Generalized scale functions and refracted processes

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Contents

1 Introduction

The scale functions play an important role in the fluctuation theory of spectrally negative Lévy processes. As a classical result, the exit time of a spectrally negative Lévy process from a finite interval is characterized by the scale functions using the Laplace transform (this characterization is called the *two-sided exit problem*). The killed potential densities upon exiting a finite interval can be represented by the scale functions. As a rather later result, Bertoin([3]) characterized in terms of the scale functions the exponential decay parameter of the tail probabilities of the exit time from the finite interval.

The scale functions are applied to the ruin theory of insurance mathematics. A spectrally negative Lévy process models the surplus of an insurance company and then the scale functions are involved in various values to expect several risks: for example, the Gerber–Shiu measure, which characterizes the joint law of the wealth prior to ruin and the deficit at ruin.

Kyprianou–Leoffen([13]) studied strong Markov processes whose positive and negative motions are spectrally negative Lévy processes different from each other where the difference between the positive and negative motions is only the drifts. They constructed such processes by proving unique existence of a strong solution for a stochastic differentiable equation. They called them the *refracted Lévy processes.* In addition, they defined the scale functions of refracted Lévy processes and represent the two-sided exit problem and the killed potential measure using the scale functions.

Kyprianou–Loeffen's refracted Lévy processes are applied to insurance mathematics. Kyprianou–Loeffen–Pérez([14]) studied the optimal dividend problem for an insurance company to pay dividends to customers. The optimal strategy can be modelled by Kyprianou–Loeffen's refracted Lévy processes, where the total amount of dividends is characterized by the scale functions.

In Noba–Yano([17]), we generalized Kyprianou–Loeffen's refracted Lévy processes to the processes whose positive and negative motions may differ in not only the drifts but also the Lévy measures, assuming that the positive motion does not have the Gaussian part. We call these the *generalized refracted Lévy processes.* To construct such a process we utilized the excursion theory. We also defined the scale functions of generalized refracted Lévy processes and represent the two-sided exit problem and the killed potential measure using the scale functions. In addition, we studied an approximation problem. For generalized refracted Lévy process with unbounded variation paths, we proved that it is the limit in distribution of a sequence of refracted Lévy processes coming from compound Poisson processes.

In this dissertation, we define functions analogous to the scale functions, for general standard processes with no positive jumps. We call them the *generalized scale functions.* In addition, we generalize refracted Lévy processes to standard processes with no positive jumps (we call them the *refracted processes*) and we study a duality problem and an approximation problem of refracted processes.

The definition of our generalized scale functions is based on the excursion measure. Our idea comes from some results of Noba–Yano([17]). We prove that the generalized scale functions characterize the two-sided exit problem and the killed potential densities of standard processes with no positive jumps. Furthermore, we study necessary and sufficient conditions of duality in terms of the generalized scale functions. We believe our study will lead in the future to some applications to the insurance mathematics.

To construct a refracted process from given two standard processes with no positive jumps, we use the excursion theory. Unlike the construction of generalized refracted Lévy processes, we need *landing functions*, which indicate the landing points at the first hitting times to $(-\infty, 0)$ of excursion processes. The landing functions play important roles in the studies of the duality problem and of the approximation problem.

The duality problem is to characterize in terms of the generalized scale functions the necessary and sufficient condition for the two refracted processes to be in duality. Let X and Y be standard processes with no positive jumps and let $\widehat{X}$ and $\widehat{Y}$ be dual processes of X and Y, respectively. Let U be the refracted process of X and Y and let $\widehat{U}$ be the refracted process of $\widehat{X}$ and $\widehat{Y}$. We will then obtain the necessary and sufficient condition that the refracted processes U and $\widehat{U}$ are in duality in terms of a certain identity involving excursion measures and landing functions. To prove duality of U and $\widehat{U}$, we require that their excursion measures are transformed into each other by time reversal. For this purpose, we utilize landing functions in order to adapt the jumps at the switching time between X and Y. We give an example of a refracted process possessing a dual. We construct it from two spectrally negative stable processes, where we will make a computation to find a suitable landing function.

The approximation problem is to associate to a given refracted process a sequence of simple refracted processes which converges to it. We assume that our new refracted process U comes from two Lévy processes X and Y. We will then prove that U is the limit in distribution on the càdlàg function space of the sequence $\{U^{(n)}\}_{n\in\mathbb{N}}$ of refracted processes coming from the drifted compound Poisson processes constructed from X and Y by removing small jumps and by adding drifts. Noba–Yano([17]) studied this problem in the special case of no Gaussian part of X, where our landing functions did not appear. In our setting, our landing functions play an important role.

The organization of the present dissertation is as follows. In Section 2 we propose some notation and recall preliminary facts about local times and excursion measures. In Section 3, we recall preliminary facts about spectrally negative Lévy processes and its scale functions. In Section 4, we recall some properties of Kyprianou–Loeffen's refracted processes and generalized refracted Lévy processes. In Section 5, we give the definition of the generalized scale functions and apply them to the exit problems. In addition, we give a property of generalized scale functions about duality. In Section 6 we give the precise definition of our new refracted processes. In Section 7 we study the duality problem and give an example of refracted processes which are in duality using stable processes. In

Section 8 we study the approximation problem.

2 Local times and excursion measures

We give some notations and recall several preliminary facts about local times and excursion measures of $\mathbb{R}$-valued standard processes.

2.1 Local times and excursion measures

Let $\mathbb{D}$ denote the set of functions $\omega : [0,\infty) \to \mathbb{R} \cup \{\partial\}$ which are càdlàg and satisfy $\omega(t) = \partial$ for $t \geq \zeta(\omega)$, where $\mathbb{R} \cup \{\partial\}$ is the one-point compactification of $\mathbb{R}$ and $\zeta(\omega) = \inf\{t > 0 : \omega(t) = \partial\}$. Let $\mathcal{B}(\mathbb{D})$ denote the class of Borel sets of $\mathbb{D}$ equipped with the Skorokhod topology. For $\omega \in \mathbb{D}$, we write

$$T_x^-(\omega) := \inf\{t > 0 : \omega(t) \leq x\}\,, \tag{2.1}$$

$$T_x^+(\omega) := \inf\{t > 0 : \omega(t) \geq x\}\,, \tag{2.2}$$

$$T_x(\omega) := \inf\{t > 0 : \omega(t) = x\}\,. \tag{2.3}$$

We sometimes write T_x^-, T_x^+, T_x simply for $T_x^-(X)$, $T_x^+(X)$, $T_x(X)$, respectively, when we consider these times for a process $(X, \mathbb{P}_x^X)$ with $X = \{X_t : t \geq 0\}$. For $\omega, \omega_1, \omega_2 \in \mathbb{D}$ and $s, t \in [0,\infty)$, we adopt the following notation:

$$\rho_x\omega(t) = \begin{cases} \omega(T_x - t-), & t < T_x < \infty, \\ x, & t \geq T_x, \\ \partial, & t \geq 0,\ T_x(\omega) = \infty, \end{cases} \tag{2.4}$$

$$k_s\omega(t) = \begin{cases} \omega(t), & t < s, \\ \partial, & t \geq s, \end{cases} \tag{2.5}$$

$$\omega_1 \circ \omega_2(t) = \begin{cases} \omega_1(t), & t < \zeta(\omega_1), \\ \omega_2(t - \zeta(\omega_1)), & t \geq \zeta(\omega_1), \end{cases} \tag{2.6}$$

$$\theta_s\omega(t) = \omega(t+s). \tag{2.7}$$

Let $\mathbb{T}$ be an interval of $\mathbb{R}$ and set $a_0 = \sup \mathbb{T}$ and $b_0 = \inf \mathbb{T}$. In this paper, a_0 may belong to $(-\infty, \infty]$ and b_0 to $[-\infty, \infty)$. We assume that the $\mathbb{T}$-valued process $(X, \mathbb{P}_x^X)$ considered in this section is a standard process with no positive jumps with $\mathbb{P}_x^X(X_0 = x) = 1$, satisfying the following conditions with a σ-finite measure m on $\mathbb{T}$:

(A1) $(x, y) \mapsto \mathbb{E}_x^X\left[e^{-T_y}\right] > 0$ is a $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable function.

(A2) X has a reference measure m on $\mathbb{T}$, i.e. for $q \geq 0$ and $x \in \mathbb{T}$, we have $R_X^{(q)} 1_{(\cdot)}(x) \ll m(\cdot)$, where

$$R_X^{(q)} f(x) := \mathbb{E}_x^X\left[\int_0^\infty e^{-qt} f(X_t)dt\right] \tag{2.8}$$

for non-negative measurable function f. Here and hereafter we use the notation $\int_b^a = \int_{(b,a]\cap\mathbb{R}}$. In particular, $\int_{b-}^a = \int_{[b,a]\cap\mathbb{R}}$.

By [7, Theorem 18.4], there exists a family of processes $\{L^{X,x}\}_{x\in\mathbb{T}}$ with $L^{X,x} = \left\{L_t^{X,x}\right\}_{t\geq 0}$ for $x \in \mathbb{T}$ which we call *local times* such that the following conditions hold: for all $q > 0$, $x \in \mathbb{T}$ and non-negative measurable function f

$$\int_0^t f(X_s)ds = \int_{\mathbb{T}} f(y)L_t^{X,y}m(dy), \quad \text{a.s.} \tag{2.9}$$

$$R_X^{(q)}f(x) = \int_{\mathbb{T}} f(y)\mathbb{E}_x^X\left[\int_0^\infty e^{-qt}dL_t^{X,y}\right]m(dy). \tag{2.10}$$

We have the following two cases:

- **Case 1.** If $x \in \mathbb{T}$ is regular for itself, this $L^{X,x}$ is the continuous local time at x ([5, pp.216]). Note that $L^{X,x}$ has no ambiguity of multiple constant because of (2.9) or (2.10).
- **Case 2.** If $x \in \mathbb{T}$ is irregular for itself, we have

$$L_t^{X,x} = l_x^X \#\{0 \leq s < t : X_s = x\}, \quad \text{a.s.} \tag{2.11}$$

for some constant $l_x^X \in (0, \infty)$.

In Case 1, let $\eta^{X,x}$ denote the inverse local time of $L^{X,x}$, i.e.,

$$\eta_t^{X,x} = \inf\{s > 0 : L_s^{X,x} > t\}. \tag{2.12}$$

Let n_x^X denote the excursion measure away from x which is associated with L_x^X (See, e.g., [9] or [4]). Then, for all $q > 0$, we have

$$-\log \mathbb{E}_0^X\left[e^{-q\eta^{X,x}(1)}\right] = \mu_x^X q + n_x^X\left[1 - e^{-qT_x}\right] \tag{2.13}$$

for a non-negative constant μ_x^X called the *stagnancy rate*. We thus have

$$\mathbb{E}_x^X\left[\int_0^\infty e^{-qt}dL_t^{X,x}\right] = \mathbb{E}_x^X\left[\int_0^\infty e^{-q\eta^{X,x}(s)}ds\right] = \frac{1}{\mu_x^X q + n_x^X[1 - e^{-qT_x}]}. \tag{2.14}$$

In Case 2, we define $n_x^X = \frac{1}{l_x^X}\mathbb{P}_x^{X^x}$, where $\mathbb{P}_x^{X^x}$ denotes the law of X started from x and stopped at x. Then we have

$$\mathbb{E}_x^X\left[\int_{0-}^\infty e^{-qt}dL_t^{X,x}\right] = l_x^X \sum_{i=0}^\infty \left(\mathbb{E}_x^X\left[e^{-qT_x}\right]\right)^i = \frac{l_x^X}{\mathbb{E}_x^X[1 - e^{-qT_x}]} = \frac{1}{n_x^X[1 - e^{-qT_x}]}. \tag{2.15}$$

Remark 2.1. Any point $x \in \mathbb{T}\backslash\{a_0\}$ cannot be a holding point. In fact, assume x is. Then X leaves x by jumps (see, e.g., [23, Theorem 1 (vi)]). But X has no positive jumps, and thus X can not exceed x, which contradicts (A1).

2.2 Duality and local times

We recall several preliminary facts about local times and excursion measures when the process considered has a dual process.

Let $(X, \mathbb{P}_x^X)$ be a $\mathbb{T}$-valued standard process with no positive jumps satisfying (A1) and (A2). Let $(\widehat{X}, \mathbb{P}_x^{\widehat{X}})$ with $\widehat{X} = \left\{ \widehat{X}_t : t \geq 0 \right\}$ be a $\mathbb{T}$-valued standard process with no negative jumps satisfying (A1) and (A2) with the same reference measure m as X. For $q \geq 0$ and non-negative measurable function f, we denote

$$R_{\widehat{X}}^{(q)} f(x) = \mathbb{E}_x^{\widehat{X}} \left[\int_0^\infty e^{-qt} f(\widehat{X}_t) dt \right]. \tag{2.16}$$

We have the local times $\{L^{\widehat{X},x}\}_{x \in \mathbb{T}}$ and the excursion measures $\{n_x^{\widehat{X}}\}_{x \in \mathbb{T}}$ of $\widehat{X}$ in the same way as X's in Section 2.1.

Definition 2.2 (See, e.g., [6]). We say that X and $\widehat{X}$ are *in duality (relative to m)* if for $q > 0$, non-negative measurable functions f and g,

$$\int_{\mathbb{T}} f(x) R_X^{(q)} g(x) m(dx) = \int_{\mathbb{T}} R_{\widehat{X}}^{(q)} f(x) g(x) m(dx). \tag{2.17}$$

Theorem 2.3 (See, e.g., [6] or [22]). *Suppose X and $\widehat{X}$ be in duality relative to m. Then, for each $q > 0$, there exists a function $r_X^{(q)} : \mathbb{T} \times \mathbb{T} \to [0, \infty)$ such that*

(i) $r_X^{(q)}$ is $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable.

(ii) $x \mapsto r_X^{(q)}(x, y)$ is q-excessive and finely continuous for each $y \in \mathbb{T}$.

(iii) $y \mapsto r_X^{(q)}(x, y)$ is q-coexcessive and cofinely continuous for each $x \in \mathbb{T}$.

(iv) For all non-negative function f,

$$R_X^{(q)} f(x) = \int_{\mathbb{T}} f(y) r_X^{(q)}(x, y) m(dy), \qquad R_{\widehat{X}}^{(q)} f(y) = \int_{\mathbb{T}} f(x) r_X^{(q)}(x, y) m(dx). \tag{2.18}$$

By [22, Proposition of Section $V.1$], if X and $\widehat{X}$ are in duality relative to m, there exist local times $\{L^{X,x}\}_{x \in \mathbb{T}}$ of X and $\{L^{\widehat{X},x}\}_{x \in \mathbb{T}}$ of $\widehat{X}$ satisfying

$$\mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right] = r_X^{(q)}(x, y), \ \mathbb{E}_y^{\widehat{X}} \left[\int_0^\infty e^{-qt} dL_t^{\widehat{X},x} \right] = r_X^{(q)}(x, y) \tag{2.19}$$

for all $q > 0$.

The following lemma shows the duality implies the time reversality of the excursion measures.

Lemma 2.4 ([8, Lemma 4.16]). *We assume that X and $\widehat{X}$ have the following conditions:*

- X *and* $\widehat{X}$ *are in duality relative to* m.
- X *and* $\widehat{X}$ *are recurrent processes.*
- $X_0 = X_{T_x-} = x$, n_x^X*-a.s. (This condition is equivalent to the counterpart of* $n_x^{\widehat{X}}$*.)*

Then we have

$$n_x^X[\ \cdot\ ] = n_x^{\widehat{X}}[\rho_x(\ \cdot\ )]\,. \tag{2.20}$$

3 Scale functions and Gerber–Shiu formulae

We recall the well-known theory of scale functions (see, e.g., [2, Section VII], [11], [12, Section 8] and [21]), for a spectrally negative Lévy process $(X, \mathbb{P}_x^X)$ with $X = \{X_t : t \geq 0\}$ and $\mathbb{P}_x^X(X_0 = x) = 1$. For $q \geq 0$, the q-scale function $W^{(q)}$ is defined via the Laplace transform

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\Psi_X(\beta) - q}, \qquad \beta > \Phi_X(q), \tag{3.1}$$

where $\Psi_X(\lambda) = \log \mathbb{E}_0^X\left[e^{\lambda X_1}\right]$ $(\lambda \geq 0)$ denotes the Laplace exponent and $\Phi_X(q) = \inf\{\lambda > 0 : \Psi_X(\lambda) > q\}$ denotes its right inverse. The Laplace transform of hitting times and the q-potential measure can be characterized as follows: for $b < x < a$

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)}, \tag{3.2}$$

$$\mathbb{E}_x^X\left[\int_0^{T_a^+ \wedge T_b^-} e^{-qt} f(X_t)dt\right] = \int_b^a f(y)\left(\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)} W^{(q)}(a-y) - W^{(q)}(x-y)\right) dy. \tag{3.3}$$

In addition, we have

$$\mathbb{E}_x^X\left[e^{-qT_b^-}; T_b^- < T_a^+\right] = Z^{(q)}(x-b) - Z^{(q)}(a-b)\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)}, \tag{3.4}$$

$$\mathbb{E}_x^X\left[e^{-qT_b^-}; T_b^- < \infty\right] = Z^{(q)}(x-b) - \frac{q}{\Phi_X(q)} W^{(q)}(x-b), \tag{3.5}$$

where $Z^{(q)}(x)$ is the *second scale function* defined by

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy. \tag{3.6}$$

In (3.5) for $q = 0$, we understand $\frac{q}{\Phi_X(q)}$ in the limit as $q \downarrow 0$. We also note that when X has unbounded variation paths, Noba–Yano([17, Theorem 3.1]) and Avram–Pérez–Yamazaki([1, pp.276]) proved the following.

Theorem 3.1. *The scale functions of spectrally negative Lévy processes satisfy*

$$W^{(q)}(x) = \frac{1}{n_0^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]}, \qquad q \geq 0, \quad x > 0, \tag{3.7}$$

where n_0^X denotes the excursion measure away from 0 subject to the normalization

$$n_0^X\left[1 - e^{-qT_0}\right] = \frac{1}{\Phi_X'(q)}, \qquad q > 0. \tag{3.8}$$

We omit the proof of Theorem 3.1.

Remark 3.2. Bertoin([3, Proposition VII.15]) gave us an identity similar to (3.7). Let $\underline{n}_0^X$ be an excursion measure away from 0 of reflected process at 0. He proved that there exists a constant $k > 0$ such that

$$W^{(0)}(x) = \frac{k}{\underline{n}_0^X[T_x^+ < \infty]}, \qquad x > 0. \tag{3.9}$$

The Laplace exponent Ψ_X has the following form:

$$\Psi_X(\lambda) = \chi_X \lambda + \frac{\sigma_X^2}{2}\lambda^2 - \int_{(-\infty,0)} (1 - e^{\lambda y} + \lambda y 1_{(-1,0)}(y))\Pi_X(dy), \quad \lambda \geq 0, \tag{3.10}$$

for some constants $\chi_X \in \mathbb{R}$, $\sigma_X \geq 0$ and some Lévy measures Π_X which satisfies $\Pi_X((0,\infty)) = 0$ and $\int_{(-\infty,0)}(1 \wedge x^2)\Pi_X(dx) < \infty$. We known that X has paths of bounded variation if and only if $\sigma_X = 0$ and $\int_{(-\infty,0)}(1 \wedge |x|)\Pi_X(dx) < \infty$; In this case, its Laplace exponent is given by

$$\Psi_X(\lambda) = \delta_X \lambda - \int_{(-\infty,0)} \left(1 - e^{\lambda y}\right)\Pi_X(dy), \qquad \lambda \geq 0, \tag{3.11}$$

where $\delta_X = \chi_X - \int_{(-1,0)} y\Pi_X(dy)$. We recall the following result for general spectrally negative Lévy processes, which are called the *Gerber–Shiu formula*.

Theorem 3.3 (See, e.g., [12, Corollary 10.2])**.** *For $q \geq 0$, $x > 0$ and non-negative measurable function f, we have*

$$\mathbb{E}_x^X\left[e^{-qT_0^-} f(X_{T_0^- -}, X_{T_0^-}); T_0^- < T_0\right] \tag{3.12}$$

$$= \int_{(0,\infty)} dv \int_{(-\infty,0)} f(v,u)(e^{-\Phi_X(q)v}W_X^{(q)}(x) - W_X^{(q)}(x-v))\Pi_X(du - v). \tag{3.13}$$

We have its analogy to the excursion measure as follows.

Theorem 3.4 ([17, Theorem 3.4])**.** *For $q \geq 0$ and non-negative measurable function f, we have*

$$n_0^X\left[e^{-qT_0^-} f(X_{T_0^- -}, X_{T_0^-}); 0 < T_0^- < T_0\right] = \int_{(0,\infty)} dv \int_{(-\infty,0)} f(v,u)e^{-\Phi_X(q)v}\Pi_X(du - v). \tag{3.14}$$

The proof of Theorem 3.4 is almost the same as that of [17, Theorem 3.4] but we give it for completeness of the paper.

Proof. By the monotone convergence theorem, the strong Markov property and since X has no positive jumps, we have

$$n_0^X\left[e^{-qT_0^-}f(X_{T_0^- -},X_{T_0^-});0<T_0^-<T_0\right] \tag{3.15}$$

$$=\lim_{\epsilon\downarrow 0} n_0^X\left[e^{-qT_0^-}f(X_{T_0^- -},X_{T_0^-});T_\epsilon^+<\infty,X_{T_0^-}>\epsilon,T_0^-<T_0\right] \tag{3.16}$$

$$=\lim_{\epsilon\downarrow 0} n_0^X\left[e^{-qT_\epsilon^+};T_\epsilon^+<\infty\right]\mathbb{E}_\epsilon^X\left[e^{-qT_0^-}f(X_{T_0^- -},X_{T_0^-});X_{T_0^-}>\epsilon,T_0^-<T_0\right]. \tag{3.17}$$

By (3.7) and Theorem 3.3, we have

$$(3.17)=\lim_{\epsilon\downarrow 0}\frac{1}{W_X^{(q)}(\epsilon)}\int_{(\epsilon,\infty)}dv\int_{(-\infty,0)}f(v,u)e^{-\Phi_X(q)v}W_X^{(q)}(\epsilon)\Pi_X(du-v), \tag{3.18}$$

and by the monotone convergence theorem, we obtain (3.14). □

We recall the following formula for the killed potential measure.

Theorem 3.5 ([12, Corollary 8.8]). *For $q\geq 0$, $x>0$ and non-negative measurable function f, we have*

$$\mathbb{E}_x^X\left[\int_0^{T_0^-}e^{-qt}f(X_t)dt\right]=\int_0^\infty f(y)(e^{-\Phi_X(q)}W_X^{(q)}(x)-W_x^{(q)}(x-y))dy. \tag{3.19}$$

We have its analogy to the excursion measure as follows.

Theorem 3.6 ([17, Lemma 3.6]). *For $q\geq 0$ and non-negative measurable function f, we have*

$$n_0^X\left[\int_0^{T_0^-}e^{-qt}f(X_t)dt\right]=\int_0^\infty f(y)e^{-\Phi_X(q)}dy. \tag{3.20}$$

Proof. Using the monotone convergence theorem and the strong Markov property, we have

$$n_0^X\left[\int_0^{T_0^-}e^{-qt}f(X_t)dt\right]=\lim_{\epsilon\downarrow 0}n_0^X\left[\int_0^{T_0^-}e^{-qt}f(X_t)1_{\{X_t\in(\epsilon,\infty)\}}dt;T_\epsilon^+<\infty\right] \tag{3.21}$$

$$=\lim_{\epsilon\downarrow 0}n_0^X\left[e^{-qT_\epsilon^+};T_\epsilon^+<\infty\right]\mathbb{E}_\epsilon^X\left[\int_0^{T_0^-}e^{-qt}f(X_t)1_{\{X_t\in(\epsilon,\infty)\}}dt\right]. \tag{3.22}$$

By (3.7) and Theorem 3.5, we have

$$(3.22)=\lim_{\epsilon\downarrow 0}\frac{1}{W_X^{(q)}(\epsilon)}\int_\epsilon^\infty f(y)e^{-\Phi_X(q)}W_X^{(q)}(\epsilon)dy, \tag{3.23}$$

and we obtain (3.20). □

The following lemma is obtained from the proof of [17, Theorem 3.1].

