Jordan blocks

In this section, we shall prove Theorem 6.1.1. We shall introduce the canonical heights for Jordan blocks whose complex absolute values are different from 0 and 1. Our canonical heights are generalizations of the canonical heights introduced by Kawaguchi and Silverman in [**KS2**] for eigenvalues whose complex absolute values are greater than 1.

Lemma 6.2.1. (a) *When $|\lambda| \geq 1$, we have $\|\Lambda^n\| \asymp n^\ell |\lambda|^n$.*

(b) *When $|\lambda| < 1$, we have $\|\Lambda^n\| \leq n^\ell |\lambda|^{n-\ell}$.*

(c) *When $|\lambda| > 1$, for a non-zero column vector $v = {}^t(x_0, \ldots, x_\ell) \in \mathbb{C}^{\ell+1} \setminus \{0\}$, we have*

$$\|\Lambda^n v\| \asymp n^t |\lambda|^n,$$

where we put

$$t := \ell - \min\{i \mid 0 \leq i \leq \ell,\ x_i \neq 0\}.$$

(d) *If $|\lambda| < 1$, for any vector $v \in \mathbb{C}^{l+1}$, we have*

$$\lim_{n \to \infty} \Lambda^n v = 0.$$

(e) *If $\lambda \neq 0$, for any nonzero vector $v \in \mathbb{C}^{l+1} \setminus \{0\}$, we have*

$$\lim_{n \to \infty} ||\Lambda^n v||^{1/n} = |\lambda|.$$

(f) *If $\lambda \neq 0$, for any nonzero vector $v \in \mathbb{C}^{l+1} \setminus \{0\}$, we have*

$$\lim_{n \to \infty} ||\Lambda^{-n} v||^{1/n} = |\lambda|^{-1}.$$

PROOF. (a), (b) Note that $\binom{n}{k} \asymp n^k$ and $\binom{n}{k} \leq n^k$. Then both assertions follow from the following equalities:

$$\begin{aligned}\|\Lambda^n\| &= \|(\lambda I + N)^n\| \\ &= \left\| \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} N^k \right\| \\ &= \left\| \sum_{k=0}^{\ell} \binom{n}{k} \lambda^{n-k} N^k \right\| && \text{since } N^{\ell+1} = 0 \\ &= \max_{0 \leq k \leq \ell} \binom{n}{k} |\lambda|^{n-k}.\end{aligned}$$

(c) We may assume $t = \ell$, so $x_0 \neq 0$. For a negative integer $j < 0$, we set $x_j := 0$. The asymptotic inequality $\|\Lambda^n v\| \preceq n^\ell |\lambda|^n$ follows from the following inequalities.

$$\begin{aligned}|(\Lambda^n v)_j| &= \left| \sum_{k=0}^{\ell} \binom{n}{k} \lambda^{n-k} (N^k v)_j \right| \\ &= \left| \sum_{k=0}^{\ell} \binom{n}{k} \lambda^{n-k} x_{j-k} \right| \\ &\leq \sum_{k=0}^{j} n^k \cdot |\lambda|^n \cdot |x_{j-k}| \\ &\leq (\ell+1) \cdot n^\ell \cdot |\lambda|^n \cdot \|v\|\end{aligned}$$

The converse asymptotic inequality $\|\Lambda^n v\| \succeq n^\ell |\lambda|^n$ follows from the following asymptotic inequalities.

$$\begin{aligned}|(\Lambda^n v)_\ell| &= \left| \sum_{k=0}^{\ell} \binom{n}{k} \lambda^{n-k} x_{\ell-k} \right| \\ &\geq \binom{n}{\ell} |\lambda^{n-\ell} x_0| - \sum_{k=0}^{\ell-1} \left| \binom{n}{k} \lambda^{n-k} x_{\ell-k} \right| \\ &\succeq n^\ell |\lambda|^n - n^{\ell-1} |\lambda|^n \\ &\succeq n^\ell |\lambda|^n.\end{aligned}$$

Hence we conclude $\|\Lambda^n v\| \asymp n^\ell |\lambda|^n$.

(d) The assertion follows from (c).

(e) See [**KS2**, Lemma 12].

(f) Since the Jordan normal form of Λ^{-1} is $\lambda^{-1} I + N$ with a nilpotent matrix N, there is an invertible matrix U such that

$$U \Lambda^{-1} U^{-1} = \lambda^{-1} I + N.$$

Since any two norms on $\mathbb{C}^{l+1}$ are equivalent to each other, there are positive real numbers $C, C' \in \mathbb{R}_{>0}$ such that for all $v \in \mathbb{C}^{l+1}$, the following inequalities hold

$$C||v|| \leq ||U^{-1} v|| \leq C'||v||.$$

Combining these inequalities with (d), we get

$$\begin{aligned}\lim_{n\to\infty} ||\Lambda^{-n}v||^{1/n} &= \lim_{n\to\infty} ||U^{-1}(\lambda^{-1}I+N)^n Uv||^{1/n}\\ &= \lim_{n\to\infty} ||(\lambda^{-1}I+N)^n(Uv)||^{1/n}\\ &= |\lambda|^{-1}.\end{aligned}$$

□

Proposition 6.2.2 (The axiomatic canonical heights for Jordan blocks). *Let $\lambda \in \mathbb{C}$ be a complex number satisfying $|\lambda| \neq 0, 1$. Let S be a set with a self-bijection $R\colon S \longrightarrow S$, and $h\colon S \longrightarrow \mathbb{C}^{l+1}$ a vector valued function satisfying*

$$||h \circ R - \Lambda^{-1}h|| = O(1).$$

Then there is a unique function $\widehat{h}\colon S \longrightarrow \mathbb{C}^{l+1}$ satisfying the functional equation

$$\widehat{h} \circ R = \Lambda^{-1}\widehat{h}$$

and the normalization condition

$$\widehat{h} = h + O(1).$$

Proof. We shall give a proof of the assertion for λ satisfying $0 < |\lambda| < 1$. If $1 < |\lambda|$, we can prove it similarly using the inverse map of R instead of R. See also [**KS2**, Theorem 13].

First, we shall define $\widehat{h}$. Let

$$E := h \circ R - \Lambda^{-1}h$$

be the error function. There is a constant $C_0 > 0$ satisfying $\|E(x)\| < C_0$ for any $x \in S$. We define

$$\widehat{h} := h + \sum_{n=0}^{\infty} \Lambda^{n+1}(E \circ R^n).$$

To prove that it is well-defined and satisfies the normalization condition, it suffices to prove that the series

$$\sum_{n=0}^{\infty} ||\Lambda^{n+1}(E \circ R^n(x))||$$

converges and is bounded by a constant which is independent of $x \in S$. These assertions follow from the following calculation.

$$\begin{aligned}&\sum_{n=0}^{\infty} ||\Lambda^{n+1}(E \circ R^n(x))||\\ &\leq \sum_{n=0}^{\infty}(l+1)\cdot||\Lambda^{n+1}||\cdot||E\circ R^n(x)||\\ &= \sum_{n=0}^{\ell-1}(l+1)\cdot||\Lambda^{n+1}||\cdot||E\circ R^n(x)|| + \sum_{n=l}^{\infty}(l+1)\cdot||\Lambda^{n+1}||\cdot||E\circ R^n(x)||\\ &\leq C_1 + C_0\cdot\sum_{n=l}^{\infty}(l+1)\cdot n^l\cdot|\lambda|^{n-l} \qquad \text{from Lemma 6.2.1 (b)}\\ &\leq C_2,\end{aligned}$$

where C_1 and C_2 are constants independent of $x \in S$.

