# A geometric study of Dynkin quiver type quantum affine Schur-Weyl duality

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# Contents

# Chapter 1

# Overview

## 1.1 Motivations

Let us start with a brief review of the previous works motivating our geometric study of Dynkin quiver type quantum affine Schur-Weyl duality.

### 1.1.1 Quantum affine Schur-Weyl duality

For a fixed pair $(n, d) \in (\mathbb{Z}_{>0})^2$ of positive integers, we have the following two fundamental objects: the complex simple Lie algebra $\mathfrak{sl}_{n+1} \equiv \mathfrak{sl}_{n+1}(\mathbb{C})$ of type $\mathsf{A}_n$ and the symmetric group $\mathfrak{S}_d$ of degree $d$. On the tensor power $(\mathbb{C}^{n+1})^{\otimes d}$, we have a right action of $\mathfrak{S}_d$ permuting tensor factors, which commutes with the natural left action of the Lie algebra $\mathfrak{sl}_{n+1}$:

$$\mathfrak{sl}_{n+1} \quad \curvearrowright \quad (\mathbb{C}^{n+1})^{\otimes d} \quad \curvearrowleft \quad \mathfrak{S}_d.$$

This natural $(\mathfrak{sl}_{n+1}, \mathfrak{S}_d)$-bimodule structure on $(\mathbb{C}^{n+1})^{\otimes d}$ produces a close relationship between their representation theories. This phenomenon is known as the classical Schur-Weyl duality and has many interesting variants.

The quantum affine Schur-Weyl duality is a variant involving their quantum affinizations: the quantum loop algebra $U_q(L\mathfrak{sl}_{n+1})$ of $\mathfrak{sl}_{n+1}$ and the affine Hecke algebra $H_d^{\mathrm{af}}(q)$ of $GL_d$. Both algebras are defined over $\Bbbk := \mathbb{Q}(q)$. Here we equip the tensor power $\mathbb{V}^{\otimes d}$ of the natural representation $\mathbb{V} := \Bbbk^{n+1}[z^{\pm 1}]$ of $U_q(L\mathfrak{sl}_{n+1})$ with a commuting right action of $H_d^{\mathrm{af}}(q^{-2})$ using the $R$-matrices:

$$U_q(L\mathfrak{sl}_{n+1}) \quad \curvearrowright \quad \mathbb{V}^{\otimes d} \quad \curvearrowleft \quad H_d^{\mathrm{af}}(q^{-2}).$$

Chari-Pressley [9] proved that the induced functor

$$H_d^{\mathrm{af}}(q^{-2})\text{-mod} \to U_q(L\mathfrak{sl}_{n+1})\text{-mod}; \quad M \mapsto \mathbb{V}^{\otimes d} \otimes_{H_d^{\mathrm{af}}(q^{-2})} M$$

gives an equivalence between suitable subcategories of finite-dimensional modules. For example, when $n \geq d$, this functor is fully faithful and its essential image is closed under extensions of modules.

### 1.1.2 Ginzburg-Reshetikhin-Vasserot's realization

The quantum affine Schur-Weyl duality has a beautiful geometric realization due to Ginzburg-Reshetikhin-Vasserot [20]. Here we recall their construction.

Let $\mu_d : \mathcal{F}_d \to \mathcal{N}_d$ be the Springer resolution of the nilpotent cone $\mathcal{N}_d$ of $\mathfrak{gl}_d(\mathbb{C})$, where $\mathcal{F}_d$ is the cotangent bundle of the full flag variety of $GL_d(\mathbb{C})$. The morphism $\mu_d$ is equivariant with respect to a natural action of the group $\mathbb{G}_d := GL_d(\mathbb{C}) \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts as the scalar multiplication on the cone $\mathcal{N}_d$. Due to Ginzburg and Kazhdan-Lusztig [33], the affine Hecke algebra $H_d^{\mathrm{af}}(q^{-2})$ is isomorphic to the convolution algebra $K^{\mathbb{G}_d}(\mathcal{Z}_d) \otimes_A \Bbbk$ of the equivariant $K$-group of the Steinberg variety $\mathcal{Z}_d := \mathcal{F}_d \times_{\mathcal{N}_d} \mathcal{F}_d$, where $A = R(\mathbb{C}^\times) = \mathbb{Z}[q^{\pm 1}]$ is the representation ring of $\mathbb{C}^\times$. On the other hand, we consider another Steinberg type variety $Z_d := \mathfrak{M}_d \times_{\mathcal{N}_d} \mathfrak{M}_d$. Here $\mathfrak{M}_d$ is the cotangent bundle of the variety of partial flags in $\mathbb{C}^d$ of length $\leq n+1$. Due to Ginzburg-Vasserot [21], there is an algebra homomorphism $\Phi : U_q(L\mathfrak{sl}_{n+1}) \to K^{\mathbb{G}_d}(Z_d) \otimes_A \Bbbk$ with some nice properties. Based on these facts, Ginzburg-Reshetikhin-Vasserot considered the intermediary fiber product $\mathfrak{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d$ and identified its equivariant $K$-group with the bimodule $\mathbb{V}^{\otimes d}$. More precisely, they established an isomorphism

$$\mathbb{V}^{\otimes d} \cong K^{\mathbb{G}_d}(\mathfrak{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d) \otimes_A \Bbbk$$

making the following diagram commute:

$$\begin{array}{ccccc} U_q(L\mathfrak{sl}_{n+1}) & \longrightarrow & \mathrm{End}\left(\mathbb{V}^{\otimes d}\right) & \longleftarrow & H_d^{\mathrm{af}}(q^2) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K^{\mathbb{G}_d}(Z_d) \otimes_A \Bbbk & \longrightarrow & \mathrm{End}\left(K^{\mathbb{G}_d}(\mathfrak{M}_d \times_{\mathcal{N}_d} \mathcal{F}_d) \otimes_A \Bbbk\right) & \longleftarrow & K^{\mathbb{G}_d}(\mathcal{Z}_d) \otimes_A \Bbbk, \end{array} \tag{1.1}$$

where horizontal arrows denote the bimodule structures.

### 1.1.3 A generalization associated with a Dynkin quiver

Recently, in a series of papers [25, 26, 27, 28], Kang, Kashiwara, Kim and Oh have established some interesting generalized versions of the quantum affine Schur-Weyl duality. One of them (treated in [26] by Kang-Kashiwara-Kim) is associated with a pair $(Q, \beta)$ of a Dynkin quiver $Q$ of type $\mathsf{ADE}$ and a sum $\beta = \sum_i d_i \alpha_i$ of simple roots, which plays a similar role as the pair $(n, d)$ in the previous paragraphs. One player here is the quantum loop algebra $U_q(L\mathfrak{g})$ of the complex simple Lie algebra $\mathfrak{g}$ whose Dynkin diagram is the underlying graph of $Q$. The other is the *quiver Hecke algebra* $H_Q(\beta)$ associated with $(Q, \beta)$, or actually its completion $\widehat{H}_Q(\beta)$ along the grading. The quiver Hecke algebra $H_Q(\beta)$ is regarded as a generalization of the affine Hecke algebra $H_d^{\mathrm{af}}(q)$ from some categorical viewpoints as we explain later.

Inspired by the work of Hernandez-Leclerc [24], Kang-Kashiwara-Kim [26] constructed on a left $U_q(L\mathfrak{g})$-module $\widehat{V}^{\otimes \beta}$ which is a direct sum of some tensor products of affinized fundamental modules a commuting right action of the algebra $\widehat{H}_Q(\beta)$ by using the normalized $R$-matrices:

$$U_q(L\mathfrak{g}) \quad \curvearrowright \quad \widehat{V}^{\otimes \beta} \quad \curvearrowleft \quad \widehat{H}_Q(\beta).$$

However, to make the $\widehat{H}_Q(\beta)$-action well-defined, we need a technical assumption on the simpleness of some specific poles of the normalized $R$-matrices. This assumption was verified for type $\mathsf{AD}$ in [26] by an explicit computation of the denominators of the normalized $R$-matrices. On the other hand, for type $\mathsf{E}$, this had remained a conjecture for a few years until a recent preprint [46] by Oh-Scrimshaw appeared. In this preprint, the assumption for type $\mathsf{E}$ was verified by explicit computations with a computer. Later in the present thesis, we give another uniform proof without a direct computation as a by-product of our geometric realization of the bimodule $\widehat{V}^{\otimes \beta}$.

When our quiver is of type $\mathsf{A}_n$ with a monotone orientation $Q = (1 \to 2 \to \cdots \to n)$, the corresponding complete quiver Hecke algebra $\widehat{H}_Q(\beta)$ is known to be isomorphic to a certain central completion of the affine Hecke algebra $H_d^{\mathrm{af}}(q^{-2})$ with $d = \mathrm{ht}\, \beta$ ([4], [47]). Under this isomorphism, we can obtain Kang-Kashiwara-Kim's bimodule $\widehat{V}^{\otimes \beta}$ for this case as the corresponding completion of the $(U_q(L\mathfrak{sl}_{n+1}), H_d^{\mathrm{af}}(q^{-2}))$-bimodule $\mathbb{V}^{\otimes d}$ in the usual quantum affine Schur-Weyl duality. In this sense, Kang-Kashiwara-Kim's construction can be seen as a generalization of the usual quantum affine Schur-Weyl duality.

### 1.1.4 From a categorical viewpoint

Under the well-definedness assumption, Kang-Kashiwara-Kim [26] proved that the induced functor between the categories of finite-dimensional modules

$$\mathscr{F}_{Q,\beta}\colon \widehat{H}_Q(\beta)\text{-mod}_{\mathrm{fd}} \to U_q(L\mathfrak{g})\text{-mod}_{\mathrm{fd}}; \quad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{\widehat{H}_Q(\beta)} M$$

enjoys several nice properties as explained below.

Associated with a Dynkin quiver $Q$, we can consider the following two interesting monoidal categories $\mathscr{M}_Q$ and $\mathscr{C}_Q$. These kinds of monoidal categories are now a subject of intensive study in connection with the monoidal categorifications of cluster algebras.

The first one $\mathscr{M}_Q$ is the direct sum of categories $\mathscr{M}_{Q,\beta} = \widehat{H}_Q(\beta)\text{-mod}_{\mathrm{fd}}$ of finite-dimensional modules of the completed quiver Hecke algebra $\widehat{H}_Q(\beta)$. The monoidal structure on $\mathscr{M}_Q$ is given by so-called convolution product (an analogue of the parabolic induction for the affine Hecke algebra). The quiver Hecke algebra (which is also known as the Khovanov-Lauda-Rouquier algebra) was introduced by Khovanov-Lauda [34] and by Rouquier [47] independently as an algebraic object which generalizes the affine Hecke algebra $H_d^{\mathrm{af}}(q)$ in the sense that it gives a categorification of the dual of the integral form $U_v^+(\mathfrak{g})_{\mathbb{Z}}$ of the positive part of the quantized enveloping algebra $U_v(\mathfrak{g})$. More precisely, the quiver Hecke algebra $H_Q(\beta)$ is equipped with a $\mathbb{Z}$-grading. Thus the Grothendieck group direct sum over $\beta$ of the categories of finite-dimensional graded $H_Q(\beta)$-modules becomes a $\mathbb{Z}[v^{\pm 1}]$-algebra, where $v$ corresponds to the grading shift. It is known to be isomorphic to the dual of $U_v^+(\mathfrak{g})_{\mathbb{Z}}$, under which the classes of self-dual simple modules correspond to the dual canonical basis elements. The category $\mathscr{M}_Q$ is obtained by forgetting the gradings of modules, which corresponds to specializing $v$ to 1 at the level of the Grothendieck ring. Therefore the complexified Grothendieck ring of the monoidal category $\mathscr{M}_Q$ is isomorphic to the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group $N$ associated with $\mathfrak{g}$:

$$K(\mathscr{M}_Q)_{\mathbb{C}} \cong \mathbb{C}[N].$$

After this specialization, the classes of simple modules correspond to the dual canonical basis elements.

The second one $\mathscr{C}_Q$ is a certain monoidal subcategory of the category of finite-dimensional modules of the quantum loop algebra $U_q(L\mathfrak{g})$, where $\mathfrak{g}$ is the complex simple Lie algebra whose Dynkin diagram is the underlying graph of the quiver $Q$. This category was introduced by Hernandez-Leclerc [24]. The definition of the category $\mathscr{C}_Q$ involves the Auslander-Reiten quiver of $Q$. Hernandez-Leclerc proved that the complexified Grothendieck ring of $\mathscr{C}_Q$ is also isomorphic to the coordinate algebra $\mathbb{C}[N]$ of the unipotent group $N$ associated with $\mathfrak{g}$:

$$K(\mathscr{C}_Q)_{\mathbb{C}} \cong \mathbb{C}[N].$$

Moreover the classes of simple modules correspond to the dual canonical basis elements also in this case.

Hence we encounter a natural question, originally asked by Hernandez-Leclerc [24], whether there is any functorial relationship between these two monoidal categories $\mathscr{M}_Q$ and $\mathscr{C}_Q$.

Interestingly, Kang-Kashiwara-Kim's functor $\mathscr{F}_{Q,\beta}$ explained above gives an affirmative answer to this question. Let us consider the direct sum functor

$$\mathscr{F}_Q := \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{F}_{Q,\beta} \colon \mathscr{M}_Q = \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{M}_{Q,\beta} \to U_q(L\mathfrak{g})\text{-mod}_{\mathrm{fd}}.$$

Note that this is a functor between monoidal categories. Actually, Kang-Kashiwara-Kim [26] proved that this functor $\mathscr{F}_Q$ is exact and monoidal. Moreover they also proved that it lands on Hernandez-Leclerc's subcategory $\mathscr{C}_Q \subset U_q(L\mathfrak{g})\text{-mod}_{\mathrm{fd}}$ and gives a bijection between simple isomorphism classes. In particular, it induces a ring isomorphism between Grothendieck rings

$$[\mathscr{F}_Q] \colon K(\mathscr{M}_Q) \xrightarrow{\cong} K(\mathscr{C}_Q).$$

Later in the paper [28], Kang-Kashiwara-Kim-Oh conjectured that this functor $\mathscr{F}_Q$ is actually an equivalence of monoidal categories. A goal of the present thesis is to verify this conjecture.

## 1.2 Main results

In the present thesis, we study Kang-Kashiwara-Kim's bimodule $\widehat{V}^{\otimes\beta}$ and Hernandez-Leclerc's category $\mathscr{C}_Q$ using a geometric technique, especially in connection with the equivariant $K$-theory of a certain graded quiver variety due to Nakajima [41]. As a result, we can prove that the functor $\mathscr{F}_Q$ is an equivalence between the monoidal categories $\mathscr{M}_Q$ and $\mathscr{C}_Q$.

First, we give an outline of our geometric realization of the bimodule $\widehat{V}^{\otimes\beta}$. This can be considered as a Dynkin quiver version of Ginzburg-Reshetikhin-Vasserot's geometric realization of the usual quantum affine Schur-Weyl duality explained in Section 1.1.2. In our case, the nilpotent cone $\mathcal{N}_d$ is replaced with the space $E_\beta := \bigoplus_{i\to j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ of representations of the quiver $Q$ over $\mathbb{C}$ of dimension vector $\beta$. The group $G_\beta := \prod_i GL_{d_i}(\mathbb{C})$ naturally acts on $E_\beta$. Instead of the Springer resolution $\mathcal{F}_d \to \mathcal{N}_d$, we consider a proper morphism $\mathcal{F}_\beta \to E_\beta$ from a "quiver flag variety" $\mathcal{F}_\beta$ introduced by Lusztig in order to construct the canonical basis of the quantum group. Varagnolo-Vasserot [49] proved that the quiver Hecke algebra $H_Q(\beta)$ is isomorphic to the convolution algebra of the equivariant Borel-Moore homology $H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)$, where $\mathcal{Z}_\beta := \mathcal{F}_\beta \times_{E_\beta} \mathcal{F}_\beta$. After completion, it is isomorphic to the completed equivariant $K$-group $\widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$. On the $U_q(L\mathfrak{g})$-side, we consider a canonical $G_\beta$-equivariant proper morphism $\mathfrak{M}^\bullet_\beta \to \mathfrak{M}^\bullet_{0,\beta}$ between certain graded quiver varieties. By Nakajima [42], we have an algebra homomorphism $\widehat{\Phi}_\beta \colon U_q(L\mathfrak{g}) \to \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$, where $Z^\bullet_\beta := \mathfrak{M}^\bullet_\beta \times_{\mathfrak{M}^\bullet_{0,\beta}} \mathfrak{M}^\bullet_\beta$. The key of our construction is a $G_\beta$-equivariant isomorphism

$$\mathfrak{M}^\bullet_{0,\beta} \cong E_\beta$$

due to Hernandez-Leclerc [24], which was originally established in order to give a geometric interpretation to the isomorphism $K(\mathscr{C}_Q)_\mathbb{C} \cong \mathbb{C}[N]$. This allows us to form the intermediary fiber product $\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta$. The following theorem is a main result of the present thesis.

**Theorem 1.2.1** (= Theorem 4.3.4 + Theorem 4.3.6)**.** There is an isomorphism

$$\widehat{V}^{\otimes\beta} \cong \widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk$$

such that the following diagram commutes (up to a twist):

$$\begin{array}{ccccc}
U_q(L\mathfrak{g}) & \longrightarrow & \mathrm{End}\left(\widehat{V}^{\otimes\beta}\right) & \longleftarrow & \widehat{H}_Q(\beta) \\
\downarrow{\scriptstyle \widehat{\Phi}_\beta} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk,
\end{array}$$

where the horizontal arrows denote the bimodule structures.

Actually, our geometric construction of the $\widehat{H}_Q(\beta)$-action is independent of that of [26], which shares the same characterization of the actions. Therefore, their comparison yields a uniform proof of:

**Corollary 1.2.2** (= Corollary 4.2.3)**.** Kang-Kashiwara-Kim's assumption [26, Conjecture 4.3.2] on the simpleness of some specific poles of normalized $R$-matrices for tensor products of fundamental modules is true for any quiver $Q$ of type $\mathsf{ADE}$.

Besides, the equivariant Chern character maps enable us to identify the convolution algebras of the equivariant $K$-groups with completed equivariant Yoneda algebras of perverse sheaves on $E_\beta$, sometimes called *geometric extension algebras*. More precisely, we can prove the following.

**Theorem 1.2.3** (=Theorem 4.3.2)**.** The equivariant Chern character maps induce the following commutative diagram:

$$\begin{array}{ccccc} \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)^{\mathrm{op}}_\Bbbk \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)^\wedge & \longrightarrow & \mathrm{End}\left(\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}_\beta)^\wedge\right) & \longleftarrow & \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)^{\wedge\mathrm{op}}, \end{array}$$

where $\mathcal{L}^\bullet_\beta$ (resp. $\mathcal{L}_\beta$) is the (derived) push-forward of the trivial local system $\underline{\Bbbk}$ along the proper morphism $\mathfrak{M}^\bullet_\beta \to E_\beta$ (resp. $\mathcal{F}_\beta \to E_\beta$). In particular, the bimodule $\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk$ induces a Morita equivalence between the convolution algebras $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ and $\widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$.

Note that, in contrast to the isomorphism $\widehat{H}_Q(\beta) \cong \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$, Nakajima's homomorphism $\widehat{\Phi}_\beta \colon U_q(L\mathfrak{g}) \to \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ is unfortunately neither injective nor surjective. Thus we need to add some more study about it and actually this is the most difficult part. On the one hand, the (modified) quantum loop algebra $\dot{U}_q(L\mathfrak{g})$ has a structure of an *affine cellular algebra*, which is characterized by a chain of ideals whose subquotients are isomorphic to global Weyl modules (Beck-Nakajima [3, 45]). This can be seen as a filtered/quantum-loop analogue of the Peter-Weyl theorem. On the other hand, the convolution algebra $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ has a chain of ideals arising from the $G_\beta$-orbit stratification of the space $E_\beta$. We compare these structures via Nakajima's homomorphism to prove:

**Theorem 1.2.4** (= Proposition 2.3.6 + Theorem 3.3.6)**.** There is a block decomposition of the category $\mathscr{C}_Q$

$$\mathscr{C}_Q \cong \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{C}_{Q,\beta}$$

corresponding to the weight decomposition of the unipotent coordinate ring $\mathbb{C}[N]$. Moreover, for each $\beta \in \mathsf{Q}^+$, the pull-back along Nakajima's homomorphism $\widehat{\Phi}_\beta \colon U_q(L\mathfrak{g}) \to \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ induces the equivalence of categories:

$$\widehat{\Phi}^*_\beta \colon \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk\text{-mod}_{\mathrm{fd}} \xrightarrow{\simeq} \mathscr{C}_{Q,\beta}.$$

Combining the above three theorems, we finally obtain the conclusion.

**Corollary 1.2.5** (= Theorem 4.3.9)**.** For each $\beta \in \mathsf{Q}^+$, Kang-Kashiwara-Kim's bimodule $\widehat{V}^{\otimes\beta}$ gives an equivalence of categories:

$$\mathscr{F}_{Q,\beta} \colon \mathscr{M}_{Q,\beta} \xrightarrow{\simeq} \mathscr{C}_{Q,\beta}.$$

Therefore, summing up over $\beta \in \mathsf{Q}^+$, we obtain the equivalence of monoidal categories:

$$\mathscr{F}_Q \colon \mathscr{M}_Q \xrightarrow{\simeq} \mathscr{C}_Q.$$

**Remark 1.2.6.** The convolution algebra $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ turns out to be an *affine quasi-hereditary algebra*, or equivalently, its module category $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk\text{-mod}_{\mathrm{fg}}$ is an *affine highest weight category*. The notion of affine highest weight category is introduced by Kleshchev [35] as a generalization of the notion of highest weight category introduced by Cline-Parshall-Scott [12]. An affine highest weight category is characterized by some special homological properties. In particular, it has *standard modules*, which filter projective modules. The quiver Hecke algebra $H_Q(\beta)$ is known to be an affine quesi-hereditary algebra by Kato [31] and also by Brundan-Kleshchev-McNamara [5]. Thus we could prove the equivalence $\mathscr{F}_{Q,\beta} \colon \mathscr{M}_{Q,\beta} \xrightarrow{\simeq} \mathscr{C}_{Q,\beta}$ as a consequence of a comparison of the affine highest weight structures. Indeed, in [18] we realized this idea by using a theory of tilting modules (see Theorem A.2.7). In the present thesis, we do not pursue this direction because it looks less elegant than our geometric realization and anyway a geometric discussion seems to be inevitable. However, we shall give a proof of the fact that the convolution algebra $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ is an affine quasi-hereditary algebra using the equivariant $K$-theory and specify the standard modules for this case as it might be interesting in itself.

## 1.3 Organization of the thesis

This thesis is written as an edited reprint of the author's two papers [18] (a preprint) and [19] (already published online). Roughly speaking, Chapter 2 and Chapter 3 are based on the 1st paper [18], and Chapter 4 is based on the 2nd paper [19]. However, a large part has been suitably modified compared with the original papers, especially the notation and the ordering of the explanation.

Chapter 2 is a review of the representation theory of the quantum loop algebras $U_q(L\mathfrak{g})$ and Hernandez-Leclerc's category $\mathscr{C}_Q$. In Section 2.1, we introduce some notation around a Dynkin quiver and its representation theory in connection with the associated root system. We also define a coordinate $\phi$ of the Auslander-Reiten quiver of the representation category $\operatorname{Rep} Q$ in Section 2.1.3, which plays an important role throughout the thesis. In Section 2.2, we collect some known facts about the representation theory of the quantum loop algebra $U_q(L\mathfrak{g})$ of type $\mathsf{ADE}$. In Section 2.3, we recall the definition of Hernandez-Leclerc's category $\mathscr{C}_Q$ and see some basic properties. We give a block decomposition of the category $\mathscr{C}_Q$ in Section 2.3.3.

Chapter 3 is a study of the category $\mathscr{C}_Q$ via the equivariant $K$-theory of the graded quiver variety $\mathfrak{M}^\bullet_\beta$. In Section 3.1, we recall the definition of Nakajima's graded quiver varieties of Dynkin types and prove some basic facts about the group actions on them. In Section 3.2, we focus on the graded quiver varieties $\mathfrak{M}^\bullet_\beta \to \mathfrak{M}^\bullet_{0,\beta}$ associated to the data $(Q, \beta)$. We give a review of Hernandez-Leclerc's isomorphism $\mathfrak{M}^\bullet_{0,\beta} \cong E_\beta$, and also study the group actions in detail. Section 3.3 contains an important part of the present thesis on the convolution algebra $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$. After recalling Nakajima's homomorphism in a general setting in Section 3.3.1 and Section 3.3.2, we focus on the case associated with $(Q, \beta)$ and give a proof of Theorem 1.2.4 in Section 3.3.3. We prove that the convolution algebra $\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ is affine quasi-hereditary in Section 3.3.4, and that it is isomorphic to a geometric extension algebra by the equivariant Chern character map in Section 3.3.5.

Chapter 4 is devoted to a geometric study of Dynkin quiver type quantum affine Schur-Weyl duality. In Section 4.1, we give a brief review of the theory of the quiver Hecke algebras and their geometric realization due to Varagnolo-Vasserot. Section 4.2 is a short account of the original algebraic construction of the bimodule $\widehat{V}^{\otimes\beta}$ by Kang-Kashiwara-Kim. Here we state the conjectures by Kang-Kashiwara-Kim-Oh precisely. In Section 4.3, we present a geometric realization of the bimodule $\widehat{V}^{\otimes\beta}$ and prove Theorem 1.2.1 and Theorem 1.2.3.

In Appendix A, we collect some general facts about the equivariant $K$-thoery and affine quasi-hereditary algebras (affine highest weight categories).

## 1.4 Overall convention

- We write the set of integers (resp. rational numbers, complex numbers) by $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{C}$) as usual;

- An algebra $A$ is always associative and unital (except for the modified quantum loop algebra $\dot{U}_q(L\mathfrak{g})$ defined in Section 2.2.2). We denote by $A^{\mathrm{op}}$ (resp. $A^{\times}$) the opposite algebra (resp. the set of invertible elements) of $A$;

- The category of left $A$-modules is denoted by $A$-mod. Its full subcategory consisting of finitely generated $A$-modules is denoted by $A\text{-mod}_{\mathrm{fg}}$;

- For a two-sided ideal $\mathfrak{a} \subset A$ and a left $A$-module $M$, we denote the quotient module $M/\mathfrak{a}M$ by $M/\mathfrak{a}$ for simplicity;

- Working over a base field $\mathbb{F}$, the symbol $\otimes$ (resp. Hom) stands for $\otimes_{\mathbb{F}}$ (resp. $\mathrm{Hom}_{\mathbb{F}}$) if there is no other clarification. If $A$ is an $\mathbb{F}$-algebra, we denote by $A\text{-mod}_{\mathrm{fd}}$ the category of finite-dimensional left $A$-modules;

- For $i = 1, 2$, let $R_i$ be a complete local commutative $\mathbb{F}$-algebra with maximal ideal $\mathfrak{r}_i \subset R_i$ with $R_i/\mathfrak{r}_i \cong \mathbb{F}$. For any $R_i$-module $M_i$ $(i = 1, 2)$, we denote by $M_1 \hat{\otimes} M_2$ the completion of the $(R_1 \otimes R_2)$-module $M_1 \otimes M_2$ with respect to the maximal ideal $\mathfrak{r}_1 \otimes R_2 + R_1 \otimes \mathfrak{r}_2 \subset R_1 \otimes R_2$. Note that $M_1 \hat{\otimes} M_2$ is a module over the complete local algebra $R_1 \hat{\otimes} R_2$;

- For an abelian (resp. additive) category $\mathscr{A}$, we denote its Grothendieck group (resp. split Grothendieck group) by $K(\mathscr{A})$. The class of an object $X \in \mathscr{A}$ in $K(\mathscr{A})$ is denoted by $[X]$. For a field $\mathbb{F}$, we define $K(\mathscr{A})_{\mathbb{F}} := K(\mathscr{A}) \otimes_{\mathbb{Z}} \mathbb{F}$.