Lemma 3.7. *We have*

$$\chi_X = (\Psi'_X(0) \wedge 0) + \int_{(-\infty,0)} \left(\int_0^{-u} e^{\Phi_X(0)(u+v)} dv + u 1_{(-1,0)}(u) \right) \Pi_X(du). \tag{3.24}$$

In particular, if X has bounded variation paths, then we have

$$\delta_X = (\Psi'_X(0) \wedge 0) + \int_{(-\infty,0)} \Pi_X(du) \int_0^{-u} e^{\Phi_X(0)(u+v)} dv. \tag{3.25}$$

Proof. The derivative of Ψ_X has the following form:

$$\Psi'_X(\lambda) = \chi_X + \sigma_X^2 \lambda + \int_{(-\infty,0)} (y e^{\lambda y} - y 1_{(-1,0)}(y)) \Pi_X(dy), \qquad \lambda > 0. \tag{3.26}$$

So we have

$$\Psi'_X(0) = \chi_X + \int_{(-\infty,-1]} y \Pi_X(dy). \tag{3.27}$$

i) Suppose that $\Psi'_X(0) > 0$. In this case, we have $\Phi(0) = 0$ and so the right hand side of (3.24) is equal to

$$\Psi'_X(0) - \int_{(-\infty,-1]} u \Pi_X(du). \tag{3.28}$$

By (3.27), we obtain (3.24).

ii) Suppose that $\Psi'_X(0) \leq 0$. In this case, we have $\Phi(0) > 0$ and so the right hand side of (3.24) is equal to

$$\frac{1}{\Phi_X(0)} \int_{(-\infty,0)} \left(1 - e^{\Phi_X(0)u} + \Phi_X(0) u 1_{(-1,0)}(u)\right) \Pi_X(du). \tag{3.29}$$

By (3.10) with $\lambda = \Phi_X(0)$, we obtain (3.24). □

4 Refracted Lévy processes and their scale functions

In this section, we recall two previous studies Kyprianou–Loeffen([13]) and Noba–Yano([17]).

4.1 Kyprianou–Loeffen's refracted Lévy processes

Let us recall some results of Kyprianou–Loeffen([13]). We fix a constant $\alpha > 0$ and let X be a general spectrally negative Lévy process, which may possibly have Gaussian component. Set $Y_t = X_t + \alpha t$. They defined a refracted Lévy process U as the pathwise unique strong solution of the stochastic differential equation

$$U_t - U_0 = X_t - X_0 + \alpha \int_0^t 1_{\{U_s<0\}} ds, \qquad t \geq 0. \tag{4.1}$$

Let Ψ_X and Ψ_Y denote Laplace exponent of X and Y, respectively. We write $\Phi_X(\theta) = \inf\{\lambda > 0 : \Psi_X(\lambda) > \theta\}$ and $\Phi_Y(\theta) = \inf\{\lambda > 0 : \Psi_Y(\lambda) > \theta\}$. For $q \geq 0$, $W_X^{(q)}$ and $W_Y^{(q)}$ are the scale functions of X and Y, respectively, in the sense of Section 3.

Theorem 4.1 ([13, Theorem 1])**.** *There exists a pathwise unique strong solution to the stochastic differential equation* (4.1)*.*

The proof of pathwise uniqueness. Suppose that $U^{(1)}$ and $U^{(2)}$ are two strong solutions to (4.1) with a common starting point $U_0^{(1)} = U_0^{(2)} = x \in \mathbb{R}$. We define

$$\Delta_t = U_t^{(1)} - U_t^{(2)} = \alpha\left(\int_0^t \left(1_{\{U_s^{(1)}<0\}} - 1_{\{U_s^{(2)}<0\}}\right) ds\right). \tag{4.2}$$

By integration by parts, we have

$$\Delta_t^2 = 2\alpha \int_0^t \Delta_s \left(1_{\{U_s^{(1)}<0\}} - 1_{\{U_s^{(2)}<0\}}\right) ds. \tag{4.3}$$

When $U_t^{(1)} \geq U_t^{(2)}$, we have $\Delta_t \geq 0$ and $\left(1_{\{U_s^{(1)}<0\}} - 1_{\{U_s^{(2)}<0\}}\right) \leq 0$. When $U_t^{(1)} \leq U_t^{(2)}$, we have $\Delta_t \leq 0$ and $\left(1_{\{U_s^{(1)}<0\}} - 1_{\{U_s^{(2)}<0\}}\right) \geq 0$. So by (4.3), we have $\Delta_2^2 \leq 0$. This implies that $\Delta_t = 0$ and $U^{(1)} = U^{(2)}$ a.s. $\square$

Sketch of the proof of existence of a strong solution. First, we assume that X has bounded variation paths. Suppose that $X_0 = 0$ for simply. We set $S_0^+ = 0$ and we put $U_t^{0,+} = X_t$ for $t \geq 0$. We set

$$S_1^- = \inf\left\{t > S_0^+ : U_t^{0,1} \leq 0\right\}, \qquad U_t^{1,-} = U_t^{0,+} + \alpha(t - S_1^-), \qquad t \geq S_1^-, \tag{4.4}$$

$$S_1^+ = \inf\left\{t > S_1^- : U_t^{1,-} \geq 0\right\}, \qquad U_t^{1,+} = U_t^{1,-} - \alpha(t - S_1^+), \qquad t \geq S_1^+. \tag{4.5}$$

We can and do define S_n^-, $\{U_t^{n,-} : t \geq 0\}$, S_n^+ and $\{U_t^{n,+} : t \geq 0\}$ for $n \geq 2$ similarly by induction. Since X has bounded variation paths and 0 is irregular for $(-\infty, 0)$ for X, we have $0 < S_1^- < S_1^+ < S_2^- < S_2^+ \dots$ and $\lim_{n\uparrow\infty} S_n^- = \lim_{n\uparrow\infty} S_n^+ = \infty$ a.s. We define

$$U_t = \begin{cases} U_t^{n-1,+}, & t \in [S_{n-1}^+, S_n^-),\ n \in \mathbb{N} \\ U_t^{n,-}, & t \in [S_n^-, S_n^+),\ n \in \mathbb{N}. \end{cases} \tag{4.6}$$

Then we easily see that U satisfies (4.1).

Second, we assume that X has unbounded variation paths. We fix $x \in \mathbb{R}$. By [2, pp.210], on the same probability space carrying X, we can construct a sequence of spectrally negative Lévy processes $X^{(n)}$ with bounded variation paths, such that for $t > 0$,

$$\lim_{n\uparrow\infty} \sup_{s\in[0,t]} \left|X_s^{(n)} - X_s\right| = 0, \qquad \text{a.s.} \tag{4.7}$$

Let $U^{(n)}$ be the strong solutions of (4.1) associated with $X^{(n)}$. We can prove that there exists a stochastic process $U^{(\infty)}$ such that for $t > 0$,

$$\lim_{n\uparrow\infty} \sup_{s\in[0,t]} \left|U_s^{(n)} - U_s^{(\infty)}\right| = 0, \qquad \text{a.s.} \tag{4.8}$$

We then see that $U^{(\infty)}$ satisfies the SDE by taking limit as $n \uparrow \infty$ in the SDE's for $U^{(n)}$. □

Let U be a solution to Kyprianou–Loeffen's stochastic differential equation (4.1).

Theorem 4.2 ([13, Theorem 4]). *For $q \geq 0$ and $x, a, b \in \mathbb{R}$ with $b < x < a$, we have*

$$\mathbb{E}_x^U\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)}, \tag{4.9}$$

where $W_U^{(q)}$ is defined by

$$W_U^{(q)}(x,y) = \begin{cases} W_Y^{(q)}(x-y) + \alpha 1_{\{x\geq 0\}} \int_0^x W_X^{(q)}(x-z) W_Y^{(q)\prime}(z-y)dz, & x \in \mathbb{R}, y < 0, \\ W_X^{(q)}(x-y), & x \in \mathbb{R}, y \geq 0. \end{cases} \tag{4.10}$$

They also calculated the potential densities with and without barriers.

Theorem 4.3 ([13, Theorem 6]). *For $q \geq 0$, $x, a, b \in \mathbb{R}$ with $b < 0 < a$, $x \in (b,a)$ and non-negative measurable function f, we have*

$$\mathbb{E}_x^U\left[\int_0^{T_a^+\wedge T_b^-} e^{-qt} f(U_t)dt\right] = \int_b^a f(y)\overline{\underline{r}}_U(x,y)dy, \tag{4.11}$$

$$\mathbb{E}_x^U\left[\int_0^{T_b^-} e^{-qt} f(U_t)dt\right] = \int_b^a f(y)\underline{r}_U(x,y)dy, \tag{4.12}$$

$$\mathbb{E}_x^U\left[\int_0^{T_a^+} e^{-qt} f(U_t)dt\right] = \int_b^a f(y)\overline{r}_U(x,y)dy, \tag{4.13}$$

$$\mathbb{E}_x^U\left[\int_0^{\infty} e^{-qt} f(U_t)dt\right] = \int_b^a f(y)r_U(x,y)dy, \tag{4.14}$$

where

$$\overline{\underline{r}}_U^{(q)}(x,y) = \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)} W_U^{(q)}(a,y) - W_U^{(q)}(x,y), \quad y \in [b,a] \tag{4.15}$$

$$\underline{r}_U^{(q)}(x,y) = \begin{cases} \frac{W_U^{(q)}(x,b)}{\alpha \underline{H}_U^{(q)}(b;b)} e^{-\Phi_X(q)(y-b)} - W_X^{(q)}(x-y) & y \in (0,\infty) \\ \frac{\underline{H}_U^{(q)}(y;b)}{\underline{H}_U^{(q)}(b;b)} W_U^{(q)}(x,b) - W_U^{(q)}(x,y) & y \in [b,0] \end{cases} \tag{4.16}$$

with $\underline{H}_U^{(q)}(y;b) = \int_0^\infty e^{-\Phi_X(q)(z-b)} W_Y^{(q)\prime}(z-y)dz$*,*

$$\overline{r}_U^{(q)}(x,y) = \begin{cases} \frac{\overline{H}_U^{(q)}(x;b)}{\overline{H}_U^{(q)}(a;b)} W_X^{(q)}(a-y) - W_X^{(q)}(x-y) & y \in (0,a] \\ \frac{\overline{H}_U^{(q)}(x;b)}{\overline{H}_U^{(q)}(a;b)} W_U^{(q)}(a,y) - W_U^{(q)}(x,y) & y \in (-\infty,0] \end{cases} \tag{4.17}$$

with $\overline{H}_U^{(q)}(x;b) = e^{\Phi_Y(q)(x-b)} + \alpha\Phi_Y(q)\int_0^x e^{\Phi_Y(q)(z-b)} W_X^{(q)}(x-z)dz$*, and*

$$r_U^{(q)}(x,y) = \begin{cases} H_U^{(q)}(x,y;b) - W_X^{(q)}(x-y) & y \in (0,\infty) \\ H_U^{(q)}(x,y;b)\int_0^\infty e^{-\Phi_Y(q)(z-b)} W_Y^{(q)\prime}(z-y)dz - W_U^{(q)}(x,y) & y \in (-\infty,0] \end{cases} \tag{4.18}$$

with $H_U^{(q)}(x,y;b) = e^{\Phi_Y(q)b}\overline{H}_U^{(q)}(x;b)\frac{\Phi_X(q)-\Phi_Y(q)}{\Phi_Y(q)} e^{-\Phi_X(q)y}$*, where* $W_U^{(q)}$ *has been given in* (4.10)*.*

4.2 Generalizations of refracted Lévy processes whose positive motions are bounded variation

Before recalling some results of Noba–Yano([17]), we discuss refracted Lévy processes whose positive motions have bounded variation paths.

Let us consider the stochastic differential equation

$$U_t - U_0 = \int_{(0,t]} 1_{\{U_{s-}\geq 0\}} dX_s + \int_{(0,t]} 1_{\{U_{s-}<0\}} dY_s. \tag{4.19}$$

Lemma 4.4 ([17])**.** *Suppose X has bounded variation paths. Then the stochastic differential equation* (4.19) *has a strong solution. Furthermore, when X and Y are compound Poisson processes with positive drifts, pathwise uniqueness for* (4.19) *holds.*

Proof. We give the proof of the existence of a strong solution. The proof is similar to that of Kyprianou–Loeffen's refracted Lévy processes. Suppose that $U_0 = 0$ for simplicity. We set $S_0^+ = 0$ and we put $U_t^{0,+} = X_t$ for $t \geq 0$. We set

$$S_1^- = \inf\{t > S_0^+ : U_t^{0,1} \leq 0\}, \qquad U_t^{1,-} = U_{S_1^-}^{0,+} + (Y_t - Y_{S_1^-}), \qquad t \geq S_1^-, \tag{4.20}$$

$$S_1^+ = \inf\{t > S_1^- : U_t^{1,-} \geq 0\}, \qquad U_t^{1,+} = U_{S_1^+}^{1,-} + (X_t - X_{S_1^+}), \qquad t \geq S_1^+. \tag{4.21}$$

We can and do define S_n^-, $\{U_t^{n,-} : t \geq S_n^-\}$, S_n^+ and $\{U_t^{n,+} : t \geq S_n^+\}$ for $n \geq 2$ similarly by induction. Since X and Y have bounded variation paths and 0 is irregular for $(-\infty, 0)$

for X and Y, we have $0 < S_1^- < S_1^+ < S_2^- < S_2^+ \dots$ and $\lim_{n\uparrow\infty} S_n^- = \lim_{n\uparrow\infty} S_n^+ = \infty$ a.s. We define

$$U_t = \begin{cases} U_t^{n-1,+}, & t \in [S_{n-1}^+, S_n^-),\ n \in \mathbb{N} \\ U_t^{n,-}, & t \in [S_n^-, S_n^+),\ n \in \mathbb{N}. \end{cases} \tag{4.22}$$

Then U satisfies (4.1). The proof in the case that $X_0 \neq 0$ is similar.

We assume that X and Y are compound Poisson processes with positive drifts. We prove the pathwise uniqueness. Let $U^{(1)}$ denote the strong solution of (4.19) with $U_0^{(1)} = x$ which is constructed as above. Let $U^{(2)}$ be another strong solutions of (4.19) with $U_0^{(2)} = x$. We define

$$T = \inf\{t > 0 : U_t^{(1)} \neq U_t^{(2)}\}. \tag{4.23}$$

By the form of (4.19), the value $U_T^{(1)}$ is determined by $\{U_t^{(1)} : t < T\}$ and the value $U_T^{(2)}$ is determined by $\{U_t^{(2)} : t < T\}$, so we have $U_T^{(1)} = U_T^{(2)}$. Suppose $U_T^{(1)} > 0$, then $U^{(1)}$ and $U^{(2)}$ behave as X for a while after time T. This fact is against the definition of T. Similarly, if $U_T^{(1)} < 0$, then $U^{(1)}$ and $U^{(2)}$ behave as Y for a while after time T and this is against the definition of T. So when $T < \infty$, it is necessary that $U_T^{(1)} = U_T^{(2)} = 0$. Since $\{X_{t+T} - X_T : t \geq 0\}$ and $\{Y_{t+T} - Y_T : t \geq 0\}$ behave as pure positive drift on $[T, T')$ where $T' = \inf\{t > T : X_{t-} \neq X_t \text{ or } Y_{t-} \neq Y_t\}$, we have $\{U_{t+T}^{(1)} - U_T^{(1)} : 0 \leq t \leq T' - T\} = \{U_{t+T}^{(2)} - U_T^{(2)} : 0 \leq t \leq T' - T\} = \{X_{t+T} - X_T : 0 \leq t \leq T' - T\}$. This is against the definition of T. So we obtain $T = \infty$ and the proof is completed. □

4.3 Generalized refracted Lévy processes

Suppose X has unbounded variation paths. In this case we do not know existence nor uniqueness for the SDE (4.19). Thus, following Noba–Yano([17]), we appeal to the excursion theory.

Let n_0^X denote the excursion measure of X away from 0 which satisfy

$$n_0^X\left[1 - e^{-qT_0}\right] = \frac{1}{\Phi'_X(q)} \tag{4.24}$$

for $q > 0$. For all non-negative measurable functional F, we define the law of the stopped process $\mathbb{P}_x^{U^0}$ by

$$\mathbb{E}_x^{U^0}[F(U)] = \begin{cases} \mathbb{E}_x^{Y^0}[F(Y^0)]\,, & x < 0, \\ \mathbb{E}_x^X\left[\mathbb{E}_{X_{T_0^-}}^{Y^0}[F(w \circ Y^0)]\,|_{w = k_{T_0^-}X}; 0 < T_0^- \leq T_0\right], & x > 0, \end{cases} \tag{4.25}$$

and the excursion measure n_0^U by

$$n_0^U[F(U)] = n_0^X\left[\mathbb{E}_{X_{T_0^-}}^{Y^0}\left[F(w \circ Y^0)\right]|_{w = k_{T_0^-}X}; 0 < T_0^- \leq T_0\right]. \tag{4.26}$$

By means of the excursion theory, we can construct from n_0^U and $\{\mathbb{P}_x^{U^0}\}_{x\in\mathbb{T}\setminus\{0\}}$ a $\mathbb{R}$-valued stochastic process U without stagnancy at 0.

Theorem 4.5 ([17, Theorme 6.4]). *Generalized refracted Lévy process U is a Feller process.*

We will later in Theorem 6.3 state and prove it in a more general form. The corresponding scale functions are introduced as follows.

Theorem 4.6 ([17, Theorem 6.2, 7.1]). *For $q \geq 0$, we define*

$$W_U^{(q)}(x,y) = \begin{cases} W_Y^{(q)}(x-y), & x \leq 0, y < 0, \\ W_X^{(q)}(x)W_Y^{(q)}(-y)(\Psi'_X(0)\vee 0) & \\ \quad + \int_0^\infty dv \int_{(-\infty,0)} \big(W_X^{(q)}(x)W_Y^{(q)}(-y)e^{\Phi_X(0)u} & \\ \quad - W_Y^{(q)}(u-y)W_X^{(q)}(x-v)\big)\Pi_X(du-v), & x > 0, y < 0 \\ W_X^{(q)}(x-y), & x\in\mathbb{R}, y \geq 0. \end{cases} \tag{4.27}$$

Then we have (4.9), (4.11) *and* (4.15) *for generalized refracted Lévy process U.*

Corollary 4.7 ([17, Corollary 6.3]). *We defined, for $x > 0$,*

$$\overline{W}_U^{(q)}(x) = W_X^{(q)}(x)(\Psi'_X(0)\wedge 0) \tag{4.28}$$

$$+ \int_0^\infty dv \int_{(-\infty,0)} \left(W_X^{(q)}(x)e^{\Psi_Y(0)u} - W_X^{(q)}(x-v)e^{\Psi_Y(q)u} \right) \Pi_X(du-v). \tag{4.29}$$

Then for $q \geq 0$ and $x < a$ with $a > 0$, we have

$$\mathbb{E}_x^U\left[e^{-qT_a^+}\right] = \frac{\overline{W}_U^{(q)}(x)}{\overline{W}_U^{(q)}(a)}. \tag{4.30}$$

In particular, $\overline{W}_U^{(q)}(x)$ is a continuous and increasing function of x.

Following [17] we also discuss an approximation problem of thier refracted Lévy processes. Let X and Y be spectrally negative Lévy processes. Suppose that X has unbounded variation paths and no Gaussian component. Let Ψ_X and Ψ_Y be Laplace exponents of X and Y, respectively, which have the following form:

$$\Psi_X(\lambda) = \chi_X\lambda - \int_{(-\infty,0)} (1 - e^{\lambda y} + \lambda y 1_{(-1,0)}(y))\Pi_X(dy), \quad \lambda \geq 0, \tag{4.31}$$

$$\Psi_Y(\lambda) = \chi_Y\lambda + \frac{\sigma_Y^2}{2}\lambda^2 - \int_{(-\infty,0)} (1 - e^{\lambda y} + \lambda y 1_{(-1,0)}(y))\Pi_Y(dy), \quad \lambda \geq 0, \tag{4.32}$$

For $n \in \mathbb{N}$, we define

$$\Psi_{X^{(n)}}(\lambda) = \delta_{X^{(n)}}\lambda - \int_{(-\infty,0)} \left(1 - e^{\lambda y}\right) \Pi_{X^{(n)}}(dy), \tag{4.33}$$

$$\Psi_{Y^{(n)}}(\lambda) = \delta_{Y^{(n)}}\lambda - \int_{(-\infty,0)} \left(1 - e^{\lambda y}\right) \Pi_{Y^{(n)}}(dy), \tag{4.34}$$

where

$$\delta_{X^{(n)}} = \chi_X + \int_{(-1,-\frac{1}{n})} (-y)\Pi_X(dy), \qquad \Pi_{X^{(n)}} = 1_{(-\infty,-\frac{1}{n})}\Pi_X, \tag{4.35}$$

$$\delta_{Y^{(n)}} = \chi_X + n\sigma_X^2 + \int_{(-1,-\frac{1}{n})} (-y)\Pi_X(dy), \quad \Pi_{Y^{(n)}} = 1_{(-\infty,-\frac{1}{n})}\Pi_X + n^2\sigma_Y^2\delta_{(-\frac{1}{n})}. \tag{4.36}$$

Let U be a generalized refracted Lévy process associated with X and Y, and let $U^{(n)}$ be a generalized refracted Lévy process associated with $X^{(n)}$ and $Y^{(n)}$.

Theorem 4.8 ([17, Theorem 8.1])**.** *The sequence of the processes $(U^{(n)}, \mathbb{P}_x^{U^{(n)}})$ converges in distribution to $(U, \mathbb{P}_x^U)$ for all $x \in \mathbb{R}$.*

We will later in Corollary 8.6 state and prove it in a more general form.

4.4 Kyprianou–Loeffen's refracted Lévy processes viewed as generalized refracted Lévy processes

Suppose that Y has the same distribution as $\{X_t + \alpha t : t \geq 0\}$. In this case we may expect that generalized refracted Lévy process U^{NY} coincides in law with Kyprianou–Loeffen's refracted Lévy process U^{KL} although these two processes are constructed in different ways. In addition, we may expect the corresponding scale functions $W_{U^{NY}}^{(q)}$ in Theorem 4.6 and $W_{U^{KL}}^{(q)}$ in Theorem 4.2 coincide, although their expressions look different. We already know that for $q \geq 0$ and $(x, y) \in \mathbb{R} \times \mathbb{R}\backslash(0,\infty) \times (-\infty, 0)$, we have

$$W_{U^{NY}}^{(q)}(x, y) = W_{U^{KL}}^{(q)}(x, y). \tag{4.37}$$

Theorem 4.9. *If Y has the same distribution as $\{X_t + \alpha t : t \geq 0\}$, then generalized refracted Lévy process $(U^{NY}, \mathbb{P}_x^{U^{NY}})$ made by X and Y and Kyprianou–Loeffen's refracted Lévy process $(U^{KL}, \mathbb{P}_x^{U^{KL}})$ made by X and α have the same distribution for all $x \in \mathbb{R}$. In addition, $W_{U^{NY}}^{(q)}$ and $W_{U^{KL}}^{(q)}$ coincide.*

Proof. *i*) We assume that X has bounded variation paths.

We prove (4.37) for $q \geq 0$ and $(x, y) \in (0,\infty) \times (-\infty, 0)$. Let Ψ_X denote to the Laplace exponent of X of the form:

$$\Psi_X(\lambda) = \delta_X\lambda - \int_{(-\infty,0)} \left(1 - e^{\lambda y}\right) \Pi_X(dy), \qquad \lambda \geq 0, \tag{4.38}$$

for $\delta_X > 0$ and Lévy measure Π_X satisfying $\Pi_X(0,\infty) = 0$ and $\int_{(-\infty,0)}(1\wedge|x|)\Pi_X(dx) < \infty$. By [13, (4.13)], for $(x,y) \in (0,\infty) \times (-\infty,0)$, we have

$$W^{(q)}_{U^{KL}}(x,y) = \delta_X W^{(q)}_X(x)W^{(q)}_Y(-y) - \int_{(0,\infty)} dv \int_{(-\infty,-v)} W^{(q)}_X(x-v)W^{(q)}_Y(-y+u+v)\Pi_X(du). \tag{4.39}$$

By (4.27) and (4.39), it suffices to show that

$$\delta_X = (\Psi'_X(0) \vee 0) + \int_0^\infty dv \int_{(-\infty,-v)} e^{\Phi_X(0)(u+v)}\Pi_X(du) \tag{4.40}$$

By (3.25), we obtain (4.40) immediately.