Next, we shall prove that $\widehat{h}$ satisfies the functional equation. It follows from the following formal calculation.

$$\begin{aligned}
\widehat{h} \circ R &= h \circ R + \sum_{n=0}^{\infty} \Lambda^{n+1}(E \circ R^{n+1}) \\
&= h \circ R - E + \sum_{n=0}^{\infty} \Lambda^{n} E \circ R^{n} \\
&= \Lambda^{-1} h + \Lambda^{-1} \sum_{n=0}^{\infty} \Lambda^{n+1} E \circ R^{n} \\
&= \Lambda^{-1} \widehat{h}.
\end{aligned}$$

Finally we shall prove the uniqueness of $\widehat{h}$. Let both $\widehat{h}$ and $\widehat{h}'$ be vector valued functions satisfying the functional equation and the normalization condition. We set $g := \widehat{h} - \widehat{h}'$. Then g is a bounded function satisfying the functional equation $g \circ R = \Lambda^{-1} g$. Assume that $g(x) \neq 0$ for some $x \in S$. Then we get

$$1 \geq \lim_{n \to \infty} ||g \circ R^n(x)||^{1/n} = ||\Lambda^{-n} g(x)||^{1/n} = |\lambda|^{-1},$$

where the first inequality follows from the boundedness of g, and the last equality follows from Lemma 6.2.1 (e). This contradicts the assumption $0 < |\lambda| < 1$. Consequently, we have $g(x) = 0$ for any $x \in S$. $\square$

Proof of Theorem 6.1.1. If $0 < |\lambda| < 1$ (resp. $1 < |\lambda|$), the assertion follows by applying Proposition 6.2.2 to the set $\varprojlim_f X(\overline{\mathbb{Q}})$, the vector valued height function

$$h_D := {}^t(h_{D_0}, h_{D_1}, \ldots, h_{D_l}) \circ \mathrm{pr}_0,$$

and the right shift operator

$$R \colon \varprojlim_f X(\overline{\mathbb{Q}}) \longrightarrow \varprojlim_f X(\overline{\mathbb{Q}})$$

(resp. left shift operator). $\square$

Proposition 6.2.3. *Let notation be the same as in Theorem 6.1.1. Moreover, assume that* $0 < |\lambda| < 1$. *Then for every* j $(0 \leq j \leq l)$ *and for every point* $P \in X(\overline{\mathbb{Q}})$, *the sequence* $\left\{h_{D_j}(f^n(P))\right\}_{n \geq 0}$ *is bounded.*

Proof. For a point $P \in X(\overline{\mathbb{Q}})$, take an f-orbit $(P_n)_{n \in \mathbb{Z}}$ of P. Let

$$\mathbf{h}_D := {}^t(h_{D_0}, h_{D_1}, \ldots, h_{D_l})$$

be the vector valued height function. Take the canonical height function $\widehat{h}_{D_j}$ as in Theorem 6.1.1 and let

$$\widehat{\mathbf{h}}_D := {}^t(\widehat{h}_{D_0}, \widehat{h}_{D_1}, \ldots, \widehat{h}_{D_l})$$

be the vector valued canonical height function. There is a real number $C \in \mathbb{R}_{>0}$ such that for every f-orbit $(P_n)_{n\in\mathbb{Z}}$ and every $m \geq 0$, we have

$$\begin{aligned}||\mathbf{h}_D(f^m(P))|| &= ||\mathbf{h}_D \circ \mathrm{pr}_0(f^m((P_n)))|| \\ &\leq ||\widehat{\mathbf{h}}_D(f^m((P_n)))|| + C \\ &= ||\Lambda^m \widehat{\mathbf{h}}_D((P_n))|| + C \\ &\leq (l+1) \cdot ||\Lambda^m|| \cdot ||\widehat{\mathbf{h}}_D((P_n))|| + C.\end{aligned}$$

When m goes to ∞, the last term converges to C by Lemma 6.2.1 (b). Hence the assertion follows. $\square$

Remark 6.2.4. It is possible to prove Proposition 6.2.3 directly without using Theorem 6.1.1. But the canonical heights for Jordan blocks are interesting themselves, and they make the proof clearer.

6.3. Preperiodicity and $\alpha_f(P) = 1$

The results of this section are motivated by the following result of Kawaguchi and Silverman: they proved that either $\alpha_f(P) = 1$, or $\alpha_f(P)$ is equal to the complex absolute value of an eigenvalue of the linear self-map

$$f^* \colon \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{NS}(X)_{\mathbb{Q}}$$

induced by f; see the proof of Theorem 6.3.1 or [**KS2**, Remark 23] for details.

It is easy to see that if P is preperiodic under f, we have $\alpha_f(P) = 1$. In this chapter, we shall provide a sufficient condition under which the converse is true.

Let V_H be the $\mathbb{Q}$-linear subspace of $\mathrm{Pic}(X)_{\mathbb{Q}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ spanned by the set

$$\{(f^n)^* H \mid n \geq 0\},$$

and $\overline{V_H}$ the image of V_H in $\mathrm{NS}(X)_{\mathbb{Q}}$. It is known that V_H and $\overline{V_H}$ are finite dimensional $\mathbb{Q}$-vector space; see Lemma 2.1.10.

In this section, we shall prove the following theorem.

Theorem 6.3.1. *Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$, and $f \colon X \longrightarrow X$ a surjective self-morphism on X over $\overline{\mathbb{Q}}$. Assume that 1 does not appear as the complex absolute value of an eigenvalue of the linear self-map*

$$f^* \colon \overline{V_H} \longrightarrow \overline{V_H}.$$

We put

$$\mu_H(f) := \min\left\{|\lambda| \mid \lambda \text{ is an eigenvalue of } f^* \colon \overline{V_H} \longrightarrow \overline{V_H} \text{ with } |\lambda| \geq 1\right\}.$$

Then for every point $P \in X(\overline{\mathbb{Q}})$, the following conditions are equivalent.

- *P is preperiodic under f.*
- *$\alpha_f(P) < \mu_H(f)$.*
- *$\alpha_f(P) = 1$.*

Remark 6.3.2. If 1 does not appear as the complex value of the linear map $f^* \colon V_H \longrightarrow V_H$, we can get the similar result of Theorem 6.3.1 for normal possibly non-smooth projective variety X.

Before giving the proof, we give easy lemmata in linear algebra which we frequently use in the proof.

Lemma 6.3.3. *Let U, V be finite dimensional vector spaces over a field, and $\varphi_U : U \longrightarrow U$, $\varphi_V : V \longrightarrow V$ be linear self-maps on U, V, respectively.*

(a) *If there is an injection $\iota : V \longrightarrow U$ satisfying $\iota \circ \varphi_V = \varphi_U \circ \iota$, then we have $\mathrm{EV}(\varphi_V) \subset \mathrm{EV}(\varphi_U)$.*

(b) *Let $\pi : V \longrightarrow U$ be a surjection satisfying $\pi \circ \varphi_V = \varphi_U \circ \pi$. Then $\mathrm{EV}(\varphi_U)$ coincides with the set*

$$\{\lambda \in \mathrm{EV}(\varphi_V) \mid \exists v \in V \text{ s.t. } \pi v \neq 0 \text{ and } \varphi_V v = \lambda v\}.$$

(c) *Let notation be as in (b). Let $\lambda \in \mathrm{EV}(\varphi_V) \setminus \mathrm{EV}(\varphi_U)$, and let $v_0, v_1, \ldots, v_r \in V$ satisfy*

$$\varphi_V v_j = v_{j-1} + \lambda v_j \quad (0 \leq j \leq r),$$

where we set $v_{-1} := 0$. Then we have $\pi v_j = 0$ for every j $(0 \leq j \leq r)$.

Proof. (a),(b) The assertions are obvious.