# Chapter 2

# Representation theory of quantum loop algebras

## 2.1 Dynkin quivers

In this section, we introduce some notation around a Dynkin quiver and its representation theory in connection with the associated root system (Gabriel's theorem). We also define a coordinate $\phi$ of the Auslander-Reiten quiver of the representation category $\operatorname{Rep} Q$ in Section 2.1.3, which plays a crucial role throughout the thesis.

### 2.1.1 Notation

Throughout this thesis, we fix a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ of type $\mathsf{ADE}$ and a quiver $Q = (I, \Omega)$ whose underlying graph is the Dynkin diagram of $\mathfrak{g}$ (see Figure 2.1 below), where $I = \{1, 2, \dots, n\}$ (resp. $\Omega$) is the set of vertices (resp. arrows). For an arrow $h \in \Omega$, let $h', h'' \in I$ denote its origin and goal respectively. We write $i \sim j$ (resp. $i \to j$) if there is an arrow $h \in \Omega$ such that $\{i, j\} = \{h', h''\}$ (resp. $(i, j) = (h', h'')$). Then the Cartan matrix $(a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

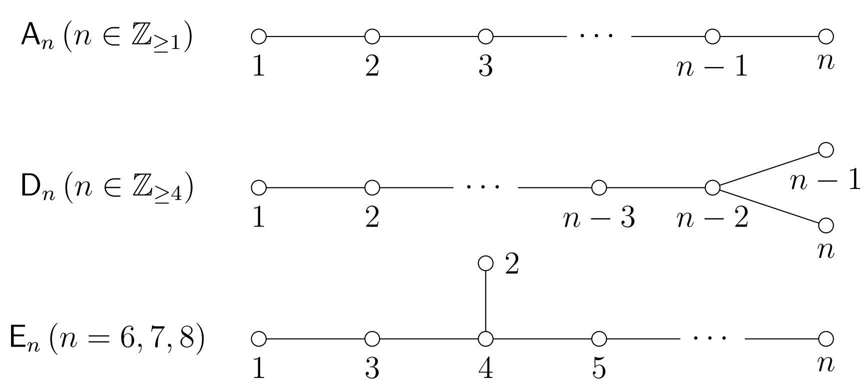



Figure 2.1: Dynkin diagrams of type $\mathsf{ADE}$

Let $\mathsf{P}^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$ be the coroot lattice of $\mathfrak{g}$. The fundamental weights $\{\varpi_i\}_{i \in I}$ form a basis of the weight lattice $\mathsf{P} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}^\vee, \mathbb{Z})$ which is dual to $\{h_i\}_{i \in I}$. Let $\alpha_i = \sum_{j \in I} a_{ij} \varpi_j$ be the $i$-th simple root and $\mathsf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathsf{P}$ be the root lattice. We put $\mathsf{P}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i$ and $\mathsf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. The Weyl group is the finite group $W$ of linear transformations on $\mathsf{P}$ generated by the set $\{r_i\}_{i \in I}$ of simple reflections, which are given by $r_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathsf{P}$. The set $\mathsf{R}^+$ of positive roots is defined by $\mathsf{R}^+ = (W\{\alpha_i\}_{i \in I}) \cap \mathsf{Q}^+$.

### 2.1.2 Representations of Dynkin quiver

For an element $\beta \in \mathsf{Q}^+$, we fix an $I$-graded $\mathbb{C}$-vector space $D = \bigoplus_{i \in I} D_i$ such that $\underline{\dim}\, D := \sum_{i \in I} (\dim D_i)\alpha_i = \beta$. Let us consider the space

$$E_\beta := \bigoplus_{h \in \Omega} \mathrm{Hom}(D_{h'}, D_{h''})$$

of representations of the quiver $Q$ of dimension vector $\beta$. On the space $E_\beta$, the group $G_\beta := \prod_{i \in I} GL(D_i)$ acts by conjugation. The set $G_\beta \backslash E_\beta$ of $G_\beta$-orbits is naturally in bijection with the set of isomorphism classes of representations of the quiver $Q$ of dimension vector $\beta$. By Gabriel's theorem, for each $\alpha \in \mathsf{R}^+$ there exists an indecomposable representation $M_\alpha$ such that $\underline{\dim}\, M_\alpha = \alpha$ uniquely up to isomorphism. The correspondence $\alpha \mapsto M_\alpha$ gives a bijection between the set $\mathsf{R}^+$ of positive roots and the set of isomorphism classes of indecomposable objects of the category $\mathrm{Rep}\, Q$ of finite-dimensional representations of $Q$. Hence, the set

$$\mathrm{KP}(\beta) = \left\{ (m_\alpha)_{\alpha \in \mathsf{R}^+} \in \mathbb{Z}_{\geq 0}^{\mathsf{R}^+} \;\middle|\; \sum_{\alpha \in \mathsf{R}^+} m_\alpha \alpha = \beta \right\}$$

of Kostant partitions of $\beta$ labels the set of $G_\beta$-orbits: $G_\beta \backslash E_\beta = \{\mathbb{O}_{\mathbf{m}}\}_{\mathbf{m} \in \mathrm{KP}(\beta)}$, where for each $\mathbf{m} = (m_\alpha) \in \mathrm{KP}(\beta)$, the $G_\beta$-orbit $\mathbb{O}_{\mathbf{m}}$ corresponds to the isomorphism class of the representation $\bigoplus_{\alpha \in \mathsf{R}^+} M_\alpha^{\oplus m_\alpha}$. We have the natural $G_\beta$-orbit stratification

$$E_\beta = \bigsqcup_{\mathbf{m} \in \mathrm{KP}(\beta)} \mathbb{O}_{\mathbf{m}}. \tag{2.1}$$

We define a partial order $\preceq$ on the set $\mathrm{KP}(\beta)$ of Kostant partitions of $\beta$ by the opposite of the orbit closure inclusion. More precisely, for $\mathbf{m}, \mathbf{m}' \in \mathrm{KP}(\beta)$, we have $\mathbf{m} \preceq \mathbf{m}'$ if and only if $\overline{\mathbb{O}_{\mathbf{m}}} \supset \mathbb{O}_{\mathbf{m}'}$.

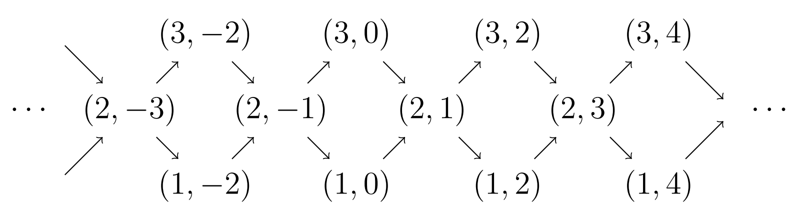
### 2.1.3 Coordinate for the Auslander-Reiten quiver

We fix a height function $\xi \colon I \to \mathbb{Z}; i \mapsto \xi_i$ of the quiver $Q$ i.e. it satisfies $\xi_i = \xi_j + 1$ if $i \to j$. Such a function $\xi$ is determined up to adding a constant. Choose a total ordering $I = \{i_1, i_2, \ldots, i_n\}$ such that $\xi_{i_1} \geq \xi_{i_2} \geq \cdots \geq \xi_{i_n}$ and define the corresponding Coxeter element $c := r_{i_1} r_{i_2} \cdots r_{i_n} \in W$.

The repetition quiver $\widehat{Q} = (\widehat{I}, \widehat{\Omega})$ is an infinite quiver defined by

$$\begin{aligned} \widehat{I} &:= \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}, \\ \widehat{\Omega} &:= \{(i,p) \to (j, p+1) \mid (i,p), (j,p+1) \in \widehat{I},\ i \sim j\}. \end{aligned}$$

**Example 2.1.1.** The following figure shows the repetition quiver $\widehat{Q}$ of a Dynkin quiver of type $\mathsf{A}_3$, which does not depends on the orientation $\Omega$ of $Q$ up to parity.



It is well-known (cf. [22]) that there exists an isomorphism $\phi$ from the Auslander-Reiten quiver of the derived category $D^b(\operatorname{Rep} Q)$ to the repetition quiver $\widehat{Q}$, which depends on the choice of $\xi$ and is described as follows. Since each indecomposable object of $D^b(\operatorname{Rep} Q)$ is isomorphic to a unique stalk complex $M_\alpha[k]$ for some $(\alpha, k) \in \mathsf{R}^+ \times \mathbb{Z}$, we have a bijection between the sets of vertices

$$\mathsf{R}^+ \times \mathbb{Z} \ni (\alpha, k) \mapsto \phi(M_\alpha[k]) \in \widehat{I},$$

which we denote by the same symbol $\phi$. This bijection $\phi\colon \mathsf{R}^+ \times \mathbb{Z} \to \widehat{I}$ is determined inductively as follows:

- For each $i \in I$, we put $\gamma_i := \sum_j \alpha_j$ where $j$ runs all the vertices $j \in I$ such that there is a path in $Q$ from $j$ to $i$. Then $M_{\gamma_i}$ is an injective hull of the 1-dimensional representation $M_{\alpha_i}$. We define $\phi(\gamma_i, 0) := (i, \xi_i)$;
- Inductively, if $\phi(\alpha, k) = (i, p)$ for $(\alpha, k) \in \mathsf{R}^+ \times \mathbb{Z}$, then we define as:

$$\begin{aligned} \phi(c^{\pm 1}(\alpha), k) &:= (i, p \mp 2) && \text{if } c^{\pm 1}(\alpha) \in \mathsf{R}^+, \\ \phi(-c^{\pm 1}(\alpha), k \mp 1) &:= (i, p \mp 2) && \text{if } c^{\pm 1}(\alpha) \in -\mathsf{R}^+. \end{aligned}$$

**Remark 2.1.2.** The action of the Coxeter element $c$ corresponds to the Auslander-Reiten translation.

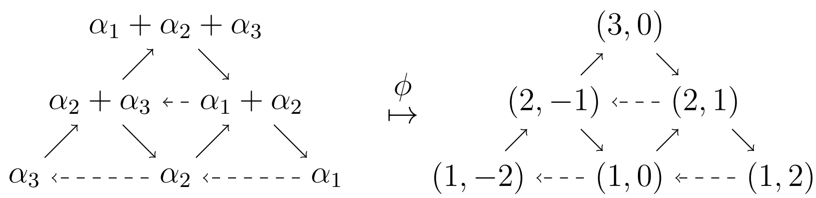
In the present thesis, we mainly consider the restriction of the bijection $\phi$ to the subset $\mathsf{R}^+ = \mathsf{R}^+ \times \{0\}$, which we denote by the same symbol (as an abuse of notation), i.e. we define a map

$$\phi\colon \mathsf{R}^+ \hookrightarrow \widehat{I}$$

by $\phi(\alpha) := \phi(\alpha, 0)$ for $\alpha \in \mathsf{R}^+$. By construction, the full subquiver of $\widehat{Q}$ whose vertex set is $\phi(\mathsf{R}^+)$ is identified with the Auslander-Reiten quiver of the abelian category $\operatorname{Rep} Q$ (the core of the natural $t$-structure of $D^b(\operatorname{Rep} Q)$).

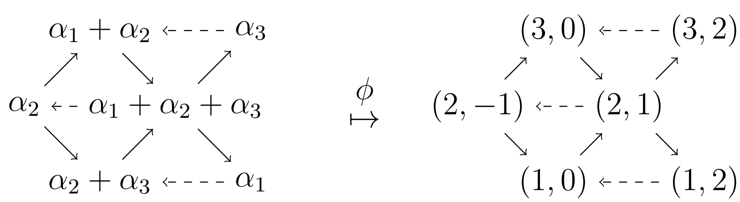
**Example 2.1.3.** Here we give two examples of type $\mathsf{A}_3$. In this case, we have six positive roots $\mathsf{R}^+ = \{\alpha_1, \alpha_2, \alpha_3, (\alpha_1 + \alpha_2), (\alpha_2 + \alpha_3), (\alpha_1 + \alpha_2 + \alpha_3)\}$. In the figures below, the arrows correspond to ones in the Auslander-Reiten quivers and the dashed arrows denote the Auslander-Reiten transformations (or the actions of the Coxeter elements $c$).

(1) For the linearly oriented quiver $Q = (1 \to 2 \to 3)$ with a height $(\xi_1, \xi_2, \xi_3) = (2, 1, 0)$, the map $\phi\colon \mathsf{R}^+ \hookrightarrow \widehat{I}$ is given as the following figure.




More generally, for the linearly oriented quiver $Q = (1 \to 2 \to \cdots \to n)$ of type $\mathsf{A}_n$ with a height $\xi_i = n - i$, we have $\phi(\alpha_i) = (1, n+1-2i)$ for all $1 \le i \le n$.

(2) For the quiver $Q = (1 \to 2 \leftarrow 3)$ with the height $(\xi_1, \xi_2, \xi_3) = (2, 1, 2)$, the map $\phi\colon \mathsf{R}^+ \hookrightarrow \widehat{I}$ is given as the following figure.




## 2.2 Quantum loop algebras

In this section, we recall and prove some facts about representation theory of quantum loop algebras $U_q(L\mathfrak{g})$ of type $\mathsf{ADE}$.

We keep the notations in the previous section. In particular, we fix a complex simple Lie algebra of type $\mathsf{ADE}$ and a Dynkin quiver $Q = (I, \Omega)$ of the same type. Let $q$ be an indeterminate and let $\Bbbk := \overline{\mathbb{Q}(q)}$ be the algebraic closure of the rational function field $\mathbb{Q}(q)$ inside $\bigcup_{m \in \mathbb{Z}_{\ge 1}} \overline{\mathbb{Q}}(\!(q^{1/m})\!)$.

### 2.2.1 Definition

Let $L\mathfrak{g} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z^{\pm 1}]$ be the loop algebra of $\mathfrak{g}$. The quantum loop algebra $U_q(L\mathfrak{g})$ defined below is regarded as a $q$-deformation of the universal enveloping algebra $U(L\mathfrak{g})$ of $L\mathfrak{g}$.

**Definition 2.2.1.** The quantum loop algebra $U_q \equiv U_q(L\mathfrak{g})$ associated to $\mathfrak{g}$ is a $\Bbbk$-algebra with the generators:

$$\{e_{i,r}, f_{i,r} \mid i \in I, r \in \mathbb{Z}\} \cup \{q^h \mid h \in \mathsf{P}^\vee\} \cup \{h_{i,m} \mid i \in I, m \in \mathbb{Z} \setminus \{0\}\}$$

satisfying the following relations:

$$q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad [q^h, h_{i,m}] = [h_{i,m}, h_{j,l}] = 0,$$

$$q^h e_{i,r} q^{-h} = q^{\alpha_i(h)} e_{i,r}, \quad q^h f_{i,r} q^{-h} = q^{-\alpha_i(h)} f_{i,r},$$

$$(z - q^{\pm a_{ij}} w)\psi_i^\varepsilon(z) x_j^\pm(w) = (q^{\pm a_{ij}} z - w) x_j^\pm(w) \psi_i^\varepsilon(z),$$

$$[x_i^+(z), x_i^-(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left( \delta\left(\frac{w}{z}\right) \psi_i^+(w) - \delta\left(\frac{z}{w}\right) \psi_i^-(z) \right),$$

$$(z - q^{\pm a_{ij}} w) x_i^\pm(z) x_j^\pm(w) = (q^{\pm a_{ij}} z - w) x_j^\pm(w) x_i^\pm(z),$$

$$\begin{aligned} \{x_i^\pm(z_1) x_j^\pm(z_2) x_j^\pm(w) - (q + q^{-1}) x_i^\pm(z_1) x_j^\pm(w) x_i^\pm(z_2) \\ + x_j^\pm(w) x_i^\pm(z_1) x_i^\pm(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \qquad \text{if } i \sim j, \end{aligned}$$

where $\varepsilon \in \{+, -\}$ and $\delta(z), \psi_i^\pm(z), x_i^\pm(z)$ are the formal series defined as follows:

$$\delta(z) := \sum_{r=-\infty}^{\infty} z^r, \quad \psi_i^\pm(z) := q^{\pm h_i} \exp\left( \pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} z^{\mp m} \right),$$

$$x_i^+(z) := \sum_{r=-\infty}^{\infty} e_{i,r} z^{-r}, \quad x_i^-(z) := \sum_{r=-\infty}^{\infty} f_{i,r} z^{-r}.$$

In the last relation, the second term $\{z_1 \leftrightarrow z_2\}$ means the exchange of $z_1$ with $z_2$ in the first term.

**Remark 2.2.2.** Let $\widehat{\mathfrak{g}}$ be the (untwisted) affine Lie algebra associated to $\mathfrak{g}$ realized as

$$\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with a suitable Lie algebra structure, where $c$ is a central element and $d := z\frac{\mathrm{d}}{\mathrm{d}z}$ is the degree operator. The derived subalgebra

$$\widehat{\mathfrak{g}}' := [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = L\mathfrak{g} \oplus \mathbb{C}c$$

is a (unique) central extension of the loop algebra $L\mathfrak{g}$.

Because the affine Lie algebra $\widehat{\mathfrak{g}}$ is a Kac-Moody algebra associated to a generalized Cartan matrix of affine type, we have the Drinfeld-Jimbo's quantum enveloping algebra $U_q(\widehat{\mathfrak{g}})$ of $\widehat{\mathfrak{g}}$. By definition, this is a Hopf algebra over $\Bbbk$ generated by the generators $\{e_i, f_i \mid i \in I \cup \{0\}\} \cup \{q^h \mid h \in \mathsf{P}^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d\}$ satisfying the well-known relations. The coproduct $\Delta\colon U_q(\widehat{\mathfrak{g}}) \to U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})$ is given by:

$$\Delta(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h$$

for $i \in I \cup \{0\}, h \in \mathsf{P}^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d$.

The subalgebra $U_q'(\widehat{\mathfrak{g}})$ generated by the Chevalley generators $\{e_i, f_i, q^{\pm h_i} \mid i \in I \cup \{0\}\}$ is a Hopf subalgebra of $U_q(\widehat{\mathfrak{g}})$. This is regarded as a $q$-deformation of the universal enveloping algebra of $\widehat{\mathfrak{g}}'$.

By Beck [2] (originally by Drinfeld), we have a $\Bbbk$-algebra isomorphism $U_q(L\mathfrak{g}) \cong U_q'(\widehat{\mathfrak{g}})/\langle q^c - 1\rangle$. Actually this isomorphism depends on a function $o\colon I \to \{\pm 1\}$ such that $o(i) = -o(j)$ if $i \sim j$. Although the choice does not affect the results of this thesis, we can choose it as $o(i) := (-1)^{\xi_i}$ for each $i \in I$ for instance, where $\xi_i$ is the height we have fixed in Section 2.1.3. Via this isomorphism, the quantum loop algebra $U_q(L\mathfrak{g})$ inherits a structure of Hopf algebra. By [13], it is known that for each $i \in I$ and $r \in \mathbb{Z}_{>0}$, we have

$$\Delta(h_{i,\pm r}) - h_{i,\pm r} \otimes 1 + 1 \otimes h_{i,\pm r} \in \bigoplus_{\gamma \in \mathsf{Q}^+ \setminus \{0\}} (U_q)_{\mp\gamma} \otimes (U_q)_{\pm\gamma}, \tag{2.2}$$

where we put $(U_q)_\gamma := \{x \in U_q \mid q^h x q^{-h} = q^{\gamma(h)} x \ (\forall h \in \mathsf{P}^\vee)\}$. The antipode $S$ is given by

$$S(e_i) = -e_i q^{h_i}, \quad S(f_i) = -q^{-h_i} f_i, \quad S(q^h) = q^{-h},$$

which we use to define dual modules.

### 2.2.2 Finite-dimensional modules

A $U_q$-module $M$ is said to be of type $\mathbf{1}$ if it has a decomposition:

$$M = \bigoplus_{\lambda \in \mathsf{P}} M_\lambda, \quad M_\lambda := \{m \in M \mid q^h m = q^{\lambda(h)} m \, (\forall h \in \mathsf{P}^\vee)\}.$$

A nonzero subspace $M_\lambda$ is called a weight space of $M$ and then $\lambda$ is called a weight of $M$. Let $\mathscr{C}_\mathfrak{g}$ denote the category of finite-dimensional $U_q$-modules of type $\mathbf{1}$. The category $\mathscr{C}_\mathfrak{g}$ becomes an abelian $\Bbbk$-linear rigid monoidal category.

We often use the modified quantum loop algebra denoted by $\dot{U}_q(L\mathfrak{g})$, which is defined by

$$\dot{U}_q \equiv \dot{U}_q(L\mathfrak{g}) := \bigoplus_{\lambda \in \mathsf{P}} U_q a_\lambda, \quad U_q a_\lambda := U_q \Big/ \sum_{h \in \mathsf{P}^\vee} U_q(q^h - q^{\lambda(h)}),$$

where $a_\lambda$ stands for the image of 1 in the quotient. The multiplication is given by

$$a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda, \quad a_\lambda x = x a_{\lambda-\gamma},$$

where $x \in (U_q)_\gamma, \gamma \in \mathsf{Q}$. By definition, considering a $\dot{U}_q(L\mathfrak{g})$-module is the same as considering a $U_q$-module of type $\mathbf{1}$. In particular, we have $\mathscr{C}_\mathfrak{g} = \dot{U}_q(L\mathfrak{g})\text{-mod}_{\mathrm{fd}}$.

In order to describe some more detailed structures of modules, we employ the following notation, which is less standard in references. Let $\mathcal{P} := \bigoplus_{(i,a) \in I \times \Bbbk^\times} \mathbb{Z}\varpi_{i,a}$ be the set of *$\ell$-weights*, which is a free abelian group with a basis $\{\varpi_{i,a} \mid i \in I, a \in \Bbbk^\times\}$. We call a basis element $\varpi_{i,a}$ a *fundamental $\ell$-weight*. An element in the submonoid $\mathcal{P}^+ := \sum \mathbb{Z}_{\geq 0}\varpi_{i,a}$ is said to be *$\ell$-dominant*. We define a $\mathbb{Z}$-linear map $\mathsf{cl} \colon \mathcal{P} \to \mathsf{P}$ by $\varpi_{i,a} \mapsto \varpi_i$. For each $(i,a) \in I \times \Bbbk^\times$, we define the corresponding *$\ell$-root* $\alpha_{i,a} \in \mathcal{P}$ by

$$\alpha_{i,a} := \varpi_{i,aq} + \varpi_{i,aq^{-1}} - \sum_{j \sim i} \varpi_{j,a}.$$

We define the *$\ell$-root lattice* by $\mathcal{Q} := \bigoplus_{(i,a) \in I \times \Bbbk^\times} \mathbb{Z}\alpha_{i,a} \subset \mathcal{P}$ and set $\mathcal{Q}^+ := \sum \mathbb{Z}_{\geq 0}\alpha_{i,a}$. Note that $\mathsf{cl} \colon \mathcal{P} \to \mathsf{P}$ induces a map $\mathsf{cl} \colon \mathcal{Q} \to \mathsf{Q}$ since $\mathsf{cl}(\alpha_{i,a}) = \alpha_i$. We define a partial order $\leq$ on $\mathcal{P}$ called the *$\ell$-dominance order* by the condition that for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}$, we have $\boldsymbol{\lambda} \leq \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu} - \boldsymbol{\lambda} \in \mathcal{Q}^+$.

Let $U_q(L\mathfrak{h})$ denote the commutative $\Bbbk$-subalgebra of $U_q(L\mathfrak{g})$ generated by elements $\{q^h \mid h \in \mathsf{P}^\vee\} \cup \{h_{i,r} \mid i \in I, r \in \mathbb{Z} \setminus \{0\}\}$. A module $M \in \mathscr{C}_\mathfrak{g}$ decomposes into a direct sum of generalized eigenspaces for $U_q(L\mathfrak{h})$ as $M = \bigoplus M_{\Psi^\pm}$, where $\Psi^\pm = (\Psi_i^\pm(z))_{i \in I} \in \Bbbk[\![z^{\mp 1}]\!]^I$ and

$$M_{\Psi^\pm} := \{m \in M \mid (\psi_i^\pm(z) - \Psi_i^\pm(z)\,\mathrm{id})^N m = 0 \text{ for any } i \in I \text{ and } N \gg 0\}.$$

It is known (see [16, Proposition 1]) that if $M_{\Psi^\pm} \neq 0$, there is a unique $\ell$-weight $\boldsymbol{\lambda} = \sum l_{i,a}\varpi_{i,a} \in \mathcal{P}$ such that we have

$$\Psi_i^\pm(z) = q^{\mathsf{cl}(\boldsymbol{\lambda})(h_i)} \left( \prod_{a \in \Bbbk^\times} \left( \frac{1 - aq^{-2}z^{-1}}{1 - az^{-1}} \right)^{l_{i,a}} \right)^\pm, \tag{2.3}$$

where $(-)^{\pm}$ denotes the formal expansion at $z = \infty$ and 0 respectively. In this case, we write $M_{\boldsymbol{\lambda}} = M_{\Psi^{\pm}}$ and call it the *$\ell$-weight space* of $\ell$-weight $\boldsymbol{\lambda}$. By definition, we get

$$M_{\lambda} = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}, \mathsf{cl}(\boldsymbol{\lambda}) = \lambda} M_{\boldsymbol{\lambda}}$$

for each $\lambda \in \mathsf{P}$.

We say a module $M \in \mathscr{C}_{\mathfrak{g}}$ is an *$\ell$-highest weight module* with $\ell$-highest weight $\boldsymbol{\lambda} \in \mathcal{P}$ if there exists a generating vector $m_0 \in M$ satisfying

$$x_i^+(z) m_0 = 0, \quad \psi_i^{\pm}(z) m_0 = q^{\mathsf{cl}(\boldsymbol{\lambda})(h_i)} \left( \prod_{a \in \Bbbk^\times} \left( \frac{1 - aq^{-2}z^{-1}}{1 - az^{-1}} \right)^{l_{i,a}} \right)^{\pm} m_0$$

for each $i \in I$. Compare the latter equation with (2.3). In this case, the $\ell$-highest weight $\boldsymbol{\lambda}$ automatically becomes $\ell$-dominant, i.e. $\boldsymbol{\lambda} \in \mathcal{P}^+$ and we have $M_{\boldsymbol{\lambda}} = \Bbbk \cdot m_0$. Any simple module in $\mathscr{C}_{\mathfrak{g}}$ is known to be an $\ell$-highest weight module and to be determined by its $\ell$-highest weight uniquely up to isomorphism. We denote by $L(\boldsymbol{\lambda})$ the simple module whose $\ell$-highest weight is $\boldsymbol{\lambda} \in \mathcal{P}^+$.

**Remark 2.2.3.** In the original classification result by Chari-Pressley [8], the simple finite-dimensional type $\mathbf{1}$-modules are parametrized by $I$-tuples of polynomials with constant terms 1, i.e. $(\pi_i(u))_{i \in I}$ with $\pi_i(u) \in 1 + u\Bbbk[u]$, which are usually referred as the *Drinfeld polynomials*. In this notation, the simple module $L(\boldsymbol{\lambda})$ of its $\ell$-highest weight $\boldsymbol{\lambda} \in \mathcal{P}^+$ corresponds the Drinfeld polynomial $(\pi_i(u))_{i \in I}$ given by

$$\pi_i(u) = \prod_{a \in \Bbbk^\times} (1 - au)^{l_{i,a}}$$

for each $i \in I$ where $\boldsymbol{\lambda} = \sum l_{i,a} \varpi_{i,a}$.

The following fundamental result is originally conjectured by Frenkel-Reshetikhin [16] and proved by Nakajima [41] (for $\mathfrak{g}$ of type $\mathsf{ADE}$ using quiver varieties) and by Frenkel-Mukhin [15] (for general finite-dimensional simple Lie algebra $\mathfrak{g}$).

**Proposition 2.2.4.** For a dominant $\ell$-weight $\boldsymbol{\lambda} \in \mathcal{P}^+$ and an $\ell$-weight $\boldsymbol{\mu} \in \mathcal{P}$, we have $L(\boldsymbol{\lambda})_{\boldsymbol{\mu}} \neq 0$ only if $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$.