Since killed potential measures of U^{NY} and U^{KL} are written by $W^{(q)}_{U^{NY}}$ and $W^{(q)}_{U^{KL}}$ as (4.15), respectively, and by (4.37), resolvents of U^{NY} and U^{KL} are same. So $(U^{NY}, \mathbb{P}^{U^{NY}}_x)$ and $(U^{KL}, \mathbb{P}^{U^{KL}}_x)$ have the same distribution for $x \in \mathbb{R}$.

$ii)$ We assume that X has unbounded variation paths.

Let $\{U^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of refracted Lévy processes which has the same definition as that in Theorem 4.8. By Theorem 4.8, the sequence of the processes $(U^{(n)}, \mathbb{P}^{U^{(n)}}_x)$ converges in distribution to $(U^{NY}, \mathbb{P}^{U^{NY}}_x)$ for all $x \in \mathbb{R}$. On the other hand, by $i)$ and the proof of Theorem 4.1, the sequence of the processes $(U^{(n)}, \mathbb{P}^{U^{(n)}}_x)$ converges in distribution to $(U^{KL}, \mathbb{P}^{U^{KL}}_x)$ for all $x \in \mathbb{R}$. So $(U^{NY}, \mathbb{P}^{U^{NY}}_x)$ and $(U^{KL}, \mathbb{P}^{U^{KL}}_x)$have the same distribution for $x \in \mathbb{R}$.

For $q \geq 0$ and x, a, b with $b < x < a$, we have

$$\mathbb{E}^{U^{NY}}_x\left[e^{-qT^+_a}; T^+_a < T^-_b\right] = \mathbb{E}^{U^{KL}}_x\left[e^{-qT^+_a}; T^+_a < T^-_b\right]. \tag{4.41}$$

Scale functions $W^{(q)}_{U^{NY}}$ and $W^{(q)}_{U^{KL}}$ satisfies (4.9) for U^{NY} and U^{KL}, respectively, so we have

$$\frac{W^{(q)}_{U^{NY}}(x,b)}{W^{(q)}_{U^{NY}}(a,b)} = \frac{W^{(q)}_{U^{KL}}(x,b)}{W^{(q)}_{U^{KL}}(a,b)}, \quad q \geq 0,\ b < x < a. \tag{4.42}$$

We already have (4.37) for $x, y \in \mathbb{R}^2\backslash(0,\infty) \times (-\infty,0)$. So we take x and b to satisfy $b < x < 0$ and then by (4.42), we have

$$W^{(q)}_{U^{NY}}(a,b) = W^{(q)}_{U^{KL}}(a,b), \qquad a > 0. \tag{4.43}$$

The proof is completed. □

5 Generalized scale functions

We define generalized scale functions for standard processes with no positive jumps using the excursion theory. In addition, we characterize the fluctuations of standard processes with no positive jumps using the generalized scale functions. The results in this section are based on [15].

5.1 Definition of generalized scale functions

We define generalized scale functions. In addition, we characterize the two-sided exit problems and the killed potential densities of standard processes with no positive jumps using the generalized scale functions.

Let X, $L^{X,x}$ and n^X_x be those in Section 2. Let us define generalized scale functions.

Definition 5.1 ([15, Definition 3.1])**.** For $q \geq 0$ and $x, y \in \mathbb{T}$, we define *generalized q-scale function* of X as

$$W^{(q)}_X(x,y) = \begin{cases} \frac{1}{n^X_y\left[e^{-qT^+_x};T^+_x<\infty\right]}, & x \geq y, \\ 0, & x < y, \end{cases} \tag{5.1}$$

where $\frac{1}{\infty} = 0$.

Remark 5.2. All $x \in \mathbb{T}\backslash\{b_0\}$ is regular for (x, ∞), i.e., $\mathbb{P}^X_x(T^+_x = 0) = 1$, thanks to the assumptions of no positive jumps and of (A1). When x is irregular for itself, we have $W^{(q)}_X(x,x) = l^X_x$ by the definition of n^X_x.

Remark 5.3. Let us characterize generalized scale functions of diffusion processes in terms of their characteristics. Let m and s be two $\mathbb{R}$-valued strictly increasing continuous functions on the interval $[0, \infty)$ satisfying $s(0) = 0$. Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion process with 0 being a reflecting boundary. Note that our n^X_0 coincides with the excursion measure defined in [24, Definition 2.1] up to scale transformation. Let $\psi^{(q)}$ denote the increasing eigenfunction $\frac{d}{dm}\frac{d}{ds}\psi^{(q)} = q\psi^{(q)}$ such that $\frac{d}{ds}\psi^{(q)}(0) = 1$. In other words, the $\psi^{(q)}$ is the unique solution of the integral equation

$$\psi^{(q)}(x) = s(x) + q\int_0^x (s(x) - s(y))\psi^{(q)}(y)dm(y), \qquad x \in [0, \infty). \tag{5.2}$$

Then, by [24, Corollary 2.4], for $q > 0$ and $x \in (0, \infty)$, we have

$$\psi^{(q)}(x) = \frac{1}{n^X_0\left[e^{-qT^+_x}; T^+_x < \infty\right]}, \tag{5.3}$$

which shows that $W^{(q)}_X(x, 0) = \psi^{(q)}(x)$. In particular, we have $W^{(0)}_X(x, 0) = s(x)$.

We fix $b, a \in \mathbb{T}$ with $b < a$. The upward exit time from the bounded interval $[b, a]$ is characterized as follows.

Theorem 5.4 ([15, Theorem 3.4])**.** *For $q \geq 0$ and $x \in (b, a)$, we have*

$$\mathbb{E}^X_x\left[e^{-qT^+_a}; T^+_a < T^-_b\right] = \frac{W^{(q)}_X(x,b)}{W^{(q)}_X(a,b)}. \tag{5.4}$$

Proof. Since $b < x < a$ and since X has no positive jumps, we have

$$n_b^X\left[e^{-qT_a^+}; T_a^+ < \infty\right] = n_b^X\left[e^{-qT_x^+}; T_x^+ < \infty\right] \mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right], \tag{5.5}$$

where we utilized the strong Markov property of n_b^X (see, e.g., [4]). □

In order to obtain killed potential density, we need the following lemma.

Lemma 5.5 ([17, Lemma 6.1], [15, Lemma 3.5])**.** *For $q \geq 0$ and $x \in (b, a)$, we have*

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = n_x^X\left[e^{-qT_a^+}; T_a^+ < \infty\right] \mathbb{E}_x^X\left[\int_{0-}^{T_a^+ \wedge T_b^-} e^{-qt} dL_t^{X,x}\right]. \tag{5.6}$$

$$= \frac{n_x^X\left[e^{-qT_a^+}; T_a^+ < \infty\right]}{\mu_x^X q + n_x^X\left[1 - e^{-qT_0} 1_{\{T_a^+ = \infty, T_b^- = \infty\}}\right]}. \tag{5.7}$$

Proof. *i*) We assume that x is regular for itself.

Let $p = \{p(t) : t \in D(p)\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ denote a Poisson point process with characteristic measure n_x^X. We write $\eta^{X,x}(s) = \sum_{u \leq s} T_x(p(u))$. We set $A = \{T_a^+ < \infty\} \cup \{T_b^- < \infty\} \cup \{\zeta < \infty\}$ and $\kappa_A = \inf\{s > 0 : p(s) \in A\}$. By the same argument as the proof of [17, Lemma 6.1], we have

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \mathbb{E}\left[e^{-q\eta^{X,x}(\kappa_A-)}\right] \frac{n_x^X\left[e^{-qT_a^+}; A\right]}{n_x^X[A]}. \tag{5.8}$$

We denote $p_{A^c} = p|_{D(p_{A^c})}$ with $D(p_{A^c}) = \{s \in D(p) : p(s) \in A^c\}$. We write $\eta_{A^c}^{X,x}(s) = \sum_{u \leq s} T_x(p_{A^c}(u))$ where $T_x(\partial) = 0$. Note that $\eta^{X,x}(\kappa_A-) = \eta_{A^c}^{X,x}(\kappa_A)$ and that $\eta_{A^c}^{X,x}$ and κ_A are independent. We thus have

$$\mathbb{E}\left[e^{-q\eta^{X,x}(\kappa_A-)}\right] = n_x^X[A]\,\mathbb{E}\left[\int_0^{\kappa_A} \exp\bigl(-q\eta^{X,x}(t)\bigr)\,dt\right] \tag{5.9}$$

$$= n_x^X[A]\,\mathbb{E}_x^X\left[\int_{0-}^{T_a^+ \wedge T_b^-} e^{-qt} dL_t^{X,x}\right], \tag{5.10}$$

where we used the fact that $\mathbb{P}[\kappa_A > t] = e^{-tn_x^X[A]}$ and the identity

$$\mathbb{E}[f(\mathbf{e}_q)] = q\mathbb{E}\left[\int_0^{\mathbf{e}_q} f(t)dt\right] \tag{5.11}$$

for an exponential variable with $\mathbb{P}[\mathbf{e}_q > t] = e^{-tq}$. Therefore we obtain (5.6). On the other hand, we have

$$(5.9) = n_x^X[A] \int_0^\infty n_x^X\left[\exp\bigl(-t\bigl(n_x^X[A] + \mu_x^X q + n_x^X\left[1 - e^{-qT_x}; A^c\right]\bigr)\bigr)\right] dt \tag{5.12}$$

$$= \frac{n_x^X[A]}{\mu_x^X q + n_x^X[1 - e^{-qT_x} 1_{A^c}]}, \tag{5.13}$$

so we obtain (5.7).

ii) We assume that x is irregular for itself.

Let $T_x^{(n)}$ denote the n-th hitting time to x and let $T_x^{(0)} = 0$. Then we have

$$\mathbb{E}_x^X\left[\int_{0-}^{T_a^+\wedge T_b^-} e^{-qt} dL_t^{X,x}\right] = l_x^X \sum_{i=0}^{\infty} \mathbb{E}_x^X\left[e^{-qT_x^{(i)}}; T_x^{(i)} < T_a^+ \wedge T_b^-\right] \tag{5.14}$$

$$= l_x^X \sum_{i=0}^{\infty} \left(\mathbb{E}_x^X\left[e^{-qT_x}; T_x < T_a^+ \wedge T_b^-\right]\right)^i. \tag{5.15}$$

On the other hand, we have

$$\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \sum_{i=0}^{\infty} \left(\mathbb{E}_x^X\left[e^{-qT_x}; T_x < T_a^+ \wedge T_b^-\right]\right)^i \mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_x \wedge T_b^-\right]. \tag{5.16}$$

Therefore we obtain (5.6). Furthermore, we have

$$(5.16) = \frac{\mathbb{E}_x^{X^x}\left[e^{-qT_a^+}; T_a^+ < \infty\right]}{1 - \mathbb{E}_x^X\left[e^{-qT_x}; T_x < T_a^+ \wedge T_b^-\right]}, \tag{5.17}$$

so we obtain (5.7). $\square$

By Theorem 5.4 and Lemma 5.5, for $q \geq 0$ and $x \in (b, a)$, we obtain

$$\mathbb{E}_x^X\left[\int_{0-}^{T_a^+\wedge T_b^-} e^{-qt} dL_t^{X,x}\right] = \frac{W_X^{(q)}(x, b) W_X^{(q)}(a, x)}{W_X^{(q)}(a, b)}. \tag{5.18}$$

For $q \geq 0$ and non-negative measurable function f, we have

$$\mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt} f(X_t) dt\right] = \int_{(b,a)} f(y) \mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt} dL_t^{X,y}\right] m(dy). \tag{5.19}$$

The following theorem represents the potential density in terms of the generalized scale functions.

Theorem 5.6 ([15, Theorem 3.6]). *For $q \geq 0$ and $x, y \in (b, a)$, we have*

$$\mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt} dL_t^{X,y}\right] = \frac{W_X^{(q)}(x, b)}{W_X^{(q)}(a, b)} W_X^{(q)}(a, y) - W_X^{(q)}(x, y). \tag{5.20}$$

Proof. *i*) Let us consider the case where $x = y$.

When x is regular for itself, the continuity of the local time implies

$$\mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right] = \mathbb{E}_x^X\left[\int_{0-}^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right] \tag{5.21}$$

and the absence of positive jumps implies

$$W_X^{(q)}(x,x) = \frac{1}{n_x^X\left[e^{-qT_x^+}; T_x^+ < \infty\right]} = \frac{1}{n_x^X[T_x^+ < \infty]} = 0. \tag{5.22}$$

Thus (5.20) follows from (5.18).

When x is irregular for itself, we have

$$\mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right] = \mathbb{E}_x^X\left[\int_{0-}^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right] - l_x^X. \tag{5.23}$$

By (5.18) and Remark 5.2, we obtain (5.20).

ii) Let us consider the case where $x \neq y$.

On one hand, we have

$$\mathbb{E}_x^X\left[\int_0^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right] = \mathbb{E}_x^X\left[e^{-qT_y}; T_y < T_b^- \wedge T_a^+\right] \mathbb{E}_y^X\left[\int_{0-}^{T_b^-\wedge T_a^+} e^{-qt}dL_t^{X,y}\right]. \tag{5.24}$$

On the other hand, we can prove

$$\mathbb{E}_x^X\left[e^{-qT_y}; T_y < T_b^- \wedge T_a^+\right] = \frac{W_X^{(q)}(a,b)}{W_X^{(q)}(y,b)}\left(\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} - \frac{W_X^{(q)}(x,y)}{W_X^{(q)}(a,y)}\right). \tag{5.25}$$

Indeed, for $x < y$, it is obvious, and, for $x > y$, the left-hand-side of (5.25) equals to

$$\frac{\mathbb{E}_x^X\left[e^{-qT_a^+}; T_y < T_a^+ < T_b^-\right]}{\mathbb{E}_y^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right]} = \frac{\mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] - \mathbb{E}_x^X\left[e^{-qT_a^+}; T_a^+ < T_y^-\right]}{\mathbb{E}_y^X\left[e^{-qT_a^+}; T_a^+ < T_b^-\right]}, \tag{5.26}$$

which leads to (5.25) by Theorem 5.4. Combining (5.24), (5.18) and (5.25), we obtain (5.20). □

The downward exit time is characterized as follows.

Corollary 5.7 ([15, Corollary 3.7])**.** *For $x, y \in (b_0, a_0)$, we define*

$$Z_X^{(q)}(x,y) = \begin{cases} 1 + q\int_{(y,x)} W_X^{(q)}(x,z)m(dz), & x > y, \\ 1, & x \leq y. \end{cases} \tag{5.27}$$

Then we have

$$\mathbb{E}_x^X\left[e^{-qT_b^-}; T_b^- < T_a^+\right] = Z_X^{(q)}(x,b) - \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} Z_X^{(q)}(a,b). \tag{5.28}$$

Proof. We have

$$\mathbb{E}_x^X\left[e^{-qT_b^-};T_b^- < T_a^+\right] = \mathbb{E}_x^X\left[e^{-q(T_b^- \wedge T_a^+)};T_b^- \wedge T_a^+ < \infty\right] - \mathbb{E}_x^X\left[e^{-qT_a^+};T_a^+ < T_b^-\right]. \quad (5.29)$$

By Theorem 5.6 and by the identity $e^{-qs} = 1 - q\int_0^s e^{-qt}dt$, we have

$$\mathbb{E}_x^X\left[e^{-q(T_b^- \wedge T_a^+)}\right] = 1 - q\int_{(b,a)}\left(\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)}W_X^{(q)}(a,y) - W_X^{(q)}(x,y)\right)m(dy). \quad (5.30)$$

By (5.29), (5.30) and Theorem 5.4, we have

$$(5.29) = 1 + q\int_{(b,a)}W_X^{(q)}(x,y)m(dy) - \frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)}\left(1 + q\int_{(b,a)}W_X^{(q)}(a,y)m(dy)\right), \quad (5.31)$$

and therefore we obtain (5.28). □

5.2 Duality in terms of generalized scale functions

In this section, we give the necessary and sufficient conditions of duality in terms of generalized scale functions.

Let X and $\widehat{X}$ be processes defined in Section 2.2. When X and $\widehat{X}$ are in duality, we always use the local times defined by [22, Proposition of Section V.1]. In other cases, we use the normalization of the local times in Section 2.2. We let scale functions $\{W_X^{(q)}\}_{q\geq 0}$ and $\{W_{-\widehat{X}}^{(q)}\}_{q\geq 0}$ be those in Section 5.1.

Theorem 5.8 ([15, Theorem 4.4]). *If X and $\widehat{X}$ are in duality relative to m, then we have*

$$W_X^{(q)}(x,y) = W_{-\widehat{X}}^{(q)}(-y,-x), \qquad x,y \in (b_0,a_0). \quad (5.32)$$

If $\mathbb{T}$ is open, then the converse is also true.

To prove Theorem 5.8, we need the following lemma, which gives us the relationship between the killed potential densities of X and $\widehat{X}$.

Lemma 5.9 ([15, Lemma 4.5]). *Suppose X and $\widehat{X}$ be in duality relative to m. Then, for all $b < a \in \mathbb{T}$ and $x,y \in (b,a)$, we have*

$$\mathbb{E}_x^X\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt}dL_t^{X,y}\right] = \mathbb{E}_y^{\widehat{X}}\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt}dL_t^{\widehat{X},x}\right]. \quad (5.33)$$

Proof. Let $X^{(b,a)}$ and $\widehat{X}^{(b,a)}$ denote the X and $\widehat{X}$ killed on exiting (b,a), respectively. We denote by $R_{X^{(b,a)}}^{(q)}$ and $R_{\widehat{X}^{(b,a)}}^{(q)}$ the q-resolvent operators of $X^{(b,a)}$ and $\widehat{X}^{(b,a)}$, respectively.

For each $q > 0$, there exists a function $r^{(q)}_{X^{(b,a)}} : (b,a) \times (b,a) \to [0,\infty)$ such that all the conditions (i)–(iv) of Theorem 2.3 hold. By definition, we have

$$R^{(q)}_{X^{(b,a)}} f(y) = \mathbb{E}^X_x \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} f(X_t) dt \right] = \int_{(b,a)} f(y) \mathbb{E}^X_x \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL^{X,y}_t \right] m(dy). \tag{5.34}$$

So, for all $x \in (b,a)$, we have $r^{(q)}_{X^{(b,a)}}(x,\cdot) = \mathbb{E}^X_x \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL^{X,\cdot}_t \right]$, m-a.e. Since

$$\mathbb{E}^X_x \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL^{X,y}_t \right] = \mathbb{E}^X_x \left[\int_0^{\infty} e^{-qt} dL^{X,y}_t \right] - \mathbb{E}^X_x \left[e^{-q(T_b^- \wedge T_a^+)} \mathbb{E}^X_{X_{T_b^- \wedge T_a^+}} \left[\int_0^{\infty} e^{-qt} dL^{X,y}_t \right] \right] \tag{5.35}$$

and the dominated convergence theorem, the function $y \mapsto \mathbb{E}^X_x \left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dL^{X,y}_t \right]$ is cofinely continuous. By cofine continuity, for all $x, y \in (b,a)$, we see that $r^{(q)}_{X^{(b,a)}}(x,y)$ coincides with the left hand side of (5.33), and also, in the same way, with the right hand side of (5.33). □

Proof of Theorem 5.8. Let us assume that X and $\widehat{X}$ are in duality relative to m. First, we fix $b, y, a \in \mathbb{T}$ with $b < y < a$. By Lemma 5.9 and Theorem 5.6, for all $q \geq 0$ and $x \in (b,y)$, we have

$$\frac{W^{(q)}_X(x,b)}{W^{(q)}_X(a,b)} W^{(q)}_X(a,y) = \frac{W^{(q)}_{-\widehat{X}}(-y,-a)}{W^{(q)}_{-\widehat{X}}(-b,-a)} W^{(q)}_{-\widehat{X}}(-b,-x). \tag{5.36}$$

Hence there exists a function $\gamma_1 : [0,\infty) \times \mathbb{T} \to (0,\infty)$ satisfying

$$W^{(q)}_X(x,b) = \gamma_1(q,b) W^{(q)}_{-\widehat{X}}(-b,-x) \qquad x \in (b, a_0). \tag{5.37}$$

Second, we fix $b, x, a \in \mathbb{T}$ with $b < x < a$. For $q \geq 0$ and $y \in (x,a)$, we have (5.36). Thus there exists a function $\gamma_2 : [0,\infty) \times \mathbb{T} \to (0,\infty)$

$$W^{(q)}_X(a,y) = \gamma_2(q,a) W^{(q)}_{-\widehat{X}}(-y,-a) \qquad y \in (b_0, a). \tag{5.38}$$

By (5.37) and (5.38), for $q \geq 0$ and $a, b \in (b_0, a_0)$, we have $\gamma_1(q,b) = \gamma_2(q,a)$, so γ_1 and γ_2 depend on only $q \geq 0$. By (5.36), $\gamma_1 = \gamma_2 \equiv 1$. Thus, for $y, x \in (b_0, a_0)$ with $y < x$, we have $W^{(q)}_X(x,y) = W^{(q)}_{-\widehat{X}}(-y,-x)$. By the fine continuity of $W^{(q)}_X$ and the cofine continuity $W^{(q)}_{-\widehat{X}}$, for $x \in (b_0, a_0)$, we have $W^{(q)}_X(x,x) = W^{(q)}_{-\widehat{X}}(-x,-x)$.

Let us assume that $\mathbb{T}$ is open and that (5.32) is satisfied. Then, for $b < a \in \mathbb{T}$ and $x, y \in (b,a)$, we have

$$\frac{W^{(q)}_X(x,b)}{W^{(q)}_X(a,b)} W^{(q)}_X(a,y) - W^{(q)}_X(x,y) = \frac{W^{(q)}_{-\widehat{X}}(-y,-a)}{W^{(q)}_{-\widehat{X}}(-b,-a)} W^{(q)}_{-\widehat{X}}(-b,-x) - W^{(q)}_{-\widehat{X}}(-y,-x). \tag{5.39}$$

By Theorem 5.6, the first terms and the second terms of the both sides of (5.39) are the potential densities of X and $\widehat{X}$ killed on exiting (b, a), respectively. We therefore conclude the duality of the killed processes, which yields that of the original processes. $\square$

Remark 5.10. We suppose that X and $\widehat{X}$ are in duality. Then at each point $x \in \mathbb{T}$, fine continuity implies right continuity and cofine continuity implies left continuity. By the proof of Lemma 5.9 and Theorem 5.6, the function $x \mapsto W^{(q)}(x, y)$ is finely (and hence right) continuous and $y \mapsto W^{(q)}(x, y)$ is cofinely (and hence left) continuous.

5.3 The case of spectrally negative Lévy processes

When X is a spectrally negative Lévy process, the definition of the usual scale function $W^{(q)}(x)$ is based on the Laplace transform

$$\int_0^\infty e^{-qt} W^{(q)}(x) dx = \frac{1}{\Psi(\beta) - q}, \qquad \beta > \Phi(q). \tag{3.1}$$

It satisfies

$$W^{(q)}(x) = e^{\Phi(q)x} r^{(q)}(0+) - r^{(q)}(-x) \tag{5.40}$$

where $r^{(q)}$ is the right-continuous potential density of X with respect to the Lebesgue measure (see [21]). Pistorius([21]) provided a potential theoretic viewpoint for the scale functions in the sense that he started from (5.40) and proved (3.1). We now provide another viewpoint.

Theorem 5.11. *For all $q \geq 0$, let us start from*

$$W^{(q)}(x) = \frac{1}{n_0^X \left[e^{-qT_x^+}; T_x^+ < \infty \right]}. \tag{3.7}$$

Then we have (3.1).

Let X be a spectrally negative Lévy process and m denote the Lebesgue measure. By Sections 5.1 and 5.2, we have local times, excursion measures and generalized scale functions $W_X^{(q)}$. Since X has the stationary independent increment property, for $q \geq 0$ and $x, y \in \mathbb{R}$, we have

$$\mathbb{E}_0^X \left[\int_0^\infty e^{-qt} dL_t^{X, y-x} \right] = \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right]. \tag{5.41}$$

If we define $r^{(q)}(x) = r_X^{(q)}(0, x)$ and $W^{(q)}(x) = W_X^{(q)}(0, x)$, then we have

$$r^{(q)}(y - x) = \mathbb{E}_x^X \left[\int_0^\infty e^{-qt} dL_t^{X,y} \right], \qquad x, y \in \mathbb{R}, \tag{5.42}$$

$$W^{(q)}(x - y) = W_X^{(q)}(x, y), \qquad x, y \in \mathbb{R}, \tag{5.43}$$

for all $q \geq 0$. Note that $r^{(q)}$ is a càglàd function. In fact, we have

$$\lim_{h \downarrow 0} r^{(q)}(x+h) = \lim_{h\downarrow 0} \mathbb{E}^X_{-h}\left[\int_0^\infty e^{-qt} dL_t^{X,x}\right] \tag{5.44}$$

$$\begin{aligned} &= \lim_{h\downarrow 0} \mathbb{E}^X_{-h}\left[e^{-qT_0^+} : T_0^+ \leq T_x\right] \mathbb{E}^X_0\left[\int_{0-}^\infty e^{-qt} dL_t^{X,x}\right] \\ &\quad + \lim_{h\downarrow 0} \mathbb{E}^X_{-h}\left[e^{-qT_x}; T_x < T_0^+\right] \mathbb{E}^X_x\left[\int_{0-}^\infty e^{-qt} dL_t^{X,x}\right] \end{aligned} \tag{5.45}$$

$$= \mathbb{E}^X_0\left[\int_{0-}^\infty e^{-qt} dL_t^{X,x}\right] \tag{5.46}$$

where in (5.46) we used

$$\mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right] = e^{-\Phi(q)x}, \qquad x > 0, \tag{5.47}$$

and $\lim_{x\downarrow 0} \mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right] = \lim_{x\downarrow 0} e^{-\Phi(q)x} = 1$ (see, e.g., [12, Theorem 3.12]).