(c) The equality $\pi v_0 = 0$ follows from (b). It is easy to see $\pi v_j = 0$ by induction. $\square$

Proof of Theorem 6.3.1. Set $(V_H)_{\mathbb{C}} := V_H \otimes_{\mathbb{Q}} \mathbb{C}$. We decompose the $\mathbb{C}$-vector space $(V_H)_{\mathbb{C}}$ to the Jordan blocks as

$$(V_H)_{\mathbb{C}} = \bigoplus_{i=1}^{\nu} V_i.$$

Here, each V_i satisfies $f^*(V_i) \subset V_i$ and $f^*|_{V_i}$ is represented by a Jordan block matrix of eigenvalue λ_i as in Definition 6.1.6. By relabeling, we may assume that

$$\begin{cases} \lambda_i \in \mathrm{EV}(f^*; \mathrm{NS}(X)_{\mathbb{Q}}) \text{ with } 1 < |\lambda_i| \text{ for } 1 \leq i \leq \sigma, \\ \lambda_i \in \mathrm{EV}(f^*; \mathrm{NS}(X)_{\mathbb{Q}}) \text{ with } 0 < |\lambda_i| < 1 \text{ for } \sigma + 1 \leq i \leq \tau, \text{ and} \\ \lambda_i \in \mathrm{EV}(f^*; \mathrm{Pic}(X)_{\mathbb{Q}}) \setminus \mathrm{EV}(f^*; \mathrm{NS}(X)_{\mathbb{Q}}) \text{ for } \tau + 1 \leq i \leq \nu. \end{cases}$$

By assumption, no $\lambda \in \mathrm{EV}(f^*; \mathrm{NS}(X)_{\mathbb{Q}})$ satisfies $|\lambda| = 1$. Let $\{D_{i,j} \mid 0 \leq j \leq l_i\}$ be the $\mathbb{C}$-basis of V_i satisfying the following linear equivalences

$$f^* D_{i,j} \sim D_{i,j-1} + \lambda_i D_{i,j} \quad (0 \leq j \leq l_j),$$

where we set $D_{i,-1} := 0$. Take the canonical height for Jordan blocks as in Theorem 6.1.1 for each $1 \leq i \leq \sigma$ and $0 \leq j \leq l_i$.

If a point $P \in X(\overline{\mathbb{Q}})$ is preperiodic point under f we have

$$\alpha_f(P) = 1 < \mu_H(f).$$

Conversely, we assume that $\alpha_f(P) < \mu_H(f)$. We shall prove that P is preperiodic under f. If $\widehat{h}_{D_{i,j}}(P) \neq 0$ for some index (i,j) with $1 \leq i \leq \sigma$, fix such an index i_0 and let j_0 be the smallest index satisfying $\widehat{h}_{D_{i_0,j_0}}(P) \neq 0$. Then we have

$$\begin{aligned} \widehat{h}_{D_{i_0,j_0}}(f^n(P)) &= \sum_{j=0}^{j_0} \binom{n}{j} \lambda_{i_0}^{n-j} \widehat{h}_{D_{i_0,j}}(P) \\ &= \lambda_{i_0}^n \widehat{h}_{D_{i_0,j_0}}(P). \end{aligned}$$

Consequently, the arithmetic degree is bounded as follows.

$$\begin{aligned}\alpha_f(P) &= \lim_{n\to\infty} \max\{h_H(f^n(P)), 1\}^{1/n} \\ &\geq \lim_{n\to\infty} |h_{D_{i_0,j_0}}(f^n(P))|^{1/n} \\ &\geq \lim_{n\to\infty} \left(|\widehat{h}_{D_{i_0,j_0}}(f^n(P))| - C\right)^{1/n} \\ &= \lim_{n\to\infty} \left(|\lambda_{i_0}^n \widehat{h}_{D_{i_0,j_0}}(P)| - C\right)^{1/n} \\ &= |\lambda_{i_0}|.\end{aligned}$$

But since $\lambda_{i_0} \in \mathrm{EV}(f^*; \mathrm{NS}(X)_{\mathbb{R}})$, we have $\alpha_f(P) \geq |\lambda_{i_0}| \geq \mu_H(f)$. This is a contradiction. Thus, we now have $\widehat{h}_{D_{i,j}}(P) = 0$ for every $1 \leq i \leq \sigma$ and $0 \leq j \leq l_i$.

Write $H = \sum_{i,j} a_{i,j} D_{i.j}$ and fix an ample height h_H with $h_H \geq 1$. Since for $\tau+1 \leq i \leq \nu$, the $\mathbb{C}$-divisors $D_{i,j}$ are algebraically equivalent to 0, the following inequality holds on $X(\overline{\mathbb{Q}})$:

$$\left| h_H - \sum_{i=1}^{\tau} \sum_{j=0}^{l_i} a_{i,j} h_{D_{i,j}} \right| \leq o(h_H)$$

(see [**HS**, Theorem B.3.2 (f)]). This is the only part that we use the smoothness of X.

Now, from the fact we proved above and Proposition 6.2.3, for $1 \leq i \leq \tau$ and $0 \leq j \leq l_i$, the heights $h_{D_{i,j}}$ are uniformly bounded by a constant on the forward f-orbit of P. Consequently, we can find a constant $C > 0$ such that the following inequalities

$$\begin{aligned} h_H(f^n(P)) - C &\leq \left| h_H(f^n(P)) - \sum_{i=1}^{\tau} \sum_{j=0}^{l_i} a_{i,j} h_{D_{i,j}}(f^n(P)) \right| \\ &\leq o(h_H(f^n(P)))\end{aligned}$$

hold for all $n \geq 0$. The finiteness of the number of elements of the set $\{h_H(f^n(P)) \mid n \geq 0\}$ follows from this inequality. Since the ample height function satisfies Northcott's property, the point P is preperiodic under f. □

6.4. The growth rate of the ample heights

In this section, we provide an explicit formula on the growth rate of ample heights of rational points under iteration of self-morphisms of smooth projective varieties over $\overline{\mathbb{Q}}$.

Theorem 6.4.1. *Let X be a normal projective variety over k and $f\colon X \longrightarrow X$ a surjective self-morphism of X over $\overline{\mathbb{Q}}$. Let H be an ample divisor on X over k. Then for any point $P \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) > 1$, there is a non-negative integer $t_f(P) \in \mathbb{Z}_{\geq 0}$, positive real numbers $C_0, C_1 > 0$, and an integer N_0 such that the inequalities*

$$C_0 n^{t_f(P)} \alpha_f(P)^n < h_H(f^n(P)) < C_1 n^{t_f(P)} \alpha_f(P)^n$$

hold for all $n \geq N_0$.

Let X be a smooth projective variety, and $f\colon X \longrightarrow X$ a surjective self-morphism of X both over $\overline{\mathbb{Q}}$.

We shall prove Theorem 6.4.1.

$$(V_H)_{\mathbb{C}} = \bigoplus_{i=1}^{\tau} V_i.$$

For each $1 \leq i \leq \tau$, let $\lambda_i \in \mathbb{C}$ be the eigenvalue of $f^*|_{V_i}$. By changing the order of the Jordan blocks if necessary, we may assume

$$\begin{aligned} &|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_\sigma| > 1 \\ &|\lambda_{\sigma+1}| = \cdots = |\lambda_\tau| = 1 \\ &1 > |\lambda_{\tau+1}| \geq \cdots \geq |\lambda_\nu| \end{aligned}$$

for some $0 < \sigma \leq \tau \leq \nu$. We put $\ell_i := \dim_{\mathbb{C}} V_i - 1$.

We take a $\mathbb{C}$-basis $\{D_{i,j}\}_{0 \leq j \leq \ell_i}$ of V_i satisfying the following linear equivalences:

$$f^* D_{i,j} \sim D_{i,j-1} + \lambda_i D_{i,j} \quad (0 \leq j \leq \ell_i), \tag{6.4.0}$$

where we set $D_{i,-1} := 0$. For each $1 \leq i \leq \sigma$, let

$$\widehat{h}_{D_{i,j}} \colon X(\overline{\mathbb{Q}}) \longrightarrow \mathbb{C} \quad (0 \leq j \leq \ell_i)$$

be unique functions satisfying the normalization condition

$$\widehat{h}_{D_{i,j}} = h_{D_{i,j}} + O(1) \quad (0 \leq j \leq \ell_i)$$

and the functional equation

$$\widehat{h}_{D_{i,j}} \circ f = \widehat{h}_{D_{i,j-1}} + \lambda_i \widehat{h}_{D_{i,j}} \quad (0 \leq j \leq \ell_i)$$

(see [**KS2**, Theorem 5] for the existence of such functions).

For each $1 \leq i \leq \nu$, we set

$$\mathbf{h}_{\mathbf{D}_i} := {}^t(h_{D_{i,0}}, h_{D_{i,1}}, \ldots, h_{D_{i,\ell_i}}).$$

For each $1 \leq i \leq \sigma$, we set

$$\widehat{\mathbf{h}}_{\mathbf{D}_i} := {}^t(\widehat{h}_{D_{i,0}}, \widehat{h}_{D_{i,1}}, \ldots, \widehat{h}_{D_{i,\ell_i}}).$$

Let Λ_i be the Jordan block matrix of size $(\ell_i + 1) \times (\ell_i + 1)$ associated with the eigenvalue λ_i.