The $q$-character $\chi_q(M)$ of a module $M \in \mathscr{C}_{\mathfrak{g}}$ is defined as an element of the group algebra $\mathbb{Z}[\mathcal{P}] = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}} \mathbb{Z} e^{\boldsymbol{\lambda}}$ by

$$\chi_q(M) := \sum_{\boldsymbol{\lambda} \in \mathcal{P}} (\dim M_{\boldsymbol{\lambda}}) e^{\boldsymbol{\lambda}}.$$

In the references, the $q$-character $\chi_q(M)$ is usually written as a Laurent polynomial in variables $\{Y_{i,a} \mid i \in I, a \in \Bbbk^{\times}\}$, where $Y_{i,a}$ is the element $e^{\varpi_{i,a}} \in \mathbb{Z}[\mathcal{P}]$ in our notation.

**Proposition 2.2.5** (Frenkel-Reshetikhin [16])**.** For $M_1, M_2 \in \mathscr{C}_{\mathfrak{g}}$, we have

$$\chi_q(M_1 \otimes M_2) = \chi_q(M_1) \cdot \chi_q(M_2).$$

Moreover, the induced ring homomorphism $\chi_q \colon K(\mathscr{C}_{\mathfrak{g}}) \to \mathbb{Z}[\mathcal{P}]$ is injective. In particular, the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}})$ is commutative.

*Proof.* This is a consequence of the formula (2.2) and the classification of simple objects in $\mathscr{C}_{\mathfrak{g}}$ described above. See [16, Section 3] for details. □

**Remark 2.2.6.** Nevertheless, $M_1 \otimes M_2 \not\cong M_2 \otimes M_1$ for general $M_1, M_2 \in \mathscr{C}_{\mathfrak{g}}$.

For each $M \in \mathscr{C}_{\mathfrak{g}}$, we define its left dual module $M^*$ (resp. right dual module ${}^*M$) as the dual space $\mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ with the left $U_q(L\mathfrak{g})$-action obtained by twisting the natural right action by using the antipode $S$ (resp. $S^{-1}$). For any $M_1, M_2 \in \mathscr{C}_{\mathfrak{g}}$, we have

$$(M_1 \otimes M_2)^* \cong M_2^* \otimes M_1^*, \quad {}^*(M_1 \otimes M_2) \cong {}^*M_2 \otimes {}^*M_1.$$

We define the following $\mathbb{Z}$-linear maps on $\mathcal{P}$:

$$\begin{aligned} (-)^* &\colon \ \mathcal{P} \to \mathcal{P}; \quad \varpi_{i,a} \mapsto \varpi_{i,a}^* := \varpi_{i^*, aq^{-h}}, \\ {}^*(-) &\colon \ \mathcal{P} \to \mathcal{P}; \quad \varpi_{i,a} \mapsto {}^*\varpi_{i,a} := \varpi_{i^*, aq^{h}}, \end{aligned}$$

where $i \mapsto i^*$ is an involution on $I$ defined by $\alpha_{i^*} := -w_0 \alpha_i$ and $h$ is the Coxeter number (the order of a Coxeter element of $W$). Under this notation, we have

$$L(\varpi_{i,a})^* \cong L(\varpi_{i,a}^*), \quad {}^*L(\varpi_{i,a}) \cong L({}^*\varpi_{i,a}).$$

### 2.2.3 Weyl modules

In this subsection, we recall the global and local Weyl modules of $U_q$ introduced by Chari-Pressley [10]. Also we define the deformed local Weyl module, which plays a role of a standard module of an affine highest weight category.

**Definition 2.2.7.** A $U_q$-module $M$ of type $\mathbf{1}$ is said to be *$\ell$-integrable* if the following property is satisfied: for each $m \in M$, there exists an integer $n_0 \geq 1$ such that we have $e_{i,r_1} e_{i,r_2} \cdots e_{i,r_N} m = f_{i,r_1} f_{i,r_2} \cdots f_{i,r_N} m = 0$ for any $N \geq n_0$ and any $i \in I$, $r_1, \ldots, r_N \in \mathbb{Z}$.

**Remark 2.2.8.** We do not impose that $\dim M_\lambda < \infty$ for $\ell$-integrability. Note that any finite-dimensional modules of type $\mathbf{1}$, i.e. any objects of the category $\mathcal{C}_{\mathfrak{g}}$ are automatically $\ell$-integrable.

First we define the global Weyl modules.

**Definition 2.2.9.** Let $\lambda \in \mathsf{P}^+$ be a dominant weight. We define the corresponding *global Weyl module* $\mathbb{W}(\lambda)$ to be the left $U_q$-module generated by a cyclic vector $w_\lambda$ satisfying the following relations:

$$e_{i,r} w_\lambda = 0, \quad q^h w_\lambda = q^{\lambda(h)} w_\lambda, \quad (f_{i,r})^{\lambda(h_i)+1} w_\lambda = 0,$$

where $i \in I, r \in \mathbb{Z}$ and $h \in \mathsf{P}^\vee$.

For a dominant weight $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathsf{P}^+$, we define the following $\Bbbk$-algebra of partially symmetric Laurent polynomials:

$$R(\lambda) := \bigotimes_{i \in I} (\Bbbk[z_i^{\pm 1}]^{\otimes l_i})^{\mathfrak{S}_{l_i}} = \bigotimes_{i \in I} \Bbbk[z_{i,1}^{\pm 1}, \ldots, z_{i,l_i}^{\pm 1}]^{\mathfrak{S}_{l_i}}. \tag{2.4}$$

**Theorem 2.2.10** (Chari-Pressley [10], Nakajima)**.** Write $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathsf{P}^+$.

(1) The global Weyl module $\mathbb{W}(\lambda)$ is $\ell$-integrable and has the following universal property: If $M$ is an $\ell$-integrable $U_q$-module with a cyclic vector $m \in M_\lambda$ of weight $\lambda$ satisfying $x_i^+(z) m = 0$ for any $i \in I$, then there is a unique $U_q$-homomorphism $\mathbb{W}(\lambda) \to M$ such that $w_\lambda \mapsto m$;

(2) $\mathrm{End}_{U_q}(\mathbb{W}(\lambda)) \cong R(\lambda)$ and $\mathbb{W}(\lambda)$ is free over $R(\lambda)$ of finite rank;

(3) For any $i \in I$, we have:

$$\psi_i^{\pm}(z) w_\lambda = q^{l_i} \prod_{k=1}^{l_i} \left( \frac{1 - q^{-2} z_{i,k} z^{-1}}{1 - z_{i,k} z^{-1}} \right)^{\pm} w_\lambda.$$

*Proof.* See [10, Section 4]. The freeness over $R(\lambda)$ in the assertion (2) is proved by the geometric realization due to Nakajima. For details, see Theorem 3.3.2 and Theorem 3.3.3 (2) below. □

**Remark 2.2.11.** The global Weyl module $\mathbb{W}(\lambda)$ is known to be isomorphic to the level 0 extremal weight module $V^{\max}(\lambda)$ of the extremal weight $\lambda$, defined by Kashiwara [29]. See [10, Proposition 4.5] and [43, Remark 2.15].

Next we consider the local Weyl modules. We identify a point of the quotient space $(\Bbbk^\times)^N / \mathfrak{S}_N$ with a $\mathbb{Z}_{\geq 0}$-linear combination of formal symbols $\{[a] \mid a \in \Bbbk^\times\}$ whose coefficients sum up to $N$. Note that we have

$$\operatorname{Specm} R(\lambda) \cong \prod_{i \in I} \left( (\Bbbk^\times)^{l_i} / \mathfrak{S}_{l_i} \right).$$

Let $\boldsymbol{\lambda} = \sum_{(i,a) \in I \times \Bbbk^\times} l_{i,a} \varpi_{i,a} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight and put $\lambda := \mathsf{cl}(\boldsymbol{\lambda}) \in \mathsf{P}^+$. We denote by $\mathfrak{m}_{\boldsymbol{\lambda}}$ the maximal ideal of $R(\lambda)$ corresponding to the point

$$\left( \sum_{a \in \Bbbk^\times} l_{i,a}[a] \right)_{i \in I} \in \prod_{i \in I} \left( (\Bbbk^\times)^{l_i} / \mathfrak{S}_{l_i} \right).$$

**Definition 2.2.12.** We define the *local Weyl module* $W(\boldsymbol{\lambda})$ corresponding to $\boldsymbol{\lambda} \in \mathcal{P}^+$ by $W(\boldsymbol{\lambda}) := \mathbb{W}(\lambda)/\mathfrak{m}_{\boldsymbol{\lambda}}$. We denote the image of the cyclic vector $w_\lambda \in \mathbb{W}(\lambda)$ by $w_{\boldsymbol{\lambda}} \in W(\boldsymbol{\lambda})$.

**Theorem 2.2.13** (Chari-Pressley [10])**.** Let $\boldsymbol{\lambda} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight.

(1) The local Weyl module $W(\boldsymbol{\lambda})$ is a finite-dimensional $\ell$-highest weight module of $\ell$-highest weight $\boldsymbol{\lambda}$ with $W(\boldsymbol{\lambda})_{\boldsymbol{\lambda}} = \Bbbk \cdot w_{\boldsymbol{\lambda}}$. Moreover it has the following universal property: If $M \in \mathscr{C}_{\mathfrak{g}}$ is an $\ell$-highest weight module of $\ell$-highest weight $\boldsymbol{\lambda}$ with $M_{\boldsymbol{\lambda}} = \Bbbk \cdot m_0$, then there is a unique $U_q$-homomorphism $W(\boldsymbol{\lambda}) \to M$ with $w_{\boldsymbol{\lambda}} \mapsto m_0$;

(2) $W(\boldsymbol{\lambda})$ has a simple head isomorphic to $L(\boldsymbol{\lambda})$.

*Proof.* Follows from Theorem 2.2.10. □

Let us introduce the deformed local Weyl modules as infinitesimal formal deformations of the local Weyl modules. Let $\boldsymbol{\lambda} = \sum l_{i,a}\varpi_{i,a} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight and set $\lambda := \mathsf{cl}(\boldsymbol{\lambda})$. We define the $\mathfrak{m}_{\boldsymbol{\lambda}}$-adic completion as

$$R(\lambda)^{\wedge}_{\boldsymbol{\lambda}} := \varprojlim_k R(\lambda)/\mathfrak{m}_{\boldsymbol{\lambda}}^k.$$

**Definition 2.2.14.** We define the *deformed local Weyl module* $\widehat{W}(\boldsymbol{\lambda})$ corresponding to $\boldsymbol{\lambda} \in \mathcal{P}^+$ by

$$\widehat{W}(\boldsymbol{\lambda}) := \mathbb{W}(\lambda) \otimes_{R(\lambda)} R(\lambda)^{\wedge}_{\boldsymbol{\lambda}} \cong \varprojlim_k \mathbb{W}(\lambda)/\mathfrak{m}_{\boldsymbol{\lambda}}^k.$$

We set $\widehat{w}_{\boldsymbol{\lambda}} := w_\lambda \otimes 1 \in \widehat{W}(\boldsymbol{\lambda})$.

We also use the following algebra:

$$R(\boldsymbol{\lambda}) := \bigotimes_{i \in I} \bigotimes_{a \in \Bbbk^\times} \left(\Bbbk[z_i^{\pm 1}]^{\otimes l_{i,a}}\right)^{\mathfrak{S}_{l_{i,a}}}. \tag{2.5}$$

Note that the algebra $R(\lambda)$ is a subalgebra of $R(\boldsymbol{\lambda})$ for $\lambda = \mathsf{cl}(\boldsymbol{\lambda})$. Let $\mathfrak{r}_{\boldsymbol{\lambda}}$ be a maximal ideal of $R(\boldsymbol{\lambda})$ corresponding the point

$$(l_{i,a}[a])_{(i,a) \in I \times \Bbbk^\times} \in \prod_{(i,a) \in I \times \Bbbk^\times} \left((\Bbbk^\times)^{l_{i,a}}/\mathfrak{S}_{l_{i,a}}\right) = \operatorname{Specm} R(\boldsymbol{\lambda}).$$

Then we have $\mathfrak{m}_{\boldsymbol{\lambda}} = R(\lambda) \cap \mathfrak{r}_{\boldsymbol{\lambda}}$ and there is a natural isomorphism

$$\widehat{R}(\boldsymbol{\lambda}) := \varprojlim_k R(\boldsymbol{\lambda})/\mathfrak{r}_{\boldsymbol{\lambda}}^k \cong R(\lambda)^{\wedge}_{\boldsymbol{\lambda}}. \tag{2.6}$$

Therefore we identify $R(\lambda)^{\wedge}_{\boldsymbol{\lambda}}$ with $\widehat{R}(\boldsymbol{\lambda})$.

**Proposition 2.2.15.** The deformed local Weyl module $\widehat{W}(\boldsymbol{\lambda})$ satisfies the following properties:

(1) For each $M \in \mathscr{C}_{\mathfrak{g}}$, taking the image of $\widehat{w}_{\boldsymbol{\lambda}}$ gives a natural isomorphism:

$$\operatorname{Hom}_{U_q}(\widehat{W}(\boldsymbol{\lambda}), M) \cong \{m \in M_{\boldsymbol{\lambda}} \mid e_{i,r}m = 0 \text{ for any } i \in I, r \in \mathbb{Z}\};$$

(2) $\operatorname{End}_{U_q}(\widehat{W}(\boldsymbol{\lambda})) = \widehat{R}(\boldsymbol{\lambda})$ and $\widehat{W}(\boldsymbol{\lambda})$ is free over $\widehat{R}(\boldsymbol{\lambda})$ of finite rank;

(3) $\widehat{W}(\boldsymbol{\lambda})/\mathfrak{r}_{\boldsymbol{\lambda}} \cong W(\boldsymbol{\lambda})$.

*Proof.* Follows from Theorem 2.2.10. □

### 2.2.4 Affine cellular structure

In this section, we briefly recall the affine cellular algebra structure (in the sense of [36]) of the modified quantum loop algebra $\dot{U}_q$ in terms of the global Weyl modules, following [3], [45].

Let $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathsf{P}^+$. By Theorem 2.2.10 (2), the global Weyl module $\mathbb{W}(\lambda)$ is regarded as a $(\dot{U}_q, R(\lambda))$-bimodule. We obtain a $(R(\lambda), \dot{U}_q)$-bimodule $\mathbb{W}(\lambda)^\sharp$ from $\mathbb{W}(\lambda)$ by twisting the actions of $\dot{U}_q$ and $R(\lambda)$ by the anti-involution $\sharp$ on $\dot{U}_q \otimes \widehat{R}(\lambda)$ determined by

$$\sharp(e_i) = f_i, \quad \sharp(f_i) = e_i, \quad \sharp(q^h) = q^h, \quad \sharp(a_\lambda) = a_\lambda, \quad \sharp(z_{j,k}) = z_{j,k}^{-1},$$

where $e_i, f_i \, (i \in I \cup \{0\})$ are Chevalley generators (see Remark 2.2.2), $h \in \mathsf{P}^\vee$, $\lambda \in \mathsf{P}$ and $z_{j,k} \, (j \in I, 1 \leq k \leq l_j)$ are as in (2.4).

Fix a dominant weight $\lambda \in \mathsf{P}^+$. Let $U_{\leq \lambda}$ be the following quotient of the modified quantum loop algebra $\dot{U}_q$ :

$$U_{\leq \lambda} := \dot{U}_q \Big/ \bigcap_{\mu \leq \lambda} \operatorname{Ann}_{\dot{U}_q} \mathbb{W}(\mu), \tag{2.7}$$

where $\operatorname{Ann}_{\dot{U}_q} M$ denotes the annihilator of a $\dot{U}_q$-module $M$. We fix a numbering $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ of the set $\mathsf{P}^+_{\leq \lambda} := \{\mu \in \mathsf{P}^+ \mid \mu \leq \lambda\}$ such that we have $\lambda_l = \lambda$ and $i < j$ whenever $\lambda_i < \lambda_j$. For each $i \in \{1, 2, \ldots, l-1\}$, we define a two-sided ideal $I_i$ of $U_{\leq \lambda}$ by

$$I_i := \bigcap_{j \leq i} \operatorname{Ann}_{U_{\leq \lambda}} \mathbb{W}(\lambda_j). \tag{2.8}$$

We also define $I_0 := U_{\leq \lambda}$ and $I_l := 0$. By definition, we have $I_i \subset I_{i-1}$ for each $i \in \{1, \ldots, l\}$.

**Theorem 2.2.16** (Beck-Nakajima [3], [45])**.** For each $i \in \{1, \ldots, l\}$, there is an isomorphism of $(U_{\leq \lambda}, U_{\leq \lambda})$-bimodules

$$I_{i-1}/I_i \cong \mathbb{W}(\lambda_i) \otimes_{R(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp.$$

Under this isomorphism, the image of the element $a_{\lambda_i} \in I_{i-1}$ corresponds to the generating vector $w_{\lambda_i} \otimes w_{\lambda_i} \in \mathbb{W}(\lambda_i) \otimes_{R(\lambda_i)} \mathbb{W}(\lambda_i)^\sharp$.

*Proof.* See [45, Section A(ii), A(iii)]. □

### 2.2.5 Normalized $R$-matrices

In this subsection, we recall some facts about $R$-matrices of $\ell$-fundamental modules following [1], [25], [30].

Let us denote $\mathrm{End}_{U_q(L\mathfrak{g})}(\mathbb{W}(\varpi_i)) = \Bbbk[z_{\varpi_i}^{\pm 1}]$ for each $i \in I$ (see Theorem 2.2.10 (2)). For any pair $(i, j) \in I^2$, there is a unique homomorphism of $(U_q(L\mathfrak{g}), \Bbbk[z_{\varpi_i}^{\pm 1}, z_{\varpi_j}^{\pm 1}])$-bimodules, called the *normalized R-matrix*

$$R_{i,j}^{\mathrm{norm}} \colon \mathbb{W}(\varpi_i) \otimes \mathbb{W}(\varpi_j) \to \Bbbk(z_{\varpi_j}/z_{\varpi_i}) \otimes_{\Bbbk[(z_{\varpi_j}/z_{\varpi_i})^{\pm 1}]} (\mathbb{W}(\varpi_j) \otimes \mathbb{W}(\varpi_i)),$$

such that $R_{i,j}^{\mathrm{norm}}(w_{\varpi_i} \otimes w_{\varpi_j}) = w_{\varpi_j} \otimes w_{\varpi_i}$. The *denominator* of the normalized $R$-matrix $R_{i,j}^{\mathrm{norm}}$ is the monic polynomial $d_{i,j}(u) \in \Bbbk[u]$ with the smallest degree among polynomials satisfying

$$\mathrm{Im}\, R_{i,j}^{\mathrm{norm}} \subset d_{i,j}(z_{\varpi_j}/z_{\varpi_i})^{-1} \otimes (\mathbb{W}(\varpi_j) \otimes \mathbb{W}(\varpi_i)).$$

It is known that zeros of the denominator $d_{i,j}(u)$ belong to $q^{1/m}\overline{\mathbb{Q}}[\![q^{1/m}]\!]$ for some $m \in \mathbb{Z}_{>0}$ (see [30, Proposition 9.3]). In particular, we have

$$d_{i,j}(1) \neq 0. \tag{2.9}$$

Moreover, it is known that for each $i, j \in I$ we have

$$d_{i,j}(u) = d_{i^*,j^*}(u). \tag{2.10}$$

See [1, Appendix A] for example.

**Theorem 2.2.17.** Let $\boldsymbol{\lambda} = \sum_{j=1}^{l} \varpi_{i_j,a_j} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight. Then the following three conditions are mutually equivalent:

(1) The tensor product module $L(\varpi_{i_1,a_1}) \otimes L(\varpi_{i_2,a_2}) \otimes \cdots \otimes L(\varpi_{i_l,a_l})$ is generated by the tensor product of $\ell$-highest weight vectors;

(2) $W(\boldsymbol{\lambda}) \cong L(\varpi_{i_1,a_1}) \otimes L(\varpi_{i_2,a_2}) \otimes \cdots \otimes L(\varpi_{i_l,a_l})$;

(3) $d_{i_j,i_k}(a_k/a_j) \neq 0$ for any $1 \leq j < k \leq l$.

*Proof.* The equivalence of (1) and (2) was proved by Chari-Moura [7, Theorem 6.4] using the results from geometry due to Nakajima [42]. The equivalence of (1) and (3) was proved by Kashiwara [30, Proposition 9.4]. □

**Definition 2.2.18.** Let $\boldsymbol{\lambda} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight. We define the *dual local Weyl module* corresponding to $\boldsymbol{\lambda}$ by $W^\vee(\boldsymbol{\lambda}) := W({}^*\boldsymbol{\lambda})^*$.

**Proposition 2.2.19.** Let $\boldsymbol{\lambda} = \sum_{j=i}^{l} \varpi_{i_j,a_j} \in \mathcal{P}^+$ be an $\ell$-dominant $\ell$-weight. Assume $W(\boldsymbol{\lambda}) \cong L(\varpi_{i_1,a_1}) \otimes L(\varpi_{i_2,a_2}) \otimes \cdots \otimes L(\varpi_{i_l,a_l})$. Then we have

$$W^\vee(\boldsymbol{\lambda}) \cong L(\varpi_{i_l,a_l}) \otimes L(\varpi_{i_{l-1},a_{l-1}}) \otimes \cdots \otimes L(\varpi_{i_1,a_1}).$$

*Proof.* Use the equivalence of (2) and (3) in Theorem 2.2.17 and (2.10). □

## 2.3 Hernandez-Leclerc's category $\mathscr{C}_Q$

In this section, we recall the monoidal subcategory $\mathscr{C}_Q \subset \mathscr{C}_\mathfrak{g}$ associated with our Dynkin quiver $Q$. This category $\mathscr{C}_Q$ was originally introduced by Hernandez-Leclerc [24]. They proved that it gives a monoidal categorification of the coordinate ring $\mathbb{C}[N]$ of the maximal uniportent subgroup associated with $\mathfrak{g}$. We also describe a block decomposition of the category $\mathscr{C}_Q$ which gives a $\mathsf{Q}^+$-grading of the monoidal category $\mathscr{C}_Q$ corresponging to the weight decomposition of $\mathbb{C}[N]$.

### 2.3.1 Definition

In Section 2.1.3, from our fixed Dynkin quiver $Q = (I, \Omega)$ and an essentially unique choice of a height function $\xi \colon I \to \mathbb{Z}$, we have defined the repetition quiver $\widehat{Q} = (\widehat{I}, \widehat{\Omega})$ with

$$\widehat{I} = \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}$$

and a map $\phi \colon \mathsf{R}^+ \hookrightarrow \widehat{I}$. Using these data, we define $\mathscr{P} := \mathbb{Z}\widehat{I}$ to be the free abelian group with its free generating set $\widehat{I}$ and $\mathscr{P}_0 := \mathbb{Z}\phi(\mathsf{R}^+)$ to be the subgroup generated by the subset $\phi(\mathsf{R}^+) \subset \widehat{I}$. Let us regard $\mathscr{P}$ as a subgroup $\mathcal{P}$ by the embedding

$$\mathscr{P} \hookrightarrow \mathcal{P}; \quad (i,p) \mapsto \varpi_{i,q^p}.$$

Thus we have $\mathscr{P}_0 \subset \mathscr{P} \subset \mathcal{P}$. We also define the following submonoids consisting of dominant $\ell$-weights:

$$\mathscr{P}_0^+ := \mathbb{Z}_{\geq 0}\phi(\mathsf{R}^+) \quad \subset \quad \mathscr{P}^+ := \mathbb{Z}_{\geq 0}\widehat{I} \quad \subset \quad \mathcal{P}^+.$$

**Definition 2.3.1** (Hernandez-Leclerc [23], [24])**.** We define the category $\mathscr{C}_Q$ (resp. $\mathscr{C}_\mathbb{Z}$) as the smallest Serre subcategory of the category $\mathscr{C}_\mathfrak{g}$ containing the simple modules $L(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathscr{P}_0^+$ (resp. $\boldsymbol{\lambda} \in \mathscr{P}^+$). In other words, a module $M \in \mathscr{C}_{\mathfrak{g}}$ belongs to the category $\mathscr{C}_Q$ (resp. $\mathscr{C}_{\mathbb{Z}}$) if and only if its composition factors are isomorphic to $L(\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} \in \mathscr{P}_0^+$ (resp. $\boldsymbol{\lambda} \in \mathscr{P}^+$).

Informally, we can understand the category $\mathscr{C}_{\mathbb{Z}}$ as the subcategory of $\mathscr{C}_{\mathfrak{g}}$ "supported on the Auslander-Reiten quiver $\widehat{Q}$ of $D^b(\operatorname{Rep} Q)$" and the category $\mathscr{C}_Q$ as the subcategory of $\mathscr{C}_{\mathbb{Z}}$"supported on the Auslander-Reiten quiver of the heart $\operatorname{Rep} Q$ of the natural $t$-structure of $D^b(\operatorname{Rep} Q)$".

**Lemma 2.3.2** (cf. [23] Proposition 5.8 and [24] Lemma 5.8)**.** The categories $\mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_Q$ are closed under the tensor product. In other words, they are monoidal subcategories.

*Proof.* By Proposition 2.2.4 and Proposition 2.2.5, it follows that $[L(\boldsymbol{\mu}) : L(\boldsymbol{\lambda}_1) \otimes L(\boldsymbol{\lambda}_2)] \neq 0$ occurs only if $\boldsymbol{\mu} \leq \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$. Therefore the assertion for $\mathscr{C}_{\mathbb{Z}}$ follows from the following fact, which is easily verified: Assume $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ occurs for $\boldsymbol{\lambda} \in \mathscr{P}^+$ and $\boldsymbol{\mu} \in \mathcal{P}^+$. Then we have $\boldsymbol{\mu} \in \mathscr{P}^+$. Similarly, the assertion for $\mathscr{C}_Q$ follows from Lemma 2.3.3 below. □

We need to introduce another notation. Let

$$\begin{aligned}
\widehat{J} &:= (I \times \mathbb{Z}) \setminus \widehat{I} \\
&= \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z} + 1\}, \\
\widehat{J}_0 &:= \{(i,p) \in \widehat{J} \mid (i,p-1), (i,p+1) \in \phi(\mathsf{R}^+)\}.
\end{aligned}$$

For each $(i,p) \in \widehat{J}$, we write

$$\langle i,p \rangle := \alpha_{i,q^p} = (i,p-1) + (i,p+1) - \sum_{j \sim i} (j,p)$$

for simplicity. We define the subgroups

$$\mathscr{Q}_0 := \bigoplus_{(i,p) \in \widehat{J}_0} \mathbb{Z}\langle i,p \rangle \quad \subset \quad \mathscr{Q} := \bigoplus_{(i,p) \in \widehat{J}} \mathbb{Z}\langle i,p \rangle \quad \subset \mathcal{Q}.$$

By definition, we have $\mathscr{Q} = \mathscr{P} \cap \mathcal{Q} \subset \mathcal{P}$. We also write the corresponding submonoids generated by $\langle i,p \rangle$'s as $\mathscr{Q}_0^+ := \mathscr{Q}_0 \cap \mathcal{Q}^+ \subset \mathscr{Q}^+ := \mathscr{Q} \cap \mathcal{Q}^+$.

We get the following simple observation.

**Lemma 2.3.3.** Under the notation above, we have:

(1) $\mathscr{Q}_0 = \mathscr{P}_0 \cap \mathscr{Q}$.

(2) Assume that $\boldsymbol{\lambda} - \boldsymbol{\nu} \in \mathcal{P}^+$ occurs for $\boldsymbol{\lambda} \in \mathscr{P}_0^+$ and $\boldsymbol{\nu} \in \mathcal{Q}$. Then we have $\boldsymbol{\nu} \in \mathscr{Q}_0^+$ and therefore $\boldsymbol{\lambda} - \boldsymbol{\nu} \in \mathscr{P}_0^+$.