Proof of Theorem 5.11. By the equation obtained from (5.6) when b limits to infinity, for $x > 0$, we have

$$W^{(q)}(x) = \frac{1}{n^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right]} \tag{5.48}$$

$$= \frac{1}{\mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right]} \mathbb{E}^X_0\left[\int_{0-}^{T_x^+} e^{-qt} dL_t^0\right] \tag{5.49}$$

$$= \frac{1}{\mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right]} \left(\mathbb{E}^X_0\left[\int_{0-}^\infty e^{-qt} dL_t^0\right] - \mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right] \mathbb{E}^X_x\left[\int_0^\infty e^{-qt} dL_t^0\right]\right) \tag{5.50}$$

$$= \frac{1}{\mathbb{E}^X_0\left[e^{-qT_x^+}; T_x^+ < \infty\right]} r^{(q)}(0+) - r^{(q)}(-x). \tag{5.51}$$

By (5.47), for $\beta > \Phi(q)$, we have

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \int_0^\infty \left(e^{-(\beta - \Phi(q))x} r^{(q)}(0+) - e^{-\beta x} r^{(q)}(-x)\right) dx \tag{5.52}$$

$$= \frac{r^{(q)}(0+)}{\beta - \Phi(q)} - \int_0^\infty e^{-\beta x} r^{(q)}(-x) dx. \tag{5.53}$$

On the other hand, for all $\beta < \Phi(q)$, we have

$$\frac{1}{q-\Psi(\beta)} = \int_0^\infty e^{-(q-\Psi(\beta))t} dt \tag{5.54}$$

$$= \int_0^\infty e^{-qt}\mathbb{E}_0^X\left[e^{\beta X_t}\right] dt \tag{5.55}$$

$$= \int_{-\infty}^\infty e^{\beta x} r^{(q)}(x)dx \tag{5.56}$$

$$= \int_0^\infty e^{\beta x}\mathbb{E}_0^X\left[e^{-qT_x^+}; T_x^+ < \infty\right] r^{(q)}(0+)dx + \int_0^\infty e^{-\beta x} r^{(q)}(-x)dx \tag{5.57}$$

$$= \int_0^\infty e^{-(\Phi(q)-\beta)x} r^{(q)}(0+)dx + \int_0^\infty e^{-\beta x} r^{(q)}(-x)dx \tag{5.58}$$

$$= \frac{r^{(q)}(0+)}{\Phi(q)-\beta} + \int_0^\infty e^{-\beta x} r^{(q)}(-x)dx. \tag{5.59}$$

By analytic extension, we have (5.59) for $\beta > \Phi(q)$. By (5.53) and (5.59), we obtain (3.1). □

Remark 5.12. The later part of the proof of Theorem 5.11 is almost the same as a part of the proof of [21, Theorem 1(i)–(iii)].

Remark 5.13. We can compute the value $l^X := l_0^X$ when X has bounded variation paths. In this case, the Laplace exponent of X can be written as the following: for $\lambda \geq 0$,

$$\Psi(\lambda) = \delta_X - \int_{(-\infty,0)} (1 - e^{\lambda x})\Pi_X(dx), \tag{5.60}$$

for some constant $\delta_X > 0$ and Lévy measure Π_X satisfying $\int_{(-\infty,0)}(|x| \wedge 1)\Pi_X(dx) < \infty$. By the definitions of generalized scale functions, n_0^X and [12, Lemma 8.6], we obtain

$$l^X = \lim_{\epsilon\downarrow 0} n_0^X\left[T_\epsilon^+ < \infty\right] = \lim_{\epsilon\downarrow 0} W^{(0)}(\epsilon) = \frac{1}{\delta_X}. \tag{5.61}$$

6 Refracted processes

In this section, we construct a refracted process from two $\mathbb{R}$-valued standard processes with no positive jumps X and Y using the excursion theory and a landing function. This section is based on [16, Section 3].

6.1 The definition of refracted processes

Let a_0, a_1, b_0 and b_1 be real numbers with $-\infty \leq b_0 \leq b_1 < 0 < a_1 \leq a_0 \leq \infty$. Let $\mathbb{T}_X$ be an interval with $\sup \mathbb{T}_X = a_0$ and $\inf \mathbb{T}_X = b_1$. Let $\mathbb{T}_Y$ be an interval with $\sup \mathbb{T}_Y = a_1$ and $\inf \mathbb{T}_X = b_0$. We let $\mathbb{T} := \mathbb{T}_X \cup \mathbb{T}_Y$. Let X and Y be $\mathbb{T}_X$ and $\mathbb{T}_Y$-valued standard processes with no positive jumps, respectively. We assume X (resp. Y) satisfying the following conditions:

(B1) $(x,y) \to \mathbb{E}^X_x\left[e^{-T_y}\right] > 0$ (resp. $(x,y) \to \mathbb{E}^Y_x\left[e^{-T_y}\right] > 0$) is a $\mathcal{B}(\mathbb{T}_X) \times \mathcal{B}(\mathbb{T}_X)$ (resp. $\mathcal{B}(\mathbb{T}_Y) \times \mathcal{B}(\mathbb{T}_Y)$)-measurable function.

(B2) We assume that $\lim_{y\uparrow x} \mathbb{E}^X_y\left[e^{-T_x}\right] = 1$ for all $x \in \mathbb{T}_X \cap (0,\infty)$ (resp. $\lim_{y\uparrow x} \mathbb{E}^Y_y\left[e^{-T_x}\right] = 1$ for all $x \in \mathbb{T}_Y \cap (-\infty, 0]$).

(B3) If $a_0 \notin \mathbb{T}_X$, we assume that $\lim_{x\uparrow a_0} \mathbb{E}^X_x\left[e^{-T_y^-}\right] = 0$ for all $y \in \mathbb{T}_X$ (resp. If $b_0 \notin \mathbb{T}_Y$, we assume that $\lim_{x\downarrow b_0} \mathbb{E}^Y_x\left[e^{-T_y^+}\right] = 0$ for all $y \in \mathbb{T}_Y$).

(B4) X (resp. Y) has a reference measure m_X on $\mathbb{T}_X$ (resp. m_Y on $\mathbb{T}_Y$).

We have the local times $\{L^{X,x}\}_{x\in\mathbb{T}_X}$ and $\{L^{Y,x}\}_{x\in\mathbb{T}_Y}$, the excursion measures $\{n^X_x\}_{x\in\mathbb{T}_X}$ and $\{n^Y_x\}_{x\in\mathbb{T}_Y}$, and the generalized scale functions $\{W^{(q)}_X\}_{q\geq 0}$ and $\{W^{(q)}_Y\}_{q\geq 0}$ of X and Y in the same way as those in Section 5.1, respectively.

Let $\psi : (0,\infty) \times (-\infty, 0) \to (-\infty, 0)$ be a measurable function satisfying

$$n^X_0\left[1 - e^{-T_0^-}\mathbb{E}^Y_{J_X}\left[e^{-T_0}\right]; 0 < T_0^- < T_0\right] < \infty, \tag{6.1}$$

where $J_X = \psi(X_{T_0^- -}, X_{T_0^-})$. We write X^0 and Y^0 for the stopped processes of X and Y upon hitting zero, respectively. Let $c_0 \geq 0$ be a constant. We define the law of stopped process $\mathbb{P}^{U^0}_x$ for $x \in \mathbb{T}\backslash\{0\}$ and the excursion measure n^U_0 away from 0 by

$$\mathbb{E}^{U^0}_x\left[F(U^0)\right] = \begin{cases} \mathbb{E}^{Y^0}_x[f(Y^0)], & x \in \mathbb{T}\cap(-\infty,0), \\ \mathbb{E}^X_x\left[\mathbb{E}^{Y^0}_{J_X}[F(w\circ Y^0)]\,|_{w=k_{T_0^-}X}; T_0^- \leq T_0\right], & x \in \mathbb{T}\cap(0,\infty), \end{cases} \tag{6.2}$$

$$\begin{aligned} n^U_0[F(U)] &= c_0 n^Y_0\left[F(Y); T_0^- = 0\right] \\ &\quad + n^X_0\left[\mathbb{E}^{Y^0}_{J_X}\left[F(w\circ Y^0)\right]|_{w=k_{T_0^-}X}; 0 < T_0^- \leq T_0\right] \end{aligned} \tag{6.3}$$

for all non-negative measurable functional F (if $\mathbb{P}^X_0[T_0 > 0] = 1$ or $\mathbb{P}^Y_0[T_0 > 0] = 1$, we assume that $c_0 = 0$).

Lemma 6.1. *The stochastic process $(U, \mathbb{P}^U_x)$ constructed by means of excursion theory from n^U_0 and $\{\mathbb{P}^{U^0}_x\}_{x\in\mathbb{T}\backslash\{0\}}$ without stagnancy at 0 is a $\mathbb{T}$-valued right continuous strong Markov process.*

We give the proof of Lemma 6.1 in Section 6.2.

Remark 6.2. The condition $c_0 = 0$ is necessary when $\mathbb{P}^X_0[T_0 > 0] = 1$. Indeed, when $\mathbb{P}^X_0[T_0 > 0] = 1$ and $c_0 > 0$, the measure n^U_0 does not satisfy the condition [23, pp.323, (vi')] and then n^U_0 is not an excursion measure.

We now prove standardness of U, or more strongly, Feller property. This property is used to define the generalized scale functions of U and to study the approximation theorem of U.

Theorem 6.3 ([16, Lemma 3.2]). *The refracted process U is a Feller process.*

Proof. Let $C_0(= C_0^{\mathbb{T}})$ denote the set of continuous functions f from $\mathbb{T}$ to $\mathbb{R}$ such that $f(x) \to 0$ as $x \downarrow b_0$ when $b_0 \notin \mathbb{T}$ and as $x \uparrow a_0$ when $a_0 \notin \mathbb{T}$. For $f \in C_0$, we write $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. It is sufficient to verify the following conditions:

(i) For all $q > 0$, $R_U^{(q)}$ is a map from C_0 to C_0.

(ii) For all $f \in C_0$, $\lim_{q\uparrow\infty} \left\| qR_U^{(q)} f - f \right\| = 0$.

1) **The proof of** (i)

First, we prove that $R_U^{(q)} f$ is continuous. We let $x \in \mathbb{T}$. By the construction of U and (B2), it is easy to check that $\lim_{y\uparrow x} \mathbb{E}_y^U \left[e^{-qT_x}\right] = \lim_{y\downarrow x} \mathbb{E}_x^U \left[e^{-qT_y}\right] = 1$. We fix $x \in \mathbb{T}$. For $y < x$, we have

$$\overline{\lim}_{y\uparrow x} \left| R_U^{(q)} f(x) - R_U^{(q)} f(y) \right| \tag{6.4}$$

$$\leq \overline{\lim}_{y\uparrow x} \left| R_U^{(q)} f(x) - \mathbb{E}_y^U \left[e^{-qT_x}\right] R_U^{(q)} f(x) \right| + \overline{\lim}_{y\uparrow x} \left| \mathbb{E}_y^U \left[\int_0^{T_x} e^{-qt} f(U_t) dt \right] \right| = 0. \tag{6.5}$$

For $y > x$, we have

$$\overline{\lim}_{y\downarrow x} \left| R_U^{(q)} f(x) - R_U^{(q)} f(y) \right| \tag{6.6}$$

$$\leq \overline{\lim}_{y\downarrow x} \left| \mathbb{E}_x^U \left[e^{-qT_y}\right] R_U^{(q)} f(y) - R_U^{(q)} f(y) \right| + \overline{\lim}_{y\downarrow x} \left| \mathbb{E}_x^U \left[\int_0^{T_y} e^{-qt} f(U_t) dt \right] \right| = 0. \tag{6.7}$$

Second, we prove that $\lim_{x\uparrow a_0} R_U^{(q)} f(x) = 0$ when $a_0 \notin \mathbb{T}$ and $\lim_{x\downarrow b_0} R_U^{(q)} f(x) = 0$ when $b_0 \notin \mathbb{T}$. We assume that $a_0 \notin \mathbb{T}$. By the assumption (B3), for all $x \in (0, a_0)$, $\lim_{y\uparrow a_0} \mathbb{E}_y^U \left[e^{-T_x^-} \right] = \lim_{y\uparrow a_0} \mathbb{E}_y^X \left[e^{-T_x^-} \right] = 0$. Since $f \in C_0$, for all $\epsilon > 0$, there exists $\delta \in (0, a_0)$ such that $\sup_{x\in(\delta,a_0)} |f(x)| < \epsilon$. So we have

$$\lim_{x\uparrow a_0} \left| R_U^{(q)} f(x) \right| \leq \lim_{x\uparrow a_0} \left(\mathbb{E}_x^X \left[\int_0^{T_\delta^-} e^{-qt} |f(X_t)| \, dt \right] + \mathbb{E}_x^U \left[\int_{T_\delta^-}^{\infty} e^{-qt} \|f\| \, dt \right] \right) \tag{6.8}$$

$$\leq \frac{\epsilon}{q} + \lim_{x\uparrow a_0} \mathbb{E}_x^X \left[e^{-qT_\delta^-} \right] \frac{\|f\|}{q} = \frac{\epsilon}{q}. \tag{6.9}$$

Therefore we have $\lim_{x\uparrow a_0} \left| R_U^{(q)} f(x) \right| = 0$. In the same way, we have $\lim_{x\downarrow b_0} \left| R_U^{(q)} f(x) \right| = 0$ when $b_0 \notin \mathbb{T}$.

2) **The proof of** (ii)

By classical arguments, it is sufficient to prove $\lim_{q\uparrow\infty} \left| qR_U^{(q)} f(x) - f(x) \right| = 0$ for $x \in \mathbb{T}$. Fix $x \in \mathbb{T}$. For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon, \qquad x, y \in \mathbb{T}. \tag{6.10}$$

We define

$$T_\delta^\uparrow = \inf\{t > 0 : |U_t - x| \geq \delta\}. \tag{6.11}$$

Then we have

$$\left|qR_U^{(q)} f(x) - f(x)\right| \leq q\mathbb{E}_x^U\left[\int_0^{T_\delta^\uparrow} e^{-qt}\left|f(U_t) - f(x)\right| dt\right] + q\mathbb{E}_x^U\left[\int_{T_\delta^\uparrow}^\infty e^{-qt}\left|f(U_t) - f(x)\right| dt\right] \tag{6.12}$$

$$\leq \epsilon\mathbb{E}_x^U\left[1 - e^{-qT_\delta^\uparrow}\right] + 2\left\|f\right\|\mathbb{E}_x^U\left[e^{-qT_\delta^\uparrow}\right]. \tag{6.13}$$

By the dominated convergence theorem, we have

$$\limsup_{q\uparrow\infty}\left|qR_U^{(q)} f(x) - f(x)\right| \leq \epsilon. \tag{6.14}$$

and so we have $\lim_{q\uparrow\infty}\left|qR_U^{(q)} f(x) - f(x)\right| = 0$. The proof is completed. □

6.2 Proof of Lemma 6.1

Let us prove Lemma 6.1. Our proof is based on [17, Appendix]. For this purpose, we use a result of Salisubury([23]), which gives the necessary and sufficient condition for a measure to be a excursion measure. For $t \geq 0$, we denote $\mathcal{D}_t = \sigma(\omega \mapsto \omega(s) : s \leq t)$.

Theorem 6.4 ([23, Theorem 2]). *Let $(Z^0, \mathbb{P}_x^{Z^0})$ be a $\mathbb{R}$-valued right-continuous strong Markov process stopped at* 0*. Suppose that a σ-finite measure n on $\mathbb{D}$ satisfies the following conditions:*

(i) *n is concentrated on $\mathbb{D}^0 := \{\omega \in \mathbb{D} : \omega(0) = 0, T_0(\omega) > 0, \omega(t) = 0$ for $t \geq T_0\}$.*

(ii) $n\left[1 - e^{-T_0}\right] < \infty$.

(iii) *For all $t > 0$, $A_1 \in \mathcal{D}_t$ with $A_1 \subset \{T_0 > t\}$ and $A_2 \in \mathcal{B}(\mathbb{D})$,*

$$n\left[A_1 \cap \theta_t^{-1}(A_2)\right] = \int_{A_1} \mathbb{P}_{\omega(t)}^{Z^0}\left[Z^0 \in A_2\right] n[d\omega], \tag{6.15}$$

where θ_t denotes the shift operator.

(iv) *If a measure n' on $\mathbb{D}$ satisfies $n \geq n' \geq 0$ and the counterpart of Condition* (iii) *for n', then n' satisfies the following:*

- *If n is a finite measure, then there exists $k \in [0, 1]$ such that $n' = kn$.*
- *If n is a infinite measure, then either $n'[\mathbb{D}^0] = 0$ or $n'[\mathbb{D}^0] = \infty$.*

Then there is a right-continuous strong Markov process Z for which n is an excursion measure away from 0 and $(Z^0, \mathbb{P}_x^{Z^0})$ is the stopped process.

To prove Theorem 6.1, we need to check that U^0 is a right-continuous strong Markov process and that n_0^U satisfies conditions of Theorem 6.4.

Lemma 6.5. *The stopped process $(U^0, \mathbb{P}_x^{U^0})$ has the Markov property.*

Proof. It is obvious that $(U^0, \mathbb{P}_x^{U^0}) = (Y^0, \mathbb{P}_x^0)$ for $x < 0$ satisfies the Markov property. We thus need to prove that $(U^0, \mathbb{P}_x^{U^0})$ satisfies the Markov property for $x > 0$. Let $A_1 \in \mathcal{D}_t$ with $A_1 \subset \{T_0 > t\}$ and $A_2 \in \mathcal{B}(\mathbb{D})$. We write $A = A_1 \cap \theta_t^{-1}(A_2)$. By the definition of $\mathbb{P}_x^{U^0}$, we have

$$\mathbb{P}_x^{U^0}\left[U^0 \in A\right] = \mathbb{E}_x^X\left[\mathbb{P}_y^{Y^0}\left[w \circ Y^0 \in A\right]\Big|_{\substack{y=J_X \\ w=(X(s))_{s<T_0^-}}} ; T_0^- \leq t\right] \tag{6.16}$$

$$+ \mathbb{E}_x^X\left[\mathbb{P}_y^{Y^0}\left[w \circ Y^0 \in A\right]\Big|_{\substack{y=J_X \\ w=(X(s))_{s<T_0^-}}} ; t > T_0^-\right], \tag{6.17}$$

where $w \circ w'$ denotes the concatenation of a path $w = (w_s)_{s<s_0}$ of finite length s_0 and a path $w' = (w'_s)_{s \geq 0}$ of infinite length:

$$(w \circ w')_s = \begin{cases} w_s & s < s_0, \\ w'_{s-s_0} & s \geq s_0. \end{cases} \tag{6.18}$$

By the Markov property of Y^0, we have

$$\mathbb{E}_x^X\left[\mathbb{P}_y^{Y^0}\left[w \circ Y^0 \in A\right]\Big|_{\substack{y=J_X \\ w=(X(s))_{s<T_0^-}}} ; T_0^- \leq t\right] \tag{6.19}$$

$$= \mathbb{E}_x^X\left[\mathbb{P}_y^{Y^0}\left[w \circ Y^0 \in A_1, \left[Y_s^0\right]_{s \geq t-u} \in A_2\right]\Big|_{\substack{y=J_X \\ w=(X(s))_{s<u} \\ u=T_0^-}} ; T_0^- \leq t\right] \tag{6.20}$$

$$= \mathbb{E}_x^X\left[\mathbb{E}_y^{Y^0}\left[1_{\{w \circ Y^0 \in A_1\}} \mathbb{P}_{y'}^{Y^0}\left[Y^0 \in A_2\right]\Big|_{y'=Y_{t-u}^0}\right]\Big|_{\substack{y=J_X \\ w=(X(s))_{s<u} \\ u=T_0^-}} ; T_0^- \leq t\right] \tag{6.21}$$

$$= \mathbb{E}_x^U\left[1_{\{U \in A_1\}} \mathbb{P}_y^{U^0}\left[U^0 \in A_2\right]\Big|_{y=U_t} ; T_0^- \leq t\right]. \tag{6.22}$$

We can do a similar argument for (6.17). So we obtain

$$\mathbb{P}_x^{U^0}\left[U^0 \in A\right] = \int_{A_1} \mathbb{P}_{\omega(t)}^{U^0}\left[U^0 \in A_2\right] \mathbb{P}_x^{U^0}\left[U^0 \in d\omega\right]. \tag{6.23}$$

The proof is complete. □

Lemma 6.6. *The stopped process U^0 has the strong Markov property.*

Proof. Fix $t > 0$. By the proof of [6, Theorem 1 of Section 2.3], it is sufficient to prove that $x \mapsto \mathbb{E}_x^{U^0}[f(U_t^0)]$ is continuous for all bounded continuous function f with $f(0) = 0$. Continuity at $x < 0$ is obvious, by the Feller property of Y^0. Left-continuity at $x = 0$ is also obvious. Right-continuity at $x = 0$ follows from the fact that $\mathbb{P}_y^{U^0}[T_0 \in \cdot] \underset{y \to 0}{\to} \delta_0$. Let us consider continuity at $x > 0$.

$$\mathbb{E}_y^{U^0}\left[f(U_t^0)\right] = \mathbb{E}_y^{U^0}\left[f(U_t^0); T_0^- \wedge t < T_x\right] + \mathbb{E}_y^X\left[f(X_t); T_x \leq t < T_0^-\right] \tag{6.24}$$

$$+ \mathbb{E}_y^X\left[\mathbb{E}_{y'}^{Y^0}\left[f(Y_{t-u}^0)\right]\Big|_{\substack{y'=J_X \\ u=T_0^-}}; T_x \leq T_0^- \leq t\right]. \tag{6.25}$$

Note that we have $\mathbb{P}_0^X[\lim_{y \to 0} T_y = 0] = 1$ by the assumption that X is spectrally negative and of bounded variation. Since X and Y^0 have càdlàg paths, we have the following identities:

$$\mathbb{E}_y^{U^0}\left[f(U_t^0); T_0^- \wedge t < T_x\right] \leq \|f\| \, \mathbb{P}_0^X\left[T_{-\frac{x}{2}}^- < T_{x-y}\right] \underset{y \to x}{\to} 0, \tag{6.26}$$

$$\mathbb{E}_y^X\left[f(X_t); T_x \leq t < T_0^-\right] = \mathbb{E}_y^X\left[\mathbb{E}_x^X\left[f(X_{t-u}); t < T_0^-\right]\Big|_{u=T_x}; T_x \leq t \wedge T_0^-\right] \tag{6.27}$$

$$\underset{y \to x}{\to} \mathbb{E}_x^X\left[f(X_t); t < T_0^-\right], \tag{6.28}$$

$$\mathbb{E}_y^X\left[\mathbb{E}_{y'}^{Y^0}\left[f(Y_{t-u}^0)\right]\Big|_{\substack{y'=J_X \\ u=T_0^-}}; T_x \leq T_0^- \leq t\right] \tag{6.29}$$

$$= \mathbb{E}_y^X\left[\mathbb{E}_x^X\left[\mathbb{E}_{y'}^{Y^0}\left[f(Y_{t-u-v}^0)\right]\Big|_{\substack{y'=J_X \\ u=T_0^-}}; T_0^- \leq t\right]\Big|_{v=T_x}; T_x \leq T_0^- \wedge t\right] \tag{6.30}$$

$$\underset{y \to x}{\to} \mathbb{E}_x^X\left[\mathbb{E}_{y'}^{Y^0}\left[f(Y_{t-u}^0)\right]\Big|_{\substack{y'=J_X \\ u=T_0^-}}; T_0^- \leq t\right]. \tag{6.31}$$

The proof is now complete. □

We have already proved the strong Markov property of U^0. So in the following theorem, we check the other conditions of Theorem 6.4.