Lemma 6.4.2. *For each $\sigma + 1 \leq i \leq \tau$ and a point $P \in X(\overline{\mathbb{Q}})$, we have*

$$\|\mathbf{h}_{\mathbf{D}_i}(f^n(P))\| \preceq n^{\ell_i + 1}.$$

Proof. By (6.4), there is a positive real number $C_0 > 0$ such that the inequality

$$\|\mathbf{h}_{\mathbf{D}_i} \circ f - \Lambda_i \cdot \mathbf{h}_{\mathbf{D}_i}\| \leq C_0$$

holds on $X(\overline{\mathbb{Q}})$. Therefore, there is a positive real number $C_1 > 0$ such that for every point $P \in X(\overline{\mathbb{Q}})$, the following inequalities hold:

$$\begin{aligned}
&\|\mathbf{h}_{\mathbf{D}_i} \circ f^n(P)\| - \|\Lambda_i^n \cdot \mathbf{h}_{\mathbf{D}_i}(P)\| \\
&\leq \|\mathbf{h}_{\mathbf{D}_i} \circ f^n(P) - \Lambda_i^n \cdot \mathbf{h}_{\mathbf{D}_i}(P)\| \\
&\leq \sum_{k=0}^{n-1} \|\Lambda_i^k \cdot (\mathbf{h}_{\mathbf{D}_i} \circ f^{n-k}(P)) - \Lambda_i^{k+1} \cdot (\mathbf{h}_{\mathbf{D}_i} \circ f^{n-k-1}(P))\| \\
&\leq \sum_{k=0}^{n-1} (\ell_i + 1)\|\Lambda_i^k\| \cdot \|\mathbf{h}_{\mathbf{D}_i}(f^{n-k}(P)) - \Lambda_i \cdot \mathbf{h}_{\mathbf{D}_i}(f^{n-k-1}(P))\| \\
&\leq \sum_{k=0}^{n-1} (\ell_i + 1)\|\Lambda_i^k\| C_0 \\
&\leq \sum_{k=0}^{n-1} (\ell_i + 1) \cdot n^{\ell_i} \cdot |\lambda_i|^{k-\ell_i} \cdot C_0 && \text{by Lemma 6.2.1} \\
&\leq C_1 n^{\ell_i + 1} && \text{because } |\lambda_i| = 1.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|\Lambda_i^n \cdot \mathbf{h}_{\mathbf{D}_i}(P)\| &\leq (\ell_i + 1) \cdot \|\Lambda_i^n\| \cdot \|\mathbf{h}_{\mathbf{D}_i}(P)\| \\
&\leq (\ell_i + 1) \cdot n^{\ell_i} \cdot |\lambda_i|^{n-\ell_i} \cdot \|\mathbf{h}_{\mathbf{D}_i}(P)\| && \text{by Lemma 6.2.1} \\
&= (\ell_i + 1) \cdot n^{\ell_i} \cdot \|\mathbf{h}_{\mathbf{D}_i}(P)\| && \text{because } |\lambda_i| = 1.
\end{aligned}$$

Combining these inequalities, the assertion is proved. □

Lemma 6.4.3. *For each $\tau + 1 \leq i \leq \nu$ and a point $P \in X(\overline{\mathbb{Q}})$, the sequence*

$$\{\ \|\mathbf{h}_{\mathbf{D}_i}(f^n(P))\|\ \}_{n \geq 0}$$

is bounded.

Proof. Consider the canonical heights as in Theorem 6.1.1. Then the assertion follows from Lemma 6.2.1 (b) and (d). □

Lemma 6.4.4 (see [KS2, Lemma 18]). *For a $\mathbb{C}$-divisor D on X and a point $P \in X(\overline{\mathbb{Q}})$, we have*

$$\max\{1, h_H(f^n(P))\} \succeq |h_D(f^n(P))|.$$

Proof. The assertion is obviously true when the forward f-orbit of P is a finite set. Hence, we may assume the forward f-orbit of P is an infinite set. Thus we may assume

$$h_H(f^n(P)) \to \infty \quad (\text{as } n \to \infty). \tag{6.4.1}$$

Write $D = D_r + \sqrt{-1}D_c$, where D_r and D_c are $\mathbb{R}$-divisors on X. By the triangle inequality, it is enough to prove the assertion for D_r and D_c. Thus we may assume D is an $\mathbb{R}$-divisor.

Take a sufficiently large positive real number $C > 0$ such that $CH \pm D$ are ample. The function $C \cdot h_H - |h_D|$ is bounded below on $X(\overline{\mathbb{Q}})$. There is a (not necessarily positive) real number $C' \in \mathbb{R}$ satisfying

$$Ch_H(f^n(P)) - |h_D(f^n(P))| \geq C' \tag{6.4.2}$$

for all $P \in X(\overline{\mathbb{Q}})$. By (6.4.1) and (6.4.2), the assertion follows. □

Proof of Theorem 6.4.1. Write H in terms of the $\mathbb{C}$-linear bases of $(V_H)_{\mathbb{C}}$:

$$H = \sum_{i=1}^{\nu} \sum_{j=0}^{\ell_i} c_{i,j} D_{i,j} \quad (c_{i,j} \in \mathbb{C}).$$

Let $P \in X(\overline{\mathbb{Q}})$ be a point with $\alpha_f(P) > 1$. If $\widehat{\mathbf{h}}_{\mathbf{D}_i}(P) = 0$ for all $1 \leq i \leq \sigma$, we have the following inequalities:

$$\begin{aligned}
&h_H(f^n(P)) \\
&\leq \left| \sum_{i=1}^{\sigma} \sum_{j=0}^{\ell_i} c_{i,j} \widehat{h}_{D_{i,j}}(f^n(P)) \right| + \left| \sum_{i=\sigma+1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| \\
&\quad + \left| \sum_{i=\tau+1}^{\nu} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1) \\
&= \left| \sum_{i=\sigma+1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1) && \text{by Lemma 6.4.3} \\
&\preceq n^{\max_i \ell_i + 1} && \text{by Lemma 6.4.2.}
\end{aligned}$$

Thus we get

$$\alpha_f(P) \leq \lim_{n \to \infty} n^{(\max_i \ell_i + 1)/n} = 1.$$

But this contradicts $\alpha_f(P) > 1$. Hence we get $\widehat{\mathbf{h}}_{\mathbf{D}_i}(P) \neq 0$ for some $1 \leq i \leq \sigma$.

We set

$$\lambda := \max \left\{ |\lambda_i| \mid 1 \leq i \leq \sigma,\ \widehat{\mathbf{h}}_{\mathbf{D}_i}(P) \neq 0 \right\}.$$

If $\widehat{h}_{D_{i,j}}(P) = 0$, we set $t_{f,i}(P) := -\infty$. Otherwise, we set

$$t_{f,i}(P) := \ell_i - \min \left\{ j \mid 0 \leq j \leq \ell_i,\ \widehat{h}_{D_{i,j}}(P) \neq 0 \right\}.$$

Finally, we set

$$t_f(P) := \max \left\{ t_{f,i}(P) \mid \lambda = |\lambda_i| \right\}.$$

Since $\widehat{\mathbf{h}}_{\mathbf{D}_i}(P) \neq 0$ holds for some $1 \leq i \leq \sigma$, we get $t_f(P) \neq -\infty$. It is enough to prove

$$h_H(f^n(P)) \asymp n^{t_f(P)} \lambda^n.$$

Note that $\lambda = \alpha_f(P)$ follows from this asymptotic inequality. We have

$$\begin{aligned}
h_H(f^n(P)) &\leq \left| \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} c_{i,j} h_{D_{i,j}}(f^n(P)) \right| + O(1) \\
&\leq \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} |c_{i,j}| \cdot |h_{D_{i,j}}(f^n(P))| + O(1) \\
&\preceq \sum_{i=1}^{\tau} \sum_{j=0}^{\ell_i} |c_{i,j}| \cdot h_H(f^n(P)) && \text{by Lemma 6.4.4} \\
&\preceq h_H(f^n(P)).
\end{aligned} \tag{6.4.3}$$