*Proof.* As mentioned in the last paragraph of Section 2.1.1, the full subquiver $\Gamma_Q$ with its vertex set $\phi(\mathsf{R}^+)$ inside $\widehat{Q}$ is isomorphic to the Auslander-Reiten quiver of the path algebra $\mathbb{C}Q$. In particular, the following properties are satisfied:

(1) If both $(i, p_1)$ and $(i, p_2)$ belong to $\phi(\mathsf{R}^+)$ with $p_1 < p_2$, then $(i, p)$ also belongs to $\phi(\mathsf{R}^+)$ for any $p$ with $p_1 < p < p_2$ and $p - \xi_i \in 2\mathbb{Z}$.

(2) If both $(i, p-1)$ and $(i, p+1)$ belong to $\phi(\mathsf{R}^+)$, then $(j, p)$ also belongs to $\phi(\mathsf{R}^+)$ for any $j$ with $i \sim j$.

From these properties, we obtain the assertions. □

### 2.3.2 Basic properties

In this section, we recall some basic properties of the monoidal subcategories $\mathscr{C}_\mathbb{Z}$ and $\mathscr{C}_Q$ in order to show that they are natural and important objects.

First, we shall see that the subcategory $\mathscr{C}_\mathbb{Z}$ is an essential part of the category $\mathscr{C}_\mathfrak{g}$. For each $t \in \Bbbk^\times$, we define the spectral shift automorphism $\sigma_t$ on $U_q(L\mathfrak{g})$ as a $\Bbbk$-Hopf algebra automorphism given by

$$\sigma_t(e_{i,r}) = t^r e_{i,r}, \quad \sigma_t(f_{i,r}) = t^r f_{i,r}, \quad \sigma_t(q^h) = q^h, \quad \sigma_t(h_{i,m}) = t^m h_{i,m},$$

where $i \in I, r \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}, h \in \mathsf{P}^\vee$. The pull-back functor $\sigma_t^*$ on the module category preserves the category $\mathscr{C}_\mathfrak{g}$ of finite-dimensional type **1** representations, i.e. it gives an auto-equivalence of $\mathscr{C}_\mathfrak{g}$. For each simple module $L(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}^+$, we have $\sigma_t^* L(\boldsymbol{\lambda}) \cong L(\sigma_t^* \boldsymbol{\lambda})$, where we define a group automorphism $\sigma_t^* \colon \mathcal{P} \xrightarrow{\cong} \mathcal{P}$ by $\sigma_t^* \varpi_{i,a} := \varpi_{i,ta}$ for each $(i,a) \in I \times \Bbbk^\times$. By definition, we have $\sigma_t^* \mathscr{P} = \mathscr{P}$ if and only if $t \in q^{2\mathbb{Z}}$, and moreover we have $\mathcal{P} = \sum_{t \in \Bbbk^\times / q^{2\mathbb{Z}}} \sigma_t^* \mathscr{P}$. Thus any $\ell$-dominant weight $\boldsymbol{\lambda} \in \mathcal{P}^+$ can be written in an essentially unique way as a sum $\boldsymbol{\lambda} = \sigma_{t_1}^* \boldsymbol{\lambda}_1 + \cdots + \sigma_{t_m}^* \boldsymbol{\lambda}_m$ with $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m \in \mathscr{P}^+$ and $t_1, \ldots, t_m \in \Bbbk^\times$ such that $t_i / t_j \notin q^{2\mathbb{Z}}$ for any $1 \le i \neq j \le m$. Then we have a factorization of simple module $L(\boldsymbol{\lambda})$ (cf. [6])

$$L(\boldsymbol{\lambda}) \cong L(\sigma_{t_1}^* \boldsymbol{\lambda}_1) \otimes \cdots \otimes L(\sigma_{t_m}^* \boldsymbol{\lambda}_m).$$

In particular, if $t_1, t_2 \in \Bbbk^\times$ satisfy $t_1/t_2 \notin q^{2\mathbb{Z}}$, we have

$$L(\sigma_{t_1}^* \boldsymbol{\lambda}_1) \otimes L(\sigma_{t_2}^* \boldsymbol{\lambda}_2) \cong L(\sigma_{t_2}^* \boldsymbol{\lambda}_2) \otimes L(\sigma_{t_1}^* \boldsymbol{\lambda}_1)$$

for any $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathscr{P}^+$. In other words, the Grothendieck ring $K(\mathscr{C}_\mathfrak{g})$ factorizes as

$$K(\mathscr{C}_\mathfrak{g}) \cong \bigotimes_{t \in \Bbbk^\times / q^{2\mathbb{Z}}} \sigma_t^* K(\mathscr{C}_\mathbb{Z})$$

where the tensor product on the RHS is over $\mathbb{Z}$ and we implicitly understand that only finitely many tensor factors are non-trivial. In this sense, we understand that the subcategory $\mathscr{C}_\mathbb{Z}$ is an essential part of the monoidal category $\mathscr{C}_\mathfrak{g}$.

We now turn to the subcategory $\mathscr{C}_Q \subset \mathscr{C}_\mathbb{Z}$. Let $G$ be a complex affine algebraic group whose Lie algebra is $\mathfrak{g}$ and $N$ be a maximal unipotent subgroup of $G$ corresponding to the positive roots $\mathsf{R}^+$. Recall that the coordinate ring $\mathbb{C}[N]$ possesses the dual canonical basis. As a main result of the paper [24], Hernandez-Leclerc proved that the category $\mathscr{C}_Q$ gives a monoidal categorification of the coordinate algebra $\mathbb{C}[N]$.

**Theorem 2.3.4** (Hernandez-Leclerc [24])**.** There is an isomorphism of $\mathbb{C}$-algebras

$$K(\mathscr{C}_Q)_\mathbb{C} \cong \mathbb{C}[N]$$

which sends the classes of simple modules to the elements of the dual canonical basis bijectively.

Actually, Hernandez-Leclerc established an isomorphism between quantizations, i.e. an isomorphism between the quantum Grothendieck ring of the category $\mathscr{C}_Q$ and the quantum coordinate ring of the maximal unipotent subgroup $N$.

### 2.3.3 Block decomposition of the category $\mathscr{C}_Q$

In this section, we give a direct sum decomposition of the category $\mathscr{C}_Q$. This decomposition corresponds to the weight space decomposition $\mathbb{C}[N] = \bigoplus_{\beta \in \mathsf{Q}^+} \mathbb{C}[N]_\beta$ of the uniportent coordinate ring under the isomorphism in Theorem 2.3.4.

By the injective map

$$\mathrm{KP}(\beta) \ni (m_\alpha) \mapsto \sum_\alpha m_\alpha \phi(\alpha) \in \mathscr{P}_0^+,$$

we regard $\mathrm{KP}(\beta)$ as a subset of $\mathscr{P}_0^+$. Then we have a disjoint union decomposition

$$\mathscr{P}_0^+ = \bigsqcup_{\beta \in \mathsf{Q}^+} \mathrm{KP}(\beta)$$

satisfying $\mathrm{KP}(\beta_1) + \mathrm{KP}(\beta_2) \subset \mathrm{KP}(\beta_1 + \beta_2)$ for any $\beta_1, \beta_2 \in \mathsf{Q}^+$.

**Definition 2.3.5.** Associated with an element $\beta \in \mathsf{Q}^+$, we define the category $\mathscr{C}_{Q,\beta}$ to be the Serre subcategory of $\mathscr{C}_Q$ full subcategory of $\mathscr{C}_Q$ containing the simple modules $L(\mathbf{m})$ for $\mathbf{m} \in \mathrm{KP}(\beta)$. In other words, the category $\mathscr{C}_{Q,\beta}$ is the full subcategory of $\mathscr{C}_Q$ consisting of modules whose composition factors are isomorphic to $L(\mathbf{m})$ for some $\mathbf{m} \in \mathrm{KP}(\beta)$.

**Proposition 2.3.6.** We have a direct sum decomposition of the category:

$$\mathscr{C}_Q \cong \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{C}_{Q,\beta}.$$

Moreover we have $\mathscr{C}_{Q,\beta_1} \otimes \mathscr{C}_{Q,\beta_2} \subset \mathscr{C}_{Q,\beta_1+\beta_2}$ for $\beta_1, \beta_2 \in \mathsf{Q}^+$.

Our proof of Proposition 2.3.6 relies on the following result by Chari-Maura [7]. Recall that for any two simple modules $M_1, M_2 \in \mathscr{C}_\mathfrak{g}$, we say that $M_1$ and $M_2$ are linked in $\mathscr{C}_\mathfrak{g}$ if there is no splitting $\mathscr{C}_\mathfrak{g} \cong \mathscr{C}_1 \oplus \mathscr{C}_2$ of abelian category such that $M_1 \in \mathscr{C}_1$ and $M_2 \in \mathscr{C}_2$. The following fact is known.

**Theorem 2.3.7** (Chari-Moura [7])**.** For any $\ell$-dominant $\ell$-weights $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^+$, the corresponding simple modules $L(\boldsymbol{\lambda})$ and $L(\boldsymbol{\mu})$ are linked in $\mathscr{C}_\mathfrak{g}$ if and only if $\boldsymbol{\lambda} - \boldsymbol{\mu} \in \mathcal{Q}$.

*Proof of Proposition 2.3.6.* Define a group surjection $\deg \colon \mathscr{P}_0 \to \mathsf{Q}$ by setting $\deg \phi(\alpha) := \alpha$ for each $\alpha \in \mathsf{R}^+$. Then we have

$$\mathrm{KP}(\beta) = \{\boldsymbol{\lambda} \in \mathscr{P}_0^+ \mid \deg(\boldsymbol{\lambda}) = \beta\}$$

by the definition of our inclusion $\mathrm{KP}(\beta) \hookrightarrow \mathscr{P}_0^+$.

Let $(i,p) \in \widehat{J}_0$. Then the indecomposable module $M(\phi^{-1}(i,p+1))$ is non-projective and its Auslander-Reiten translation is $M(\phi^{-1}(i,p-1))$, where $\phi^{-1}$ is the inverse map of the bijection $\phi \colon \mathsf{R}^+ \to \phi(\mathsf{R}^+)$ (see Remark 2.1.2). By the Auslander-Reiten theory, there is an almost split sequence:

$$0 \to M(\phi^{-1}(i,p-1)) \to \bigoplus_{j \sim i} M(\phi^{-1}(j,p)) \to M(\phi^{-1}(i,p+1)) \to 0.$$

Recall that we have $\underline{\dim} M(\alpha) = \alpha \in \mathsf{Q}$ by definition for each $\alpha \in \mathsf{R}^+$. Because the dimension vector function $\underline{\dim}(-)$ is additive, we compute in $\mathsf{Q}$ to get

$$\deg\langle i, p\rangle = \phi^{-1}(i, p-1) + \phi^{-1}(i, p+1) - \sum_{j\sim i} \phi^{-1}(j, p) = 0$$

for each $(i, p) \in \widehat{J}_0$. Therefore, for any $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathscr{P}_0^+$ with $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \in \mathscr{Q}_0$, we have $\deg \boldsymbol{\lambda}_1 = \deg \boldsymbol{\lambda}_2$. Combining this observation with Theorem 2.3.7 and Lemma 2.3.3, we obtain the direct sum decomposition $\mathscr{C}_Q = \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{C}_{Q,\beta}$.

The latter assertion $\mathscr{C}_{Q,\beta_1} \otimes \mathscr{C}_{Q,\beta_2} \subset \mathscr{C}_{Q,\beta_1+\beta_2}$ follows from Proposition 2.2.5. □

**Remark 2.3.8.** The decomposition $\mathscr{C}_Q = \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{C}_{Q,\beta}$ turns out to be a block decomposition i.e. $L(\mathbf{m}_1)$ and $L(\mathbf{m}_2)$ are linked in $\mathscr{C}_{Q,\beta}$ for any $\mathbf{m}_1, \mathbf{m}_2 \in \mathrm{KP}(\beta)$. Indeed, the composition multiplicity of the simple module $L(\mathbf{m})$ in the local Weyl module $W(\boldsymbol{\lambda}_\beta)$ is non-zero for each $\mathbf{m} \in \mathrm{KP}(\beta)$. This follows from the geometric fact $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta - \mathbf{m}, \boldsymbol{\lambda}_\beta) \neq \varnothing$ (see Lemma 3.2.3 below).

# Chapter 3

# Quiver varieties and the category $\mathscr{C}_Q$

## 3.1 Quiver varieties

In this section, we collect definitions and some properties of (graded) quiver varieties associated to a Dynkin quiver $Q$. Basic references are [39], [40], [42]. We keep the notation in Section 2.2.

### 3.1.1 Quiver varieties of Dynkin types

Fix an element $\nu = \sum_{i \in I} n_i \alpha_i \in \mathsf{Q}^+$ and a dominant weight $\lambda = \sum_{i \in I} l_i \varpi_i \in \mathsf{P}^+$. Consider $I$-graded $\mathbb{C}$-vector spaces $V^\nu = \bigoplus_{i \in I} V_i^\nu, W^\lambda = \bigoplus_{i \in I} W_i^\lambda$ such that $\dim V_i^\nu = n_i, \dim W_i^\lambda = l_i$ for each $i \in I$. We form the following space of linear maps:

$$\mathbf{N}(V^\nu, W^\lambda) := \left( \bigoplus_{i \to j \in \Omega} \mathrm{Hom}(V_i^\nu, V_j^\nu) \right) \oplus \left( \bigoplus_{i \in I} \mathrm{Hom}(W_i^\lambda, V_i^\nu) \right),$$

which is considered as the space of framed representations of the quiver $Q$ of dimension vector $(\nu, \lambda)$. On the space $\mathbf{N}(V^\nu, W^\lambda)$, the group $G(\nu) := \prod_{i \in I} GL(V_i^\nu)$ acts by conjugation. Let

$$\mathbf{M}(V^\nu, W^\lambda) := T^*\mathbf{N}(V^\nu, W^\lambda) = \mathbf{N}(V^\nu, W^\lambda) \oplus \mathbf{N}(V^\nu, W^\lambda)^*$$

be the cotangent bundle of the space $\mathbf{N}(V^\nu, W^\lambda)$, which is naturally regarded as a symplectic vector space. More explicitly, the space $\mathbf{M}(V^\nu, W^\lambda)$ is naturally identified with a direct sum

$$\left(\bigoplus_{(i,j);i\sim j}\mathrm{Hom}(V_j^\nu, V_i^\nu)\right)\oplus\left(\bigoplus_{i\in I}\mathrm{Hom}(W_i^\lambda, V_i^\nu)\right)\oplus\left(\bigoplus_{i\in I}\mathrm{Hom}(V_i^\nu, W_i^\lambda)\right).$$

According to this direct sum expression, we write an element of $\mathbf{M}(V^\nu, W^\lambda)$ as a triple $(B, a, b)$ of linear maps $B = \bigoplus B_{ij}$, $a = \bigoplus a_i$ and $b = \bigoplus b_i$. Let $\mu = \bigoplus_{i\in I}\mu_i \colon \mathbf{M}(V^\nu, W^\lambda) \to \bigoplus_{i\in I}\mathfrak{gl}(V_i^\nu)$ be the moment map with respect to the $G(\nu)$-action. Explicitly, it is given by the formula

$$\mu_i(B, a, b) = a_i b_i + \sum_{j\sim i}\varepsilon(i,j)B_{ij}B_{ji},$$

where $\varepsilon(i,j) := 1$ (resp. $-1$) if $j \to i \in \Omega$ (resp. $i \to j \in \Omega$). A point $(B, a, b) \in \mu^{-1}(0)$ is said to be stable if there exists no non-zero $I$-graded subspace $V' \subset V^\nu$ such that $B(V') \subset V'$ and $V' \subset \operatorname{Ker} b$. Let $\mu^{-1}(0)^{\mathrm{st}}$ be the set of stable points, on which $G(\nu)$ acts freely. Then we consider a set-theoretic quotient $\mathfrak{M}(\nu,\lambda) := \mu^{-1}(0)^{\mathrm{st}}/G(\nu)$. It is known that this quotient has a structure of a non-singular quasi-projective variety which is isomorphic to a quotient in the geometric invariant theory. On the other hand, we also consider the affine algebro-geometric quotient $\mathfrak{M}_0(\nu,\lambda) := \mu^{-1}(0)/\!/G(\nu) = \operatorname{Spec}\mathbb{C}[\mu^{-1}(0)]^{G(\nu)}$, together with the canonical projective morphism $\pi\colon \mathfrak{M}(\nu,\lambda) \to \mathfrak{M}_0(\nu,\lambda)$. We refer to these varieties $\mathfrak{M}(\nu,\lambda), \mathfrak{M}_0(\nu,\lambda)$ as *quiver varieties*.

Note that, on the linear space $\mathbf{M}(V^\nu, W^\lambda)$, the group $G(\lambda) := \prod_{i\in I} GL(W_i^\lambda)$ acts by conjugation and $\mathbb{C}^\times$ acts as the scalar multiplication. The combined action of the group $\mathbb{G}(\lambda) := G(\lambda)\times\mathbb{C}^\times$ on $\mathbf{M}(V^\nu, W^\lambda)$ commutes with the action of the group $G(\nu)$. Thus we have the induced $\mathbb{G}(\lambda)$-action on the quotients $\mathfrak{M}(\nu,\lambda), \mathfrak{M}_0(\nu,\lambda)$ which makes the canonical morphism $\pi$ into a $\mathbb{G}(\lambda)$-equivariant morphism.

For $\nu, \nu' \in \mathsf{Q}^+$ with $\nu \le \nu'$, we fix a direct sum decomposition $V^{\nu'} = V^\nu \oplus V^{\nu'-\nu}$. Extending by 0 on $V^{\nu'-\nu}$, we have an injective linear map $\mathbf{M}(V^\nu, W^\lambda) \hookrightarrow \mathbf{M}(V^{\nu'}, W^\lambda)$. This induces a natural closed embedding $\mathfrak{M}_0(\nu,\lambda) \hookrightarrow \mathfrak{M}_0(\nu',\lambda)$, which does not depend on the choice of decomposition $V^{\nu'} = V^\nu \oplus V^{\nu'-\nu}$. Via this natural embedding, we regard $\mathfrak{M}_0(\nu,\lambda)$ as a closed subvariety of $\mathfrak{M}_0(\nu',\lambda)$. We consider the union of them and obtain the following combined morphism:

$$\pi\colon \mathfrak{M}(\lambda) := \bigsqcup_\nu \mathfrak{M}(\nu,\lambda) \to \mathfrak{M}_0(\lambda) := \bigcup_\nu \mathfrak{M}_0(\nu,\lambda).$$

For each $x \in \mathfrak{M}_0(\lambda)$, let $\mathfrak{M}(\lambda)_x := \pi^{-1}(x)$ denote the fiber of $x$. The fiber $\mathfrak{L}(\lambda) := \pi^{-1}(0)$ of the origin $0 \in \mathfrak{M}_0(\lambda)$ is called the *central fiber*. We also set $\mathfrak{M}(\nu, \lambda)_x := \mathfrak{M}(\lambda)_x \cap \mathfrak{M}(\nu, \lambda)$ and $\mathfrak{L}(\nu, \lambda) := \mathfrak{L}(\lambda) \cap \mathfrak{M}(\nu, \lambda)$.

Recall that the geometric points of $\mathfrak{M}_0(\nu, \lambda)$ correspond to closed $G(\nu)$-orbits in $\mu^{-1}(0)$. Let $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$ be the subset of $\mathfrak{M}_0(\nu, \lambda)$ consisting of closed $G(\nu)$-orbits containing elements $\mathbf{x} = (B, a, b) \in \mu^{-1}(0)$ with trivial stabilizers (i.e. $\mathrm{Stab}_{G(\nu)} \mathbf{x} = \{1\}$). This is a (possibly empty) non-singular open set of $\mathfrak{M}_0(\nu, \lambda)$, on which the morphism $\pi$ becomes an isomorphism $\pi^{-1}(\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)) \xrightarrow{\cong} \mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$. It is known that $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda) \neq \varnothing$ if and only if $\lambda - \nu$ is a dominant weight appearing in the finite dimensional irreducible $\mathfrak{g}$-module of highest weight $\lambda$. They form a stratification:

$$\mathfrak{M}_0(\lambda) = \bigsqcup_{\nu \in \mathsf{Q}^+; \lambda - \nu \in \mathsf{P}^+} \mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda). \tag{3.1}$$

We have $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda) \subset \overline{\mathfrak{M}_0^{\mathrm{reg}}(\nu', \lambda)}$ only if $\nu \leq \nu'$.

### 3.1.2 Graded quiver varieties

Fix an element $\boldsymbol{\nu} = \sum_{(i,p) \in \widehat{J}} n_{i,p} \langle i, p \rangle \in \mathscr{Q}^+$ and an $\ell$-dominant $\ell$-weight $\boldsymbol{\lambda} = \sum_{(i,p) \in \widehat{I}} l_{i,p} (i, p) \in \mathscr{P}^+$. Consider a $\widehat{J}$-graded $\mathbb{C}$-vector space $V^{\boldsymbol{\nu}} = \bigoplus_{(i,p) \in \widehat{J}} V_i^{\boldsymbol{\nu}}(p)$ with $\dim V_i^{\boldsymbol{\nu}}(p) = n_{i,p}$ for $(i, p) \in \widehat{J}$, and an $\widehat{I}$-graded $\mathbb{C}$-vector space $W^{\boldsymbol{\lambda}} = \bigoplus_{(i,p) \in \widehat{I}} W_i^{\boldsymbol{\lambda}}(p)$ with $\dim W_i^{\boldsymbol{\lambda}}(p) = l_{i,p}$ for $(i, p) \in \widehat{I}$. We form the following space of linear maps:

$$\mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}}) := \left( \bigoplus_{(i,p) \in \widehat{J}, j \in I; i \sim j} \mathrm{Hom}(V_i^{\boldsymbol{\nu}}(p), V_j^{\boldsymbol{\nu}}(p-1)) \right)$$
$$\oplus \left( \bigoplus_{(i,p) \in \widehat{I}} \mathrm{Hom}(W_i^{\boldsymbol{\lambda}}(p), V_i^{\boldsymbol{\nu}}(p-1)) \right) \oplus \left( \bigoplus_{(i,p) \in \widehat{J}} \mathrm{Hom}(V_i^{\boldsymbol{\nu}}(p), W_i^{\boldsymbol{\lambda}}(p-1)) \right).$$

According to this direct sum expression, we write an element of $\mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}})$ as a triple $(B, a, b)$ of linear maps $B = \bigoplus B_{ji}(p)$, $a = \bigoplus a_i(p)$ and $b = \bigoplus b_i(p)$. Let $\mu^\bullet = \bigoplus_{(i,p) \in \widehat{J}} \mu_{i,p} \colon \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}}) \to \bigoplus_{(i,p) \in \widehat{J}} \mathrm{Hom}(V_i^{\boldsymbol{\nu}}(p), V_i^{\boldsymbol{\nu}}(p-2))$ be the map defined by the formula

$$\mu_{i,p}^\bullet(B, a, b) = a_i(p-1) b_i(p) + \sum_{j \sim i} \varepsilon(i, j) B_{ij}(p-1) B_{ji}(p),$$

where $\varepsilon(i,j)$ is the same as in Section 3.1.1. The map $\mu^\bullet$ is equivariant with respect to the conjugate action of the group $G(\boldsymbol{\nu}) := \prod_{(i,p)\in\widehat{J}} GL(V_i^{\boldsymbol{\nu}}(p))$. A point $(B,a,b) \in \mu^{\bullet\,-1}(0)$ is said to be stable if there exists no non-zero $\widehat{J}$-graded subspace $V' \subset V^{\boldsymbol{\nu}}$ such that $B(V') \subset V'$ and $V' \subset \operatorname{Ker} b$. Let $\mu^{\bullet\,-1}(0)^{\mathrm{st}}$ be the set of stable points. Similarly as in Section 3.1.1, we consider two kinds of quotients $\mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) := \mu^{\bullet\,-1}(0)^{\mathrm{st}}/G(\boldsymbol{\nu})$ and $\mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) := \mu^{\bullet\,-1}(0)/\!/G(\boldsymbol{\nu})$, together with the canonical projective morphism $\pi^\bullet\colon \mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) \to \mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda})$. We refer to these varieties $\mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}), \mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda})$ as *graded quiver varieties.*

On the space $\mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}})$, we have the conjugation action of the group $G(\boldsymbol{\lambda}) := \prod_{(i,p)\in\widehat{I}} GL(W_i^{\boldsymbol{\lambda}}(p))$ and the scalar action of $\mathbb{C}^\times$. The combined action of the group $\mathbb{G}(\boldsymbol{\lambda}) := G(\boldsymbol{\lambda}) \times \mathbb{C}^\times$ on $\mathbf{M}(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}})$ induces actions on the quotients $\mathfrak{M}(\boldsymbol{\nu},\boldsymbol{\lambda}), \mathfrak{M}_0(\boldsymbol{\nu},\boldsymbol{\lambda})$ which make the canonical morphism $\pi^\bullet$ into a $\mathbb{G}(\boldsymbol{\lambda})$-equivariant morphism. As in Section 3.1.1, we can form the unions:

$$\pi^\bullet\colon \mathfrak{M}^\bullet(\boldsymbol{\lambda}) := \bigsqcup_{\boldsymbol{\nu}} \mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) \to \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}) := \bigcup_{\boldsymbol{\nu}} \mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}).$$

Let $\mathfrak{M}^\bullet(\boldsymbol{\lambda})_x := \pi^{\bullet\,-1}(x)$ denote the fiber of a point $x \in \mathfrak{M}_0(\lambda)$. We set $\mathfrak{L}^\bullet(\boldsymbol{\lambda}) := \pi^{\bullet\,-1}(0)$.

### 3.1.3 Identification with fixed point subvarieties

Let $\boldsymbol{\lambda} = \sum l_{i,p}(i,p) \in \mathscr{P}^+$ be an $\ell$-dominant $\ell$-weight. In this subsection, we realize the graded quiver varieties $\mathfrak{M}^\bullet(\boldsymbol{\lambda}), \mathfrak{M}_0^\bullet(\boldsymbol{\lambda})$ as subvarieties of fixed points for a certain torus action in the usual quiver varieties $\mathfrak{M}(\lambda), \mathfrak{M}_0(\lambda)$ with $\lambda := \mathsf{cl}(\boldsymbol{\lambda})$.

We have $\lambda = \sum_{i\in I} l_i \varpi_i$ with $l_i = \sum_{p\in 2\mathbb{Z}+\xi_i} l_{i,p}$ by the definition of $\mathsf{cl}$. For each $i \in I$, we choose a direct sum decomposition $W_i^\lambda = \bigoplus_{p\in 2\mathbb{Z}+\xi_i} W_i^{\boldsymbol{\lambda}}(p)$ such that $\dim W_i^{\boldsymbol{\lambda}}(p) = l_{i,p}$. Note that this choice specifies a group embedding $G(\boldsymbol{\lambda}) \hookrightarrow G(\lambda)$. Define a group homomorphism $\rho_i\colon \mathbb{C}^\times \to GL(W_i^\lambda)$ by $\rho_i(t)|_{W_i^{\boldsymbol{\lambda}}(p)} := t^p \cdot \mathrm{id}_{W_i^{\boldsymbol{\lambda}}(p)}$ for $t \in \mathbb{C}^\times$. Let

$$\rho_{\boldsymbol{\lambda}} = (\prod_{i\in I} \rho_i \times \mathrm{id})\colon \mathbb{C}^\times \to G(\lambda) \times \mathbb{C}^\times = \mathbb{G}(\lambda)$$

be a 1-parameter subgroup and put $\mathbb{T}(\boldsymbol{\lambda}) := \rho_{\boldsymbol{\lambda}}(\mathbb{C}^\times)$. Then we consider the subvarieties $\mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}, \mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$ consisting of $\mathbb{T}(\boldsymbol{\lambda})$-fixed points and the induced canonical morphism $\pi^{\mathbb{T}(\boldsymbol{\lambda})}\colon \mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})} \to \mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$. Since the centralizer of $\mathbb{T}(\boldsymbol{\lambda})$ in $\mathbb{G}(\lambda)$ is identical to the subgroup $\mathbb{G}(\boldsymbol{\lambda}) = G(\boldsymbol{\lambda}) \times \mathbb{C}^\times \subset \mathbb{G}(\lambda)$, we have an induced action of $\mathbb{G}(\boldsymbol{\lambda})$ on the $\mathbb{T}(\boldsymbol{\lambda})$-fixed point subvarieties $\mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}, \mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$. The morphism $\pi^{\mathbb{T}(\boldsymbol{\lambda})}$ is $\mathbb{G}(\boldsymbol{\lambda})$-equivariant.