Lemma 6.7. *The measure $n = n_0^U$ satisfies Conditions* (i), (ii), (iii) *and* (iv) *of Theorem 6.4.*

Proof. It is obvious by definition and (6.1) that n_0^U satisfies (i) and (ii).

The proof of (iii) is the same as that of the Markov property of $(U^0, \mathbb{P}_x^{U^0})$ for $x > 0$ in Lemma 6.5.

Let us prove (iv). First, we prove in the case that X has unbounded variation paths. We define the σ-finite measure n'^{ϵ} by

$$n'^{\epsilon}[F(U^{\epsilon})] = n'\left[\mathbb{E}_{\epsilon}^{X^0}\left[F(\omega \circ X^0)\right]|_{\omega=\{U_t\}_{t\in[0,T_{\epsilon}^+)}}; T_{\epsilon}^+ < \infty\right], \tag{6.32}$$

for all non-negative measurable functional F. Then for $0 < \delta < \epsilon$ and non-negative measurable functional F, we have

$$\begin{aligned} n'^{\delta}\left[F(U^{\delta})\right] =& n'^{\epsilon}[F(U^{\epsilon})] + n'\left[\mathbb{E}_{\delta}^{X^0}\left[F(\omega \circ X^0); T_{\epsilon}^+ = \infty\right]|_{\omega=\{U_t\}_{t\in[0,T_{\delta}^+)}}; T_{\delta}^+ < \infty\right] & (6.33)\\ \geq& n'^{\epsilon}[F(U^{\epsilon})]. & (6.34)\end{aligned}$$

So we can define a measure n'^0 by $n'^0 = \lim_{\epsilon\downarrow 0} n'^{\epsilon}$ as the increasing limit. Then n'^0 satisfies the Markov property for $\{\mathbb{P}_x^{X^0}\}_{x\in\mathbb{R}\setminus\{0\}}$ and for non-negative measurable functional F, we have

$$n'^0\left[F(\{U_t^0\}_{t<T_0^-})\right] = n'\left[F(\{U_t\}_{t<T_0^-})\right]. \tag{6.35}$$

By the definition of n_0^U, we have $n_0^X \geq n'^0 \geq 0$. By [23, Proposition 1], n_0^X satisfies Condition (iv) and $n'^0\left[T_0^- > 0\right]$ is equal to either 0 or ∞. By (6.35), $n'\left[T_0^- > 0\right]$ is equal to either 0 or ∞. If $n'\left[T_0^- = 0\right] > 0$, then $n'|_{\{T_0^-=0\}}$ satisfies the Markov property for Y^0 and $n_0^Y = c_0 n|_{\{T_0^-=0\}} \geq c_0 n'|_{\{T_0^-=0\}} \geq 0$, so by [23, Proposition 1], $n'\left[T_0^- = 0\right]$ is equal to either 0 or ∞. From the above, we obtain either $n'[\mathbb{D}_0] = 0$ or $n'[\mathbb{D}_0] = \infty$.

Second, we prove in the case that X has bounded variation paths. We construct a measure n'^0 in the same way as that in the unbounded variation case. Then by [23, Proposition 1], there exists $k \in [0,1]$ such that $n'^0 = k n_0^X$. By (6.35) and the definition of n_0^U, we obtain $n' = k n_0^U$.

The proof is completed. □

7 Duality problem of refracted processes

We deal with the duality problem of refracted processes. This section follows [16, Section 6 and Section 7].

7.1 Duality problem of refracted processes

In this section, we obtain the necessary and sufficient condition that the refracted processes U and $\widehat{U}$ are in duality in terms of an identity involving excursion measures and landing functions.

We assume that $\mathbb{T}$ is an open set. Let X and Y be recurrent standard processes which are same as those in Section 6. We assume that 0 is irregular for itself for X and Y or 0 is regular for itself for X and Y. Let $\widehat{X}$ and $\widehat{Y}$ be $\mathbb{T}_X$-valued and $\mathbb{T}_Y$-valued standard processes with no negative jumps which satisfy the following conditions:

(B̂1) $(x,y) \to \mathbb{E}_x^{\widehat{X}}\left[e^{-T_y}\right] > 0$ (resp. $(x,y) \to \mathbb{E}_x^{\widehat{Y}}\left[e^{-T_y}\right] > 0$) is a $\mathcal{B}(\mathbb{T}_X) \times \mathcal{B}(\mathbb{T}_X)$ (resp. $\mathcal{B}(\mathbb{T}_Y) \times \mathcal{B}(\mathbb{T}_Y)$)-measurable function.

(B̂2) We assume that $\lim_{y\downarrow x} \mathbb{E}_y^{\widehat{X}}\left[e^{-T_x}\right] = 1$ for all $x \in \mathbb{T}_X \cap [0,\infty)$ (resp. $\lim_{y\downarrow x} \mathbb{E}_y^{\widehat{Y}}\left[e^{-T_x}\right] = 1$ for all $x \in \mathbb{T}_Y \cap (-\infty, 0)$).

(B̂3) We assume that $\lim_{x\uparrow a_0} \mathbb{E}_x^{\widehat{X}}\left[e^{-T_y^-}\right] = 0$ for all $y \in \mathbb{T}_X$ (resp. We assume that $\lim_{x\downarrow b_0} \mathbb{E}_x^{\widehat{Y}}\left[e^{-T_y^+}\right] = 0$ for all $y \in \mathbb{T}_Y$).

(B̂4) $\widehat{X}$ (resp. $\widehat{Y}$) has a reference measure m_X on $\mathbb{T}_X$ (resp. m_Y on $\mathbb{T}_Y$).

In addition we assume the following conditions:

- $X_0 = X_{T_0-} = 0$, n_0^X-a.s. $Y_0 = Y_{T_0-} = 0$, n_0^Y-a.s.
- X and $\widehat{X}$(resp. Y and $\widehat{Y}$) are in duality relative to m_X (resp. m_Y).

We take the local times $\{L^{X,x}\}_{x\in\mathbb{T}_X}$, $\{L^{Y,x}\}_{x\in\mathbb{T}_Y}$, $\{L^{\widehat{X},x}\}_{x\in\mathbb{T}_X}$ and $\{L^{\widehat{Y},x}\}_{x\in\mathbb{T}_Y}$, the excursion measures $\{n_x^X\}_{x\in\mathbb{T}_X}$, $\{n_x^Y\}_{x\in\mathbb{T}_Y}$, $\{n_x^{\widehat{X}}\}_{x\in\mathbb{T}_X}$ and $\{n_x^{\widehat{Y}}\}_{x\in\mathbb{T}_Y}$, and the generalized scale functions $\{W_X^{(q)}\}_{q\geq 0}$, $\{W_Y^{(q)}\}_{q\geq 0}$, $\{W_{-\widehat{X}}^{(q)}\}_{q\geq 0}$, and $\{W_{-\widehat{Y}}^{(q)}\}_{q\geq 0}$ as those in Section 5.2. As the landing functions, let $\psi : (0,\infty) \times (-\infty,0) \to (-\infty,0)$ be a measurable function satisfying (6.1) and $\phi : (-\infty,0) \times (0,\infty) \to (0,\infty)$ be a measurable function satisfying

$$n_0^Y\left[1 - e^{-T_0^+}\mathbb{E}^X_{\phi(Y_{T_0^+-}, Y_{T_0^+})}\left[e^{-T_0}\right]; 0 < T_0^+ < T_0\right] < \infty. \tag{7.1}$$

Let $\mathbb{P}_x^{U^0}$ and n_0^U be those in Section 6. By the excursion theory, we can construct a $\mathbb{T}$-valued right continuous strong Markov processes U from n_0^U and $\{\mathbb{P}_x^{U^0}\}_{x\in\mathbb{T}\setminus\{0\}}$. Let $\widehat{c}_0 \geq 0$ and $\widehat{c}_1 > 0$ be constants. We define the law of stopped process $\mathbb{P}_x^{\widehat{U}^0}$ for $x \neq 0$ and an excursion measure $n_0^{\widehat{U}}$ away from 0 by the following identities:

$$\mathbb{P}_x^{\widehat{U}^0}\left[F(\widehat{U}^0)\right] = \begin{cases} \mathbb{E}_x^{\widehat{X}^0}\left[F(\widehat{X}^0)\right], & x > 0, \\ \mathbb{E}_x^{\widehat{Y}}\left[\mathbb{E}^{\widehat{X}^0}_{\phi(\widehat{Y}_{T_0^+-}, \widehat{Y}_{T_0^+})}\left[F(\omega \circ \widehat{X}^0)\right]|_{w=k_{T_0^+}\widehat{Y}}; T_0^+ \leq T_0\right], & x < 0, \end{cases} \tag{7.2}$$

$$\begin{aligned} n_0^{\widehat{U}}\left[F(\widehat{U})\right] &= \widehat{c}_0 n_0^{\widehat{X}}\left[F(\widehat{X}); T_0^+ = 0\right] \\ &\quad + \widehat{c}_1 n_0^{\widehat{Y}}\left[\mathbb{E}^{\widehat{X}^0}_{\phi(\widehat{Y}_{T_0^+-}, \widehat{Y}_{T_0^+})}\left[F(\omega \circ \widehat{X}^0)\right]|_{w=k_{T_0^+}\widehat{Y}}; 0 < T_0^+ \leq T_0\right] \end{aligned} \tag{7.3}$$

for all positive measurable functional F. By the excursion theory, we can construct a $\mathbb{T}$-valued right continuous strong Markov processes $\widehat{U}$ from $n_0^{\widehat{U}}$ together with $\{\mathbb{P}_x^{\widehat{U}}\}_{x\in\mathbb{T}\setminus\{0\}}$.

We may and do assume $c_0 = \widehat{c}_0 = \widehat{c}_1 = 1$ without loss of generality. Let us explain the reason. We discuss positivity of c_0. By Lemma 2.4, the excursion measures n_0^U and $n_0^{\widehat{U}}$ need to satisfy $n_0^U[\ \cdot\ ] = c_2 n_0^{\widehat{U}}[\rho_x(\ \cdot\ )]$ for some constant $c_2 > 0$.

This means that $n_0^U\left[\ \cdot\ ; T_0^- = 0\right] = c_0 n_0^Y\left[\ \cdot\ ; T_0^- = 0\right] = c_2\widehat{c}_1 n_0^{\widehat{Y}}\left[\rho_x(\ \cdot\ ); T_0^+ = T_0\right] = c_2 n_0^{\widehat{U}}\left[\rho_x(\ \cdot\ ); T_0^+ = T_0\right]$. So c_0 needs to be equal to $c_2\widehat{c}_1$ unless $n_0^{\widehat{Y}}\left[\rho_x(\ \cdot\ ); T_0^+ = T_0\right]$ is the zero measure. When $n_0^{\widehat{Y}}\left[\rho_x(\ \cdot\ ); T_0^+ = T_0\right]$ is the zero measure, so is $n_0^Y\left[\ \cdot\ ; T_0^- = 0\right]$ by Lemma 2.4, which allows us to take $c_0 > 0$. For the same reason, we may assume that $1 = c_2\widehat{c}_0$. By changing the normalization of m_Y, n_0^Y and $n_0^{\widehat{U}}$, we may assume $c_0 = c_2 = 1$ without loss of generality, which yields $\widehat{c}_0 = \widehat{c}_1 = 1$.

Furthermore, when $n_0^X\left[\ \cdot\ ; T_0^- = T_0\right]$ and $n_0^Y\left[\ \cdot\ ; T_0^- = 0\right]$ are the zero measures, we fix $\kappa \in \mathbb{T}_X \cap (0,\infty)$ and we change the normalization of m_Y, n_0^Y and $n_0^{\widehat{U}}$ to satisfy $n_0^U\left[\sup_{t\in(0,\zeta)} U_t > \kappa\right] = n_0^{\widehat{U}}\left[\sup_{t\in(0,\zeta)} \widehat{U} > \kappa\right]$.

We define $m_U = m_X|_{[0,\infty)} + m_Y|_{(-\infty,0)}$. The following theorem gives an identity which characterizes the duality.

Theorem 7.1 ([16, Theorem 6.1])**.** *If n_0^X, n_0^Y, ψ and ϕ satisfy*

$$n_0^X\left[h(X_{T_0^- -}, \psi(X_{T_0^- -}, X_{T_0^-})); 0 < T_0^- < T_0\right] = n_0^Y\left[h(\phi(Y_{T_0^-}, Y_{T_0^- -}), Y_{T_0^-}); 0 < T_0^- < T_0\right] \tag{7.4}$$

for all non-negative measurable function h, or equivalently,

$$n_0^{\widehat{X}}\left[h(\widehat{X}_{T_0^+}, \psi(\widehat{X}_{T_0^+}, \widehat{X}_{T_0^+ -})); 0 < T_0^+ < T_0\right] = n_0^{\widehat{Y}}\left[h(\phi(\widehat{Y}_{T_0^+ -}, \widehat{Y}_{T_0^+}), \widehat{Y}_{T_0^+ -}); 0 < T_0^+ < T_0\right] \tag{7.5}$$

for all h, then U and $\widehat{U}$ are in duality relative to m_U. The converse is also true.

To prove Theorem 7.1, we need the following lemma about the time reversality.

Lemma 7.2 ([16, Theorem 6.2])**.** *If* (7.4) *is true, then we have*

$$n_0^U[\ \cdot\ ] \overset{d}{=} n_0^{\widehat{U}}[\rho_0(\ \cdot\ )]\,. \tag{7.6}$$

Proof. By (6.3) and Lemma 2.4, for non-negative measurable functional F, we have

$$\begin{aligned}
n_0^U[F(U)] &= n_0^X\left[F(X); T_0^- = T_0\right] \\
&\quad + n_0^X\left[\mathbb{E}^{Y^0}_{\psi(X_{T_0^- -}, X_{T_0^-})}\left[F(\omega \circ Y^0)\right]|_{w = k_{T_0^-}X}; 0 < T_0^- < T_0\right] \\
&\quad + n_0^Y\left[F(Y); T_0^- = 0\right] \qquad\qquad (7.7)\\
&= n_0^{\widehat{X}}\left[F(\rho_0\widehat{X}); T_0^+ = 0\right] \\
&\quad + n_0^{\widehat{X}}\left[\mathbb{E}^{Y^0}_{\psi(\widehat{X}_{T_0^+}, \widehat{X}_{T_0^+ -})}\left[F(\omega \circ Y^0)\right]|_{w = k_{T_0}\rho_0\theta_{T_0^+}\widehat{X}_t}; 0 < T_0^+ < T_0\right] \\
&\quad + n_0^{\widehat{Y}}\left[F(\rho_0\widehat{Y}); T_0^+ = T_0\right] \qquad\qquad (7.8)
\end{aligned}$$

By (7.5), Lemma 2.4 and Fubini's theorem, we have

$$\begin{aligned}
& n_0^{\widehat{X}}\left[\mathbb{E}^{Y^0}_{\psi(\widehat{X}_{T_0^+},\widehat{X}_{T_0^+-})}\left[F(\omega\circ Y^0)\right]\Big|_{w=k_{T_0}\rho_0\theta_{T_0^+}\widehat{X}_t};0<T_0^+<T_0\right] \\
&= n_0^{\widehat{X}}\left[\int \mathbb{P}^{\widehat{X}^0}_{\widehat{X}_{T_0^+}}\left[\widehat{X}^0\in d\omega\right]\mathbb{E}^{Y^0}_{\psi(\widehat{X}_{T_0^+},\widehat{X}_{T_0^+-})}\left[F(k_{T_0}\rho_0\omega\circ Y^0)\right];0<T_0^+<T_0\right] && (7.9)\\
&= n_0^{\widehat{Y}}\left[\int \mathbb{P}^{\widehat{X}^0}_{\phi(\widehat{Y}_{T_0^+-},\widehat{Y}_{T_0^+})}\left[\widehat{X}^0\in d\omega\right]\mathbb{E}^{Y^0}_{\widehat{Y}_{T_0^+-}}\left[F(k_{T_0}\rho_0\omega\circ Y^0)\right];0<T_0^+<T_0\right] && (7.10)\\
&= n_0^{\widehat{Y}}\left[\mathbb{E}^{Y^0}_{\widehat{Y}_{T_0^+-}}\left[\int F(\omega\circ Y^0)\mathbb{P}^{\widehat{X}^0}_y\left[k_{T_0}\rho_0\widehat{X}\in d\omega\right]\right]\Big|_{y=\phi(\widehat{Y}_{T_0^+-},\widehat{Y}_{T_0^+})};0<T_0^+<T_0\right]. && (7.11)
\end{aligned}$$

By the strong Markov property and Lemma 2.4, we have

$$\begin{aligned}
(7.11) &= n_0^{Y}\left[\mathbb{E}^{Y^0}_{Y_{T_0^-}}\left[\int F(\omega\circ Y^0)\mathbb{P}^{\widehat{X}^0}_y\left[k_{T_0}\rho_0\widehat{X}\in d\omega\right]\right]\Big|_{y=\phi(Y_{T_0^-},Y_{T_0^--})};0<T_0^-<T_0\right] \\
& && (7.12)\\
&= n_0^{Y}\left[\int F(\omega\circ\theta_{T_0^-}Y)\mathbb{P}^{\widehat{X}^0}_{\phi(Y_{T_0^-},Y_{T_0^--})}\left[k_{T_0}\rho_0\widehat{X}\in d\omega\right];0<T_0^-<T_0\right] && (7.13)\\
&= n_0^{Y}\left[\mathbb{E}^{\widehat{X}^0}_{\phi(Y_{T_0^-},Y_{T_0^--})}\left[F(k_{T_0}\rho_0\widehat{X}\circ\omega')\right]\Big|_{\omega'=\theta_{T_0^-}Y};0<T_0^-<T_0\right] && (7.14)\\
&= n_0^{\widehat{Y}}\left[\mathbb{E}^{\widehat{X}^0}_{\phi(\widehat{Y}_{T_0^+-},\widehat{Y}_{T_0^+})}\left[F(\rho_0(\omega\circ\widehat{X}^0))\right]\Big|_{\omega=k_{T_0^+}\widehat{Y}};0<T_0^+<T_0\right]. && (7.15)
\end{aligned}$$

By (7.15), we have

$$\begin{aligned}
(7.8) &= n_0^{\widehat{Y}}\left[F(\rho_0\widehat{Y});T_0^+=T_0\right] \\
&\quad + n_0^{\widehat{Y}}\left[\mathbb{E}^{\widehat{X}^0}_{\phi(\widehat{Y}_{T_0^+-},\widehat{Y}_{T_0^+})}\left[F(\rho_0(\omega\circ\widehat{X}^0))\right]\Big|_{\omega=k_{T_0^+}\widehat{Y}};0<T_0^+<T_0\right] \\
&\quad + n_0^{\widehat{X}}\left[F(\rho_0\widehat{X});T_0^+=0\right] && (7.16)\\
&= n_0^{\widehat{U}}\left[F(\rho_0\widehat{U})\right] && (7.17)
\end{aligned}$$

So we obtain (7.6). □

To study the duality of U, we need to know a reference measure of U. The following lemma proves that m_U is a reference measure of U.

Lemma 7.3 ([16, Lemma 6.3]). *For all $q>0$ and $x\in\mathbb{T}$, the measure $R_U^{(q)}1_{(\cdot)}(x)$ is absolutely continuous with respect to $m_U(\cdot)$.*

Proof. Let A be a set in $\mathcal{B}(\mathbb{T})$ which satisfies $m_X(A\cap[0,\infty))=0$ and $m_Y(A\cap(-\infty,0))=0$. It is sufficient to prove that $\mathbb{E}_0^U\left[\int_0^\infty e^{-qt}1_A(U_t)dt\right]=0$. By the compensation theorem of excursion point processes, we have

$$q n_0^U \left[1 - e^{-qT_0}\right] \mathbb{E}_0^U \left[\int_0^\infty e^{-qt} 1_A(U_t) dt\right] \tag{7.18}$$

$$= n_0^U \left[\int_0^{T_0} e^{-qt} 1_A(U_t) dt\right] \tag{7.19}$$

$$= n_0^X \left[\int_0^{T_0^-} e^{-qt} 1_A(X_t) dt\right] + n_0^X \left[\mathbb{E}_{J_X}^Y \left[\int_0^{T_0^+} e^{-qt} 1_A(Y_t) dt\right]\right] + n_0^Y \left[\int_0^{T_0} e^{-qt} 1_A(Y_t) dt; T_0^- = 0\right]. \tag{7.20}$$

By the assumption of A, we have

$$n_0^X \left[\int_0^{T_0^-} e^{-qt} 1_A(X_t) dt\right] = q n_0^X \left[1 - e^{-qT_0}\right] \mathbb{E}_0^X \left[\int_0^\infty e^{-qt} 1_{A\cap[0,\infty)}(X_t) dt\right] = 0, \tag{7.21}$$

$$\mathbb{E}_{J_X}^Y \left[\int_0^{T_0^+} e^{-qt} 1_A(Y_t) dt\right] \le \mathbb{E}_{J_X}^Y \left[\int_0^\infty e^{-qt} 1_{A\cap(-\infty,0)}(Y_t) dt\right] = 0. \tag{7.22}$$

and

$$n_0^Y \left[\int_0^{T_0} e^{-qt} 1_A(Y_t) dt; T_0^- = 0\right] \le q n_0^Y \left[1 - e^{-qT_0}\right] \mathbb{E}_0^Y \left[\int_0^\infty e^{-qt} 1_{A\cap(-\infty,0)}(Y_t) dt\right] = 0 \tag{7.23}$$

So we obtain $\mathbb{E}_0^U \left[\int_0^\infty e^{-qt} 1_A(U_t) dt\right] = 0$. □

By the same argument as the proof of Lemma 7.3, we can prove that m_U is a reference measure of $\widehat{U}$.

To prove Theorem 7.1, we use the generalized scale functions of U. So we want to find suitable normalization of local times of U. By [7, Theorem 18.4], we let local times $\{L^{U,x\prime}\}_{x\in\mathbb{T}\setminus\{0\}}$ of U be those in Section 2. We set $n_0^{U\prime} = n_0^U$ and let $n_x^{U\prime}$ for $x \in \mathbb{T}\setminus\{0\}$ be the excursion measure associated to $L^{U,x\prime}$. Then there exists the positive function $c(x)$ such that $c(0) = 1$ (by the definition of U^0) and for all non-negative functional F:

$$n_x^{U\prime}\left[F(\{U_t\}_{t<T_0^-})\right] = c(x) n_x^X \left[F(\{X_t\}_{t<T_0^-})\right], \quad x \in \mathbb{T}\cap[0,\infty), \tag{7.24}$$

$$n_x^{U\prime}\left[F(\{U_t\}_{t<T_0^+})\right] = c(x) n_x^Y \left[F(\{Y_t\}_{t<T_0^+})\right], \quad x \in \mathbb{T}\cap(-\infty,0]. \tag{7.25}$$

Then we have $c(x) = 1$ m_U-a.e. Indeed, for all $q > 0$, $x, y \in \mathbb{T}\cap[0,\infty)$ and non-negative measurable function f, we have

$$\mathbb{E}_x^U \left[\int_0^{T_0^-} e^{-qt} dL_t^{U,y\prime}\right] = \frac{1}{c(y)} \mathbb{E}_x^X \left[\int_0^{T_0^-} e^{-qt} dL_t^{X,y}\right] \tag{7.26}$$

and

$$\int_{\mathbb{T}\cap[0,\infty)} f(y)\mathbb{E}^U_x\left[\int_0^{T_0^-} e^{-qt}dL^{U,y\prime}_t\right] m_U(dy) = \mathbb{E}^U_x\left[\int_0^{T_0^-} e^{-qt}f(U_t)dt\right] \tag{7.27}$$

$$= \mathbb{E}^X_x\left[\int_0^{T_0^-} e^{-qt}f(X_t)dt\right] \tag{7.28}$$

$$= \int_{\mathbb{T}\cap[0,\infty)} f(y)\mathbb{E}^X_x\left[\int_0^{T_0^-} e^{-qt}dL^{X,y}_t\right] m_U(dy). \tag{7.29}$$

So $c(x) = 1$ on $\mathbb{T}\cap[0,\infty)$ m_U-a.e. Similarly, $c(x) = 1$ on $\mathbb{T}\cap(-\infty,0)$ m_U-a.e. We now set $L^{U,x} = c(x)L^{U,x,\prime}$ and $n^U_x = \frac{1}{c(x)}n^{U,\prime}_x$. This local times satisfy (2.9) and (2.10) since $c(x) = 1$ m_U-a.e.