By the equality $\widehat{\mathbf{h}}_{\mathbf{D}_i}(f(P)) = \Lambda_i \widehat{\mathbf{h}}_{\mathbf{D}_i}(P)$ and Lemma 6.2.1, we get

$$\|\widehat{\mathbf{h}}_{\mathbf{D}_i}(f^n(P))\| \asymp n^{t_{f,i}(P)}|\lambda_i|^n \quad (1 \leq i \leq \sigma). \tag{6.4.4}$$

Combining Lemma 6.4.2, Lemma 6.4.3, and (6.4.4), we conclude

$$\sum_{i=1}^{\nu}\sum_{j=0}^{\ell_i} |c_{i,j}| \cdot |h_{D_{i,j}}(f^n(P))| \asymp n^{t_f(P)}\lambda^n.$$

The assertion follows from this asymptotic equality and (6.4.3). $\square$

6.5. Applications to the Kawaguchi-Silverman Conjecture

In this section, we provide an application of Theorem 6.3.1 to the conjecture stated by Kawaguchi and Silverman in [**KS3**, Conjecture 6].

Recall that V_H is a $\mathbb{Q}$-linear subspace of $\mathrm{Pic}(X)_{\mathbb{Q}}$ spanned by the set $\{(f^n)^*H \mid n \geq 0\}$, and $\overline{V_H}$ is the image of V_H in $\mathrm{NS}(X)_{\mathbb{Q}}$.

Lemma 6.5.1. *The map $f_*\colon \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$ induces the self-morphism on V_H.*

Proof. Since f_*f^* is the multiplication by $\deg f$ on $\mathrm{Pic}(X)$, the map f^* is injective on $\mathrm{Pic}(X)_{\mathbb{Q}}$. In particular, the map $f^*|_{V_H}\colon V_H \longrightarrow V_H$ is injective. Because V_H is a finite dimensional $\mathbb{Q}$-vector space, the map $f^*|_{V_H}$ is surjective. For a $\mathbb{Q}$-divisor $D \in V_H$, there is a $\mathbb{Q}$-divisor $D' \in V_H$ such that $f^*D' = D$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$. Hence we get

$$f_*D = f_*f^*D' = \deg f \cdot D' \in V_H.$$

$\square$

Proposition 6.5.2. *Let X be a smooth projective variety defined over $\overline{\mathbb{Q}}$. Let H be an ample divisor on X defined over $\overline{\mathbb{Q}}$, and $f\colon X \longrightarrow X$ a surjective self-morphism on X over $\overline{\mathbb{Q}}$. Assume that $\delta_f > (\deg f)^2$ and $\dim_{\mathbb{Q}} \overline{V_H} \leq 2$. Then for every point $P \in X(\overline{\mathbb{Q}})$, P is wandering under f (i.e., the forward f-orbit of P is an infinite set) if and only if $\alpha_f(P) = \delta_f$. In particular, Conjecture 3.1.1 is true in this case.*

Remark 6.5.3. As a special case of Proposition 6.5.2, Conjecture 3.1.1 is true for any automorphisms of any smooth projective varieties of Picard number 2.

Proof. By Theorem 2.1.11, the dynamical degree δ_f appears as an eigenvalue of $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$. If we have $\dim_{\mathbb{Q}} \overline{V_H} = 1$, the ample $\mathbb{R}$-divisor H satisfies the following numerical equivalence

$$f^*H \equiv \delta_f H.$$

In this case, it is well-known that P is wandering under f if and only if $\alpha_f(P) = \delta_f$. See [**KS1**, Proposition 7] for details.

So we may assume $\dim_{\mathbb{Q}} \overline{V_H} = 2$. Since $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$ and $f_*\colon \overline{V_H} \longrightarrow \overline{V_H}$ come from the $\mathbb{Z}$-linear self-maps on $\mathrm{NS}(X)$, their eigenvalues are algebraic integers. Let $\det(f^*|_{\overline{V_H}})$ and $\det(f_*|_{\overline{V_H}})$ be the determinants of the $\mathbb{Q}$-linear maps $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$ and $f_*\colon \overline{V_H} \longrightarrow \overline{V_H}$, respectively. Since $\det(f_*|_{\overline{V_H}})$ is a non-zero rational number which is also an algebraic integer, we have

$$\det(f_*|_{\overline{V_H}}) \geq 1.$$

Thus, we have

$$\begin{aligned}\det(f^*|_{\overline{V_H}}) &\le \det(f_*|_{\overline{V_H}}) \cdot \det(f^*|_{\overline{V_H}}) \\ &= \det(\deg f \cdot \mathrm{id}_{\overline{V_H}}) \\ &= (\deg f)^2.\end{aligned}$$

Let $\{\delta_f, \lambda\}$ be the eigenvalues of $f^*\colon \overline{V_H} \longrightarrow \overline{V_H}$. Since we have

$$\delta_f \cdot \lambda = \det(f^*|_{\overline{V_H}}),$$

we obtain

$$\lambda = \frac{\det(f^*|_{\overline{V_H}})}{\delta_f} \le \frac{(\deg f)^2}{\delta_f} < 1$$

by the assumption $\delta_f > (\deg f)^2$. Hence, we conclude that $\mu_H(f) = \delta_f > 1$.

Now let $P \in X(\overline{\mathbb{Q}})$ be a wandering point under f. By Theorem 6.3.1, we get $\alpha_f(P) \ge \mu_H(f) = \delta_f$. The opposite inequality $\alpha_f(P) \le \delta_f$ is known in general; see [**KS3**, Theorem 4], [**Matz**, Theorem 1.4] for dominant rational maps, and see also the last sentence of the proof of Theorem 3 in [**KS2**] for surjective self-morphisms. Hence the assertion $\alpha_f(P) = \delta_f$ follows. □

Remark 6.5.4. If $f\colon X \longrightarrow X$ does not satisfy the assumption $\delta_f > (\deg f)^2$, the assertion that P is wandering under f if and only if $\alpha_f(P) = \delta_f$ may not be true. For example, consider the elliptic curves E and E' over $\overline{\mathbb{Q}}$, and a non-torsion point $P_0 \in E'(\overline{\mathbb{Q}})$. Then for the self-morphism $f\colon E \times E' \longrightarrow E \times E'$ defined by

$$f(Q, R) = ([2]Q, R + P_0),$$

we have

$$\deg f = \delta_f = 4.$$

Every rational point $(Q, R) \in E(\overline{\mathbb{Q}}) \times E'(\overline{\mathbb{Q}})$ is wandering under f. On the other hand, the arithmetic degree $\alpha_f(Q, R)$ is equal to 4 if and only if Q is non-torsion.

CHAPTER 7

An application to a variant of the Dynamical Mordell-Lang Conjecture

7.1. Outline of this chapter

In this chapter, as an application, we give a positive answer to a variant of the Dynamical Mordell-Lang conjecture for pairs of étale self-morphisms, which is also a variant of the original one stated by Bell, Ghioca, and Tucker in their monograph (see Section 7.2 for details).

The main theorem is as follows.

Theorem 7.1.1. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and $f, g\colon X \longrightarrow X$ étale self-morphisms of X over $\overline{\mathbb{Q}}$. Let $P, Q \in X(\overline{\mathbb{Q}})$ be points satisfying the following two conditions:*

- $\alpha_f(P)^p = \alpha_g(Q)^q > 1$ *for some* $p, q \in \mathbb{Z}_{\geq 1}$, *and*
- $t_f(P) = t_g(Q)$, *where* $t_f(P)$ *and* $t_g(Q)$ *are as in Theorem 6.4.1.*

Then the set

$$S_{f,g}(P, Q) := \{(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \mid f^m(P) = g^n(Q)\}$$

is a finite union of the sets of the form

$$\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$.