On the other hand, for each $\boldsymbol{\nu} = \sum n_{i,p}\langle i,p\rangle \in \mathscr{Q}^+$, we fix a direct sum decomposition $V_i^\nu = \bigoplus_{p\in 2\mathbb{Z}+\xi_i+1} V_i^{\boldsymbol{\nu}}(p)$ of $I$-graded vector space $V^\nu$ with $\nu := \mathsf{cl}(\boldsymbol{\nu})$ such that $\dim V_i(p) = n_{i,p}$, just as we have done for $W^\lambda$ in the last paragraph. These direct sum decompositions induce an embedding $\iota_{\boldsymbol{\nu},\boldsymbol{\lambda}} \colon \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}}) \hookrightarrow \mathbf{M}(V^\nu, W^\lambda)$. After taking quotients, this embedding $\iota_{\boldsymbol{\nu},\boldsymbol{\lambda}}$ yields the morphisms $\mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) \to \mathfrak{M}(\nu,\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$ and $\mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) \to \mathfrak{M}_0(\nu,\lambda)^{\rho_{\boldsymbol{\lambda}}}$.

**Lemma 3.1.1.** The morphisms constructed above induce $\mathbb{G}(\boldsymbol{\lambda})$-equivariant isomorphisms $\mathfrak{M}^\bullet(\boldsymbol{\lambda}) \xrightarrow{\cong} \mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}, \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}) \xrightarrow{\cong} \mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$ which make the following diagram commute:

$$\begin{array}{ccc} \mathfrak{M}^\bullet(\boldsymbol{\lambda}) & \xrightarrow{\cong} & \mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})} \\ \downarrow{\scriptstyle \pi^\bullet} & & \downarrow{\scriptstyle \pi^{\mathbb{T}(\boldsymbol{\lambda})}} \\ \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}) & \xrightarrow{\cong} & \mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}. \end{array}$$

In particular, we have a $\mathbb{G}(\boldsymbol{\lambda})$-equivariant isomorphism $\mathfrak{L}^\bullet(\boldsymbol{\lambda}) \cong \mathfrak{L}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$.

*Proof.* See [41, Section 4] □

Under these isomorphisms, we identify graded quiver varieties $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}), \mathfrak{M}^\bullet(\boldsymbol{\lambda})$ with $\mathbb{T}(\boldsymbol{\lambda})$-fixed point subvarieties $\mathfrak{M}_0(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}, \mathfrak{M}(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}$. Then we have

$$\mathfrak{M}(\nu,\lambda)^{\mathbb{T}(\boldsymbol{\lambda})} = \bigsqcup_{\boldsymbol{\nu}\in\mathscr{Q}^+;\mathsf{cl}(\boldsymbol{\nu})=\nu} \mathfrak{M}^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}), \quad \mathfrak{M}_0(\nu,\lambda)^{\mathbb{T}(\boldsymbol{\lambda})} = \bigsqcup_{\boldsymbol{\nu}\in\mathscr{Q}^+;\mathsf{cl}(\boldsymbol{\nu})=\nu} \mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}). \tag{3.2}$$

We define $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu},\boldsymbol{\lambda}) := \mathfrak{M}_0^\bullet(\boldsymbol{\nu},\boldsymbol{\lambda}) \cap \mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda)$. It is known (cf. [41, Theorem 14.3.2]) that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu},\boldsymbol{\lambda}) \neq \varnothing$ if and only if $\boldsymbol{\lambda}-\boldsymbol{\nu}$ is an $\ell$-dominant $\ell$-weight appearing in the local Weyl module $W(\boldsymbol{\lambda})$. By (3.1) and (3.2), we get a stratification:

$$\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}) = \bigsqcup_{\boldsymbol{\nu}\in\mathscr{Q}^+;\boldsymbol{\lambda}-\boldsymbol{\nu}\in\mathscr{P}^+} \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu},\boldsymbol{\lambda}). \tag{3.3}$$

It is known that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_1,\boldsymbol{\lambda}) \subset \overline{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_2,\boldsymbol{\lambda})}$ only if $\boldsymbol{\nu}_1 \le \boldsymbol{\nu}_2$.

### 3.1.4 Structure of non-central fibers

In this subsection, we recall the structures of (non-central) fibers of canonical morphisms $\pi$ and $\pi^\bullet$. Our exposition is based on [39, Section 6], [41, Section 3] and [44, Section 2.7] with some more details about group actions.

Let $(\nu, \lambda) \in \mathsf{Q}^+ \times \mathsf{P}^+$ be a pair. For any triple $\mathbf{x} = (B, a, b) \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$, we consider the following two kinds of complexes of vector spaces:

$$C_i(\nu, \lambda)_{\mathbf{x}}: \quad V_i^\nu \xrightarrow{\sigma_i} W_i^\lambda \oplus \bigoplus_{j \sim i} V_j^\nu \xrightarrow{\tau_i} V_i^\nu \quad \text{for each } i \in I, \tag{3.4}$$

where we define $\sigma_i := b_i \oplus \bigoplus_j B_{ji}$ and $\tau_i := a_i + \sum_j \varepsilon(i,j) B_{ij}$;

$$\mathscr{C}(\nu, \lambda)_{\mathbf{x}}: \quad \bigoplus_{i \in I} \mathrm{End}(V_i^\nu) \xrightarrow{\iota} \mathbf{M}(V^\nu, W^\lambda) \xrightarrow{\mathrm{d}\mu} \bigoplus_{i \in I} \mathrm{End}(V_i^\nu), \tag{3.5}$$

where $\iota$ is given by $\iota(\xi) = (B\xi - \xi B) \oplus (-\xi a) \oplus (b\xi)$ and $\mathrm{d}\mu$ is the differential of the moment map $\mu = \bigoplus_i \mu_i$ at the point $\mathbf{x} = (B, a, b)$. Note that the middle cohomology $H^0(\mathscr{C}(\nu, \lambda)_{\mathbf{x}})$ of the complex (3.5) is identical to the quotient space $(T_{\mathbf{x}} G(\nu)\mathbf{x})^\perp / T_{\mathbf{x}} G(\nu)\mathbf{x}$, where $(T_{\mathbf{x}} G(\nu)\mathbf{x})^\perp$ is the symplectic perpendicular of the tangent space $T_{\mathbf{x}} G(\nu)\mathbf{x}$ of the $G(\nu)$-orbit of $\mathbf{x}$. In particular, if $\mathbf{x}$ is stable, then the space $H^0(\mathscr{C}(\nu, \lambda)_{\mathbf{x}})$ is isomorphic to the tangent space $T_x \mathfrak{M}(\nu, \lambda)$ of the point $x \in \mathfrak{M}(\nu, \lambda)$ corresponding to $\mathbf{x}$.

Let $(\nu, \lambda) \in \mathsf{Q}^+ \times \mathsf{P}^+$ be a pair such that $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda) \neq \varnothing$. Recall that we have $\lambda - \nu \in \mathsf{P}^+$ in this case. We fix a point $x_\nu \in \mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$ and its lift $\mathbf{x}_\nu \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$ whose $G(\nu)$-orbit is closed. Then in the complex $C_i(\nu, \lambda)_{\mathbf{x}_\nu}$, the map $\sigma_i$ is injective ([40, Proposition 3.24]) and the map $\tau_i$ is surjective ([40, Lemma 4.7]). In particular, the dimension of the middle cohomology $H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu}) = \mathrm{Ker}\, \tau_i / \mathrm{Im}\, \sigma_i$ is equal to $(\lambda - \nu)(h_i)$. Therefore we can identify $W_i^{\lambda - \nu} = H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu})$.

We pick an arbitrary element $\nu'$ such that $\nu \le \nu'$. In order to construct the natural embedding $\mathfrak{M}_0(\nu, \lambda) \hookrightarrow \mathfrak{M}_0(\nu', \lambda)$, we fix a direct sum decomposition $V^{\nu'} = V^\nu \oplus V^{\nu' - \nu}$. Extending by 0 on $V^{\nu' - \nu}$, we have an injective linear map $\mathbf{M}(V^\nu, W^\lambda) \hookrightarrow \mathbf{M}(V^{\nu'}, W^\lambda)$, by which our fixed element $\mathbf{x}_\nu = (B, a, b)$ is regarded as an element of $\mu^{-1}(0) \subset \mathbf{M}(V^{\nu'}, W^\lambda)$. Then we can calculate as

$$H^0(\mathscr{C}(\nu', \lambda)_{\mathbf{x}_\nu}) \cong \mathbf{M}(V^{\nu' - \nu}, W^{\lambda - \nu}) \oplus H^0(\mathscr{C}(\nu, \lambda)_{\mathbf{x}_\nu}), \tag{3.6}$$

where we have $W_i^{\lambda-\nu} = H^0(C_i(\nu,\lambda)_{\mathbf{x}_\nu})$. We also see that the space $H^0(\mathscr{C}(\nu,\lambda)_{\mathbf{x}_\nu})$ is isomorphic to the tangent space $T := T_{x_\nu}\mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda)$.

The stabilizer $\mathrm{Stab}_{G(\nu')}\,\mathbf{x}_\nu$ is naturally isomorphic to $G(\nu'-\nu)$. Under this isomorphism, the action of $\mathrm{Stab}_{G(\nu')}\,\mathbf{x}_\nu$ on the LHS of (3.6) coincides with the action of $G(\nu'-\nu)$ on the RHS of (3.6), which is the direct sum of the natural action on $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$ and the trivial action on $T \cong H^0(\mathscr{C}(\nu,\lambda)_{\mathbf{x}_\nu})$.

An appropriate Hamiltonian reduction with respect to the action of the group $\mathrm{Stab}_{G(\nu')}\,\mathbf{x}_\nu \cong G(\nu'-\nu)$ on the RHS of (3.6) yields the following canonical map:

$$\pi \times \mathrm{id}\colon \mathfrak{M}(\nu'-\nu, \lambda-\nu) \times T \to \mathfrak{M}_0(\nu'-\nu, \lambda-\nu) \times T.$$

According to the discussion in [41, Section 3], this gives a local description of $\pi\colon \mathfrak{M}(\nu',\lambda) \to \mathfrak{M}_0(\nu',\lambda)$ around the point $x_\nu \in \mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda) \subset \mathfrak{M}_0(\nu',\lambda)$. More precisely, we have the following theorem.

**Theorem 3.1.2** (Nakajima)**.** Let $x_\nu \in \mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda) \subset \mathfrak{M}_0(\nu',\lambda)$. Then there exist neighborhoods $U, U_S, U_T$ of $x_\nu \in \mathfrak{M}_0(\nu',\lambda)$, $0 \in \mathfrak{M}_0(\nu'-\nu, \lambda-\nu)$, $0 \in T := T_{x_\nu}\mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda)$ respectively and biholomorphic maps $U \xrightarrow{\cong} U_S \times U_T; x_\nu \mapsto (0,0)$ and $\pi^{-1}(U) \xrightarrow{\cong} \pi^{-1}(U_S) \times U_T$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathfrak{M}(\nu',\lambda) & \supset & \pi^{-1}(U) & \xrightarrow{\cong} & \pi^{-1}(U_S) \times U_T & \subset & \mathfrak{M}(\nu'-\nu, \lambda-\nu) \times T \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\times\mathrm{id}} & & \\
\mathfrak{M}_0(\nu',\lambda) & \supset & U & \xrightarrow{\cong} & U_S \times U_T & \subset & \mathfrak{M}_0(\nu'-\nu, \lambda-\nu) \times T.
\end{array}$$

*Proof.* See [41, Theorem 3.3.2]. □

Now let us consider the action of the group $\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$ on the fiber $\mathfrak{M}(\lambda)_{x_\nu}$. By the definition of $\mathfrak{M}_0^{\mathrm{reg}}(\nu,\lambda)$, we have $\mathrm{Stab}_{G(\nu)}\,\mathbf{x}_\nu = \{1\}$. Therefore the second projection $G(\nu) \times \mathbb{G}(\lambda) \to \mathbb{G}(\lambda)$ restricts to an isomorphism $r\colon \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \xrightarrow{\cong} \mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$. Via the fixed direct sum decomposition $V^{\nu'} = V^{\nu} \oplus V^{\nu'-\nu}$, we can regard the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ as a subgroup of $\mathrm{Stab}_{G(\nu')\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$. In fact we have a decomposition

$$\mathrm{Stab}_{G(\nu')\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \cong \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \times G(\nu'-\nu). \tag{3.7}$$

Thus the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ acts on the vector space $\mathbf{M}(\nu',\lambda)$. Note that the action of the stabilizer $\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$ on the quiver varieties $\mathfrak{M}(\nu',\lambda)$ and

$\mathfrak{M}_0(\nu', \lambda)$ comes from this action of the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ on the vector space $\mathbf{M}(\nu', \lambda)$.

On the other hand, via the decomposition (3.7), the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ acts also on the complex $\mathscr{C}(\nu', \lambda)_{\mathbf{x}_\nu}$ and hence on its middle cohomology (3.6). Note that this induced action preserves each summand of the RHS of (3.6). In particular, we obtain an action of the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ on the vector space $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$. By the construction, we can easily see that this action factors through the natural action of $\mathbb{G}(\lambda-\nu)$ on $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$. The corresponding group homomorphism $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \to \mathbb{G}(\lambda-\nu) = G(\lambda-\nu)\times\mathbb{C}^\times$ is the direct product of two homomorphisms $\varphi : \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \to G(\lambda-\nu)$ and $\psi : \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \to \mathbb{C}^\times$. The homomorphism $\varphi$ is given as the induced action of the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ on the middle cohomology of the complex $C_i(\nu, \lambda)_{\mathbf{x}_\nu}$ under the identification $W_i^{\lambda-\nu} = H^0(C_i(\nu, \lambda)_{\mathbf{x}_\nu})$ and $G(\lambda-\nu) = \prod_{i\in I} GL(W_i^{\lambda-\nu})$. The homomorphism $\psi$ is obtained by the projection,

$$\psi\colon\ \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \hookrightarrow G(\nu)\times\mathbb{G}(\lambda) = G(\nu)\times G(\lambda)\times\mathbb{C}^\times \xrightarrow{\mathrm{pr}_3} \mathbb{C}^\times.$$

The action of the group $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu$ on the space $\mathbf{M}(V^{\nu'-\nu}, W^{\lambda-\nu})$ commutes with the action of group $G(\nu'-\nu)$. After taking the Hamiltonian reductions with respect to the action of the group $G(\nu'-\nu)$, we obtain an action of the group $\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$ on the central fiber $\mathfrak{L}(\nu'-\nu, \lambda-\nu)$. The above argument says that this action factors through the group homomorphism

$$\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu \xrightarrow[\cong]{r^{-1}} \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\,\mathbf{x}_\nu \xrightarrow{\varphi\times\psi} \mathbb{G}(\lambda-\nu). \tag{3.8}$$

This homomorphism (3.8) does not depend on $\nu'$.

By Theorem 3.1.2, there is an isomorphism $\mathfrak{M}(\nu', \lambda)_{x_\nu} \xrightarrow{\cong} \mathfrak{L}(\nu'-\nu, \lambda-\nu)$. As stated in [41, Remark 3.3.3], this isomorphism can be made equivariant with respect to the actions of the group $\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$. Summing up over $\nu'$, we obtain the following.

**Lemma 3.1.3.** Let $(\nu, \lambda) \in \mathsf{Q}^+ \times \mathsf{P}^+$ be a pair such that $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda) \neq \varnothing$ and $\pi\colon \mathfrak{M}(\lambda) \to \mathfrak{M}_0(\lambda)$ be the canonical morphism. Then for each point $x_\nu \in \mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$, there exists a ($\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$)-equivariant isomorphism

$$\mathfrak{M}(\lambda)_{x_\nu} \cong \mathfrak{L}(\lambda-\nu),$$

where the group $\mathrm{Stab}_{\mathbb{G}(\lambda)}\,x_\nu$ acts on the RHS $\mathfrak{L}(\lambda-\nu)$ via the group homomorphism $(\varphi\times\psi)\circ r^{-1}$ in (3.8).

Next we consider graded versions. Let $(\boldsymbol{\nu}, \boldsymbol{\lambda}) \in \mathscr{Q}^+ \times \mathscr{P}^+$ be a pair. For any triple $\mathbf{x} = (B, a, b) \in \mu^{\bullet -1}(0) \subset \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}})$ and $(i,p) \in \widehat{I}$, we can consider a complex of vector spaces

$$C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}}: \; V_i^{\boldsymbol{\nu}}(p+1) \xrightarrow{\sigma_{i,p}} W_i^{\boldsymbol{\lambda}}(p) \oplus \bigoplus_{j \sim i} V_j^{\boldsymbol{\nu}}(p) \xrightarrow{\tau_{i,p}} V_i^{\boldsymbol{\nu}}(p-1),$$

where we define $\sigma_{i,p} := b_i(p+1) \oplus \bigoplus_j B_{ji}(p+1)$ and $\tau_{i,p} := a_i(p) + \sum_j \varepsilon(i,j) B_{ij}(p)$.

Now we assume that $\mathfrak{M}^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}) \neq \varnothing$. In particular, we have $\boldsymbol{\lambda} - \boldsymbol{\nu} \in \mathscr{P}^+$. We fix a point $x_{\boldsymbol{\nu}} \in \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda})$ and its lift $\mathbf{x}_{\boldsymbol{\nu}} \in \mu^{\bullet -1}(0) \subset \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}})$ whose $G(\boldsymbol{\nu})$-orbit is closed. By the same reason as in the non-graded case, in the complex $C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}}$, the map $\sigma_{i,p}$ is injective and the map $\tau_{i,p}$ is surjective. Therefore the dimension vector of the $\widehat{I}$-graded vector space $\bigoplus_{(i,p) \in \widehat{I}} H^0(C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}})$ is equal to $\boldsymbol{\lambda} - \boldsymbol{\nu}$. This allows us to identify $W_i^{\boldsymbol{\lambda}-\boldsymbol{\nu}}(p)$ with $H^0(C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}})$ for each $(i,p) \in \widehat{I}$. Similarly as in (3.8), we consider the following group homomorphism

$$\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}} \xrightarrow[\cong]{\hat{r}^{-1}} \mathrm{Stab}_{G(\boldsymbol{\nu}) \times \mathbb{G}(\boldsymbol{\lambda})}\, \mathbf{x}_{\boldsymbol{\nu}} \xrightarrow{\hat{\varphi} \times \hat{\psi}} \mathbb{G}(\boldsymbol{\lambda} - \boldsymbol{\nu}) \tag{3.9}$$

where $\hat{r}$ is the isomorphism obtained as the restriction of the projection $G(\boldsymbol{\nu}) \times \mathbb{G}(\boldsymbol{\lambda}) \to \mathbb{G}(\boldsymbol{\lambda})$, $\hat{\varphi}$ is given as the induced action of the group $\mathrm{Stab}_{G(\boldsymbol{\nu}) \times \mathbb{G}(\boldsymbol{\lambda})}\, \mathbf{x}_{\boldsymbol{\nu}}$ on $W_i^{\boldsymbol{\lambda}-\boldsymbol{\nu}}(p) = H^0(C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}})$ and $\hat{\psi}$ is the restriction of the projection $G(\boldsymbol{\nu}) \times G(\boldsymbol{\lambda}) \times \mathbb{C}^\times \to \mathbb{C}^\times$.

**Lemma 3.1.4.** Let $(\boldsymbol{\nu}, \boldsymbol{\lambda}) \in \mathscr{Q}^+ \times \mathscr{P}^+$ be a pair such that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}) \neq \varnothing$ and $\pi^\bullet \colon \mathfrak{M}^\bullet(\boldsymbol{\lambda}) \to \mathfrak{M}_0^\bullet(\boldsymbol{\lambda})$ be the canonical morphism. Then for each point $x_{\boldsymbol{\nu}} \in \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda})$, there exists a $(\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}})$-equivariant isomorphism

$$\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}} \cong \mathfrak{L}(\boldsymbol{\lambda} - \boldsymbol{\nu}),$$

where the group $\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}}$ acts on the RHS $\mathfrak{L}^\bullet(\boldsymbol{\lambda} - \boldsymbol{\nu})$ via the group homomorphism $(\hat{\varphi} \times \hat{\psi}) \circ \hat{r}^{-1}$ in (3.9).

*Proof.* We put $\nu := \mathsf{cl}(\boldsymbol{\nu}), \lambda := \mathsf{cl}(\boldsymbol{\lambda})$. We make identifications of vector spaces: $W_i^\lambda = \bigoplus_{p \in 2\mathbb{Z}+\xi_i} W_i^{\boldsymbol{\lambda}}(p), V_i^\nu = \bigoplus_{p \in 2\mathbb{Z}+\xi_i+1} V_i^{\boldsymbol{\nu}}(p)$, which specifies an embedding $\iota \equiv \iota_{\boldsymbol{\nu},\boldsymbol{\lambda}} \colon \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}}) \hookrightarrow \mathbf{M}(V^\nu, W^\lambda)$. Using these direct sum decompositions, we define a group homomorphism $\rho_i \colon \mathbb{C}^\times \to \prod_p GL(W_i^{\boldsymbol{\lambda}}(p)) \hookrightarrow GL(W_i^\lambda)$ (resp. $\rho_i' \colon \mathbb{C}^\times \to \prod_p GL(V_i^{\boldsymbol{\nu}}(p)) \hookrightarrow GL(V_i^\nu)$)

for each $i \in I$ by $\rho_i(t)|_{W_i^{\boldsymbol{\lambda}}(p)} := t^p \cdot \mathrm{id}_{W_i^{\boldsymbol{\lambda}}(p)}$ (resp. $\rho'_i(t)|_{V_i^{\boldsymbol{\nu}}(p)} := t^p \cdot \mathrm{id}_{V_i^{\boldsymbol{\nu}}(p)}$). Recall we have $\mathfrak{M}^{\bullet}(\boldsymbol{\lambda}) \cong \mathfrak{M}(\lambda)^{\rho_{\boldsymbol{\lambda}}}$ and $\mathfrak{M}_0^{\bullet}(\boldsymbol{\lambda}) \cong \mathfrak{M}_0(\lambda)^{\rho_{\boldsymbol{\lambda}}}$ by Lemma 3.1.1, where $\rho_{\boldsymbol{\lambda}} := (\prod_{i\in I} \rho_i \times \mathrm{id})\colon \mathbb{C}^\times \to \mathbb{G}(\lambda)$ is the 1-parameter subgroup corresponding to $\boldsymbol{\lambda}$. Under this identification, we also regard $x_{\boldsymbol{\nu}}$ is a point of $\mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$. We can easily see that the image $\iota(\mathbf{x}_{\boldsymbol{\nu}}) \in \mu^{-1}(0) \subset \mathbf{M}(\nu, \lambda)$ has a closed $G(\nu)$-orbit corresponding to the point $x_{\boldsymbol{\nu}} \in \mathfrak{M}_0^{\mathrm{reg}}(\nu, \lambda)$ and in particular $\mathrm{Stab}_{G(\nu)}\, \iota(\mathbf{x}_{\boldsymbol{\nu}}) = \{1\}$. Let $\widetilde{\rho} := \left(\prod_{i\in I} \rho'_i \times \prod_{i \in I} \rho_i \times \mathrm{id}\right) : \mathbb{C}^\times \to \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\, \iota(\mathbf{x}_{\boldsymbol{\nu}})$ be a 1-parameter subgroup. By the restriction homomorphism $r\colon \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\, \iota(\mathbf{x}_{\boldsymbol{\nu}}) \xrightarrow{\cong} \mathrm{Stab}_{\mathbb{G}(\lambda)}\, x_{\boldsymbol{\nu}}$, the torus $\widetilde{\mathbb{T}} := \widetilde{\rho}(\mathbb{C}^\times)$ is isomorphic to $\mathbb{T}(\boldsymbol{\lambda}) = \rho_{\boldsymbol{\lambda}}(\mathbb{C}^\times)$. In fact, we have $r \circ \widetilde{\rho} = \rho_{\boldsymbol{\lambda}}$.

On the other hand, we have a decomposition $C_i(\nu, \lambda)_{\iota(\mathbf{x}_{\boldsymbol{\nu}})} = \bigoplus_{p \in 2\mathbb{Z}+\xi_i} C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}}$ of complexes and hence $H^0(C_i(\nu, \lambda)_{\iota(\mathbf{x}_{\boldsymbol{\nu}})}) = \bigoplus_{p \in 2\mathbb{Z}+\xi_i} H^0(C_{i,p}(\boldsymbol{\nu}, \boldsymbol{\lambda})_{\mathbf{x}_{\boldsymbol{\nu}}})$, which is identified with $W_i^{\lambda-\nu} = \bigoplus_{p \in 2\mathbb{Z}+\xi_i} W_i^{\boldsymbol{\lambda}-\boldsymbol{\nu}}(p)$. Then we can easily see that

$$(\varphi \times \psi) \circ r^{-1} \circ \rho_{\boldsymbol{\lambda}} = (\varphi \times \psi) \circ \widetilde{\rho} = \rho_{\boldsymbol{\lambda}-\boldsymbol{\nu}}.$$

Therefore, under the isomorphism in Lemma 3.1.3, the action of torus $\mathbb{T}(\boldsymbol{\lambda})$ on $\mathfrak{M}(\lambda)_{x_{\boldsymbol{\nu}}}$ coincides with the action of the torus $\mathbb{T}(\boldsymbol{\lambda} - \boldsymbol{\nu})$ on $\mathfrak{L}(\lambda - \nu)$. Therefore, by Lemma 3.1.1, we have

$$\mathfrak{M}^{\bullet}(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}} = \mathfrak{M}(\lambda)_{x_{\boldsymbol{\nu}}}^{\mathbb{T}(\boldsymbol{\lambda})} \cong \mathfrak{L}(\lambda - \nu)^{\mathbb{T}(\boldsymbol{\lambda}-\boldsymbol{\nu})} = \mathfrak{L}^{\bullet}(\boldsymbol{\lambda} - \boldsymbol{\nu}). \tag{3.10}$$

It remains to show that this isomorphism (3.10) is $\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}}$-equivariant. Note that the centralizer of torus $\mathbb{T}(\boldsymbol{\lambda})$ (resp. $\widetilde{\mathbb{T}}$ , $\mathbb{T}(\boldsymbol{\lambda} - \boldsymbol{\nu})$) in $\mathrm{Stab}_{\mathbb{G}(\lambda)}\, x_{\boldsymbol{\nu}}$ (resp. $\mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\, \iota(\mathbf{x}_{\boldsymbol{\nu}})$, $\mathbb{G}(\lambda-\nu)$) is the subgroup $\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}}$ (resp. $\mathrm{Stab}_{G(\boldsymbol{\nu})\times\mathbb{G}(\boldsymbol{\lambda})}\, \mathbf{x}_{\boldsymbol{\nu}}$, $\mathbb{G}(\boldsymbol{\lambda} - \boldsymbol{\nu})$). We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Stab}_{\mathbb{G}(\lambda)}\, x_{\boldsymbol{\nu}} & \xleftarrow[\cong]{r} & \mathrm{Stab}_{G(\nu)\times\mathbb{G}(\lambda)}\, \iota(\mathbf{x}_{\boldsymbol{\nu}}) & \xrightarrow{\varphi\times\psi} & \mathbb{G}(\lambda-\nu) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}} & \xleftarrow[\cong]{\hat{r}} & \mathrm{Stab}_{G(\boldsymbol{\nu})\times\mathbb{G}(\boldsymbol{\lambda})}\, \mathbf{x}_{\boldsymbol{\nu}} & \xrightarrow{\hat{\varphi}\times\hat{\psi}} & \mathbb{G}(\boldsymbol{\lambda}-\boldsymbol{\nu}).
\end{array}$$

Because the isomorphism in Lemma 3.1.3 is ($\mathrm{Stab}_{\mathbb{G}(\lambda)}\, x_{\boldsymbol{\nu}}$)-equivariant via the homomorphism $(\varphi \times \psi) \circ r^{-1}$, the induced isomorphism (3.10) on the torus fixed parts is $\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda})}\, x_{\boldsymbol{\nu}}$-equivariant via the homomorphism $(\hat{\varphi} \times \hat{\psi}) \circ \hat{r}^{-1}$ from the above commutative diagram. □

## 3.2 Graded quiver variety associated with $(Q, \beta)$

Henceforth, we fix a pair $(Q, \beta)$ of Dynkin quiver $Q = (I, \Omega)$ and a sum $\beta = \sum_{i \in I} d_i \alpha_i \in \mathsf{Q}^+$ of simple roots with an essentially unique height $\xi = (\xi_i)_{i \in I} \in \mathbb{Z}^I$. In the following Section 3.2.1 we define a graded quiver variety associated with these data and identify it with the space $E_\beta$ of representations of our quiver $Q$ of dimension vector $\beta$. Originally, this identification was established by Hernandez-Leclerc in order to give a geometric interpretation of their monoidal categorification theorem (= Theorem 2.3.4). In Section 3.2.2, we further study the group action on this quiver variety.