In the same way, let the excursion measures $\{n^{\widehat{U}}_x\}_{x\in\mathbb{T}}$ of $\widehat{U}$ be those in Section 2 satisfying the following conditions;

$$n^{\widehat{U}}_x\left[F(\{\widehat{U}_t\}_{t<T_0^-})\right] = n^{\widehat{X}}_x\left[F(\{\widehat{X}_t\}_{t<T_0^-})\right], \quad x\in\mathbb{T}\cap[0,\infty), \tag{7.30}$$

$$n^{\widehat{U}}_x\left[F(\{\widehat{U}_t\}_{t<T_0^+})\right] = n^{\widehat{Y}}_x\left[F(\{\widehat{Y}_t\}_{t<T_0^+})\right], \quad x\in\mathbb{T}\cap(-\infty,0]. \tag{7.31}$$

We let the scale functions $\left\{W^{(q)}_U\right\}_{q\geq 0}$ and $\left\{W^{(q)}_{-\widehat{U}}\right\}_{q\geq 0}$ be those in (5.1).

Proof of Theorem 7.1. Let us assume that we have (7.4) for all non-negative measurable function h. By Theorem 5.8 and Lemma 7.3, it is sufficient to prove that

$$W^{(q)}_U(x,y) = W^{(q)}_{-\widehat{U}}(-y,-x), \tag{7.32}$$

for $q \geq 0$ and $x,y\in\mathbb{T}$. For $0\leq y<x$, we have

$$W^{(q)}_U(x,y) = n^U_y\left[e^{-qT^+_x}; T^+_x<\infty\right]^{-1} = n^X_y\left[e^{-qT^+_x}; T^+_x<\infty\right]^{-1} = W^{(q)}_X(x,y) \tag{7.33}$$

$$= W^{(q)}_{-\widehat{X}}(-y,-x) = n^{-\widehat{X}}_{-x}\left[e^{-qT^+_{-y}}; T^+_{-y}<\infty\right]^{-1} = n^{-\widehat{U}}_{-x}\left[e^{-qT^+_{-y}}; T^+_{-y}<\infty\right]^{-1} = W^{(q)}_{-\widehat{U}}(-y,-x) \tag{7.34}$$

by the definitions of n^U_y, $n^{-\widehat{U}}_{-x}$ and Theorem 5.8. Similarly, for $y<x\leq 0$, we have

$$W^{(q)}_U(x,y) = W^{(q)}_Y(x,y) = W^{(q)}_{-\widehat{Y}}(-y,-x) = W^{(q)}_{-\widehat{U}}(-y,-x). \tag{7.35}$$

When $y<0<x$, by (5.7), (5.1) and (5.4), we have

$$W^{(q)}_U(x,y) = W^{(q)}_U(0,y)W^{(q)}_U(x,0)n^U_0\left[1-e^{-qT_0}1_{\{T^-_y=\infty,T^+_x=\infty\}}\right] \tag{7.36}$$

and

$$W^{(q)}_{-\widehat{U}}(-y,-x) = W^{(q)}_{-\widehat{U}}(-y,0)W^{(q)}_{-\widehat{U}}(0,-x)n_0^{-\widehat{U}}\left[1-e^{-qT_0}1_{\{T_y^-=\infty,T_x^+=\infty\}}\right]. \tag{7.37}$$

By Lemma 7.2, (7.34), (7.35), (7.36) and (7.37), we obtain (7.32).

We assume that U and $\widehat{U}$ are in duality relative to m_U. By Lemma 2.4 and the definitions of n_0^U and $n_0^{\widehat{U}}$, we have (7.6). We have

$$n_0^X\left[h(X_{T_0^- -},\psi(X_{T_0^- -},X_{T_0^-}));0<T_0^-<T_0\right] = n_0^U\left[h(U_{T_0^- -},U_{T_0^-});0<T_0^-<T_0\right] \tag{7.38}$$

and

$$n_0^Y\left[h(\phi(Y_{T_0^-},Y_{T_0^- -}),Y_{T_0^-});0<T_0^-<T_0\right] = n_0^{\widehat{Y}}\left[h(\phi(\widehat{Y}_{T_0^+ -},Y_{T_0^+}),\widehat{Y}_{T_0^+ -});0<T_0^+<T_0\right] \tag{7.39}$$

$$= n_0^{\widehat{U}}\left[h(\widehat{U}_{T_0^+},\widehat{U}_{T_0^+ -});0<T_0^+<T_0\right]. \tag{7.40}$$

By (7.6), (7.38) and (7.40), we obtain (7.4). The proof is completed. □

7.2 An example of the duality problem

In this section, we construct refracted processes in duality from spectrally negative stable processes.

Let X be a spectrally negative strictly α_1-stable process whose Lévy measure is

$$\Pi_X(dx) = c_X 1_{\{x<0\}}|x|^{-\alpha_1-1}dx \tag{7.41}$$

for a constant $c_X > 0$, and Y be a spectrally negative strictly α_2-stable process whose Lévy measure is

$$\Pi_Y(dx) = c_Y 1_{\{x<0\}}|x|^{-\alpha_2-1}dx \tag{7.42}$$

where $c_Y > 0$. Then it is known that

$$\widehat{X} \text{ (under } \mathbb{P}_x^{\widehat{X}}) \overset{d}{=} -X \text{ (under } \mathbb{P}_{-x}^X) \tag{7.43}$$

and

$$\widehat{Y} \text{ (under } \mathbb{P}_x^{\widehat{Y}}) \overset{d}{=} -Y \text{ (under } \mathbb{P}_{-x}^Y). \tag{7.44}$$

We set reference measure $m_X(dx)$ as $\frac{\alpha_1-1}{c_X}dx$ and reference measure $m_Y(dx)$ as $\frac{\alpha_2-1}{c_Y}dx$. Let n_0^X and n_0^Y be those in Section 7.1. We want to find suitable landing functions such that U and $\widehat{U}$ are in duality. So we need to find ψ and ϕ satisfying (7.4).

Proposition 7.4 ([16, Proposition 7.1])**.** *Suppose $\alpha_1 > \alpha_2$. We let $\psi(x,y) = y(x-y)^{\frac{\alpha_1-1}{\alpha_2-1}-1}$ and $\phi(x,y) = y(y-x)^{\frac{\alpha_2-1}{\alpha_1-1}-1}$. Then U constructed from X, Y, ψ and $c_0 = 0$ and $\widehat{U}$ constructed from $\widehat{X}$, $\widehat{Y}$, $\widehat{\psi}$ and $c_0 = 0$ are well-defined and in duality relative to m_U.*

Proof. Let us prove (7.4). By Theorem 3.4, we have

$$
\begin{aligned}
&n_0^X\Big[h(X_{T_0^- -}, \psi(X_{T_0^- -}, X_{T_0^-})); 0 < T_0^- < T_0\Big] \\
&= \frac{\alpha_1 - 1}{c_X}\int_0^\infty dv \int_{(-\infty,0)} h(v, \psi(v,u))\Pi_X(du - v) &(7.45)\\
&= (\alpha_1 - 1)\int_0^\infty dv \int_0^\infty h(v, \psi(v,-u))(u+v)^{-1-\alpha_1}du &(7.46)
\end{aligned}
$$

and

$$
\begin{aligned}
&n_0^Y\Big[h(\phi(Y_{T_0^-}, Y_{T_0^- -}), Y_{T_0^-}); 0 < T_0^- < T_0\Big] \\
&= \frac{\alpha_2 - 1}{c_Y}\int_0^\infty dv \int_{(-\infty,0)} h(\phi(u,v),u)\Pi_Y(du - v) &(7.47)\\
&= (\alpha_2 - 1)\int_0^\infty dv \int_0^\infty h(\phi(-u,v),-u)(u+v)^{-1-\alpha_2}du. &(7.48)
\end{aligned}
$$

We set $s = \frac{u}{u+v}$, $t = u+v$, $t_1 = t^{-\alpha_1+1}$ and $t_2 = t^{-\alpha_2+1}$. Then we have $u = st$, $v = t(1-s)$ and $\left|\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial u}{\partial t}\frac{\partial v}{\partial s}\right| = t$. So we have

$$
\begin{aligned}
(7.46) &= (\alpha_1 - 1)\int_0^1 ds \int_0^\infty h(t(1-s), \psi(t(1-s), -st))t^{-\alpha_1}dt &(7.49)\\
&= \int_0^1 ds \int_0^\infty h(t_1^{-\frac{1}{\alpha_1-1}}(1-s), \psi(t_1^{-\frac{1}{\alpha_1-1}}(1-s), -st_1^{-\frac{1}{\alpha_1-1}}))dt_1 &(7.50)
\end{aligned}
$$

and

$$
\begin{aligned}
(7.48) &= (\alpha_2 - 1)\int_0^1 ds \int_0^\infty h(\phi(-st, t(1-s)), -st)t^{-\alpha_2}dt &(7.51)\\
&= \int_0^1 ds \int_0^\infty h(\phi(-st_2^{-\frac{1}{\alpha_2-1}}, t_2^{-\frac{1}{\alpha_2-1}}(1-s)), -st_2^{-\frac{1}{\alpha_2-1}})dt_2. &(7.52)
\end{aligned}
$$

Since $\psi(x,y) = y(x-y)^{\frac{\alpha_1-1}{\alpha_2-1}-1}$ and $\phi(x,y) = y(y-x)^{\frac{\alpha_2-1}{\alpha_1-1}-1}$, we have

$$
\psi(t^{-\frac{1}{\alpha_1-1}}(1-s), -st^{-\frac{1}{\alpha_1-1}}) = -st^{-\frac{1}{\alpha_2-1}}, \qquad s,t>0 \tag{7.53}
$$

and

$$
\phi(-st^{-\frac{1}{\alpha_2-1}}, t^{-\frac{1}{\alpha_2-1}}(1-s)) = t^{-\frac{1}{\alpha_1-1}}(1-s), \qquad s,t>0. \tag{7.54}
$$

By (7.50), (7.52), (7.53) and (7.54), we obtain (7.4).

Let us prove (6.1) and (7.1). Let Φ_X and Φ_Y be those in Section 8. By [12, Theorem 3.12], we have

$$
\begin{aligned}
&n_0^X\left[1 - e^{-T_0^-}\mathbb{E}^Y_{\psi(X_{T_0^- -}, X_{T_0^-})}\left[e^{-T_0}\right]; 0 < T_0^- < T_0\right] &(7.55)\\
=&n_0^X\left[1 - e^{-T_0^-}e^{\Phi_Y(1)\psi(X_{T_0^- -}, X_{T_0^-})}; 0 < T_0^- < T_0\right] &(7.56)\\
\le&n_0^X\left[1 - e^{-T_0^-}e^{\Phi_Y(1)X_{T_0^-}}; 0 < T_0^- < T_0, X_{T_0^- -} - X_{T_0^-} \le 1\right] &(7.57)\\
&+ n_0^X\left[1 - e^{-T_0^-}e^{\Phi_Y(1)\psi(X_{T_0^- -}, X_{T_0^-})}; 0 < T_0^- < T_0, X_{T_0^- -} - X_{T_0^-} > 1\right] &(7.58)
\end{aligned}
$$

where the inequality (7.57) uses $\alpha_1 > \alpha_2$. There is a constant $q \geq 1$ such that $\Phi_X(q) \geq \Phi_Y(1)$. By [12, Theorem 3.12] and the strong Markov property, we have

$$(7.57) \leq n_0^X \left[1 - e^{-qT_0^-} e^{\Phi_X(q) X_{T_0^-}}\right] = n_0^X \left[1 - e^{-qT_0}\right] < \infty. \tag{7.59}$$

By the property of excursion measures, we have

$$(7.58) \leq n_0^X \left[X_{T_0^- -} - X_{T_0^-} > 1\right] < \infty. \tag{7.60}$$

By (7.59) and (7.60), we obtain (6.1). Since we have (7.4) and (6.1), in the same way as the proof of Lemma 7.2, we obtain (7.1). So the refracted processes U and $\widehat{U}$ are well-defined. By (7.4) and Theorem 7.1, the proof is completed. □

8 Refracted processes coming from Lévy processes

In this section we confine ourselves to the study of refracted processes coming from Lévy processes. We discuss representations of the generalized scale functions and study the approximation problems.

Let X, Y be spectrally negative Lévy processes which have Laplace exponents

$$\Psi_X(\lambda) = \chi_X \lambda + \frac{\sigma_X^2}{2}\lambda^2 - \int_{(-\infty,0)} (1 - e^{\lambda y} + \lambda y 1_{(-1,0)}(y))\Pi_X(dy), \quad \lambda \geq 0, \tag{8.1}$$

$$\Psi_Y(\lambda) = \chi_Y \lambda + \frac{\sigma_Y^2}{2}\lambda^2 - \int_{(-\infty,0)} (1 - e^{\lambda y} + \lambda y 1_{(-1,0)}(y))\Pi_Y(dy), \quad \lambda \geq 0, \tag{8.2}$$

for some constants $\chi_X, \chi_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y \geq 0$ and some Lévy measures Π_X, Π_Y, respectively. We let $\Phi_X(\theta) = \inf\{\lambda > 0 : \Psi_X(\lambda) > \theta\}$ and $\Phi_Y(\theta) = \inf\{\lambda > 0 : \Psi_Y(\lambda) > \theta\}$. We adopt the notation in Section 6: the reference measures m_X, m_Y are Lebesgue measures and we have the local times $\{L^{X,x}\}_{x\in\mathbb{R}}$ and $\{L^{Y,x}\}_{x\in\mathbb{R}}$, and the excursion measures $\{n_x^X\}_{x\in\mathbb{R}}$ and $\{n_x^Y\}_{x\in\mathbb{R}}$ of X and Y. For $q \geq 0$, let $W_X^{(q)}$ and $W_Y^{(q)}$ denote the scale functions of X and Y defined by the Laplace transform (3.1), respectively. Let ψ be a continuous landing function which satisfies (6.1). Let c_0 be a non-negative constant such that $c_0 = 0$ when $\sigma_X = 0$ or $\sigma_Y = 0$.

In [12, Corollary 8.9], the potential densities of X is given as $r_X^{(q)}(x, y) = r^{(q)}(y - x)$ with

$$r_X^{(q)}(x) = \Phi_X'(q) e^{-\Phi_X(q)x} - W_X^{(q)}(-x), \tag{8.3}$$

in particular, by (5.61) and [12, Lemma 8.6], we have

$$r_X^{(q)}(0) = \Phi_X'(q) - W_X^{(q)}(0) = \Phi_X'(q) - l^X. \tag{8.4}$$

So we have

$$n_x^X \left[1 - e^{-qT_0}\right] = \frac{1}{\mathbb{E}_x^X\left[\int_{0-}^\infty e^{-qt} dL_t^{X,x}\right]} = \frac{1}{r_X^{(q)}(0) + l^X} = \frac{1}{\Phi_X'(q)}. \tag{8.5}$$

Let U be a refracted process constructed from X, Y, ψ and c_0 in (C0). In this section, we normalize local times $\{L^{U,x}\}_{x\in\mathbb{R}\setminus\{0\}}$ and excursion measures $\{n_x^U\}_{x\in\mathbb{R}\setminus\{0\}}$ to satisfy, for non-negative measurable functional F,

$$n_x^U\left[F(\{U_t\}_{t<T_0^-})\right] = n_x^X\left[F(\{X_t\}_{t<T_0^-})\right], \quad x\in(0,\infty), \tag{8.6}$$

$$n_x^U\left[F(\{U_t\}_{t<T_0^+})\right] = n_x^Y\left[F(\{Y_t\}_{t<T_0^+})\right], \quad x\in(-\infty,0). \tag{8.7}$$

Then by the same argument as that of the discussion after the proof of Lemma 7.3, the local times $\{L^{U,x}\}_{x\in\mathbb{R}}$ satisfies the occupation formula with respect to the Lebesgue measure.

8.1 Generalized scale functions of refracted processes

Let us discuss representations of the generalized scale functions. This section follows [17, Section 6]. For $q\geq 0$, let $W_U^{(q)}$ be the generalized q-scale function of U. In this section, we give a representation of genetralized scale functions of refracted processes using the Laplace exponents and Lévy measures of X and Y.

Theorem 8.1. *For $q>0$, we have*

$$W_U^{(q)}(x,y) = \begin{cases} W_Y^{(q)}(x-y), & y<x\leq 0, \\ W_X^{(q)}(x-y), & 0\leq y<x. \end{cases} \tag{8.8}$$

In addition, for $q\geq 0$ and $y<0<x$, we have

$$W_U^{(q)}(x,y) = \frac{q}{\Phi_X(q)}W_Y^{(q)}(-y)W_X^{(q)}(x) + \frac{\sigma_X^2}{2}W_Y^{(q)}(-y)\left(W_X^{(q)\prime}(x) - \Phi_X(q)W_X^{(q)}(x)\right) \tag{8.9}$$

$$+ c_0\frac{\sigma_Y^2}{2}W_X^{(q)}(x)\left(\Phi_Y(q)W_Y^{(q)}(-y) - \frac{\sigma_Y^2}{2}e^{\Phi_Y(q)y}\left(W_Y^{(q)\prime\prime}(-y)W_Y^{(q)}(-y) - W_Y^{(q)\prime 2}(-y)\right)\right. \tag{8.10}$$

$$\left. + \int_0^{-y} dv\int_{(0,\infty)} e^{\Phi_Y(q)(u+y)}\left(W_Y^{(q)\prime}(-y-v)W_Y^{(q)}(-y) - W_Y^{(q)}(-y-v)W_Y^{(q)\prime}(-y)\right)\Pi_Y(du-v)\right) \tag{8.11}$$

$$+ \int_{(0,\infty)} dv\int_{(-\infty,0)}\left(e^{-\Phi_X(q)v}W_Y^{(q)}(-y)W_X^{(q)}(x) - W_Y^{(q)}(-y+\psi(v,u))W_X^{(q)}(x-v)\right)\Pi_X(du-v). \tag{8.12}$$

Proof. By (8.6), (8.7) and the same argument as (7.33) and (7.36), we have,

$$W_U^{(q)}(x,y) = W_Y^{(q)}(x-y), \qquad y<x\leq 0, \tag{8.13}$$

$$W_U^{(q)}(x,y) = W_X^{(q)}(x-y), \qquad 0\leq x<y, \tag{8.14}$$

$$W_U^{(q)}(x,y) = W_Y^{(q)}(-y)W_X^{(q)}(x)n_0^U\left[1-e^{-qT_0}1_{\{T_y^-=\infty, T_x^+=\infty\}}\right], \qquad y<0<x. \tag{8.15}$$

Let us compute $n_0^U\left[1-e^{-qT_0}1_{\{T_y^-=\infty,T_x^+=\infty\}}\right]$ for $q>0$ and $y<0<x$. We divide $n_0^U\left[1-e^{-qT_0}1_{\{T_y^-=\infty,T_x^+=\infty\}}\right]$ into the following sum:

$$n_0^U\left[1-e^{-qT_0}\right]+n_0^U\left[e^{-qT_0};T_x^+<\infty,T_y^-=\infty\right]+n_0^U\left[e^{-qT_0};T_y^-<\infty\right]. \tag{8.16}$$

For the first term, for $q>0$, we have

$$\begin{aligned}n_0^U\left[1-e^{-qT_0}\right]&=qn_0^U\left[\int_0^{T_0^-}e^{-qt}dt\right]+n_0^U\left[e^{-qT_0^-}(1-e^{-q(T_0-T_0^-)})\right] &(8.17)\\ &=qn_0^X\left[\int_0^{T_0^-}e^{-qt}dt\right]+n_0^X\left[e^{-qT_0^-}\mathbb{E}^Y_{\psi(X_{T_0^--},X_{T_0^-})}\left[1-e^{-qT_0^+}\right];X_{T_0^-}<0\right]\\ &\quad+c_0n_0^Y\left[1-e^{-qT_0};T_0^-=0\right]. &(8.18)\end{aligned}$$

By Theorem 3.4, Theorem 3.6, [12, Theorem 3.12] and [19, Lemma 2, (iv)], we have

$$\begin{aligned}(8.18)=&q\int_0^\infty e^{-\Phi_X(q)x}dx+\int_{(0,\infty)}dv\int_{(-\infty,0)}\left(1-e^{\Phi_Y(q)\psi(v,u)}\right)e^{-\Phi_X(q)v}\Pi_X(du-v)+c_0\frac{\sigma_Y^2}{2}\Phi_Y(q) &(8.19)\\ =&\frac{q}{\Phi_X(q)}+\int_{(0,\infty)}dv\int_{(-\infty,0)}\left(1-e^{\Phi_Y(q)\psi(v,u)}\right)e^{-\Phi_X(q)v}\Pi_X(du-v)+c_0\frac{\sigma_Y^2}{2}\Phi_Y(q). &(8.20)\end{aligned}$$

For the second term, we have

$$\begin{aligned}n_0^U\left[e^{-qT_0};T_x^+<\infty,T_y^-=\infty\right]=&n_0^U\left[e^{-qT_x^+}\left(e^{-qT_0}1_{\{T_0^-<\infty,T_y^-=\infty\}}\right)\circ\theta_{T_x^+};T_x^+<\infty\right] &(8.21)\\ =&n_0^X\left[e^{-qT_x^+};T_x^+<\infty\right]\Bigg(\mathbb{E}_x^X\left[e^{-qT_0};T_0=T_0^-\right] &(8.22)\\ &+\mathbb{E}_x^X\left[e^{-qT_0^-}\mathbb{E}^Y_{\psi(X_{T_0^--},X_{T_0^-})}\left[e^{-qT_0^+};T_0^+<T_y^-\right];X_{T_0^-}<0\right]\Bigg). &(8.23)\end{aligned}$$

By [11, Theorem 1.4], for $x>0$, we have

$$\mathbb{E}_x^X\left[e^{-qT_0};T_0=T_0^-\right]=\frac{\sigma_X^2}{2}\left(W_X^{(q)\prime}(x)-\Phi_X(q)W_X^{(q)}(x)\right). \tag{8.24}$$

By (3.7), (8.24), Theorem 3.3 and (3.2), we have

$$\begin{aligned}(8.23)=&\frac{1}{W_X^{(q)}(x)}\Bigg(\frac{\sigma_X^2}{2}\left(W_X^{(q)\prime}(x)-\Phi_X(q)W_X^{(q)}(x)\right) &(8.25)\\ &+\int_{(0,\infty)}dv\int_{(-\infty,0)}\frac{W_Y^{(q)}(-y+\psi(v,u))}{W_Y^{(q)}(-y)}(e^{-\Phi_X(q)v}W_X^{(q)}(x)-W_X^{(q)}(x-v))\Pi_X(du-v)\Bigg). &(8.26)\end{aligned}$$

For the third term, we have

$$n_0^U\left[e^{-qT_0};T_y^-<\infty\right] \tag{8.27}$$

$$=n_0^U\left[e^{-qT_0};T_0^-=0,T_y^-<\infty\right]+n_0^U\left[e^{-qT_0^-}\mathbb{E}^U_{U_{T_0^-}}\left[e^{-qT_0^+};T_y^-<T_0^+\right];U_{T_0^-}<0\right] \tag{8.28}$$

$$=c_0n_0^Y\left[e^{-qT_0};T_0^-=0,T_y^-<\infty\right]+n_0^X\left[e^{-qT_0^-}\mathbb{E}^Y_{\psi(X_{T_0^-}-,X_{T_0^-})}\left[e^{-qT_0^+};T_y^-<T_0^+\right];X_{T_0^-}<0\right]. \tag{8.29}$$

By [11, Theorem 3.10], $W^{(q)}\in C^2(0,\infty)$ when $\sigma_Y>0$. By [18, Lemma 5], for $y<0$, we know that

$$n_0^Y\left[e^{-qT_0};T_0^-=0,T_y^-<\infty\right]=-\frac{\sigma_Y^4}{4}e^{\Phi_Y(q)y}\left(W_Y^{(q)\prime\prime}(-y)-\frac{W_Y^{(q)\prime 2}(-y)}{W_Y^{(q)}(-y)}\right) \tag{8.30}$$

$$+\frac{\sigma_Y^2}{2}\int_0^{-y}dv\int_{(0,\infty)}e^{\Phi_Y(q)(u+y)}\left(W_Y^{(q)\prime}(-y-v)-\frac{W_Y^{(q)}(-y-v)W_Y^{(q)\prime}(-y)}{W_Y^{(q)}(-y)}\right)\Pi_Y(du-v), \tag{8.31}$$

and by [12, Theorem 3.12] and (3.2), for $y<x<0$, we have

$$\mathbb{E}_x^Y\left[e^{-qT_0^+};T_y^-<T_0^+\right]=\mathbb{E}_x^Y\left[e^{-qT_0^+}\right]-\mathbb{E}_x^Y\left[e^{-qT_0^+};T_0^+<T_y^-\right] \tag{8.32}$$

$$=e^{\Phi_Y(q)x}-\frac{W_Y^{(q)}(-y+x)}{W_Y^{(q)}(-y)}. \tag{8.33}$$

and Theorem 3.4, we have

$$(8.29)=-c_0\frac{\sigma_Y^4}{4}e^{\Phi_Y(q)y}\left(W_Y^{(q)\prime\prime}(-y)-\frac{W_Y^{(q)\prime 2}(-y)}{W_Y^{(q)}(-y)}\right) \tag{8.34}$$

$$+c_0\frac{\sigma_Y^2}{2}\int_0^{-y}dv\int_{(0,\infty)}e^{\Phi_Y(q)(u+y)}\left(W_Y^{(q)\prime}(-y-v)-\frac{W_Y^{(q)}(-y-v)W_Y^{(q)\prime}(-y)}{W_Y^{(q)}(-y)}\right)\Pi_Y(du-v) \tag{8.35}$$

$$+\int_{(0,\infty)}dv\int_{(-\infty,0)}\left(e^{\Phi_Y(q)\psi(v,u)}-\frac{W_Y^{(q)}(-y+\psi(v,u))}{W_Y^{(q)}(-y)}\right)e^{-\Phi_X(q)v}\Pi_X(du-v). \tag{8.36}$$

By (8.15), (8.16), (8.20), (8.26) and (8.36), we obtain (8.12). $\square$

In the same way as above, the Laplace transforms of hitting times of U can be represented in the following lemma using the Laplace exponents and the scale functions X and Y.