We now briefly sketch the plan of this chapter. In Section 7.2, we provide backgrounds of Theorem 7.1.1, and recall known results related to this theorem. In Section 7.3, we prove Theorem 7.1.1. It seems plausible that we can generalize the assertion of Theorem 7.1.1 further. We give a conjecture (Conjecture 7.2.6) generalizing Theorem 7.1.1, and some evidence in Section 7.4. Furthermore, to see that the results given in Section 7.4 support our conjecture, we give a definition of the double canonical height in a special case (see the proof of Theorem 7.4.1).

7.2. Backgrounds and general conjectures

Theorem 7.1.1 gives a positive answer to a variant of the Dynamical Mordell-Lang conjecture for pairs of étale self-morphisms, which is a variant of the original one stated by Bell, Ghioca, and Tucker (see [**BGT2**, Question 5.11.0.4]). In [**GTZ1**] and [**GTZ2**], Ghioca, Tucker, and Zieve studied similar problems for polynomial maps and got deeper results. Moreover, they introduced some of reductions including Lemma 7.3.2 we use. In [**GN**], Ghioca and Nguyen also studied similar problems for self-maps of semi-abelian varieties.

First, we recall a version of the Dynamical Mordell-Lang Conjecture. Note that there are several variants of the Dynamical Mordell-Lang Conjecture. Many results are obtained in various situations (see [**BGT2**] for details).

Conjecture 7.2.1 (the Dynamical Mordell-Lang Conjecture [GT, Conjecture 1.7]). *Let X be a quasi-projective variety over $\mathbb{C}$. Let $f\colon X \longrightarrow X$ be an self-morphism of X over $\mathbb{C}$. For any $\mathbb{C}$-rational point $P \in X(\mathbb{C})$ and any closed subvariety $Y \subset X$, the set*

$$S_f(P,Y) := \{n \in \mathbb{Z}_{\geq 0} \mid f^n(P) \in Y(\mathbb{C})\}$$

is a union of finitely many sets of the form

$$\{a_i + b_i m \mid m \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i \in \mathbb{Z}_{\geq 0}$.

In Section 7.3, we shall use the following result proved by Bell, Ghioca, and Tucker, which is a special case of the Dynamical Mordell-Lang conjecture.

Theorem 7.2.2 ([BGT1, Theorem 1.3]). *If $f\colon X \longrightarrow X$ is an étale self-morphism, Conjecture 7.2.1 holds.*

Furthermore, in [**BGT2**, Question 5.11.0.4], the following conjecture is stated.

Conjecture 7.2.3 ([BGT2, Question 5.11.0.4]). *Let X be a projective variety over $\overline{\mathbb{Q}}$. Let H be an ample $\mathbb{R}$-divisor on X over $\overline{\mathbb{Q}}$. Let $f, g\colon X \longrightarrow X$ be étale self-morphisms of X over $\overline{\mathbb{Q}}$ such that $f^*H \equiv \delta_f H$ and $g^*H \equiv \delta_g H$ hold in $\mathrm{NS}(X)_{\mathbb{R}}$ for some $\delta_f, \delta_g \in \mathbb{R}_{>1}$. Then for any points $P, Q \in X(\overline{\mathbb{Q}})$, the set*

$$S_{f,g}(P,Q) := \{(m,n) \mid f^m(P) = g^n(Q)\}$$

is a union of finitely many sets of the form

$$\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$

Bell, Ghioca, and Tucker proved a special case of Conjecture 7.2.3.

Theorem 7.2.4 ([BGT2, Theorem 5.11.0.1]). *Conjecture 7.2.3 holds if $\delta_f = \delta_g$.*

Remark 7.2.5 (see [KS1, Theorem 2 (a), Proposition 7] for details). If an ample $\mathbb{R}$-divisor H satisfies $f^*H \equiv dH$ in $\mathrm{NS}(X)_{\mathbb{R}}$ for some $d \in \mathbb{R}_{>1}$, the limit

$$\widehat{h}_{f,H}(P) := \lim_{n\to\infty} \frac{h_H(f^n(P))}{d^n}$$

converges for all $P \in X(\overline{\mathbb{Q}})$ and satisfies

$$\widehat{h}_{f,H}(f(P)) = d\widehat{h}_{f,H}(P)$$

and

$$\widehat{h}_{f,H} - h_H = O(\sqrt{h_H}). \tag{7.2.1}$$

The function $\widehat{h}_{f,H}$ is called the canonical height. Furthermore, the following conditions are equivalent to each other:

- $\alpha_f(P) > 1$,
- $\alpha_f(P) = d$,
- $\widehat{h}_{f,H}(P) \neq 0$, and
- the forward f-orbit of P is an infinite set.

In the setting of Theorem 7.2.4, when the forward f-orbit of P or the forward g-orbit of Q is finite, the assertion of Theorem 7.2.4 is obviously true. By Remark 7.2.5, we have

$$\alpha_f(P) = \delta_f = \delta_g = \alpha_g(Q) > 1$$

in the remaining case. Thus, when X is smooth, Theorem 7.1.1 is a generalization of Theorem 7.2.4. The assumption of Conjecture 7.2.3 seems too strong. We propose a more general conjecture as follows.

Conjecture 7.2.6. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and let $f, g\colon X \longrightarrow X$ be étale self-morphisms of X over $\overline{\mathbb{Q}}$. For points $P, Q \in X(\overline{\mathbb{Q}})$ with $\alpha_f(P) > 1$ and $\alpha_g(Q) > 1$, the following statements hold.*

(a) *The set $S_{f,g}(P,Q)$ is a finite union of the sets of the form*

$$\{(a_i + b_i\ell, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$$

for some non-negative integers $a_i, b_i, c_i, d_i \in \mathbb{Z}_{\geq 0}$.

(b) *If $\log_{\alpha_f(P)} \alpha_g(Q)$ is irrational or $t_f(P) \neq t_g(Q)$, the set $S_{f,g}(P,Q)$ is finite.*

The part (*b*) asserts that the hypotheses from Theorem 7.1.1 regarding $\alpha_f(P)$, $\alpha_g(Q)$, $t_f(P)$, $t_g(Q)$ must met if the set $S_{f,g}(P,Q)$ were infinite. In Section 7.4, we give some examples of self-morphisms for which Conjecture 7.2.6 (b) hold.

7.3. Proof of Theorem 7.1.1

In this section we prove Theorem 7.1.1.

Lemma 7.3.1. *To prove Theorem 7.1.1, we may assume $p = q = 1$.*

Proof. We see that

$$S_{f,g}(P,Q) = \bigcup_{\substack{0 \leq i \leq p-1 \\ 0 \leq j \leq q-1}} \{(i + pm, j + qn) \mid (m,n) \in S_{f^p, g^q}(f^i(P), g^j(Q))\}.$$

Thus to prove Theorem 7.1.1, it is enough to prove it for f^p and g^q. By using Theorem 6.4.1 twice, we have

$$\begin{aligned} n^{t_{f^p}(P)} \alpha_{f^p}(P)^n &\asymp h_H(f^{pn}(P)) \\ &\asymp (pn)^{t_f(P)} \alpha_f(P)^{pn} \\ &\asymp n^{t_f(P)} \alpha_f(P)^{pn}. \end{aligned}$$

Similarly, we obtain

$$n^{t_{g^q}(Q)} \alpha_{g^q}(Q)^n \asymp n^{t_g(Q)} \alpha_g(Q)^{qn}.$$

Hence combining with the assumption of Theorem 7.1.1 for f and g, we get

$$\begin{aligned} t_{f^p}(P) &= t_f(P) = t_g(Q) = t_{g^q}(Q), \\ \alpha_{f^p}(P) &= \alpha_f(P)^p = \alpha_g(Q)^q = \alpha_{g^q}(Q). \end{aligned}$$

Hence our assertion follows. $\square$

Lemma 7.3.2. *To prove Theorem 7.1.1 in the case $p = q = 1$, it is enough to prove*

$$\sup_{(m,n) \in S_{f,g}(P,Q)} |m - n| < \infty.$$

PROOF. We set

$$M := \sup_{(m,n)\in S_{f,g}(P,Q)} |m-n|.$$

Then we have

$$\begin{aligned} S_{f,g}(P,Q) = & \bigcup_{0\le k\le M} \{(m, m+k) \in S_{f,g}(P,Q) \mid m \in \mathbb{Z}_{\ge 0}\} \\ & \cup \bigcup_{1\le k\le M} \{(n+k, n) \in S_{f,g}(P,Q) \mid n \in \mathbb{Z}_{\ge 0}\}. \end{aligned}$$