### 3.2.1 Hernandez-Leclerc's isomorphism

Recall that in Section 2.1.2 we have defined the space

$$E_\beta := \bigoplus_{h \in \Omega} \mathrm{Hom}(D_{h'}, D_{h''})$$

of the representation of the Dynkin quiver $Q$ of dimension vector $\beta$, where $D_i = \mathbb{C}^{d_i}$ for each $i \in I$. It is equipped with the natural action of the group $G_\beta = \prod_{i \in I} GL(D_i)$, yielding the $G_\beta$-orbit stratification (2.1) $E_\beta = \bigsqcup_{\mathbf{m} \in \mathrm{KP}(\beta)} \mathbb{O}_{\mathbf{m}}$.

Associated with our fixed pair $(Q, \beta)$, we set

$$\boldsymbol{\lambda}_\beta := \sum_{i \in I} d_i \phi(\alpha_i) \in \mathscr{P}_0^+,$$

where $\phi$ is the bijection $\mathsf{R}^+ \to \phi(\mathsf{R}^+)$ defined in Section 2.1.1. We consider the corresponding graded quiver variety $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta)$. We identify $G(\boldsymbol{\lambda}_\beta)$ with $G_\beta$ via an isomorphism $D_i = \mathbb{C}^{di} = W_{j_i}^{\boldsymbol{\lambda}_\beta}(p_i)$ for each $i \in I$, where we set $(j_i, p_i) := \phi(\alpha_i)$.

We also define a homomorphism $f_i \colon \mathbb{C}^\times \to GL(D_i)$ for each $i \in I$ by $f_i(t) := t^{-p_i} \cdot \mathrm{id}_{D_i}$. Then we have a group surjection

$$m \circ (\mathrm{id} \times \prod_{i \in I} f_i) \colon \mathbb{G}(\boldsymbol{\lambda}_\beta) = G_\beta \times \mathbb{C}^\times \to G_\beta \times G_\beta \to G_\beta$$

where $m$ is the multiplication in $G_\beta$, via which $E_\beta$ is equipped with a $\mathbb{G}(\boldsymbol{\lambda}_\beta)$-action.

In [24], Hernandez-Leclerc constructed a $\mathbb{G}(\boldsymbol{\lambda}_\beta)$-equivariant isomorphism $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) \xrightarrow{\cong} E_\beta$. We recall their construction. By Lemma 2.3.3 (2), it is enough to consider the graded quiver varieties $\mathfrak{M}_0^\bullet(\boldsymbol{\nu}, \boldsymbol{\lambda}_\beta)$ with $\boldsymbol{\nu} \in \mathscr{Q}_0^+$. We define a $\mathbb{C}$-algebra $\widetilde{\Lambda}_Q$ given by the following quiver $\widetilde{\Gamma}_Q$ with relations. The quiver $\widetilde{\Gamma}_Q$ consists of two types of vertices $\{v_j(p) \mid (j,p) \in \widehat{J}_0\} \cup \{w_j(p) \mid (j,p) = \phi(\alpha_i) \text{ for some } i \in I\}$ and three types of arrows:

$$\mathbf{a}_i(p)\colon w_i(p) \to v_i(p-1), \quad \mathbf{b}_i(p)\colon v_i(p) \to w_i(p-1),$$
$$\mathbf{B}_{ji}(p)\colon v_i(p) \to v_j(p-1) \text{ for } i \sim j.$$

The relations are

$$\mathbf{a}_i(p-1)\mathbf{b}_i(p) + \sum_{j \sim i} \varepsilon(i,j)\mathbf{B}_{ij}(p-1)\mathbf{B}_{ji}(p) = 0 \quad \text{for each } i \in I.$$

For each $i \in I$, let $\epsilon_i \in \widetilde{\Lambda}_Q$ be the idempotent corresponding to the vertex $w_j(p)$ with $(j,p) = \phi(\alpha_i)$. Then Hernandez-Leclerc [24, Lemma 9.6] proved that the algebra $\bigoplus_{i,j \in I} \epsilon_i \widetilde{\Lambda}_Q \epsilon_j$ is identical to the path algebra $\mathbb{C}Q$. By definition, each element $\mathbf{x} = (B, a, b) \in \mu^{\bullet\,-1}(0) \subset \mathbf{M}^\bullet(V^{\boldsymbol{\nu}}, W^{\boldsymbol{\lambda}_\beta})$ gives a representation of $\widetilde{\Lambda}_Q$. Then restricted to $\bigoplus_{i,j\in I} \epsilon_i \widetilde{\Lambda}_Q \epsilon_j$, it gives a representation of $\mathbb{C}Q$ of dimension vector $\beta$. This defines a morphism $\mathfrak{M}_0^\bullet(\boldsymbol{\nu}, \boldsymbol{\lambda}_\beta) \to E_\beta$.

**Theorem 3.2.1** (Hernandez-Leclerc [24] Theorem 9.11)**.** The morphism constructed above induces a $\mathbb{G}(\boldsymbol{\lambda}_\beta)$-equivariant isomorphism of varieties

$$\Psi_\beta \colon \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) \xrightarrow{\cong} E_\beta.$$

**Remark 3.2.2.** Recall that in Section 2.3.3 we have identified the set $\mathrm{KP}(\beta)$ of Kostant partitions of $\beta$ with a subset of $\mathscr{P}_0^+$ via the injection $\mathrm{KP}(\beta) \ni (m_\alpha) \mapsto \sum m_\alpha \phi(\alpha) \in \mathscr{P}_0^+$. From Lemma 2.3.3, we have

$$\{\boldsymbol{\mu} \in \mathscr{P}^+ \mid \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\lambda}_\beta - \boldsymbol{\mu}, \boldsymbol{\lambda}_\beta) \neq \varnothing\} = \{\mathbf{m} \in \mathrm{KP}(\beta) \mid \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\lambda}_\beta - \mathbf{m}, \boldsymbol{\lambda}_\beta) \neq \varnothing\}.$$

In particular, this is a subset of $\mathrm{KP}(\beta)$.

We give a proof of the following lemma, which is implicit in [24].

**Lemma 3.2.3.** For the isomorphism $\Psi_\beta$ in Theorem 3.2.1, we have

$$\Psi_\beta(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\lambda}_\beta - \mathbf{m}, \boldsymbol{\lambda}_\beta)) = \mathbb{O}_{\mathbf{m}}$$

for each $\mathbf{m} \in \mathrm{KP}(\beta)$. In particular, we have $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta - \mathbf{m}, \boldsymbol{\lambda}_\beta) \neq \varnothing$ for any $\mathbf{m} \in \mathrm{KP}(\beta)$.

*Proof.* For any $\mathbf{m} \in \mathrm{KP}(\beta)$, there is a unique $\boldsymbol{\nu} \in \mathscr{Q}_0^+$ (see Remark 3.2.2 above) such that $\Psi_\beta(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}_\beta)) \supset \mathbb{O}_{\mathbf{m}}$ since each stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}_\beta)$ is stable under the action of $G(\beta)$. We need to prove that $\boldsymbol{\nu} = \boldsymbol{\lambda}_\beta - \mathbf{m}$.

First we consider the case when $\beta = \alpha \in \mathsf{R}^+$ and $\mathbf{m}$ is the Kostant partition $\mathbf{m}_\alpha := (\delta_{\alpha,\alpha'})_{\alpha' \in \mathsf{R}^+}$ consisting of the single root $\alpha$. In this case, the orbit $\mathbb{O}_{\mathbf{m}_\alpha}$ is the unique open dense orbit of $E_\alpha$. Recall that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}_\beta) \subset \overline{\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}', \boldsymbol{\lambda}_\beta)}$ implies $\boldsymbol{\lambda}_\beta - \boldsymbol{\nu} \geq \boldsymbol{\lambda}_\beta - \boldsymbol{\nu}'$. Since the $\ell$-weight $\mathbf{m}_\alpha (= \varpi_{\phi(\alpha)})$ is minimal in $\mathscr{P}^+$, the corresponding stratum $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_\alpha, \boldsymbol{\lambda}_\alpha)$ is maximal, where we put $\boldsymbol{\nu}_\alpha := \boldsymbol{\lambda}_\alpha - \mathbf{m}_\alpha$. Therefore we have $\Psi_\beta(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_\alpha, \boldsymbol{\lambda}_\alpha)) \supset \mathbb{O}_{\mathbf{m}_\alpha}$.

Next we consider general $\mathbf{m} = (m_\alpha)_{\alpha \in \mathsf{R}^+} \in \mathrm{KP}(\beta)$. For each $\alpha \in \mathsf{R}^+$, we fix an element $\mathbf{y}_\alpha \in \mu^{\bullet\,-1}(0) \subset \mathbf{M}^\bullet(V^{\boldsymbol{\nu}_\alpha}, W^{\boldsymbol{\lambda}_\alpha})$ such that $\mathbf{y}_\alpha$ has a closed $G(\boldsymbol{\nu}_\alpha)$-orbit and $\mathrm{Stab}_{G(\boldsymbol{\nu}_\alpha)} \mathbf{y}_\alpha = \{1\}$ holds. By the previous paragraph, the element $\mathbf{y}_\alpha$, which is regarded as a representation of the algebra $\widetilde{\Lambda}_Q$, restricts to give the indecomposable representation $M(\alpha)$ of $\mathbb{C}Q$. We put

$$\mathbf{x}_{\mathbf{m}} := \bigoplus_{\alpha \in \mathsf{R}^+} \mathbf{y}_\alpha^{\oplus m_\alpha} \in \mu^{\bullet\,-1}(0)$$

$$\subset \mathbf{M}^\bullet\left( \bigoplus_{\alpha \in \mathsf{R}^+} (V^{\boldsymbol{\nu}_\alpha})^{\oplus m_\alpha}, \bigoplus_{\alpha \in \mathsf{R}^+} (W^{\boldsymbol{\lambda}_\alpha})^{\oplus m_\alpha} \right) = \mathbf{M}^\bullet(V^{\boldsymbol{\nu}_{\mathbf{m}}}, W^{\boldsymbol{\lambda}_\beta}),$$

where $\boldsymbol{\nu}_{\mathbf{m}} := \sum_{\alpha \in \mathsf{R}^+} m_\alpha \boldsymbol{\nu}_\alpha = \boldsymbol{\lambda}_\beta - \mathbf{m}$. Then $\mathbf{x}_{\mathbf{m}}$ defines a closed $G(\boldsymbol{\nu}_{\mathbf{m}})$-orbit and has a trivial stabilizer. Hence, the corresponding point $x_{\mathbf{m}}$ belongs to $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_{\mathbf{m}}, \boldsymbol{\lambda}_\beta)$. On the other hand, the element $\mathbf{x}_{\mathbf{m}}$, which is regarded as a representation of $\widetilde{\Lambda}_Q$, restricts to give a representation $\bigoplus_{\alpha \in \mathsf{R}^+} M(\alpha)^{\oplus m_\alpha}$ of $\mathbb{C}Q$. This means that $\Psi_\beta(x_{\mathbf{m}}) \in \mathbb{O}_{\mathbf{m}}$. Therefore we have $\Psi_\beta(\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_{\mathbf{m}}, \boldsymbol{\lambda}_\beta)) \supset \mathbb{O}_{\mathbf{m}}$. □

From the Lemma 3.2.3 above, we conclude that the $\ell$-dominance order $\leq$ on $\mathrm{KP}(\beta)$ coming from the inclusion $\mathrm{KP}(\beta) \hookrightarrow \mathscr{P}^+$ refines the opposite of the orbit closure ordering $\preceq$ defined in Section 2.1.2.

**Example 3.2.4.** If our quiver $Q$ is a monotone quiver of type $\mathsf{A}_n$, i.e. if $Q = (1 \to 2 \to \cdots \to n)$, we can see the isomorphism $\Psi_\beta \colon \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) \xrightarrow{\cong} E_\beta$ explicitly. In this case, the coordinate map $\phi \colon \mathsf{R}^+ \hookrightarrow \widehat{I}$ is given by $\phi(\alpha_i) = (1, n+1-2i)$ as we have seen in Example 2.1.3 (1). Thus we have $\lambda_\beta := \mathsf{cl}(\boldsymbol{\lambda}_\beta) = d\varpi_1$ with $d = \sum_{i=1}^n d_i = \mathrm{ht}\,\beta$. By Nakajima [39, Theorem 8.4], the quiver variety $\mathfrak{M}_0(\lambda_\beta)$ is isomorphic to the nilpotent cone

$$\mathcal{N}_d := \{x \in \mathrm{End}(D) \mid x^d = 0\}$$

where $D := \mathbb{C}^d$. This isomorphism $\mathfrak{M}_0^\bullet(\lambda) \xrightarrow{\cong} \mathcal{N}_d$ is induced from a $G(\nu)$-invariant map

$$\mathbf{M}(V^\nu, W^{\lambda_\beta}) \to \mathrm{End}(D);\ (B, a, b) \mapsto b_1 a_1$$

under an identification $W_1^{\lambda_\beta} = D$. The action of the group $\mathbb{G}(\lambda_\beta) = GL(D) \times \mathbb{C}^\times =: \mathbb{G}_d$ is given by $(g, t) \colon x \mapsto t^2 \mathrm{Ad}(g) x$. We fix an $I$-grading $D = \bigoplus_{i \in I} D_i$ with $D_i \cong \mathbb{C}^{d_i}$. The 1-parameter subgroup $\rho_{\boldsymbol{\lambda}_\beta} \colon \mathbb{C}^\times \to \mathbb{G}_d$ is defined by $\rho_{\boldsymbol{\lambda}_\beta} = (\prod_{i \in I} \rho_i \times \mathrm{id})$ with $\rho_i(t) = t^{n+1-2i} \cdot \mathrm{id}_{D_i} \in GL(D_i)$. Therefore, an element $x \in \mathcal{N}_d$ is fixed by the action of the torus $\mathbb{T}_\beta := \rho_{\boldsymbol{\lambda}_\beta}(\mathbb{C}^\times)$ if and only if it satisfies $x(D_i) \subset D_{i+1}$ for each $i \in I$, where we set $D_{n+1} = 0$. Thus we obtain

$$\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) = \mathcal{N}_d^{\mathbb{T}_\beta} = E_\beta.$$

### 3.2.2 Remarks on stabilizers

Keep the notation in the previous section. The following observation is crucially used in Section 3.3.3 below.

**Lemma 3.2.5.** Fix a Kostant partition $\mathbf{m} \in \mathrm{KP}(\beta)$ and choose an arbitrary point $x_{\mathbf{m}} \in \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\lambda}_\beta - \mathbf{m}, \boldsymbol{\lambda}_\beta)$. Then the maximal reductive quotient (= the quotient by the unipotent radical) of the group $\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}}$ is isomorphic to $\mathbb{G}(\mathbf{m})$. Moreover the group morphism $(\hat{\varphi} \times \hat{\psi}) \circ \hat{r}^{-1} \colon \mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}} \to \mathbb{G}(\mathbf{m})$ defined in (3.9) is identical to the canonical quotient map.

*Proof.* Define $\boldsymbol{\nu}_{\mathbf{m}} := \boldsymbol{\lambda}_\beta - \mathbf{m}$ as in the proof of Lemma 3.2.3 to simplify the notation. By the same Lemma 3.2.3, we know the point $\Psi_\beta(x_{\mathbf{m}})$ corresponds to a $\mathbb{C}Q$-module $M(\mathbf{m}) \cong \bigoplus_{\alpha \in \mathsf{R}^+} M(\alpha)^{\oplus m_\alpha}$. Then we have $\mathrm{Stab}_{G(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}} = \mathrm{Stab}_{G_\beta} \Psi_\beta(x_{\mathbf{m}}) = \mathrm{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times$. We consider a subgroup

$$G_1 := \prod_{\alpha \in \mathsf{R}^+} \mathrm{End}_{\mathbb{C}Q}(M(\alpha)^{\oplus m_\alpha})^\times \subset \mathrm{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times.$$

Note that we have $\mathrm{End}_{\mathbb{C}Q}(M(\alpha)) = \mathbb{C}$ for any root $\alpha \in \mathsf{R}^+$ and hence $G_1 \cong \prod_{\alpha \in \mathsf{R}^+} GL_{m_\alpha}(\mathbb{C})$. We see that this subgroup $G_1$ is a Levi subgroup of $\mathrm{End}_{\mathbb{C}Q}(M(\mathbf{m}))^\times$ and therefore $G_1 \times \mathbb{T}(\boldsymbol{\lambda}_\beta) \xhookrightarrow{m} \mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}}$ is a Levi subgroup, where $m$ is the multiplication.

Corresponding to the decomposition $M(\mathbf{m}) \cong \bigoplus_{\alpha \in \mathsf{R}^+} M(\alpha)^{\oplus m_\alpha}$, we choose the element $\mathbf{x}_{\mathbf{m}} = \bigoplus_{\alpha \in \mathsf{R}^+} \mathbf{y}_\alpha^{\oplus m_\alpha}$ as a lift of the point $x_{\mathbf{m}}$, where $\mathbf{y}_\alpha$'s are the same as in the proof of Lemma 3.2.3 above. Then we have $\mathrm{Stab}_{G(\boldsymbol{\nu}_{\mathbf{m}})\times G(\boldsymbol{\lambda}_\beta)}\,\mathbf{x}_{\mathbf{m}} = \mathrm{End}_{\widetilde{\Lambda}_Q}(\mathbf{x}_{\mathbf{m}})^\times$. We consider a subgroup

$$\widetilde{G}_1 := \prod_{\alpha\in\mathsf{R}^+} GL_{m_\alpha}(\mathbb{C}\cdot \mathrm{id}_{\mathbf{y}_\alpha}) \subset \prod_{\alpha\in\mathsf{R}^+} \mathrm{End}_{\widetilde{\Lambda}_Q}(\mathbf{y}_\alpha^{\oplus m_\alpha})^\times \subset \mathrm{End}_{\widetilde{\Lambda}_Q}(\mathbf{x}_{\mathbf{m}})^\times.$$

Note that the homomorphism $\hat{r}$ gives an isomorphism $\widetilde{G}_1 \xrightarrow{\cong} G_1$. On the other hand, we can easily see that the homomorphism $\hat{\varphi}\colon \mathrm{Stab}_{G(\boldsymbol{\nu}_{\mathbf{m}})\times \mathbb{G}(\boldsymbol{\lambda}_\beta)}\,\mathbf{x}_{\mathbf{m}} \to G(\mathbf{m})$ induces the isomorphism

$$\widetilde{G}_1 \xrightarrow{\cong} \prod_{\alpha\in\mathsf{R}^+} GL(H^0(C_{\phi(\alpha)}(\boldsymbol{\nu}_{\mathbf{m}},\boldsymbol{\lambda}_\beta)_{\mathbf{x}_{\mathbf{m}}})) \cong G(\mathbf{m}).$$

As a result, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)}\, x_{\mathbf{m}} & \xleftarrow[\cong]{\hat{r}} & \mathrm{Stab}_{G(\boldsymbol{\nu}_{\mathbf{m}})\times\mathbb{G}(\boldsymbol{\lambda}_\beta)}\,\mathbf{x}_{\mathbf{m}} & \xrightarrow{\hat{\varphi}\times\hat{\psi}} & \mathbb{G}(\mathbf{m}) \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle m} & & \cong\uparrow{\scriptstyle m} \\ G_1\times\mathbb{T}(\boldsymbol{\lambda}_\beta) & \xleftarrow{\cong} & \widetilde{G}_1\times\widetilde{\mathbb{T}} & \xrightarrow{\cong} & G(\mathbf{m})\times\mathbb{T}(\mathbf{m}) \end{array} \tag{3.11}$$

where $\widetilde{\mathbb{T}}$ is the 1-dimensional torus defined in the proof of Lemma 3.1.4 and $m$ stands for the multiplication. Furthermore the lower horizontal arrows are isomorphisms for each factor. This diagram completes a proof. □

For a linear algebraic group $G$, we denote its representation ring by $R(G)$. For $G = \mathbb{C}^\times$, we make an identification $R(\mathbb{C}^\times) = A := \mathbb{Z}[q^{\pm 1}]$, where $q$ is an indeterminate.

For any $\boldsymbol{\lambda} \in \mathscr{P}^+$, the 2nd projection

$$\mathbb{G}(\boldsymbol{\lambda}) = G(\boldsymbol{\lambda}) \times \mathbb{C}^\times \to \mathbb{C}^\times$$

induces an algebra homomorphism $A \hookrightarrow R(\mathbb{G}(\boldsymbol{\lambda}))$ via which we regard $R(\mathbb{G}(\boldsymbol{\lambda}))$ as an $A$-algebra. We also have the natural inclusion $A \hookrightarrow \Bbbk = \overline{\mathbb{Q}(q)}$. Under this notation, we shall make the standard identification:

$$R(\mathbb{G}(\boldsymbol{\lambda})) \otimes_A \Bbbk = R(\boldsymbol{\lambda}). \tag{3.12}$$

Similarly we identify $R(\mathbb{G}(\lambda)) \otimes_A \Bbbk = R(\lambda)$ for $\lambda \in \mathsf{P}^+$. (See Section 2.2.3 for the definition of $R(\boldsymbol{\lambda})$ and $R(\lambda)$.) As special cases of (3.12) when $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\beta$

or $\boldsymbol{\lambda} = \mathbf{m} \in \mathrm{KP}(\beta)$, we have

$$R(\mathbb{G}(\boldsymbol{\lambda}_\beta)) \otimes_A \Bbbk = \bigotimes_{i \in I} \left(\Bbbk[z_i^{\pm 1}]^{\otimes d_i}\right)^{\mathfrak{S}_{d_i}} = R(\boldsymbol{\lambda}_\beta), \tag{3.13}$$

$$R(\mathbb{G}(\mathbf{m})) \otimes_A \Bbbk = \bigotimes_{\alpha \in \mathsf{R}^+} \left(\Bbbk[z_\alpha^{\pm 1}]^{\otimes m_\alpha}\right)^{\mathfrak{S}_{m_\alpha}} = R(\mathbf{m}). \tag{3.14}$$

Now we return to the setting of Lemma 3.2.5. We have the following group homomorphisms:

$$\mathbb{G}(\mathbf{m}) \xleftarrow{(\hat{\varphi} \times \hat{\psi}) \circ \hat{r}^{-1}} \mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}} \hookrightarrow \mathbb{G}(\boldsymbol{\lambda}_\beta),$$

which induces the following homomorphism:

$$\theta_{\mathbf{m}} \colon R(\boldsymbol{\lambda}_\beta) \overset{(3.13)}{=} R(\mathbb{G}(\boldsymbol{\lambda}_\beta))_{\Bbbk} \to R(\mathrm{Stab}_{\mathbb{G}(\boldsymbol{\lambda}_\beta)} x_{\mathbf{m}}) \cong R(\mathbb{G}(\mathbf{m})) \overset{(3.14)}{=} R(\mathbf{m}). \tag{3.15}$$

From the proof of Lemmas 3.1.4 and 3.2.5, we obtain the following.

**Corollary 3.2.6.** For a positive root $\alpha = \sum_{i \in I} c_i \alpha_i \in \mathsf{R}^+$, we define the following $\Bbbk$-algebra homomorphism:

$$\theta_\alpha \colon \bigotimes_{i \in I} \Bbbk[z_i^{\pm 1}]^{\otimes c_i} \to \Bbbk[z_\alpha^{\pm 1}];\ z_i \mapsto q^{\mathsf{p}(\alpha_i) - \mathsf{p}(\alpha)} z_\alpha,$$

where $\mathsf{p} := \mathrm{pr}_2 \circ \phi \colon \mathsf{R}^+ \to I \times \mathbb{Z} \to \mathbb{Z}$. The homomorphism $\theta_{\mathbf{m}} \colon R(\boldsymbol{\lambda}_\beta) \to R(\mathbf{m})$ defined by (3.15) is obtained as the restriction of the homomorphism

$$\bigotimes_{\alpha \in \mathsf{R}^+} \theta_\alpha^{\otimes m_\alpha} \colon \bigotimes_{i \in I} \Bbbk[z_i^{\pm 1}]^{\otimes d_i} \to \bigotimes_{\alpha \in \mathsf{R}^+} \Bbbk[z_\alpha^{\pm 1}]^{\otimes m_\alpha}.$$

## 3.3 Analysis of convolution algebra

Our aim is to study the structure of Hernandez-Leclerc's category $\mathscr{C}_Q$ using geometry of graded quiver varieties based on Nakajima's framework [41]. Thanks to the block decomposition $\mathscr{C}_Q = \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{C}_{Q,\beta}$ (see Section 2.3.3), we can concentrate on a direct summand $\mathscr{C}_{Q,\beta}$. We start with recalling Nakajima's geometric construction of an algebra homomorphism from the quantum loop algebra $U_q(L\mathfrak{g})$ to the convolution algebra of the equivariant $K$-group of the Steinberg type quiver variety and its properties in Section 3.3.1.

By completing this homomorphism along an ideal corresponding to the a 1-dimensional torus, we obtain an algebra homomorphism to a central completion of the convolution algebra of the equivariant $K$-group of the Steinberg type graded quiver variety (Section 3.3.2). Our main interest in Section 3.3.3 is the case when the graded quiver variety is associated with $(Q, \beta)$. In this case, we can use Hernandez-Leclerc's isomorphism in Section 3.2. Our main theorem (=Theorem 3.3.6) says that the corresponding completed Nakajima homomorphism induces a fully faithful functor on the category $\mathscr{C}_{Q,\beta}$. We also discuss a structure of affine highest weight category in Section 3.3.4 and a comparison with a geometric extension algebra in Section 3.3.5.

### 3.3.1 Nakajima's homomorphism

In this section, we recall Nakajima's homomorphism based on Nakajima's original paper [41]. See Appendix A.1.1 for the notation around the equivarint $K$-groups.

Fix a dominant weight $\lambda \in \mathsf{P}^+$ and consider the corresponding quiver variety $\pi\colon \mathfrak{M}(\lambda) \to \mathfrak{M}_0(\lambda)$. We define the *Steinberg type variety* as

$$Z(\lambda) := \mathfrak{M}(\lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\lambda) = \bigsqcup_{\nu_1, \nu_2 \in \mathsf{Q}^+} \mathfrak{M}(\nu_1, \lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\nu_2, \lambda),$$

together with the canonical map $\pi\colon Z(\lambda) \to \mathfrak{M}_0(\lambda)$. By the convolution product (see Appendix A.1.3), the equivariant $K$-group

$$K^{\mathbb{G}(\lambda)}(Z(\lambda)) = \bigoplus_{\nu_1, \nu_2 \in \mathsf{Q}^+} K^{\mathbb{G}(\lambda)}(\mathfrak{M}(\nu_1, \lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\nu_2, \lambda))$$

becomes an algebra over the commutative algebra $R(\mathbb{G}(\lambda))$.