Corollary 8.2. *For $q>0$ and $a,x\in\mathbb{R}$ with $x<a$, we have*

$$\mathbb{E}_x^U\left[e^{-qT_a^+}\right]=\frac{\overline{W}_U^{(q)}(x)}{\overline{W}_U^{(q)}(a)}, \tag{8.37}$$

where

$$\overline{W}_U^{(q)}(x) = \begin{cases} e^{\Phi_Y(q)x}, & x \le 0, \\ \frac{q}{\Phi_X(q)} W_X^{(q)}(x) + \frac{\sigma_X^2}{2}(W_X^{(q)\prime}(x) - \Phi_X(q) W_X^{(q)}(x)) & \\ + \int_{(0,\infty)} dv \int_{(-\infty,0)} \left(e^{-\Phi_X(q)v} W_X^{(q)}(x) - e^{\Phi_Y(q)\psi(v,u)} W_X^{(q)}(x-v) \right) \Pi_X(du - v), & x > 0. \end{cases} \tag{8.38}$$

In particular, $\overline{W}_U^{(q)}(x)$ is a continuous and increasing function of x.

Proof. By (3.2), we have

$$\mathbb{E}_x^U\left[e^{-qT_a^+}\right] = \lim_{b\downarrow-\infty} \mathbb{E}_x^U\left[e^{-qT_a^+}; T_a^+ < T_b^-\right] = \lim_{b\downarrow-\infty} \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)} = \lim_{b\downarrow-\infty} \frac{W_U^{(q)}(x,b)/W_Y^{(q)}(-b)}{W_U^{(q)}(a,b)/W_Y^{(q)}(-b)}. \tag{8.39}$$

By (8.8) and [12, Theorem 3.12], for $x \le 0$, we have

$$\lim_{b\downarrow-\infty} \frac{W_U^{(q)}(x,b)}{W_Y^{(q)}(-b)} = \lim_{b\downarrow-\infty} \frac{W_Y^{(q)}(x-b)}{W_Y^{(q)}(-b)} = \lim_{b\downarrow-\infty} \mathbb{E}_0^Y\left[e^{-qT_{-x}^+}; T_{-x}^+ < T_{b-x}^-\right] = e^{\Phi_Y(q)x}. \tag{8.40}$$

By (8.15), for $x > 0$, we have

$$\lim_{b\downarrow-\infty} \frac{W_U^{(q)}(x,b)}{W_Y^{(q)}(-b)} = \lim_{b\downarrow-\infty} W_X^{(q)}(x) n_0^U\left[1 - e^{-qT_0} 1_{\{T_b^- = \infty, T_x^+ = \infty\}}\right] = W_X^{(q)}(x) n_0^U\left[1 - e^{-qT_0} 1_{\{T_x^+ = \infty\}}\right]. \tag{8.41}$$

We devide the demoninator $n_0^U\left[1 - e^{-qT_0} 1_{\{T_x^+ = \infty\}}\right]$ into the following sum:

$$n_0^U\left[1 - e^{-qT_0}\right] + n_0^U\left[e^{-qT_0}; T_x^+ < \infty\right]. \tag{8.42}$$

For the second term in (8.42), by the strong Markov property, (3.7), (8.24), [12, Theorem 3.12] and Theorem 3.3, we have

$$n_0^U\left[e^{-qT_0}; T_x^+ < \infty\right] = n_0^U\left[e^{-qT_x^+}; T_x^+ < \infty\right] \mathbb{E}_x^U\left[e^{-qT_0}\right] \tag{8.43}$$

$$= n_0^U\left[e^{-qT_x^+}; T_x^+ < \infty\right] \left(\mathbb{E}_x^U\left[e^{-qT_0}; T_0 = T_0^-\right] + \mathbb{E}_x^U\left[e^{-qT_0^-} \mathbb{E}_{U_{T_0^-}}^U\left[e^{-qT_0^+}\right]; T_0^- < T_0\right] \right) \tag{8.44}$$

$$= \frac{1}{W_X^{(q)}(x)} \left(\frac{\sigma_x^2}{2} \left(W_X^{(q)\prime}(x) - \Phi_X(q) W_X^{(q)}(x) \right) \right. \tag{8.45}$$

$$\left. + \int_{(0,\infty)} dv \int_{(-\infty,0)} e^{\Phi_Y(q)\psi(v,u)} (e^{-\Phi_X(q)v} W_X^{(q)}(x) - W_X^{(q)}(x-v)) \Pi_X(du - v) \right). \tag{8.46}$$

By (8.40), (8.41), (8.42), (8.20) and (8.46), we obtain (8.38).

Next, we prove that $\overline{W}_U^{(q)}$ is increasing and continuous. It is obvious that $\overline{W}_U^{(q)}$ is increasing and continuous on $(-\infty,0]$, since $\overline{W}_U^{(q)}(x) = e^{\Phi_Y(q)x}$. Using the dominated convergence theorem, we have

$$\lim_{\epsilon\downarrow 0} \overline{W}_U^{(q)}(\epsilon) = \lim_{\epsilon\downarrow 0} \frac{1}{\mathbb{E}_0^U\left[e^{-qT_\epsilon^+}\right]} = \frac{1}{\mathbb{E}_0^U\left[\lim_{\epsilon\downarrow 0} e^{-qT_\epsilon^+}\right]} = 1, \tag{8.47}$$

so that we see $\overline{W}_U^{(q)}$ is continuous at 0. Since

$$\overline{W}_U^{(q)}(x) = \frac{1}{\mathbb{E}_0^U\left[e^{-qT_x^+}\right]}, \tag{8.48}$$

it is thus sufficient to prove that $\mathbb{E}_0^U\left[e^{-qT_x^+}\right]$ is decreasing and continuous on $(0,\infty)$. For $0 < x < y$, we have

$$\mathbb{E}_0^U\left[e^{-qT_x^+}\right] - \mathbb{E}_0^U\left[e^{-qT_y^+}\right] = \mathbb{E}_0^U\left[e^{-qT_x^+}\right]\left(1 - \mathbb{E}_x^U\left[e^{-qT_y^+}\right]\right) \geq 0. \tag{8.49}$$

Using (3.2), for $x > 0$, we have

$$\begin{aligned}
\limsup_{\epsilon\downarrow 0}\left|\mathbb{E}_0^U\left[e^{-qT_{x-\epsilon}^+}\right] - \mathbb{E}_0^U\left[e^{-qT_{x+\epsilon}^+}\right]\right| &= \limsup_{\epsilon\downarrow 0} \mathbb{E}_0^U\left[e^{-qT_{x-\epsilon}^+}\right]\left(1 - \mathbb{E}_{x-\epsilon}^U\left[e^{-qT_{x+\epsilon}^+}\right]\right) && (8.50)\\
&\leq \limsup_{\epsilon\downarrow 0}\left(1 - \mathbb{E}_{x-\epsilon}^X\left[e^{-qT_{x+\epsilon}^+}; T_{x+\epsilon}^+ < T_0^-\right]\right) && (8.51)\\
&= \left(1 - \lim_{\epsilon\downarrow 0} \frac{W_X^{(q)}(x-\epsilon)}{W_X^{(q)}(x+\epsilon)}\right) = 0. && (8.52)
\end{aligned}$$

The proof is complete. □

8.2 Approximation problem

In this section, we discuss the approximation problem. This section follows [16, Section 4].

We impose the following conditions:

(C0) There exist $k, l > 0$ such that

$$\psi(x,y) \geq l(y-x), \text{ for } x - y < k. \tag{8.53}$$

(Note that (8.53) implies (6.1).)

(C1) Let $\{\epsilon_n^X\}_{n\in\mathbb{N}}$ and $\{\epsilon_n^Y\}_{n\in\mathbb{N}}$ be sequences of strictly positive numbers satisfying

$$\lim_{n\uparrow\infty} \epsilon_n^X = \lim_{n\uparrow\infty} \epsilon_n^Y = 0. \tag{8.54}$$

When $c_0 > 0$ (and consequently $\sigma_X \sigma_Y > 0$), we assume that

$$\lim_{n \uparrow \infty} \frac{\epsilon_n^Y}{\epsilon_n^X} = \frac{\sigma_Y^2}{\sigma_X^2} c_0. \tag{8.55}$$

For $n \in \mathbb{N}$, we define

$$\begin{aligned}\Psi_{X^{(n)}}(\lambda) = \chi_X \lambda - \frac{\sigma_X^2}{(\epsilon_n^X)^2}\Big(1 - e^{\lambda(-\epsilon_n^X)} + \lambda\big(-\epsilon_n^X\big)\Big) \\ - \int_{(-\infty,-\epsilon_n^X)} \big(1 - e^{\lambda y} + \lambda y 1_{(-1,-\epsilon_n^X)}(y)\big)\, \Pi_X(dy) \quad (8.56) \\ = \delta_{X^{(n)}} \lambda - \int_{(-\infty,0)} \big(1 - e^{\lambda y}\big)\, \Pi_{X^{(n)}}(dy) \quad (8.57)\end{aligned}$$

where

$$\delta_{X^{(n)}} = \chi_X + \frac{\sigma_X^2}{\epsilon_n^X} + \int_{(-1,-\epsilon_n^X)} (-y) \Pi_X(dy) \tag{8.58}$$

$$\Pi_{X^{(n)}} = 1_{(-\infty,-\epsilon_n^X)} \Pi_X + \frac{\sigma_X^2}{(\epsilon_n^X)^2} \delta_{(-\epsilon_n^X)}. \tag{8.59}$$

Let $X^{(n)}$ be a compound Poisson process with positive drift which has Laplace exponent $\Psi_{X^{(n)}}$. We let $\Phi_{X^{(n)}}$ denote the right inverse of $\Psi_{X^{(n)}}$. We note that $\Psi_{X^{(n)}}(\lambda) \to \Psi_X(\lambda)$ for all $\lambda \geq 0$, so that we have $X^{(n)} \to X$ in law on $\mathbb{D}$. More preciously, by [2, pp.210], we see that there exists a coupling of $X^{(n)}$'s such that $X^{(n)} \to X$ uniformly on compact intervals almost surely. We define $\Psi_{Y^{(n)}}$, $\delta_{Y^{(n)}}$, $\Pi_{Y^{(n)}}$, $\Phi_{Y^{(n)}}$ and $Y^{(n)}$ in the same way as those for X.

It is known that Ψ_X is a strictly convex function with $\Psi_X(0) = 0$. The function $\Psi_{X^{(n)}}$ satisfies the same facts. For X and $X^{(n)}$ satisfying the conditions in (C1), we have

$$\Phi_{X^{(n)}}(\lambda) \to \Phi_X(\lambda), \qquad \lambda \geq 0. \tag{8.60}$$

For the proof of (8.60), we prove the following lemma, which was omitted in [16]. Let $\mathbb{F}$ denote the set of strictly convex functions $f : [0,\infty) \to \mathbb{R}$ which satisfies $f(0) = 0$ and $\lim_{x \uparrow \infty} f(x) = \infty$. For $f \in \mathbb{F}$, we denote

$$f^{-1}(\lambda) = \sup\{x \geq 0 : f(x) \leq \lambda\}. \tag{8.61}$$

Lemma 8.3. *Assume that the sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathbb{F}$ and $f \in \mathbb{F}$ satisfy*

$$f_n(x) \to f(x), \qquad \text{as } n \uparrow \infty, \text{ for } x \geq 0. \tag{8.62}$$

Then we have

$$f_n^{-1}(\lambda) \to f^{-1}(\lambda), \qquad \text{as } n \uparrow \infty, \quad \text{for } \lambda \geq 0. \tag{8.63}$$

Proof. First, we assume that $f^{-1}(0) > 0$. In this case, by the strict convexity of f, there exists a constant $a_f > 0$ such that f si strictly decreasing on $[0, a_f]$ and f is strictly increasing on $[a_f, \infty)$. Let us fix $\lambda \geq 0$ and write $b_\lambda = f^{-1}(\lambda)$. Note that $0 = f(0) > f(a_\lambda)$ and that $b_\lambda > a_\lambda$. For any $\epsilon \in (0, b_\lambda - a_\lambda)$, we see that f is strictly increasing on $[b_\lambda - \epsilon, b_\lambda + \epsilon]$. By (8.62), there exists $N > 0$ such that for all $n > N$, we have

$$|f_n(b_\lambda + \epsilon) - f(b_\lambda + \epsilon)| < f(b_\lambda + \epsilon) - \lambda, \tag{8.64}$$

$$|f_n(b_\lambda - \epsilon) - f(b_\lambda - \epsilon)| < \lambda - f(b_\lambda - \epsilon). \tag{8.65}$$

Then we have $f_n(b_\lambda - \epsilon) < \lambda < f_n(b_\lambda + \epsilon)$, which implies $b_\lambda - \epsilon \leq f_n^{-1}(\lambda) \leq b_\lambda + \epsilon$.

Second, we assume that $f^{-1}(0) = 0$. In this case f is a strictly increasing function on $[0, \infty)$, and so we obtain $\lim_{n\uparrow\infty} f_n^{-1}(\lambda) = f^{-1}(\lambda)$ for $\lambda > 0$ in the same way as above. We have $\lim_{n\uparrow\infty} f_n^{-1}(0) = f^{-1}(0)$ similarly by ignoring (8.65). The proof is now completed. $\square$

We need the following lemma for the resolvent convergence when Y has the Gaussian part.

Lemma 8.4 ([16, Lemma 4.1]). *We assume that $\sigma_Y > 0$. Then for all $q > 0$ and all bounded continuous function f, we have*

$$\lim_{n\uparrow\infty} \frac{\sigma_Y^2}{(\epsilon_n^Y)^2} \int_0^{\epsilon_n^Y} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(-v)\, dv = n_0^Y \left[\int_0^{T_0} e^{-qt} f(Y_t) dt; T_0^- = 0 \right]. \tag{8.66}$$

Proof. By the definition of $\{Y^{(n)}\}_{n\in\mathbb{N}}$, we have that for all $q > 0$, $u < 0$ and for $g = f1_{(-\infty,0)}$ or $f1_{(0,\infty)}$,

$$\lim_{n\uparrow\infty} \frac{n_0^{Y^{(n)}}\left[\int_0^\infty e^{-qt} g(Y_t^{(n)}) dt\right]}{n_0^{Y^{(n)}}\left[\int_0^\infty e^{-qt} dt\right]} = \lim_{n\uparrow\infty} R^{(q)}_{Y^{(n)}} g(0) = R^{(q)}_Y g(0) = \frac{n_0^Y\left[\int_0^\infty e^{-qt} g(Y_t) dt\right]}{n_0^Y\left[\int_0^\infty e^{-qt} dt\right]} \tag{8.67}$$

and $\lim_{n\uparrow\infty} R^{(q)}_{Y^{(n)}0} f(u) = R^{(q)}_{Y0} f(u)$. By Theorem 3.6 and by $\lim_{n\uparrow\infty} \Phi_{Y^{(n)}}(\lambda) = \Phi_Y(\lambda)$ on for all $\lambda \geq 0$, we have, for all $q > 0$,

$$\begin{aligned}
\lim_{n\uparrow\infty} n_0^{Y^{(n)}} \left[\int_0^\infty e^{-qt} f(Y_t^{(n)}) 1_{(0,\infty)}(Y_t^{(n)}) dt \right] &= \lim_{n\uparrow\infty} \int_0^\infty f(x) e^{-\Phi_{Y^{(n)}}(q)x} dx && (8.68) \\
&= \int_0^\infty f(x) e^{-\Phi_Y(q)x} dx && (8.69) \\
&= n_0^Y \left[\int_0^\infty e^{-qt} f(Y_t) 1_{(0,\infty)}(Y_t) dt \right], && (8.70)
\end{aligned}$$

and thus by (8.67), we have $\lim_{n\uparrow\infty} n_0^{Y^{(n)}}\left[\int_0^\infty e^{-qt} dt\right] = n_0^Y\left[\int_0^\infty e^{-qt} dt\right]$. Again by (8.67), we obtain

$$\lim_{n\uparrow\infty} n_0^{Y^{(n)}} \left[\int_0^\infty e^{-qt} f(Y_t^{(n)}) 1_{(-\infty,0)}(Y_t^{(n)}) dt \right] = n_0^Y \left[\int_0^\infty e^{-qt} f(Y_t) 1_{(-\infty,0)}(Y_t) dt \right]. \tag{8.71}$$

By Theorem 3.4, we have

$$
\begin{aligned}
& n_0^Y\left[\int_0^{T_0} e^{-qt} f(Y_t) 1_{(-\infty,0)}(Y_t) dt\right] && (8.72)\\
=& n_0^Y\left[\int_0^{T_0} e^{-qt} f(Y_t) dt; T_0^- = 0\right] + n_0^Y\left[e^{-qT_0^-}\mathbb{E}^Y_{Y_{T_0^-}}\left[\int_0^{T_0} e^{-qt} f(Y_t) dt\right]; 0 < T_0^- < T_0\right] && (8.73)\\
=& n_0^Y\left[\int_0^{T_0} e^{-qt} f(Y_t) dt; T_0^- = 0\right] + \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{k_{T_0^+} Y} f(u) e^{-\Phi_Y(q)v}\Pi_Y(du - v). && (8.74)
\end{aligned}
$$

and

$$
\begin{aligned}
& n_0^{Y^{(n)}}\left[\int_0^{T_0} e^{-qt} f(Y^{(n)}_t) 1_{(-\infty,0)}(Y^{(n)}_t) dt\right] \\
=& n_0^{Y^{(n)}}\left[e^{-qT_0^-}\mathbb{E}^{Y^{(n)}}_{Y^{(n)}_{T_0^-}}\left[\int_0^{T_0} e^{-qt} f(Y^{(n)}_t) dt\right]; T_0^- < T_0\right] && (8.75)\\
=& \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(u) e^{-\Phi_{Y^{(n)}}(q)v}\Pi_{Y^{(n)}}(du - v) && (8.76)\\
=& \frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{\epsilon_n^Y} R^{(q)}_{k_{T_0^+} Y^{(n)}} f\left(v - \epsilon_n^Y\right) e^{-\Phi_{Y^{(n)}}(q)v} dv \\
& + \int_0^\infty dv \int_{(-\infty,0\wedge(-\epsilon_n^Y+v))} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(u) e^{-\Phi_{Y^{(n)}}(q)v}\Pi_Y(du - v). && (8.77)
\end{aligned}
$$

By the same argument as that of the proof of [17, Theorem 8.4], we have

$$
\begin{aligned}
& \lim_{n\uparrow\infty}\int_0^\infty dv \int_{(-\infty,0\wedge(-\epsilon_n^Y+v))} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(u) e^{-\Phi_{Y^{(n)}}(q)v}\Pi_Y(du - v) \\
=& \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{k_{T_0^+} Y} f(u) e^{-\Phi_Y(q)v}\Pi_Y(du - v) && (8.78)
\end{aligned}
$$

By (8.71), (8.74), (8.77) and (8.78), we obtain

$$
\lim_{n\uparrow\infty}\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{\epsilon_n^Y} R^{(q)}_{k_{T_0^+} Y^{(n)}} f\left(v - \epsilon_n^Y\right) e^{-\Phi_{Y^{(n)}}(q)v} dv = n_0^Y\left[\int_0^{T_0} e^{-qt} f(Y_t) dt; T_0^- = 0\right]. \quad (8.79)
$$

By a simple argument, we can see that the left hand side of (8.79) coincides with that of (8.66), which leads to the desired conclusion. □

(C2) Let $\{\psi^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of functions satisfying

$$
\psi^{(n)}(x, y) = \psi(x, y) 1_{\{x-y>\epsilon_n^X\}} - \frac{\sigma_Y^2}{\sigma_X^2} c_0 x 1_{\{x-y=\epsilon_n^X\}} \quad (8.80)
$$

for all $x > 0$, $y < 0$ and $n \in \mathbb{N}$ where we understand $\frac{0}{0} = 0$.

Let $X^{(n)}$ and $Y^{(n)}$ be those in (C1) and let $\psi^{(n)}$ be that in (C2). Let $U^{(n)}$ be the refracted process constructed by $X^{(n)}$, $Y^{(n)}$, $\psi^{(n)}$ and $c_0^{(n)} = 0$. Then, we obtain the following theorem for the resolvent convergence.