Let $f \times g \colon X \times X \longrightarrow X \times X$ be the product of the self-morphisms f, g. Let $\Delta \subset X \times X$ be the diagonal. Then we have

$$\begin{aligned} (m, m+k) \in S_{f,g}(P,Q) &\Leftrightarrow f^m(P) = g^{m+k}(Q) \\ &\Leftrightarrow (f\times g)^m(P, g^k(Q)) \in \Delta. \end{aligned}$$

Since $f \times g$ is étale, the Dynamical Mordell-Lang conjecture is true for $f \times g$ by Theorem 7.2.2. Hence the set

$$\{(m, m+k) \in S_{f,g}(P,Q) \mid m \in \mathbb{Z}_{\ge 0}\}$$

is a finite union of the sets of the form

$$\{(a_i + b_i\ell, a_i + b_i\ell + k) \mid \ell \in \mathbb{Z}_{\ge 0}\}$$

for some non-negative integers $a_i, b_i \in \mathbb{Z}_{\ge 0}$. Similarly, the set

$$\{(n+k, n) \in S_{f,g}(P,Q) \mid n \in \mathbb{Z}_{\ge 0}\}$$

is a finite union of the sets of the form

$$\{(c_i + d_i\ell + k, c_i + d_i\ell) \mid \ell \in \mathbb{Z}_{\ge 0}\}$$

for some non-negative integers $c_i, d_i \in \mathbb{Z}_{\ge 0}$. Thus the assertion follows. □

Remark 7.3.3. Lemma 7.3.2 is the only part where the assumption of the étaleness of f, g is used. So Theorem 7.1.1 is true if the Dynamical Mordell-Lang conjecture (Conjecture 7.2.1) is true for the self-morphism

$$f^p \times g^q \colon X \times X \longrightarrow X \times X$$

and the diagonal $\Delta \subset X \times X$.

PROOF OF THEOREM 7.1.1. By Lemma 7.3.1, we may assume $p = q = 1$. By Theorem 6.4.1, there are positive real numbers $C_0, C_1, C_2, C_3 > 0$ such that the inequalities

$$\begin{aligned} C_0 m^{t_f(P)} \alpha_f(P)^m \le h_H(f^m(P)) \le C_1 m^{t_f(P)} \alpha_f(P)^m, \\ C_2 n^{t_g(Q)} \alpha_g(Q)^n \le h_H(g^n(Q)) \le C_3 n^{t_g(Q)} \alpha_g(Q)^n \end{aligned}$$

hold except for finitely many m and n. Suppose $f^m(P) = g^n(Q)$ with $m \ge n$. Then we get

$$(m/n)^{t_f(P)} \alpha_f(P)^{m-n} \le C_3/C_0$$

because $t_f(P) = t_g(Q)$ and $\alpha_f(P) = \alpha_g(Q)$ by assumption. Hence we get

$$\alpha_f(P)^{m-n} \le C_3/C_0. \tag{7.3.1}$$

Since $\alpha_f(P) > 1$, the inequality (7.3.1) holds only for finitely many values of $m - n$. By the same argument for the case $m < n$, we conclude

$$\sup_{(m,n)\in S_{f,g}(P,Q)} |m-n| < \infty.$$

Hence by Lemma 7.3.2, the proof of Theorem 7.1.1 is complete. □

7.4. Some evidence for Conjecture 7.2.6

In this section, we prove the following theorem which gives some evidence for Conjecture 7.2.6 (b).

Theorem 7.4.1. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Let $f, g\colon X \longrightarrow X$ be surjective self-morphisms on X over $\overline{\mathbb{Q}}$. Assume that f commutes with g, and there is an ample $\mathbb{R}$-divisor H on X over $\overline{\mathbb{Q}}$ such that $f^*H \equiv dH$ in $\mathrm{NS}(X)_{\mathbb{R}}$ for some $d \in \mathbb{R}_{>1}$. Let $P, Q \in X(\overline{\mathbb{Q}})$ be points with $\alpha_f(P) > 1$ and $\alpha_g(Q) > 1$. Assume that $\log_{\alpha_f(P)} \alpha_g(Q)$ is an irrational real number. Then the set $S_{f,g}(P, Q)$ is finite.*

Proof of Theorem 7.4.1. It is enough to prove that once $f^{m_0}(P) = g^{n_0}(Q)$ is satisfied for some m_0 and n_0, we have $f^{m+m_0}(P) \neq g^{n+n_0}(Q)$ for all $(m, n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$. Therefore, it is enough to prove that for every point $R \in X(\overline{\mathbb{Q}})$, we have $f^m(R) \neq g^n(R)$ for all $(m, n) \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$.

Fix a Weil height function h_H associated with H so that $h_H \geq 1$. By Theorem 6.4.1, there are positive real numbers $C_0 > 0$ and $C_1 > 0$ satisfying

$$C_0 n^{t_g(R)} \alpha_g(R)^n \leq h_H(g^n(R)) \leq C_1 n^{t_g(R)} \alpha_g(R)^n \tag{7.4.1}$$

for all $n \in \mathbb{Z}_{\geq 1}$. We set

$$\underline{\widehat{h}}_{f,g,H}(R) := \liminf_{n\to\infty} \frac{\widehat{h}_{f,H}(g^n(R))}{n^{t_g(R)}\alpha_g(R)^n}.$$

From (7.2.1) and (7.4.1), we have

$$\underline{\widehat{h}}_{f,g,H}(R) = \liminf_{n\to\infty} \frac{h_H(g^n(R))}{n^{t_g(R)}\alpha_g(R)^n} \geq C_0 > 0.$$

Since the asymptotic behavior of $h_H(g^n(R))$ does not depend on the choice of an ample divisor H and a height function h_H, one can see that

$$\begin{aligned} h_H(g^n \circ f(R)) &= h_H(f \circ g^n(R)) \\ &= h_{f^*H}(g^n(R)) + O(1) \\ &\asymp n^{t_g(R)}\alpha_g(R)^n, \end{aligned}$$

where the first equality follows from the commutativity of f and g. This asymptotic equality means that we have $t_g(f(R)) = t_g(R)$ and $\alpha_g(f(R)) = \alpha_g(R)$. Hence the functional equations

$$\begin{aligned} \underline{\widehat{h}}_{f,g,H}(f(R)) &= \alpha_f(R)\underline{\widehat{h}}_{f,g,H}(R), \\ \underline{\widehat{h}}_{f,g,H}(g(R)) &= \alpha_g(R)\underline{\widehat{h}}_{f,g,H}(R) \end{aligned}$$

hold (see Remark 7.2.5). Thus, if we have $f^m(R) = g^n(R)$, we get

$$\begin{aligned} \alpha_f(R)^m\underline{\widehat{h}}_{f,g,H}(R) &= \underline{\widehat{h}}_{f,g,H}(f^m(R)) \\ &= \underline{\widehat{h}}_{f,g,H}(g^n(R)) \\ &= \alpha_g(R)^n\underline{\widehat{h}}_{f,g,H}(R). \end{aligned}$$

Hence $\alpha_f(R)^m = \alpha_g(R)^n$. This equality can hold only when $m = n = 0$ since we are assuming $\log_{\alpha_f(R)} \alpha_g(R)$ is an irrational real number. □

Bibliography

[Ame] Amerik, E., *On endomorphisms of projective bundle*, Man. Math. **111** (2003), 17–28.

[Bad] Badescu, L., *Algebraic surfaces*, Springer-Verlag, New York, 2001.

[BGT1] Bell, J. P., Ghioca, D., Tucker, T. J., *The dynamical Mordell-Lang problem for étale maps*, Amer. J. Math. **132** (2010), no. 6, 1655–1675.

[BGT2] Bell, J. P., Ghioca, D., Tucker, T. J., *The Dynamical Mordell-Lang Conjecture*, Mathematical Surveys and Monographs 210, Amer. Math. Soc., Providence, RI, (2016).