For any $\mathbb{G}(\lambda)$-variety $X$, we define

$$\mathsf{K}^{\mathbb{G}(\lambda)}(X) := K^{\mathbb{G}(\lambda)}(X) \otimes_A \Bbbk, \quad \mathsf{K}^{\mathbb{G}(\lambda)}_{i,\mathrm{top}}(X) := K^{\mathbb{G}(\lambda)}_{i,\mathrm{top}}(X) \otimes_A \Bbbk$$

for brevity of notation. When $X = Z(\lambda)$, the $K$-group $\mathsf{K}^{\mathbb{G}(\lambda)}(Z(\lambda))$ becomes an algebra over $R(\lambda) = R(\mathbb{G}(\lambda)) \otimes_A \Bbbk$ with respect to the convolution product.

We consider the following tautological vector bundles on $\mathfrak{M}(\nu, \lambda)$. The vector bundle $\mathcal{V}_i^\nu$ is defined by $\mathcal{V}_i^\nu := \mu^{-1}(0)^{\mathrm{st}} \times_{G(\nu)} V_i^\nu$ for each $i \in I$. We regard $\mathcal{V}_i^\nu$ as a $\mathbb{G}(\lambda)$-equivariant vector bundle with the trivial action. On the other hand, we consider a trivial vector bundle $\mathcal{W}_i^\lambda := \mathfrak{M}(\nu, \lambda) \times W_i^\lambda$ with fiber $W_i^\lambda$ for each $i \in I$. We regard $\mathcal{W}_i^\lambda$ as a $\mathbb{G}(\lambda)$-equivariant vector bundle with the natural $G(\lambda)$-action and the trivial $\mathbb{C}^\times$-action. Recall the complex of vector spaces $C_i(\nu,\lambda)_{\mathbf{x}}$ for each $\mathbf{x} \in \mu^{-1}(0) \subset \mathbf{M}(V^\nu, W^\lambda)$ defined in (3.4). This complex yields the complex $\mathcal{C}_i(\nu,\lambda)$ of $\mathbb{G}(\lambda)$-equivariant vector bundles on $\mathfrak{M}(\nu,\lambda)$:

$$\mathcal{C}_i(\nu,\lambda):\ q^{-2}\mathcal{V}_i^\nu \xrightarrow{\sigma_i} q^{-1}\left(\mathcal{W}_i^\lambda \oplus \bigoplus_{j\sim i} \mathcal{V}_j^\nu\right) \xrightarrow{\tau_i} \mathcal{V}_i^\nu.$$

Note that the class of the complex $\mathcal{C}_i(\nu,\lambda)$ in $\mathsf{K}^{\mathbb{G}(\lambda)}(\mathfrak{M}(\nu,\lambda))$ is calculated as

$$[\mathcal{C}_i(\nu,\lambda)] = q^{-1}\left([\mathcal{W}_i^\lambda] - (q+q^{-1})[\mathcal{V}_i^\nu] + \sum_{j\sim i}[\mathcal{V}_j^\nu]\right).$$

Then we have the following fundamental result due to Nakajima [42].

**Theorem 3.3.1** (Nakajima [42] Theorem 9.4.1)**.** There is a $\Bbbk$-algebra homomorphism

$$\Phi_\lambda\colon \dot{U}_q(L\mathfrak{g}) \to \mathsf{K}^{\mathbb{G}(\lambda)}(Z(\lambda)),$$

such that

$$\Phi_\lambda(a_\mu) = \begin{cases} \Delta_*[\mathcal{O}_{\mathfrak{M}(\nu,\lambda)}] & \text{if } \nu := \lambda - \mu \in \mathsf{Q}^+; \\ 0 & \text{otherwise,} \end{cases}$$

and it sends the series $\psi_i^\pm(z)a_\mu$ with $\nu := \lambda - \mu \in \mathsf{Q}^+$, $i \in I$ to the series

$$q^{\mu(h_i)}\Delta_*\left(\frac{\bigwedge_{-1/qz}[\mathcal{C}_i(\nu,\lambda)]}{\bigwedge_{-q/z}[\mathcal{C}_i(\nu,\lambda)]}\right)^\pm,$$

where $\Delta\colon \mathfrak{M}(\nu,\lambda) \to \mathfrak{M}(\nu,\lambda) \times_{\mathfrak{M}_0(\lambda)} \mathfrak{M}(\nu,\lambda)$ is the diagonal embedding and $(-)^\pm$ denotes the formal expansion at $z = \infty$ and $0$ respectively.

We refer to the homomorphism $\Phi_\lambda$ as *Nakajima's homomorphism.*

By construction, the equivariant $K$-group $K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ of the central fiber $\mathfrak{L}(\lambda) = \mathfrak{M}(\lambda) \times_{\mathfrak{M}_0(\lambda)} \{0\}$ becomes a left module over the convolution algebra $K^{\mathbb{G}(\lambda)}(Z(\lambda))$. Via the Nakajima homomorphism $\Phi_\lambda$, we regard $\mathsf{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ as a $(\dot{U}_q(L\mathfrak{g}), R(\lambda))$-bimodule.

**Theorem 3.3.2** (Nakajima)**.** As a $(\dot{U}_q(L\mathfrak{g}), R(\lambda))$-bimodule, the equivariant $K$-group $\mathsf{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ is isomorphic to the global Weyl module $\mathbb{W}(\lambda)$. The element $[\mathcal{O}_{\mathfrak{L}(0,\lambda)}] \in \mathsf{K}^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda))$ corresponds to the cyclic vector $w_\lambda \in \mathbb{W}(\lambda)$.

*Proof.* See [43, Theorem 2]. □

For future references, we collect some important properties of equivariant $K$-groups of central fibers.

**Theorem 3.3.3** (Nakajima)**.** For any closed reductive subgroup $G'$ of $\mathbb{G}(\lambda)$, the following holds true.

(1) We have $K^{G'}_{1,\mathrm{top}}(\mathfrak{L}(\lambda)) = 0$;

(2) $K^{G'}_{0,\mathrm{top}}(\mathfrak{L}(\lambda))$ is a free $R(G')$-module and the comparison map $K^{G'}(\mathfrak{L}(\lambda)) \to K^{G'}_{0,\mathrm{top}}(\mathfrak{L}(\lambda))$ is an isomorphism;

(3) The natural map $K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \otimes_{R(\mathbb{G}(\lambda))} R(G') \to K^{G'}(\mathfrak{L}(\lambda))$ is an isomorphism;

(4) The Künneth homomorphisms

$$K^{G'}(\mathfrak{L}(\lambda)) \otimes_{R(G')} K^{G'}(\mathfrak{L}(\lambda)) \to K^{G'}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda)),$$
$$K^{G'}_{0,\mathrm{top}}(\mathfrak{L}(\lambda)) \otimes_{R(G')} K^{G'}_{i,\mathrm{top}}(\mathfrak{L}(\lambda)) \to K^{G'}_{i,\mathrm{top}}(\mathfrak{L}(\lambda) \times \mathfrak{L}(\lambda))$$

are isomorphisms, where $i = 0, 1$.

*Proof.* The properties (1), (2), (3) are the same as the property $(T_{\mathbb{G}(\lambda)})$ in [41, Section 7]. The assertion for $K^{\mathbb{G}(\lambda)}$ in (4) follows from [42, Theorem 3.4]. The assertion for $K^{\mathbb{G}(\lambda)}_{i,\mathrm{top}}$ in (4) follows from the properties (1), (2) and the property (n3) in [33, Section 1.2]. □

### 3.3.2 Central completion

Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ be an $\ell$-dominant $\ell$-weight. We consider the corresponding graded quiver varieties $\pi^\bullet \colon \mathfrak{M}^\bullet(\boldsymbol{\lambda}) \to \mathfrak{M}^\bullet_0(\boldsymbol{\lambda})$ and form the Steinberg type variety

$$Z^\bullet(\boldsymbol{\lambda}) := \mathfrak{M}^\bullet(\boldsymbol{\lambda}) \times_{\mathfrak{M}^\bullet_0(\boldsymbol{\lambda})} \mathfrak{M}^\bullet(\boldsymbol{\lambda}).$$

The $\mathbb{G}(\boldsymbol{\lambda})$-equivariant $K$-group $K^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))$ becomes an algebra over $R(\mathbb{G}(\boldsymbol{\lambda}))$ by the convolution product.

Set $\lambda := \mathsf{cl}(\boldsymbol{\lambda}) \in \mathsf{P}^+$ and consider the corresponding 1-dimensional subtorus $\mathbb{T}(\boldsymbol{\lambda}) \subset \mathbb{G}(\boldsymbol{\lambda}) \subset \mathbb{G}(\lambda)$ as in Lemma 3.1.1. Then we have

$$Z^\bullet(\boldsymbol{\lambda}) \cong Z(\lambda)^{\mathbb{T}(\boldsymbol{\lambda})}. \tag{3.16}$$

In the sequel, for any $\mathbb{G}(\boldsymbol{\lambda})$-variety $X$, we define

$$\mathsf{K}^{\mathbb{G}(\boldsymbol{\lambda})}(X) := K^{\mathbb{G}(\boldsymbol{\lambda})}(X) \otimes_A \Bbbk, \quad \mathsf{K}^{\mathbb{G}(\boldsymbol{\lambda})}_{i,\mathrm{top}}(X) := K^{\mathbb{G}(\boldsymbol{\lambda})}_{i,\mathrm{top}}(X) \otimes_A \Bbbk,$$

for brevity of notation. They are $R(\boldsymbol{\lambda})$-modules. We also define

$$\widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(X) := \mathsf{K}^{\mathbb{G}(\boldsymbol{\lambda})}(X) \otimes_{R(\boldsymbol{\lambda})} \widehat{R}(\boldsymbol{\lambda}), \quad \widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}_{i,\mathrm{top}}(X) := \mathsf{K}^{\mathbb{G}(\boldsymbol{\lambda})}_{i,\mathrm{top}}(X) \otimes_{R(\boldsymbol{\lambda})} \widehat{R}(\boldsymbol{\lambda}).$$

When $X$ is the Steinberg type variety $Z^\bullet(\boldsymbol{\lambda})$, the completed $K$-group $\widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))$ is an algebra over $\widehat{R}(\boldsymbol{\lambda})$ with respect to the convolution product.

**Definition 3.3.4.** We define the completed Nakajima homomorphism $\widehat{\Phi}_{\boldsymbol{\lambda}} \colon \dot{U}_q(L\mathfrak{g}) \to \widehat{\mathsf{K}}^{G(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))$ as a $\Bbbk$-algebra homomorphism given by the following composition:

$$\begin{aligned}
\dot{U}_q(L\mathfrak{g}) &\xrightarrow{\Phi_\lambda} \mathsf{K}^{\mathbb{G}(\lambda)}(Z(\lambda)) \\
&\to \mathsf{K}^{\mathbb{G}(\boldsymbol{\lambda})}(Z(\lambda)) && \text{(restriction to } \mathbb{G}(\boldsymbol{\lambda}) \subset \mathbb{G}(\lambda)) \\
&\to \widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z(\lambda)) && (\mathfrak{r}_{\boldsymbol{\lambda}}\text{-adic completion}) \\
&\cong \widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda})). && \text{(localization theorem and (3.16))}
\end{aligned}$$

Let $\boldsymbol{\nu} \in \mathscr{Q}^+$ be an element such that $\mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda}) \neq \varnothing$. We pick an arbitrary point $x_{\boldsymbol{\nu}} \in \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}, \boldsymbol{\lambda})$ and consider the (non-equivariant) $K$-group $K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk$ of the fiber $\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}}$. This is a module over the convolution algebra $K(Z^\bullet(\boldsymbol{\lambda}))_\Bbbk$. We regard $K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk$ as a $\dot{U}_q$-module via the following composition:

$$\dot{U}_q \xrightarrow{\widehat{\Phi}_{\boldsymbol{\lambda}}} \widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda})) \to \widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))/\mathfrak{r}_{\boldsymbol{\lambda}} \to K^{\mathbb{T}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda})) \otimes_A \Bbbk \cong K(Z^\bullet(\boldsymbol{\lambda}))_\Bbbk.$$

**Proposition 3.3.5.** The $\dot{U}_q$-module $K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk$ is isomorphic to the local Weyl module $W(\boldsymbol{\lambda} - \boldsymbol{\nu})$.

*Proof.* When $\boldsymbol{\nu} = 0$, we have

$$\begin{aligned}
K(\mathfrak{L}^\bullet(\boldsymbol{\lambda}))_\Bbbk &\cong K^{\mathbb{T}(\boldsymbol{\lambda})}(\mathfrak{L}(\lambda)) \otimes_A \Bbbk && \text{(localization theorem)} \\
&\cong \big(K^{\mathbb{G}(\lambda)}(\mathfrak{L}(\lambda)) \otimes_{R(\mathbb{G}(\lambda))} R(\mathbb{T}(\boldsymbol{\lambda}))\big) \otimes_A \Bbbk && \text{(Theorem 3.3.3 (3))} \\
&\cong \mathbb{W}(\lambda)/\mathfrak{m}_{\boldsymbol{\lambda}} && \text{(Theorem 3.3.2)} \\
&= W(\boldsymbol{\lambda}),
\end{aligned}$$

where we should note that the maximal ideal $\mathfrak{m}_{\boldsymbol{\lambda}} \subset R(\lambda)$ is identical to the kernel of the restriction $R(\lambda) = R(\mathbb{G}(\lambda)) \otimes_A \Bbbk \to R(\mathbb{T}(\boldsymbol{\lambda})) \otimes_A \Bbbk = \Bbbk$.

For a general $\boldsymbol{\nu}$, we know that the $\dot{U}_q$-module $K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk$ is a quotient of $W(\boldsymbol{\lambda} - \boldsymbol{\nu})$ by [42, Proposition 13.3.1] and by the universality of the local Weyl module. Since there is an isomorphism $\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}} \cong \mathfrak{L}^\bullet(\boldsymbol{\lambda} - \boldsymbol{\nu})$ by Lemma 3.1.4, we have $\dim K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk = \dim K(\mathfrak{L}^\bullet(\boldsymbol{\lambda} - \boldsymbol{\nu}))_\Bbbk = \dim W(\boldsymbol{\lambda} - \boldsymbol{\nu})$ and hence the isomorphism $W(\boldsymbol{\lambda} - \boldsymbol{\nu}) \xrightarrow{\cong} K(\mathfrak{M}^\bullet(\boldsymbol{\lambda})_{x_{\boldsymbol{\nu}}})_\Bbbk$. □

### 3.3.3 Completion associated with $(Q, \beta)$

Associated with our fixed pair $(Q, \beta)$ of a Dynkin quiver $Q$ and an element $\beta = \sum_{i \in I} d_i \alpha_i$ we define $\boldsymbol{\lambda}_\beta := \sum_{i \in I} d_i \varpi_{\phi(\alpha_i)} \in \mathrm{KP}(\beta) \subset \mathscr{P}_0^+$ as before in Section 3.2.1 and set $\lambda_\beta := \mathsf{cl}(\boldsymbol{\lambda}_\beta)$. Henceforth we identify the graded quiver variety $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta)$ with the space $E_\beta$ via Hernandez-Leclerc's isomorphism $\Psi_\beta \colon \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) \cong E_\beta$ (Theorem 3.2.1) and often use the following abbreviations:

$$
\begin{aligned}
\mathbb{G}_\beta &:= G_\beta \times \mathbb{C}^\times = \mathbb{G}(\boldsymbol{\lambda}_\beta) \subset \mathbb{G}(\lambda_\beta);\\
\mathbb{T}_\beta &:= \mathbb{T}(\boldsymbol{\lambda}_\beta) \subset \mathbb{G}_\beta;\\
\mathfrak{M}_\beta^\bullet &:= \mathfrak{M}^\bullet(\boldsymbol{\lambda}_\beta) = \mathfrak{M}(\lambda_\beta)^{\mathbb{T}_\beta};\\
\pi_\beta &:= \pi^{\mathbb{T}_\beta} \colon \mathfrak{M}_\beta^\bullet \to E_\beta;\\
Z_\beta^\bullet &:= \mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathfrak{M}_\beta^\bullet = Z^\bullet(\boldsymbol{\lambda}_\beta);\\
\widehat{\Phi}_\beta &:= \widehat{\Phi}_{\boldsymbol{\lambda}_\beta} \colon \dot{U}_q(L\mathfrak{g}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet);\\
\mathfrak{r}_\beta &:= \mathfrak{r}_{\boldsymbol{\lambda}_\beta} \subset R(\boldsymbol{\lambda}_\beta).
\end{aligned}
$$

The main theorem of this section is the following.

**Theorem 3.3.6.** The pull-back along the completed Nakajima homomorphism $\widehat{\Phi}_\beta \colon \dot{U}_q(L\mathfrak{g}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)$ induces an equivalence of categories:

$$\widehat{\Phi}_\beta^* \colon \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)\text{-mod}_{\mathrm{fd}} \xrightarrow{\simeq} \mathscr{C}_{Q,\beta}.$$

A proof of Theorem 3.3.6 is given after Corollary 3.3.15. We need some preparation.

Fix a Kostant partition $\mathbf{m} \in \mathrm{KP}(\beta)$. Via the $\Bbbk$-algebra homomorphism $\theta_{\mathbf{m}} \colon R(\boldsymbol{\lambda}_\beta) \to R(\mathbf{m})$, we regard $R(\mathbf{m})$ to be an $R(\boldsymbol{\lambda}_\beta)$-algebra.

**Lemma 3.3.7.** The ideal $\langle \theta_{\mathbf{m}}(\mathfrak{r}_\beta) \rangle \subset R(\mathbf{m})$ generated by the image of $\mathfrak{r}_\beta$ is a primary ideal whose associated prime is the maximal ideal $\mathfrak{r}_{\mathbf{m}}$. In particular, we have

$$R(\mathbf{m}) \otimes_{R(\boldsymbol{\lambda}_\beta)} \widehat{R}(\boldsymbol{\lambda}_\beta) \cong \widehat{R}(\mathbf{m}).$$

*Proof.* This is a direct consequence of Corollary 3.2.6. □

We set $\boldsymbol{\nu}_{\mathbf{m}} := \boldsymbol{\lambda}_\beta - \mathbf{m} \in \mathscr{Q}_0^+$ and put

$$\mu_{\mathbf{m}} := \mathsf{cl}(\mathbf{m}) \in \mathsf{P}^+, \quad \nu_{\mathbf{m}} := \mathsf{cl}(\boldsymbol{\nu}_{\mathbf{m}}) \in \mathsf{Q}^+.$$

Consider the inverse image $\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}} = \mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathbb{O}_{\mathbf{m}}$ of the orbit $\mathbb{O}_{\mathbf{m}} = \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_{\mathbf{m}}, \boldsymbol{\lambda}_\beta)$ along the canonical morphism $\pi_\beta \colon \mathfrak{M}^\bullet_\beta \to E_\beta$. Its completed equivariant $K$-group $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}})$ is a $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\boldsymbol{\lambda}_\beta))$-bimodule via the completed Nakajima homomorphism $\widehat{\Phi}_\beta$.

**Proposition 3.3.8.** Then we have the following isomorphism of $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\boldsymbol{\lambda}_\beta))$-bimodules:

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}}) \cong \widehat{W}(\mathbf{m}),$$

where the action of $\widehat{R}(\boldsymbol{\lambda}_\beta)$ on the RHS is given via the homomorphism $\theta_{\mathbf{m}}$.

*Proof.* Pick an arbitrary point $x \in \mathbb{O}_{\mathbf{m}}$. Since the morphism $\pi_\beta \colon \mathfrak{M}^\bullet_\beta \to E_\beta$ is $\mathbb{G}_\beta$-equivariant, we have an isomorphism $\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}} \cong \mathbb{G}_\beta \times^{(\mathrm{Stab}_{\mathbb{G}_\beta} x)} (\mathfrak{M}^\bullet_\beta)_x$. Then we have

$$\begin{aligned} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}}) &\cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}\left(\mathbb{G}_\beta \times^{(\mathrm{Stab}_{\mathbb{G}_\beta} x)} (\mathfrak{M}^\bullet_\beta)_x\right) \\ &\cong \mathsf{K}^{(\mathrm{Stab}_{\mathbb{G}_\beta} x)}((\mathfrak{M}^\bullet_\beta)_x) \otimes_{R(\boldsymbol{\lambda}_\beta)} \widehat{R}(\boldsymbol{\lambda}_\beta) \\ &\cong \mathsf{K}^{\mathbb{G}(\mathbf{m})}(\mathfrak{L}^\bullet(\mathbf{m})) \otimes_{R(\mathbf{m})} \widehat{R}(\mathbf{m}) \\ &\cong \mathsf{K}^{\mathbb{G}(\mu_{\mathbf{m}})}(\mathfrak{L}(\mu_{\mathbf{m}})) \otimes_{R(\mu_{\mathbf{m}})} \widehat{R}(\mathbf{m}), \end{aligned}$$

where the second isomorphism is by the induction (see [11, 5.2.16]), the third is due to Lemma 3.1.4, Lemma 3.2.5 and Lemma 3.3.7, the last is due to the localization and Theorem 3.3.3 (3). Through this isomorphism $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}}) \cong \mathsf{K}^{\mathbb{G}(\mu_{\mathbf{m}})}(\mathfrak{L}(\mu_{\mathbf{m}})) \otimes_{R(\mu_{\mathbf{m}})} \widehat{R}(\mathbf{m})$, we see that the action of $\widehat{R}(\boldsymbol{\lambda}_\beta)$ on $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}})$ extends to an action of $\widehat{R}(\mathbf{m})$, which commutes with the action of $\dot{U}_q(L\mathfrak{g})$. By Proposition 3.3.5, the module $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_{\mathbf{m}}})/\mathfrak{r}_{\mathbf{m}} \cong K(\mathfrak{M}^\bullet(\boldsymbol{\lambda}_\beta)_x)_{\Bbbk}$ is isomorphic to the local Weyl module $W(\mathbf{m})$. Therefore, by Nakayama's lemma, we see that the vector in $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$ which corresponds to $[\mathcal{O}_{\mathfrak{L}(0,\mu_\mathbf{m})}]$ in $\mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}(\mathfrak{L}(\mu_\mathbf{m})) \otimes_{R(\mu_\mathbf{m})} \widehat{R}(\mathbf{m})$ generates $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$ as $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\mathbf{m}))$-bimodule. Moreover, from the construction of isomorphism $\mathfrak{M}(\lambda)_x \cong \mathfrak{L}(\mu_\mathbf{m})$ in Lemma 3.1.3, we see that the restriction of the class $[\mathcal{C}_i(\lambda,\nu)]$ in $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$ corresponds to the class $[\mathcal{W}_i^{\lambda-\nu}]|_{\mathfrak{L}(0,\mu_\mathbf{m})}$ in $\mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}(\mathfrak{L}(\mu_\mathbf{m})) \otimes_{R(\mu_\mathbf{m})} \widehat{R}(\mathbf{m})$. Therefore, by the universal property of the global Weyl module, we find a surjection $\widehat{W}(\mathbf{m}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$ of $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\mathbf{m}))$-bimodules. Since both are free over $\widehat{R}(\mathbf{m})$ of the same rank $\dim W(\mathbf{m})$, this surjection should be an isomorphism $\widehat{W}(\mathbf{m}) \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathfrak{M}^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$. □

**Lemma 3.3.9.** Let us consider the inverse image $Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}$ of the orbit $\mathbb{O}_\mathbf{m}$ along the canonical morphism $Z^\bullet_\beta \to E_\beta$ and regard its equivariant $K$-group $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_\beta})$ as a $(\dot{U}_q, \dot{U}_q)$-bimodule via the completed Nakajima homomorphism $\widehat{\Phi}_\beta$. Then the following statements hold.

(1) As a $(\dot{U}_q, \dot{U}_q)$-bimodule, we have $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}) \cong \widehat{W}(\mathbf{m}) \otimes_{\widehat{R}(\mathbf{m})} \widehat{W}(\mathbf{m})^\sharp$;

(2) We have $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}) = 0$;

(3) The comparison map gives an isomorphism $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}) \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}})$.

*Proof.* Pick an arbitrary point $x \in \mathbb{O}_\mathbf{m}$. Then we have an isomorphism

$$Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}} \cong \mathbb{G}_\beta \times^{(\mathrm{Stab}_{\mathbb{G}_\beta} x_\mathbf{m})} \left((\mathfrak{M}^\bullet_\beta)_x \times (\mathfrak{M}^\bullet_\beta)_x\right).$$

A similar computation as in the proof of Lemma 3.3.8 yields:

$$\begin{aligned}
\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}) &\cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}\left(\mathbb{G}_\beta \times^{(\mathrm{Stab}_{\mathbb{G}_\beta} x)} \left((\mathfrak{M}^\bullet_\beta)_x \times (\mathfrak{M}^\bullet_\beta)_x\right)\right) \\
&\cong \mathsf{K}^{\mathbb{G}(\mathbf{m})}(\mathfrak{L}(\mu_\mathbf{m}) \times \mathfrak{L}(\mu_\mathbf{m})) \otimes_{R(\mathbf{m})} \widehat{R}(\mathbf{m}) \\
&\cong \mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}(\mathfrak{L}(\mu_\mathbf{m})) \otimes_{R(\mu_\mathbf{m})} \widehat{R}(\mathbf{m}) \otimes_{R(\mu_\mathbf{m})} \mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}(\mathfrak{L}(\mu_\mathbf{m})),
\end{aligned}$$

where the last isomorphism is due to Theorem 3.3.3 (4). Then Proposition 3.3.8 proves the assertion (1). Because the same computation is valid for equivariant $K$-homologies, we have

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{i,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_\mathbf{m}}) \cong \mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}_{0,\mathrm{top}}(\mathfrak{L}(\mu_\mathbf{m})) \otimes_{R(\mu_\mathbf{m})} \widehat{R}(\mathbf{m}) \otimes_{R(\mu_\mathbf{m})} \mathsf{K}^{\mathbb{G}(\mu_\mathbf{m})}_{i,\mathrm{top}}(\mathfrak{L}(\mu_\mathbf{m})),$$

for $i = 0, 1$. Then Theorem 3.3.3 (1), (2) prove the assertions (2), (3). □

Let $\lambda_\beta := \mathsf{cl}(\boldsymbol{\lambda}_\beta)$. We fix a numbering $\{\mu_1, \mu_2, \ldots, \mu_l\}$ of the set $\{\mu \in \mathsf{P}^+ \mid \mu \le \lambda_\beta\}$ such that $\lambda_\beta = \mu_l$ and $i < j$ whenever $\mu_i < \mu_j$. Let $\nu_i := \lambda_\beta - \mu_i \in \mathsf{Q}^+$ and $\mathfrak{N}_i := \mathfrak{M}_0^{\mathrm{reg}}(\nu_i, \lambda_\beta)$ for $i \in \{1, \ldots, l\}$. Then the stratification (3.1) is written as:

$$\mathfrak{M}_0(\lambda_\beta) = \mathfrak{N}_1 \sqcup \mathfrak{N}_2 \sqcup \cdots \sqcup \mathfrak{N}_l$$

with $\mathfrak{N}_l = \{0\}$. For each $i \in \{1, \ldots, l\}$, we set $\mathfrak{N}_{\le i} := \bigsqcup_{j \le i} \mathfrak{N}_j \subset \mathfrak{M}_0(\lambda_\beta)$. Note that $\mathfrak{N}_i$ is a closed subvariety of $\mathfrak{N}_{\le i}$ and its complement is $\mathfrak{N}_{\le i-1}$.