Theorem 8.5 ([16, Theorem 4.2]). *For all $q > 0$, $x \in \mathbb{R}$ and bounded continuous function f, we have*

$$\lim_{n\uparrow\infty} R^{(q)}_{U^{(n)}} f(x) = R^{(q)}_U f(x). \tag{8.81}$$

Proof. *i*) We prove (8.81) for $x = 0$. For this purpose we shall prove that

$$\lim_{n\uparrow\infty} n_0^{U^{(n)}}\left[\int_0^{T_0} e^{-qt} f(U_t^{(n)})dt\right] = n_0^U\left[\int_0^{T_0} e^{-qt} f(U_t)dt\right]. \tag{8.82}$$

for all $q > 0$ and bounded continuous function f. By Theorem 3.6 and $\lim_{n\uparrow\infty} \Phi_{X^{(n)}}(\lambda) = \Phi_X(\lambda)$ for all $\lambda \geq 0$, we have

$$\lim_{n\uparrow\infty} n_0^{U^{(n)}}\left[\int_0^{T_0^-} e^{-qt} f(U_t^{(n)})dt\right] = \lim_{n\uparrow\infty} n_0^{X^{(n)}}\left[\int_0^{T_0^-} e^{-qt} f(X_t^{(n)})dt\right] \tag{8.83}$$

$$= n_0^X\left[\int_0^{T_0^-} e^{-qt} f(X_t)dt\right] \tag{8.84}$$

$$= n_0^U\left[\int_0^{T_0^-} e^{-qt} f(U_t)dt\right]. \tag{8.85}$$

By the definition of $n_0^{U^{(n)}}$ and Theorem 3.4, we have

$$n_0^{U^{(n)}}\left[\int_{T_0^-}^{T_0} e^{-qt} f(U_t^{(n)})dt\right]$$

$$= n_0^{X^{(n)}}\left[e^{-qT_0^-}\mathbb{E}^{Y^{(n)}}_{\psi^{(n)}(X_{T_0^- -}, X_{T_0^-})}\left[\int_0^{T_0} e^{-qt} f(U_t^{(n)})dt\right]; T_0^- < T_0\right] \tag{8.86}$$

$$= \int_0^\infty dv \int_{(-\infty,0)} R^{(q)}_{k_{T_0^+}Y^{(n)}} f(\psi^{(n)}(v,u)) e^{-\Phi_{X^{(n)}}(q)v}\Pi_{X^{(n)}}(du - v) \tag{8.87}$$

$$= \frac{\sigma_X^2}{(\epsilon_n^X)^2}\int_0^{\epsilon_n^X} R^{(q)}_{k_{T_0^+}Y^{(n)}} f\left(\psi^{(n)}(v, v - \epsilon_n^X)\right) e^{-\Phi_{X^{(n)}}(q)v} dv$$

$$+ \int_0^\infty dv \int_{(-\infty, 0\wedge(-\epsilon_n^X + v))} R^{(q)}_{k_{T_0^+}Y^{(n)}} f(\psi(v,u)) e^{-\Phi_{X^{(n)}}(q)v}\Pi_X(du - v) \tag{8.88}$$

$$= (\mathrm{I}) + (\mathrm{II}). \tag{8.89}$$

Let us compute the limit of (II). We have

$$(\mathrm{II}) = \int_{(-\infty,0)} \Pi_X(du) 1_{\{u < -\epsilon_n^Y\}} \int_0^{-u} R^{(q)}_{k_{T_0^+}Y^{(n)}} f(\psi(v, u+v)) e^{-\Phi_{X^{(n)}}(q)v} dv \tag{8.90}$$

To use the dominated convergence theorem, we dominate the integrand as

$$\left| 1_{\{u<-\epsilon_n^Y\}} \int_0^{-u} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(\psi(v,u+v)) e^{-\Phi_{X^{(n)}}(q)v} dv \right| \tag{8.91}$$

$$\leq \|f\| \int_0^{-u} e^{-\Phi_X^{\inf}(q)v} \mathbb{E}^{Y^{(n)}}_{\psi(v,u+v)} \left[\int_0^{T_0} e^{-qt} dt \right] dv \tag{8.92}$$

$$= \frac{\|f\|}{q} \int_0^{-u} e^{-\Phi_X^{\inf}(q)v} (1 - e^{\Phi_Y^{\inf}(q)\psi(v,u+v)}) dv, \tag{8.93}$$

where $\Phi_X^{\inf}(q) = \inf_{n\in\mathbb{N}} \Phi_{X^{(n)}}(q)$ and $\Phi_Y^{\inf}(q) = \inf_{n\in\mathbb{N}} \Phi_{Y^{(n)}}(q)$. By (8.53), we have

$$(8.93) \leq \frac{\|f\|}{q} \int_0^{-u} e^{-\Phi_X^{\inf}(q)v} (1 - 1_{\{u>-k\}} e^{\Phi_Y^{\inf}(q)lu}) dv \tag{8.94}$$

$$= \frac{\|f\|}{q\Phi_X^{\inf}(q)} (1 - e^{\Phi_X^{\inf}(q)u})(1 - 1_{\{u>-k\}} e^{\Phi_Y^{\inf}(q)lu}) \in L^1(\Pi_X). \tag{8.95}$$

By (8.95) and the dominated convergence theorem, we have

$$\lim_{n\uparrow\infty} (8.90) = \int_{(-\infty,0)} \Pi_X(du) \int_0^{-u} R^{(q)}_{k_{T_0^+} Y} f(\psi(v,u+v)) e^{-\Phi_X(q)v} dv \tag{8.96}$$

$$= \int_0^{\infty} dv \int_{(-\infty,0)} R^{(q)}_{k_{T_0^+} Y} f(\psi(v,u)) e^{-\Phi_X(q)v} \Pi_X(du - v). \tag{8.97}$$

By the definition of n_0^U and Theorem 3.4, we have

$$(8.97) = n_0^X \left[e^{-qT_0^-} \mathbb{E}^Y_{\psi(X_{T_0^-}-, X_{T_0^-})} \left[\int_0^{T_0} e^{-qt} f(Y_t) dt \right]; 0 < T_0^- < T_0 \right] \tag{8.98}$$

$$= n_0^U \left[\int_{T_0^-}^{T_0} e^{-qt} f(U_t) dt; 0 < T_0^- < T_0 \right]. \tag{8.99}$$

Let us compute the limit of (I). Let $c_1 = \frac{\sigma_Y^2}{\sigma_X^2} c_0$. By the definition of $\psi^{(n)}$, we have

$$(\mathrm{I}) = \frac{\sigma_X^2}{(\epsilon_n^X)^2} \int_0^{\epsilon_n^X} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(-c_1 v)\, e^{-\Phi_{X^{(n)}}(q)v} dv. \tag{8.100}$$

When $c_1 = 0$, we have $(8.100) = 0$. When $c_1 > 0$, we have

$$\lim_{n\uparrow\infty} (8.100) = \lim_{n\uparrow\infty} c_0 c_1 \frac{\sigma_Y^2}{(\epsilon_n^Y)^2} \int_0^{\epsilon_n^X} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(-c_1 v)\, dv, \tag{8.101}$$

if the right hand side of (8.101) has the limit. By the change of variables, we have

$$c_0 c_1 \frac{\sigma_Y^2}{(\epsilon_n^Y)^2} \int_0^{\epsilon_n^X} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(-c_1 v)\, dv = c_0 \frac{\sigma_Y^2}{(\epsilon_n^Y)^2} \int_0^{c_1\epsilon_n^X} R^{(q)}_{k_{T_0^+} Y^{(n)}} f(-v)\, dv. \tag{8.102}$$

We prove

$$\lim_{n\uparrow\infty}\left|c_0\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{c_1\epsilon_n^X} R^{(q)}_{k_{T_0^+}Y^{(n)}}f(-v)\,dv - c_0\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{\epsilon_n^Y} R^{(q)}_{k_{T_0^+}Y^{(n)}}f(-v)\,dv\right| = 0. \tag{8.103}$$

Let $M_Y(q) = \sup_{n\in\mathbb{N}}\Phi_{Y^{(n)}}(q)\times(1\vee\sup_{n\in\mathbb{N}}\frac{c_1\epsilon_n^X}{\epsilon_n^Y})$. We have

$$\left|c_0\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{c_1\epsilon_n^X} R^{(q)}_{k_{T_0^+}Y^{(n)}}f(-v)\,dv - c_0\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\int_0^{\epsilon_n^Y} R^{(q)}_{k_{T_0^+}Y^{(n)}}f(-v)\,dv\right| \tag{8.104}$$

$$\leq c_0\frac{\sigma_Y^2}{(\epsilon_n^Y)^2}\left|c_1\epsilon_n^X - \epsilon_n^Y\right|\|f\|\sup_{0\leq v\leq(c_1\epsilon_n^X)\vee\epsilon_n^Y}\mathbb{E}^{Y^{(n)}}_{-v}\left[\int_0^{T_0}e^{-qt}dt\right] \tag{8.105}$$

$$\leq \frac{c_0\sigma_Y^2\,\|f\|}{q}\left|c_1\frac{\epsilon_n^X}{\epsilon_n^Y}-1\right|\frac{1-e^{-M_Y(q)\epsilon_n^Y}}{\epsilon_n^Y}. \tag{8.106}$$

By the definition of c_1, we have

$$\frac{c_0\sigma_Y^2\,\|f\|}{q}\left|c_1\frac{\epsilon_n^X}{\epsilon_n^Y}-1\right|\frac{1-e^{-M_Y(q)\epsilon_n^Y}}{\epsilon_n^Y} \to \frac{c_0\sigma_Y^2\,\|f\|}{q}\times 0\times M_Y(q) = 0, \quad \text{as } n\uparrow\infty. \tag{8.107}$$

So we have (8.103). By (8.100), (8.101), (8.102), (8.103) and Lemma 8.4, we have

$$\lim_{n\uparrow\infty}\frac{\sigma_X^2}{(\epsilon_n^X)^2}\int_0^{\epsilon_n^X} R^{(q)}_{k_{T_0^+}Y^{(n)}}f\left(\psi^{(n)}(v, v-\epsilon_n^X)\right)e^{-\Phi_{X^{(n)}}(q)v}dv \tag{8.108}$$

$$= c_0 n_0^Y\left[\int_0^{T_0}e^{-qt}f(Y_t)dt; T_0^- = 0\right] \tag{8.109}$$

$$= n_0^U\left[\int_0^{T_0}e^{-qt}f(U_t)dt; T_0^- = 0\right]. \tag{8.110}$$

By (8.85), (8.89), (8.99) and (8.110), we obtain (8.82).

ii) We prove (8.81) for $x<0$. By the strong Markov property and the definition of $U^{(n)}$, we have

$$R^{(q)}_{U^{(n)}}f(x) = R^{(q)}_{Y^{(n)}0}f(x) + \mathbb{E}^{Y^{(n)}0}_x\left[e^{-qT_0^+}\right]R^{(q)}_{U^{(n)}}f(0) \tag{8.111}$$

$$= R^{(q)}_{Y^{(n)}0}f(x) + e^{\Phi_{Y^{(n)}}(q)x}R^{(q)}_{U^{(n)}}f(0). \tag{8.112}$$

By [17, Lemma 8.3] and *i*), we obtain

$$(8.112) \to R^{(q)}_{Y0}f(x) + e^{\Phi_Y(q)x}R^{(q)}_U f(0) = R^{(q)}_U f(x) \tag{8.113}$$

as $n\uparrow\infty$.

iii) We prove (8.81) for $x>0$. We divide

$$R^{(q)}_U f(x) = \mathbb{E}^U_x\left[\int_0^{T_0^-}e^{-qt}f(U_t)dt\right] + \mathbb{E}^U_x\left[\int_{T_0^-}^{T_0}e^{-qt}f(U_t)dt\right] + \mathbb{E}^U_x\left[\int_{T_0}^{\infty}e^{-qt}f(U_t)dt\right] \tag{8.114}$$

and we can divide $R^{(q)}_{U^{(n)}}f(x)$ similarly. By the definition of U and [12, Theorem 3.12], we have the following:

$$\mathbb{E}^U_x\left[\int_0^{T_0^-} e^{-qt}f(U_t)dt\right] = \mathbb{E}^X_x\left[\int_0^{T_0^-} e^{-qt}f(X_t)dt\right], \tag{8.115}$$

$$\mathbb{E}^U_x\left[\int_{T_0^-}^{T_0} e^{-qt}f(U_t)dt\right] = \mathbb{E}^X_x\left[e^{-qT_0^-}R^{(q)}_{k_{T_0^+}Y}f(\psi(X_{T_0^-–},X_{T_0^-}));T_0^-<\infty\right], \tag{8.116}$$

$$\mathbb{E}^U_x\left[\int_{T_0}^{\infty} e^{-qt}f(U_t)dt\right] = \mathbb{E}^X_x\left[e^{-qT_0^-}e^{\Phi_Y(q)\psi(X_{T_0^-–},X_{T_0^-})};T_0^-<\infty\right]R^{(q)}_U f(0), \tag{8.117}$$

where we understand $\psi(0,0) = 0$. We have similar identities also for $U^{(n)}$. By the dominated convergence theorem, by the uniformly convergent coupling, by [17, Lemma 8.3], $i)$ and by $\lim_{n\uparrow\infty}\Phi_{Y^{(n)}} = \Phi_Y$, it is sufficient to prove that

$$\begin{cases} T_0^-(X^{(n)}) \to T_0^-(X), \\ 1_{\{T_0^-(X^{(n)})<\infty\}}\psi^{(n)}(X^{(n)}_{T_0^-(X^{(n)})-}, X^{(n)}_{T_0^-(X^{(n)})}) \to 1_{\{T_0^-(X)<\infty\}}\psi(X_{T_0^-(X)-}, X_{T_0^-(X)}) \end{cases} \tag{8.118}$$

hold as $n \uparrow \infty$ almost surely.

First, we prove (8.118) on $A := \{T_0^-(X) = \infty\} \cup \{T_0^-(X) < \infty, X_{T_0^-(X)} < 0\}$. We have

$$\inf_{t\in[0,T_0^-(X))} X_t > 0 \text{ and } X_{T_0^-(X)} < 0 \text{ a.s. on } A. \tag{8.119}$$

For almost every sample path with A based on the uniformly convergent coupling of [2, pp.210], we have

$$\inf_{t\in[0,T_0^-(X))} X^{(n)}_t \underset{n\uparrow\infty}{\to} \inf_{t\in[0,T_0^-(X))} X_t \text{ on } A \tag{8.120}$$

$$X^{(n)}_{T_0^-(X)} \underset{n\uparrow\infty}{\to} X_{T_0^-(X)} \text{ on } \left(T_0^-(X) < \infty\right), \tag{8.121}$$

so that we have

$$T_0^-(X) = T_0^-(X^{(n)}) \text{ for large } n \text{ on } A \tag{8.122}$$

and

$$\lim_{n\uparrow\infty} X^{(n)}_{T_0^-(X^{(n)})-} = X_{T_0^-(X)-} \text{ and } \lim_{n\uparrow\infty} X^{(n)}_{T_0^-(X^{(n)})} = X_{T_0^-(X)} \text{ on } A\cap\{T_0^-(X)<\infty\}. \tag{8.123}$$

By (8.123) and the definition of $\psi^{(n)}$, we obtain (8.118) on A.

Second, we prove (8.118) on $A^c = \{T_0^-(X) < \infty, X_{T_0^-(X)} = 0\}$. Let $\epsilon > 0$ and let us argue on A^c. Set $I_\epsilon := [T_0^-(X)-\epsilon, T_0^-(X)+\epsilon]$ and $\epsilon' := \left(\inf_{t\in[0,T_0^-(X)-\epsilon]} X_t\right) \wedge |\inf_{t\in I_\epsilon} X_t|$. Then there exists $N(\epsilon) > 0$ such that for all $n > N(\epsilon)$, we have

$$\sup_{t\in[0,T_0^-(X)+\epsilon]} \left|X^{(n)}_t - X_t\right| < \epsilon'. \tag{8.124}$$

By (8.124), (8.53) and the definition of $\psi^{(n)}$, for $n > N(\epsilon)$, we have

$$T_0^-(X) - \epsilon < T_0^-(X^{(n)}) < T_0^-(X) + \epsilon, \tag{8.125}$$

$$\psi^{(n)}(X^{(n)}_{T_0^-(X^{(n)})-}, X^{(n)}_{T_0^-(X^{(n)})}) < 2(l \vee c_1)\left(\sup_{t\in I_\epsilon} X_t - \inf_{t\in I_\epsilon} X_t\right). \tag{8.126}$$

By (8.125) and (8.126), we have (8.118) on A^c.

The proof is therefor completed. $\square$

We now obtain the following corollary for the convergence in distribution. Let $\overline{W}_U^{(q)}$ be the same as that in Corollary 8.2.

Corollary 8.6 ([16, Corollary 4.3])**.** *Under the same assumption of Theorem 8.5, the process* $(U^{(n)}, \mathbb{P}_x^{U^{(n)}})$ *converges in distribution to* $(U, \mathbb{P}_x^U)$ *for all* $x \in \mathbb{R}$.

Proof. This proof is almost the same as that of [17, Theorem 8.1 and 8.5].

We prove, for $f \in C_0$,

$$R^{(q)}_{U^{(n)}} f \to R^{(q)}_U f \tag{8.127}$$

uniformly as $n \uparrow \infty$. Then using Theorem 6.3 and [20, Theorem 3.4.2], we have, for $t > 0$ and $f \in C_0$,

$$P_t^{U^{(n)}} f \to P_t^U f \tag{8.128}$$

uniformly as $n \uparrow \infty$, where

$$P_t^Z f(x) = \mathbb{E}_x^Z[f(Z_t)] \tag{8.129}$$

for a Markov process Z and $t > 0$. Using [10, Theorem 19.25], we can conclude that $(U^{(n)}, \mathbb{P}_x^{U^{(n)}})$ converges in distribution to $(U, \mathbb{P}_x^U)$ for all $x \in \mathbb{R}$.

Let us prove (8.127). We divide the proof of (8.127) into three steps.

Step.1 Let $k > 0$ be a constant. We prove $\{\overline{W}^{(q)}_{U^{(n)}}(x)\}_{n\in\mathbb{N}}$ is equicontinuous in $x \in [-k, k]$. For this, we prove pointwise convergence $\lim_{n\uparrow\infty} \overline{W}^{(q)}_{U^{(n)}}(x) = \overline{W}^{(q)}_U(x)$. Since $\{\overline{W}^{(q)}_{U^{(n)}}\}_{n\in\mathbb{N}}$ is increasing and continuous by Corollary 8.2, the pointwise convergence implies convergence in $x \in [-k, k]$ uniformly, thus $\{\overline{W}^{(q)}_{U^{(n)}}(x)\}_{n\in\mathbb{N}}$ is equicontinuous in $x \in [-k, k]$. The desired convergence is obvious for $x \le 0$ by the definition of $\overline{W}^{(q)}_U(x)$.

For $x > 0$, it suffices to show

$$\lim_{n\uparrow\infty} \mathbb{E}_0^{U^{(n)}}\left[e^{-qT_x^+}\right] = \mathbb{E}_0^U\left[e^{-qT_x^+}\right] \tag{8.130}$$

by Corollary 8.2. By the strong Markov property, we have

$$R_{U^{(n)}}^{(q)} 1_{(-\infty,x)}(0) = \frac{1}{q}\left(1 - \mathbb{E}_0^{U^{(n)}}\left[e^{-qT_x^+}\right]\right) + \mathbb{E}_0^{U^{(n)}}\left[e^{-qT_x^+}\right] R_{U^{(n)}}^{(q)} 1_{(-\infty,x)}(x). \tag{8.131}$$

As $f^- := 1_{(-\infty,x)}$ is not continuous, we take bounded continuous functions such that f_m^- and f_m^+ such that $f_m^- \uparrow f^-$ and $f_m^+ \downarrow f^+ := 1_{(-\infty,x]}$. Using Theorem 8.5, we have $R_{U^{(n)}}^{(q)} f_m^\pm \to R_U^{(q)} f_m^\pm$. It is obvious that $R_{U^{(n)}}^{(q)} f^\pm \to R_U^{(q)} f^\pm$. Thus we obtain (8.130).

Step.2 We may assume without loss of generality that $\|f\| = 1$. Let us prove

$$R_{U^{(n)}}^{(q)} f(x) \to R_U^{(q)} f(x) \text{ uniformly in } x \in [-k,k]. \tag{8.132}$$

Since we have the pointwise convergence by Theorem 8.5, it is sufficient to prove $\{R_{U^{(n)}}^{(q)} f\}_{n\in\mathbb{N}}$ is equicontinuous. For all x, $y \in \mathbb{R}$ with $x < y$, making a computation similar to **1)** of the proof of Theorem 6.3, we have

$$\left| R_{U^{(n)}}^{(q)} f(y) - R_{U^{(n)}}^{(q)} f(x) \right| \le \frac{2}{q} \|f\| \left(1 - \frac{\overline{W}_{U^{(n)}}^{(q)}(x)}{\overline{W}_{U^{(n)}}^{(q)}(y)} \right). \tag{8.133}$$

Let $\epsilon > 0$ be a constant. By **Step.1** and since $\inf_{n\in\mathbb{N}} \overline{W}_{U^{(n)}}^{(q)}(-k) = \inf_{n\in\mathbb{N}} e^{-\Phi_{Y^{(n)}}(q)k} > 0$, we see that there exists $\xi > 0$ such that for all $x, y \in [-k,k]$ with $0 < y - x < \xi$

$$\sup_{n\in\mathbb{N}} \left| \overline{W}_{U^{(n)}}^{(q)}(y) - \overline{W}_{U^{(n)}}^{(q)}(x) \right| \le \epsilon \inf_{n\in\mathbb{N}} \overline{W}_{U^{(n)}}^{(q)}(-k). \tag{8.134}$$

Then we have

$$(8.133) \le \frac{2}{q} \|f\| \frac{\epsilon \inf_{n\in\mathbb{N}} \overline{W}_{U^{(n)}}^{(q)}(-k)}{\overline{W}_{U^{(n)}}^{(q)}(y)} \le \frac{2}{q} \|f\| \epsilon, \tag{8.135}$$

where we used the fact that $\overline{W}_{U^{(n)}}^{(q)}$ is increasing. Therefore we conclude that $\{R_{U^{(n)}}^{(q)} f\}_{n\in\mathbb{N}}$ is equicontinuous.

Step.3 We prove that for any $\epsilon > 0$ there is $k > 0$ such that

$$\sup_{x\in(-\infty,-k)\cup(k,\infty)} \sup_{n\in\mathbb{N}} \left| R_{U^{(n)}}^{(q)} f(x) \right| < \epsilon. \tag{8.136}$$

For all $x < y < 0$ we have

$$\begin{aligned} \left| R_{U^{(n)}}^{(q)} f(x) \right| =& \left| \mathbb{E}_x^{U^{(n)}} \left[\int_0^{T_y^+} e^{-qt} f(U_t^{(n)}) dt \right] + \mathbb{E}_x^{U^{(n)}} \left[e^{-qT_y^+} \right] R_{U^{(n)}}^{(q)} f(y) \right| && (8.137) \\ \le& \frac{1}{q} \sup_{z<y} |f(z)| + \frac{1}{q} \sup_{m\in\mathbb{N}} \mathbb{E}_x^{Y^{(m)}} \left[e^{-qT_y^+} \right] \|f\|. && (8.138) \end{aligned}$$

By the same argument, for all $x > y > 0$, we have

$$\left| R_{U^{(n)}}^{(q)} f(x) \right| \le \frac{1}{q} \sup_{z>y} |f(z)| + \sup_{m\in\mathbb{N}} \mathbb{E}_x^{X^{(m)}} \left[e^{-qT_y^-} \right] \|f\|. \tag{8.139}$$

Since $f \in C_0$, there exists $k_1 > 0$ such that

$$\sup_{|z|>k_1} |f(z)| < \frac{1}{3} q\epsilon. \tag{8.140}$$

Using the uniformly convergence coupling, we have for $x > y > 0$

$$\lim_{n\uparrow\infty} \mathbb{E}^{Y^{(n)}}_{-x}\left[e^{-qT^+_{-y}}\right] = \mathbb{E}^{Y}_{-x}\left[e^{-qT^+_{-y}}\right] \quad \text{and} \quad \lim_{n\uparrow\infty} \mathbb{E}^{X^{(n)}}_{x}\left[e^{-qT^-_{y}}\right] = \mathbb{E}^{X}_{x}\left[e^{-qT^-_{y}}\right] \tag{8.141}$$

and

$$\lim_{x\uparrow\infty} \mathbb{E}^{Y}_{-x}\left[e^{-qT^+_{-y}}\right] = 0 \quad \text{and} \quad \lim_{x\uparrow\infty} \mathbb{E}^{X}_{x}\left[e^{-qT^-_{y}}\right] = 0. \tag{8.142}$$

By (8.142), there exists $k_2 > k_1$ such that

$$\mathbb{E}^{Y}_{-k_2}\left[e^{-qT^+_{-k_1}}\right] < \frac{\epsilon}{3\,\|f\|} \quad \text{and} \quad \mathbb{E}^{X}_{k_2}\left[e^{-qT^-_{k_1}}\right] < \frac{\epsilon}{3\,\|f\|} \tag{8.143}$$

By (8.141), there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\left|\mathbb{E}^{Y^{(n)}}_{-k_2}\left[e^{-qT^+_{-k_1}}\right] - \mathbb{E}^{Y}_{-k_2}\left[e^{-qT^+_{-k_1}}\right]\right| < \frac{\epsilon}{3\,\|f\|} \tag{8.144}$$

and

$$\left|\mathbb{E}^{X^{(n)}}_{k_2}\left[e^{-qT^+_{k_1}}\right] - \mathbb{E}^{X}_{k_2}\left[e^{-qT^+_{k_1}}\right]\right| < \frac{\epsilon}{3\,\|f\|}. \tag{8.145}$$

By (8.142) again, there exists $k_3 > k_2$ such that for all $n \le N$

$$\mathbb{E}^{Y^{(n)}}_{-k_3}\left[e^{-qT^+_{-k_1}}\right] < \frac{\epsilon}{3\,\|f\|} \quad \text{and} \quad \mathbb{E}^{X^{(n)}}_{k_3}\left[e^{-qT^-_{k_1}}\right] < \frac{\epsilon}{3\,\|f\|} \tag{8.146}$$

Thus we obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{Y^{(n)}}_{-k_3}\left[e^{-qT^+_{-k_1}}\right] < \frac{2\epsilon}{3\,\|f\|} \quad \text{and} \quad \sup_{n\in\mathbb{N}} \mathbb{E}^{X^{(n)}}_{k_3}\left[e^{-qT^-_{k_1}}\right] < \frac{2\epsilon}{3\,\|f\|}. \tag{8.147}$$

By (8.138), (8.139), (8.140) and (8.147), we obtain (8.136).

The proof is complete. □

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