[Bir] Birkhoff, G., *Linear transformations with invariant cones*, Amer. Math. Monthly **74** (1967), 274-276.

[Bri] Brion, M., *Some structure theorems for algebraic groups*, preprint, 2016, https://arxiv.org/abs/1509.03059

[CS] Call, G. S., Silverman, J. H., *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), no. 2, 163–205.

[Dan] Dang, N.-B., *Degrees of iterates of rational maps on normal projective varieties*, preprint, 2017, https://arxiv.org/abs/1701.07760

[DF] Diller, J., Favre, C., *Dynamics of bimeromorphic maps of surfaces*, Amer. J. Math. **123** (2001), no. 6, 1135–1169.

[DN] Dinh, T-C., Nguyên, V-A., *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. **86** (2011), no. 4, 817–840.

[DS1] Dinh, T.-C., Sibony, N., *Regularization of currents and entropy*, Ann. Sci. École Norm. Sup. (4), **37** (2004), no. 6, 959–971.

[DS2] Dinh, T.-C., Sibony, N., *Une borne supérieure de l'entropie topologique d'une application rationnelle*, Ann. of Math. (2) **161** (2005), no. 3, 1637–1644.

[DS3] Dinh, T-C., Sibony, N., *Equidistribution problems in complex dynamics of higher dimension*, Internat. J. Math. **28**, 1750057 (2007).

[FAG] Fantechi, B., Gttsche, L., Illusie, L., Kleiman, S. L., Nitsure, N., Vistoli, A., *Fundamental algebraic geometry. Grothendieck' s FGA explained*, Mathematical Surveys and Monographs **123**, Amer. Math. Soc. 2005.

[Fuj] Fujimoto, Y., *Endomorphisms of smooth projective 3-folds with nonnegative Kodaira dimension*, Publ. RIMS, Kyoto Univ., **38** (2002), 33–92.

[FN1] Fujimoto, Y., Nakayama, N., *Compact complex surfaces admitting non-trivial surjective endomorphisms*, Tohoku Math. J. **57** (2005), 395–426.

[FN2] Fujimoto, Y., Nakayama, N., *Complex projective manifolds which admit non-isomorphic surjective endomorphisms* RIMS Kokyuroku Bessatsu. **B9** (2008), 51–79.

[Ful] W. Fulton, *Intersection theory second edition*, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1998.

[GN] Ghioca, D., Nguyen K. D., *The orbit intersection problem for linear spaces and semiabelian varieties*, Math. Res. Lett., **24** (2017), no. 5, 1263–1283.

[GT] Ghioca, D., Tucker, T. J., *Periodic points, linearizing maps, and the dynamical Mordell-Lang problem*, J. Number Theory **129** (2009), 1392-1403.

[GTZ1] Ghioca, D., Tucker, T. J., Zieve, M. E., *Intersections of polynomial orbits, and a dynamical Mordell-Lang conjecture*, Invent. math. 171 (2008), 463–483.

[GTZ2] Ghioca, D., Tucker, T. J., Zieve, M. E., *Linear relations between polynomial orbits*, Duke Math. J., 161 (2012), no. 7, 1379-1410.

[Gue] Guedj, V., *Ergodic properties of rational mappings with large topological degree*, Ann. of Math. (2) **161** (2005), no. 3, 1589–1607.

[Har] Hartshorne, R., *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HS] Hindry, M., Silverman, J. H., *Diophantine geometry. An introduction*, Graduate Texts in Mathematics, No. 201. Springer-Verlag, New York, 2000.

[Hom1] Homma, Y., *On finite morphisms of rational ruled surfaces*, Math. Nachr., **158**, (1992), 263–281.

[Hom2] Homma, Y., *On finite morphisms of ruled surfaces*, Geom. Dedicata, **78**, (1999), 3, 259–269.

[Iit] Iitaka, S., *Algebraic geometry. An introduction to birational geometry of algebraic varieties*, Graduate Texts in Mathematics, 76. North-Holland Mathematical Library, 24. Springer-Verlag, New York-Berlin, 1982.

[Kaw] Kawaguchi, S., *Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint*, Amer. J. Math. **130** (2008), no. 1, 159–186.

[KS1] Kawaguchi, S., Silverman, J. H., *Examples of dynamical degree equals arithmetic degree*, Michigan Math. J. **63** (2014), no. 1, 41–63.

[KS2] Kawaguchi, S., Silverman, J. H., *Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties*, Trans. Amer. Math. Soc. **368** (2016), 5009–5035.

[KS3] Kawaguchi, S., Silverman, J. H., *On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties*, J. Reine Angew. Math. **713** (2016), 21–48.

[KM] Kollár, K., Mori, S., *Birational geometry of algebraic varieties*, Cambridge Univ. Press, 1998.

[Lan] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, New York, 1983.

[LS1] Lesieutre, J., Satriano, M., *Canonical heights on hyper-Kähler varieties and the Kawaguchi-Silverman conjecture*, preprint, 2018, https://arxiv.org/abs/1802.07388

[LS2] Lesieutre, J., Satriano, M., *A rational map with infinitely many points of distinct arithmetic degrees*, preprint, 2018, https://arxiv.org/abs/1809.00047

[Lin] Lin., J.-L., *On the Arithmetic dynamics of monomial maps*, Ergodic Theory Dynam. Systems (2018), to appear, http://arxiv.org/abs/1704.02661

[Matm] Matsumura, H., *On algebraic groups of birational transformations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 34 (1963), 151–155.

[Matz] Matsuzawa, Y., *On upper bounds of arithmetic degrees*, preprint, 2016, https://arxiv.org/abs/1606.00598

[MS] Matsuzawa, Y., Sano, K., *Arithmetic and dynamical degrees of self-morphisms of semi-abelian varieties*, Ergodic Theory Dynam. Systems (2018), to appear, online version: http://dx.doi.org/10.1017/etds.2018.117.

[MSS1] Matsuzawa, Y., Sano, K., Shibata, T., *Arithmetic degrees and dynamical degrees of endomorphisms on surfaces*, Algebra Number Theory 12 (2018), no. 7, 1635–1657.

[MSS2] Matsuzawa, Y., Sano, K., Shibata, T., *Arithmetic degrees for dynamical systems over function fields of characteristic zero*, Math. Z. 290 (2018), no. 3-4, 1063-1083.

[Muk] Mukai, S., *An introduction to invariants and moduli*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2002.

[Mum] Mumford, D., *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[Nak] Nakayama, N., *Ruled surfaces with non-trivial surjective endomorphisms*, Kyushu J. Math., **56** (2002), 433–446.

[NZ] Nakayama, N., Zhang, D.-Q., *Building blocks of étale endomorphisms of complex projective manifolds*, Proc. Lond. Math. Soc. (3) **99** (2009), no. 3, 725–756.

[San1] Sano, K., *Dynamical degree and arithmetic degree of endomorphisms on product varieties*, 2018, Tohoku math. J., to appear

[San2] Sano, K., *The canonical heights for Jordan blocks of small eigenvalues, preperiodic points, and the arithmetic degrees*, preprint, 2017, https://arxiv.org/pdf/1712.07533

[San3] Sano, K., *Growth rate of ample heights and the dynamical Mordell-Lang type conjecture*, Int. J. Number Theory, 14 (2018), no. 10 (2018), 2673-2685.

[Ser] Serre, J.-P., *Lectures on the Mordell-Weil theorem*, Third edition, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Sil1] Silverman, J. H., *Dynamical degree, arithmetic entropy, and canonical heights for dominant rational self-maps of projective space*, Ergodic Theory Dynam. Systems **34** (2014), no. 2, 647–678.

[Sil2] Silverman, J. H., *Arithmetic and dynamical degrees on abelian varieties*, J. Théor. Nombres Bordeaux, **29**, (2017), no. 1, 151–167.

[Tru] Truong, T. T., *(Relative) dynamical degrees of rational maps over an algebraic closed field*, preprint, 2015, https://arxiv.org/abs/1501.01523

[Voj] Vojta, P., *Integral points on subvarieties of semiabelian varieties, I*, Invent. Math. **126** (1996), no. 1, 133–181.