We set $\mathfrak{N}_i^\bullet := \mathfrak{N}_i \cap \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) (= \mathfrak{N}_i \cap E_\beta)$ and $\mathfrak{N}_{\le i}^\bullet := \mathfrak{N}_{\le i} \cap \mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta)$ for each $i \in \{1, \ldots, l\}$. We fix a numbering $\{\mathbf{m}_{i,1}, \mathbf{m}_{i,2}, \ldots, \mathbf{m}_{i,k_i}\}$ of the set $\mathsf{cl}^{-1}(\lambda_i) \cap \mathrm{KP}(\beta)$, where we define $k_i = 0$ if $\mathsf{cl}^{-1}(\lambda_i) \cap \mathrm{KP}(\beta) = \varnothing$. We shall simplify the notation by setting $R_{i,s} := R(\mathbf{m}_{i,s})$, $\widehat{R}_{i,s} := \widehat{R}(\mathbf{m}_{i,s})$, $\theta_{i,s} := \theta_{\mathbf{m}_{i,s}}$ for each $i \in I$ and $s \in \{1, \ldots, k_i\}$. Set $\boldsymbol{\nu}_{i,s} := \boldsymbol{\nu}_{\mathbf{m}_{i,s}} \in \mathscr{Q}_0^+$ and $\mathbb{O}_{i,s} := \mathbb{O}_{\mathbf{m}_{i,s}} = \mathfrak{M}_0^{\bullet\,\mathrm{reg}}(\boldsymbol{\nu}_{i,s}, \boldsymbol{\lambda}_\beta)$. Note that the orbit $\mathbb{O}_{i,s}$ is a connected component of $\mathfrak{N}_i^\bullet$ for each $s \in \{1, \ldots, k_i\}$. Namely, we get the decomposition

$$\mathfrak{N}_i^\bullet = \bigsqcup_{s=1}^{k_i} \mathbb{O}_{i,s} \tag{3.17}$$

of $\mathfrak{N}_i^\bullet$ into connected components. We define a subvariety $Z_i^\bullet$ (resp. $Z_{\le i}^\bullet$) of $Z_\beta^\bullet$ to be the inverse image of the subvariety $\mathfrak{N}_i^\bullet$ (resp. $\mathfrak{N}_{\le i}^\bullet$) along the canonical morphism $Z_\beta^\bullet \to E_\beta$. From the decomposition (3.17), we have

$$Z_i^\bullet = \bigsqcup_{s=1}^{k_i} Z_\beta^\bullet|_{\mathbb{O}_{i,s}}. \tag{3.18}$$

By construction, $Z_i^\bullet$ is a closed subvariety of $Z_{\le i}^\bullet$ and its complement is $Z_{\le i-1}^\bullet$ for each $i \in \{2, \ldots, l\}$. From (A.1), we have an exact sequence:

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet) \xrightarrow{\imath_*} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\le i}^\bullet) \xrightarrow{\jmath^*} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\le i-1}^\bullet) \longrightarrow 0, \tag{3.19}$$

where $\imath : Z_i^\bullet \hookrightarrow Z_{\le i}^\bullet$ and $\jmath : Z_{\le i-1}^\bullet \hookrightarrow Z_{\le i}^\bullet$ are the inclusions.

**Lemma 3.3.10.** The map $\imath_*$ in the sequence (3.19) is injective. Therefore we have the following short exact sequence:

$$0 \longrightarrow \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet) \xrightarrow{\imath_*} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\le i}^\bullet) \xrightarrow{\jmath^*} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\le i-1}^\bullet) \longrightarrow 0,$$

for each $i \in \{2, \ldots, l\}$.

*Proof.* We shall prove that $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_{\le i}) = 0$ and the comparison map $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_{\le i}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_{\le i})$ is an isomorphism for each $i \in \{1, \dots, l\}$. If we prove this, the exact hexagon (A.2) completes a proof. We proceed by induction on $i$.

When $i = 1$, from (3.18), we have

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{j,\mathrm{top}}(Z^\bullet_{\le 1}) = \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{j,\mathrm{top}}(Z^\bullet_1) = \bigoplus_{s=1}^{k_1} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{j,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_{i,s}}),$$

where $j = 0, 1$. From Lemma 3.3.9, we know that $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_{1,s}}) = 0$ and $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{1,s}}) \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_\beta|_{\mathbb{O}_{1,s}})$ for each $s \in \{1, \dots, k_1\}$. Therefore we are done in this case.

Let $i > 1$ and assume that we know that $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_{\le i-1}) = 0$ and $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_{\le i-1}) \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_{\le i-1})$. By the same reason for the case $i = 1$ above, we have $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_i) = 0$ and $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_i) \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_i)$. Then we see that $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{1,\mathrm{top}}(Z^\bullet_{\le i}) = 0$ from the exact hexagon (A.2). Moreover we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_i) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_{\le i}) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_{\le i-1}) & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_i) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_{\le i}) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_{\le i-1}) & \longrightarrow & 0, \end{array}$$

where the upper row is the exact sequence (3.19), the lower row is the exact sequence coming from the exact hexagon (A.2). All vertical arrows are the comparison maps. Applying the five lemma, we see that the middle comparison map $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_{\le i}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}_{0,\mathrm{top}}(Z^\bullet_{\le i})$ is also an isomorphism. □

Recall we have defined a quotient algebra $U_{\le\lambda}$ of the modified quantum loop algebra $\dot{U}_q(L\mathfrak{g})$ for each $\lambda \in \mathsf{P}^+$ in Section 2.2.4 by (2.7).

**Lemma 3.3.11.** The completed Nakajima homomorphism $\widehat{\Phi}_\beta \colon \dot{U}_q(L\mathfrak{g}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$ factor through the quotient $\dot{U}_q(L\mathfrak{g}) \to U_{\le\lambda_\beta}$.

*Proof.* Since we have $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) \cong \varprojlim \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathfrak{r}_\beta^N$, it is enough to prove that the composition

$$\dot{U}_q(L\mathfrak{g}) \xrightarrow{\widehat{\Phi}_\beta} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathfrak{r}_\beta^N \tag{3.20}$$

factors through the quotient $U_{\leq\lambda_\beta}$ for every $N \in \mathbb{Z}_{>0}$. We can discuss the composition factors of $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathfrak{r}_\beta^N$ as left $\dot{U}_q(L\mathfrak{g})$-module because it is finite dimensional. By Lemma 3.3.9 (1) and the exact sequence (3.19), we see that every composition factor is a subquotient of the global Weyl modules $\mathbb{W}(\mu)$ for some $\mu \leq \lambda_\beta$. Therefore the ideal $\bigcap_{\mu\leq\lambda_\beta} \operatorname{Ann}_{\dot{U}_q(L\mathfrak{g})} \mathbb{W}(\mu)$ is included in the kernel of the map (3.20). □

By Lemma 3.3.11 above, we have the induced homomorphism $\widehat{\Phi}_\beta \colon U_{\leq\lambda_\beta} \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$, which we denote by the same symbol $\widehat{\Phi}_\beta$.

**Proposition 3.3.12.** For each $N \in \mathbb{Z}_{>0}$, the homomorphism $\widehat{\Phi}_\beta^N$ given by the composition

$$\widehat{\Phi}_\beta^N : U_{\leq\lambda_\beta} \xrightarrow{\widehat{\Phi}_\beta} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathfrak{r}_\beta^N$$

is surjective. In particular, the forgetful functor from the category of left $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$-modules to the category of $(U_{\leq\lambda_\beta}, \widehat{R}(\boldsymbol{\lambda}_\beta))$-bimodules is fully faithful.

*Proof.* Fix $N \in \mathbb{Z}_{>0}$. Using the homomorphism $\widehat{\Phi}_\beta$, we compare the affine cellular structure of the algebra $U_{\leq\lambda_\beta}$ with the filtration of the algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$ coming from the geometric stratification of $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda}_\beta) = E_\beta$ as in Lemma 3.3.10. First, for each $i$, we observe that there is the following isomorphism of $(U_{\leq\lambda_\beta}, U_{\leq\lambda_\beta})$-bimodules by Lemma 3.3.9 (1):

$$\begin{aligned}
\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_i)/\mathfrak{r}_\beta^N &\cong \bigoplus_{s=1}^{k_i} \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{i,s}})/\mathfrak{r}_\beta^N \\
&\cong \bigoplus_{s=1}^{k_i} \widehat{W}(\mathbf{m}_{i,s}) \otimes_{\widehat{R}_{i,s}} \left(\frac{\widehat{R}_{i,s}}{\langle \theta_{i,s}(\mathfrak{r}_\beta)^N\rangle}\right) \otimes_{\widehat{R}_{i,s}} \widehat{W}(\mathbf{m}_{i,s})^\sharp. \\
&\cong \bigoplus_{s=1}^{k_i} \mathbb{W}(\mu_i) \otimes_{R(\mu_i)} \left(\frac{R(\mu_i)}{R(\mu_i)\cap\langle \theta_{i,s}(\mathfrak{r}_\beta)^N\rangle}\right) \otimes_{R(\mu_i)} \mathbb{W}(\mu_i)^\sharp \\
&\cong \mathbb{W}(\mu_i) \otimes_{R(\mu_i)} \left(\frac{R(\mu_i)}{\prod_{s=1}^{k_i} R(\mu_i)\cap\langle \theta_{i,s}(\mathfrak{r}_\beta)^N\rangle}\right) \otimes_{R(\mu_i)} \mathbb{W}(\mu_i)^\sharp,
\end{aligned} \tag{3.21}$$

where we apply the Chinese remainder theorem for the last isomorphism. This is possible because maximal ideals associated to primary ideals $R(\mu_i) \cap$

$\langle \theta_{i,s}(\mathfrak{r}_\beta)^N \rangle$ are distinct by Lemma 3.3.7. By (3.21), we see that the $K$-group $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet)$ is cyclic as $(U_{\leq \lambda_\beta}, U_{\leq \lambda_\beta})$-bimodule. By construction, we can easily see that the class in $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet)$ obtained as the restriction of the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(\nu_i,\lambda_\beta)}]$ corresponds to the cyclic vector $w_{\mu_i} \otimes 1 \otimes w_{\mu_i}$ of the RHS of (3.21).

Recall the ideals $I_i$ of $U_{\leq \lambda}$ defined by (2.8). By downward induction on $i \in \{1, \dots, l\}$, we construct algebra homomorphisms $f_i^N \colon U_{\leq \lambda_\beta}/I_i \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i}^\bullet)/\mathfrak{r}_\beta^N$ and $(U_{\leq \lambda_\beta}, U_{\leq \lambda_\beta})$-bimodule homomorphisms $g_i^N \colon I_{i-1}/I_i \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet)/\mathfrak{r}_\beta^N$, which make following diagrams commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{W}(\mu_i) \otimes_{R(\mu_i)} \mathbb{W}(\mu_i)^\sharp & \xrightarrow{a_i} & U_{\leq \lambda_\beta}/I_i & \xrightarrow{b_i} & U_{\leq \lambda_\beta}/I_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_i^N} & & \downarrow{\scriptstyle f_i^N} & & \downarrow{\scriptstyle f_{i-1}^N} & & \\
 & & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet)/\mathfrak{r}_\beta^N & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i}^\bullet)/\mathfrak{r}_\beta^N & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i-1}^\bullet)/\mathfrak{r}_\beta^N & \longrightarrow & 0,
\end{array} \tag{3.22}$$

where the upper row is the exact sequence coming from the ideal chain and the lower row is exact sequence coming from (3.19).

We start from $i = l$. Define $f_l^N$ to be the homomorphism $\widehat{\Phi}_\beta^N$. Recall that the Nakajima homomorphism $\Phi_{\lambda_\beta}$ sends the element $a_{\lambda_\beta}$ to the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(0,\lambda_\beta)}]$ (see Theorem 3.3.1). Therefore, by our observation in the previous paragraph, if we define the homomorphism $g_l^N$ to give the quotient map from $\mathbb{W}(\lambda_\beta) \otimes_{R(\lambda_\beta)} \mathbb{W}(\lambda_\beta)^\sharp$ to the RHS of (3.21) via the isomorphism (3.21), the left square in (3.22) commutes. Then we have the induced homomorphism $f_{l-1}^N$ between the cokernels.

The induction step is similar. Assume that we have defined $f_i^N$. By construction, $f_i^N$ sends the image of element $a_{\mu_i}$ to the restriction of the class $\Delta_*[\mathcal{O}_{\mathfrak{M}(\nu_i,\lambda_\beta)}]$. Then if we define $g_i^N$ to be the quotient map via the isomorphism (3.21), the left square in (3.22) commutes. We get $f_{i-1}^N$ as the induced homomorphism between cokernels.

Note that we have $f_1^N = g_1^N$ and all the homomorphisms $g_i^N$ are surjective by construction. Therefore we can apply the five lemma to diagram (3.22) inductively, starting from the case $i = 2$, to prove that every $f_i^N$ is also surjective. Eventually we see that the homomorphism $f_l^N = \widehat{\Phi}_\beta^N$ is surjective. □

Using the notation in the above proof of Proposition 3.3.12, we define $K_{\leq i}^N := \operatorname{Ker} f_i^N, K_i^N := \operatorname{Ker} g_i^N$ for each $i \in \{1, \dots, l\}$ and $N \in \mathbb{Z}_{>0}$.

**Proposition 3.3.13.** For each $i \in \{1, \dots, l\}$ and for any $N_1, N_2 \in \mathbb{Z}_{>0}$, there exists a positive integer $N > N_1 + N_2$ satisfying:

(1) $K_i^N \subset K_i^{N_1} \cdot K_i^{N_2}$;

(2) $K_{\leq i}^N \subset K_{\leq i}^{N_1} \cdot K_{\leq i}^{N_2}$.

*Proof.* We first prove the assertion (1). Assume that $k_i = 0$. Thus we have $g_i^N = 0$ and hence $K_i^N = I_{i-1}/I_i$ for any $N \in \mathbb{Z}_{>0}$. In this case, the assertion (1) is equivalent to the assertion $(I_{i-1}/I_i)^2 = I_{i-1}/I_i$, which follows from Theorem 2.2.16. Next we consider the case $k_i \neq 0$. By (3.21), we have

$$K_i^N = \mathbb{W}(\mu_i) \otimes_{R(\mu_i)} \left( \prod_{s=1}^{k_i} \left( R(\mu_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\beta)^N \rangle \right) \right) \otimes_{R(\mu_i)} \mathbb{W}(\mu_i)^\sharp,$$

for each $N \in \mathbb{Z}_{>0}$. By Lemma 3.3.7, the ideal $R(\mu_i) \cap \langle \theta_{i,a}(\mathfrak{r}_\beta)^N \rangle$ is a primary ideal whose associated prime is the maximal ideal $\mathfrak{r}_{\mu_i, \boldsymbol{\lambda}_{i,a}}$. Thus for a sufficiently large $N > 0$, we have $R(\mu_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\beta)^N \rangle \subset (R(\mu_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\beta)^{N_1} \rangle)(R(\mu_i) \cap \langle \theta_{i,s}(\mathfrak{r}_\beta)^{N_2} \rangle)$. Then we obtain the assertion $K_i^N \subset K_i^{N_1} \cdot K_i^{N_2}$.

We prove the assertion (2) by induction on $i$. The case $i = 1$ follows from (1) since $f_1^N = g_1^N$. We assume that $i > 1$ and the assertion (2) is true for $i - 1$. For given $N_1, N_2 \in \mathbb{Z}_{>0}$, we can find an integer $M > N_1 + N_2$ such that $K_i^M \subset K_i^{N_1} \cdot K_i^{N_2}$ by (1). Applying Lemma 3.3.14 below to the injection $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_i^\bullet) \hookrightarrow \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i}^\bullet)$, we find an integer $N' \in \mathbb{Z}_{>0}$ such that we have

$$\operatorname{Ker}(f_i^{N'} \circ a_i) \subset K_i^M \subset K_i^{N_1} \cdot K_i^{N_2}, \tag{3.23}$$

where $a_i$ is the inclusion $I_{i-1}/I_i \hookrightarrow U_{\leq \lambda}/I_i$ as in the diagram (3.22). Applying the snake lemma to the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{W}(\mu_i) \otimes_{R(\mu_i)} \mathbb{W}(\mu_i)^\sharp & \xrightarrow{a_i} & U_{\leq \lambda_\beta}/I_i & \xrightarrow{b_i} & U_{\leq \lambda_\beta}/I_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_i^{N'} \circ a_i} & & \downarrow{\scriptstyle f_i^{N'}} & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Im}(f_i^{N'} \circ a_i) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i}^\bullet)/\mathfrak{r}_{\boldsymbol{\lambda}_\beta}^{N'} & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_{\leq i-1}^\bullet)/\mathfrak{r}_{\boldsymbol{\lambda}_\beta}^{N'} & \longrightarrow & 0,
\end{array}$$

we get an exact sequence:

$$0 \longrightarrow \operatorname{Ker}(f_i^{N'} \circ a_i) \longrightarrow K_{\leq i}^{N'} \longrightarrow K_{\leq i-1}^{N'} \longrightarrow 0. \tag{3.24}$$

Let $N'_1, N'_2$ be any two integers larger than $N'$. By induction hypothesis, there is an integer $N > N'_1 + N'_2$ such that $K^N_{\leq i-1} \subset K^{N'_1}_{\leq i-1} \cdot K^{N'_2}_{\leq i-1}$. We shall prove that the assertion (2) holds for this $N$. Let $x \in K^N_{\leq i}$ be an arbitrary element. Note that we have $b_i(x) \in K^N_{\leq i-1} \subset K^{N'_1}_{\leq i-1} \cdot K^{N'_2}_{\leq i-1}$. Since the quotient map $b_i : U_{\leq\lambda}/I_i \to U_{\leq\lambda}/I_{i-1}$ induces the surjection $K^{N'_1}_{\leq i} \cdot K^{N'_2}_{\leq i} \twoheadrightarrow K^{N'_1}_{\leq i-1} \cdot K^{N'_2}_{\leq i-1}$, we can choose an element $y \in K^{N'_1}_{\leq i} \cdot K^{N'_2}_{\leq i}$ so that $b_i(x-y) = 0$. By the exact sequence (3.24), there is an element $y' \in \operatorname{Ker}(f_i^{N'} \circ a_i)$ such that $x = y + a_i(y')$. By (3.23), we see that $a_i(y') \in K^{N_1}_{\leq i} \cdot K^{N_2}_{\leq i}$. Therefore we have $x \in K^{N_1}_{\leq i} \cdot K^{N_2}_{\leq i}$ as desired. □

**Lemma 3.3.14.** Let $R$ be a commutative Noetherian complete local algebra over a field $\Bbbk$ with the maximal ideal $\mathfrak{r}$. Let $\varphi : M_1 \hookrightarrow M_2$ be a injective homomorphism between finitely generated $R$-modules $M_1, M_2$. Put $M_i^n := \mathfrak{r}^n M_i$ for $i = 1, 2$ and for each $n \in \mathbb{Z}_{>0}$. We denote the kernel of the induced map $\varphi^n \colon M_1/M_1^n \to M_2/M_2^n$ by $K^n$. Then for any $n \in \mathbb{Z}_{>0}$, there is an integer $n_0 > n$ such that we have $K^N \subset M_1^n/M_1^N$ for any $N \geq n_0$.

*Proof.* Assume the contrary to deduce a contradiction. Namely we assume that there is an integer $n \in \mathbb{Z}_{>0}$ such that $K^N \not\subset M_1^n/M_1^N$ for any $N > n$. For each $a, b \in \mathbb{Z}_{>0}$ with $a < b$, let $\kappa^{a,b} \colon K^b \to K^a$ be the homomorphism induced from the quotient homomorphism $M_i/M_i^b \to M_i/M_i^a$. These homomorphisms $\kappa^{a,b}$ define a projective system $\{\kappa^{a,b} : K^b \to K^a \mid a < b\}$. Note that $\varprojlim K^N = \operatorname{Ker}\varphi = 0$.

By our assumption, we have $\kappa^{n,N} \neq 0$ for any $N > n$. Since $\kappa^{n,N+1} = \kappa^{n,N} \circ \kappa^{N,N+1}$, we get a decreasing sequence of subspaces:

$$K^n \supset \operatorname{Im}\kappa^{n,n+1} \supset \operatorname{Im}\kappa^{n,n+2} \supset \cdots.$$

The fact $\dim K^n < \infty$ ensures that there exists a non-zero subspace $L^n \subset K^n$ which is included in $\operatorname{Im}\kappa^{n,N}$ for all $N > n$. Define $L^N := (\kappa^{n,N})^{-1}(L^n) \subset K^N$. then we get a projective subsystem of $\{K^N\}_{N\in\mathbb{Z}_{>0}}$:

$$\cdots \to L^N \to \cdots \to L^{n+1} \to L^n \to 0 \to 0 \to \cdots \to 0,$$

whose limit is non-zero by construction. Thus we have $0 \neq \varprojlim L^N \subset \varprojlim K^N = 0$, which is a contradiction. □

As the special case $i = l$ of Proposition 3.3.13 (2), we obtain the following.

**Corollary 3.3.15.** For any $N_1, N_2 \in \mathbb{Z}_{>0}$, there is an positive integer $N$ such that $\operatorname{Ker} \widehat{\Phi}_\beta^N \subset (\operatorname{Ker} \widehat{\Phi}_\beta^{N_1}) \cdot (\operatorname{Ker} \widehat{\Phi}_\beta^{N_2})$.

*Proof of Theorem 3.3.6.* From Lemmas 3.3.9 and 3.3.10, we see that the pullback $\widehat{\Phi}_\beta^* \left( \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet) \right)$ is stratified as a left $\dot{U}_q(L\mathfrak{g})$-module by various deformed Weyl modules $\widehat{W}(\mathbf{m})$ with $\mathbf{m} \in \mathrm{KP}(\beta)$. Therefore the pullback functor $\widehat{\Phi}_\beta^* : \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)\text{-mod}_{\mathrm{fd}} \to \mathscr{C}_\mathfrak{g}$ lands in the full subcategory $\mathscr{C}_{Q,\beta}$. Note that we have

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)\text{-mod}_{\mathrm{fd}} = \bigcup_{N \in \mathbb{Z}_{>0}} \left( \widehat{\mathsf{K}}^{\mathbb{G}(\beta)}(Z_\beta^\bullet)/\mathfrak{r}_\beta^N \right)\text{-mod}_{\mathrm{fd}},$$

where $\left( \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)/\mathfrak{r}_\beta^N \right)$-mod$_{\mathrm{fd}}$ is identical to the full subcategory of $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)$-mod$_{\mathrm{fd}}$ consisting of modules $M$ satisfying $\mathfrak{r}_{\boldsymbol{\lambda}_\beta}^N M = 0$. Thus, by Proposition 3.3.12, we see that the pullback functor $\widehat{\Phi}_\beta^* : \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)\text{-mod}_{\mathrm{fd}} \to \mathscr{C}_{Q,\beta}$ is fully faithful. To prove that it is essentially surjective, it is enough to show that for each module $M \in \mathscr{C}_{Q,\beta}$ there is an positive integer $N \in \mathbb{Z}_{>0}$ such that $(\operatorname{Ker} \widehat{\Phi}_\beta^N)M = 0$. We proceed by induction on the length of $M$. When $M = L(\mathbf{m})$ is a simple module of $\mathscr{C}_{Q,\beta}$ with $\mathbf{m} \in \mathrm{KP}(\beta)$, we see that $(\operatorname{Ker} \widehat{\Phi}_\beta^1)L(\mathbf{m}) = 0$. For induction step, we write the module $M$ as an extension of two non-zero modules $M_1, M_2 \in \mathscr{C}_{Q,\beta}$. By induction hypothesis, there are integers $N_1, N_2 \in \mathbb{Z}_{>0}$ such that $(\operatorname{Ker} \widehat{\Phi}_\beta^{N_1})M_1 = (\operatorname{Ker} \widehat{\Phi}_\beta^{N_2})M_2 = 0$. We can find an integer $N \in \mathbb{Z}_{>0}$ such that $\operatorname{Ker} \widehat{\Phi}_\beta^N \subset (\operatorname{Ker} \widehat{\Phi}_\beta^{N_1}) \cdot (\operatorname{Ker} \widehat{\Phi}_\beta^{N_2})$ by Corollary 3.3.15. Then we have $(\operatorname{Ker} \widehat{\Phi}_\beta^N)M = 0$. □

**Remark 3.3.16.** As a generalization of Theorem 3.2.1, Leclerc-Plamondon [37] established some equivariant isomorphisms between graded quiver varieties $\mathfrak{M}_0^\bullet(\boldsymbol{\lambda})$ associated with certain $\ell$-dominant weights $\boldsymbol{\lambda} \in \mathscr{P}^+$ and the spaces of representations of the repetitive algebra $\widehat{\mathbb{C}Q}$ of the quiver $Q$ whose dimension vector corresponds to $\boldsymbol{\lambda}$. For this generalized choice of $\boldsymbol{\lambda} \in \mathscr{P}^+$, the completed convolution algebra $\widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))$ still becomes affine quasi-hereditary. In fact, it is isomorphic to the completion of the geometric extension algebra associated to $\pi\colon \mathfrak{M}^\bullet(\boldsymbol{\lambda}) \to \mathfrak{M}_0^\bullet(\boldsymbol{\lambda})$ discussed as above, and we can apply [38, Theorem 4.7]. However, in this generalized setting, Theorem 3.3.6 does not hold. In particular, standard modules of $\widehat{\mathsf{K}}^{\mathbb{G}(\boldsymbol{\lambda})}(Z^\bullet(\boldsymbol{\lambda}))$ are not always isomorphic to the deformed local Weyl modules in such a general setting. This happens because an analogue of Lemma 3.2.5 does not hold in general. In this sense, Theorem 3.3.6 is a special phenomenon for our setting.

### 3.3.4 Affine highest weight structure

Thanks to Theorem 3.3.6, we can regard the category

$$\widehat{\mathscr{C}}_{Q,\beta} := \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)\text{-mod}_{\mathrm{fg}}$$

as a "completion" of the category $\mathscr{C}_{Q,\beta} \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)\text{-mod}_{\mathrm{fd}}$ so that we have enough projective modules (and hence the notation above). In this section, we see that the category $\widehat{\mathscr{C}}_{Q,\beta}$ has a structure of affine highest weight category, or equivalently, the algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$ is an affine quasi-hereditary algebra. For the notion of affine highest weight category and affine quasi-hereditary algebra in general, see Appendix A.2. Recall that we defined a partial order $\preceq$ on the set $\mathrm{KP}(\beta)$ to be the opposite of the orbit closure inclusion.

**Theorem 3.3.17.** The algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$ is affine quasi-hereditary for the poset $(\mathrm{KP}(\beta), \preceq)$. Via the completed Nakajima homomorphism $\widehat{\Phi}_\beta \colon \dot{U}_q(L\mathfrak{g}) \to \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$, the standard (resp. proper standard, proper costandard) module associated with $\mathbf{m} \in \mathrm{KP}(\beta)$ is identified with the deformed local Weyl module $\widehat{W}(\mathbf{m})$ (resp. local Weyl module $W(\mathbf{m})$, dual local Weyl module $W^\vee(\mathbf{m})$).

*Proof.* We construct an affine quasi-heredity chain of the algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$. Let $\{\mathbf{m}_1, \dots, \mathbf{m}_r\}$ be a total ordering of the set $\mathrm{KP}(\beta)$ refining the partial order $\preceq$, i.e. it satisfies $j \le k$ whenever $\mathbf{m}_j \preceq \mathbf{m}_k$. Then we have $\mathbf{m}_r = \boldsymbol{\lambda}_\beta$. We set $\mathbb{O}_i := \mathbb{O}_{\mathbf{m}_i}$, $\mathbb{O}_{\ge i} := \bigsqcup_{j \ge i} \mathbb{O}_j$, $\mathbb{O}_{\le i} := \bigsqcup_{j \le i} \mathbb{O}_j$. We denote the inverse image of $\mathbb{O}_i$ (resp. $\mathbb{O}_{\ge i}, \mathbb{O}_{\le i}$) along the canonical morphism $Z^\bullet_\beta \to E_\beta$ by $Z^\bullet_\beta|_{\mathbb{O}_i}$ (resp. $Z^\bullet_\beta|_{\mathbb{O}_{\ge i}}, Z^\bullet_\beta|_{\mathbb{O}_{\le i}}$). By construction, $Z^\bullet_\beta|_{\mathbb{O}_{\ge i+1}}$ (resp. $Z^\bullet_\beta|_{\mathbb{O}_i}$) is a closed subvariety of $Z^\bullet_\beta|_{\mathbb{O}_{\ge i}}$ (resp. $Z^\bullet_\beta|_{\mathbb{O}_{\le i}}$) and its complement is $Z^\bullet_\beta|_{\mathbb{O}_i}$ (resp. $Z^\bullet_\beta|_{\mathbb{O}_{\le i-1}}$). Also $Z^\bullet_\beta|_{\mathbb{O}_{\ge i}}$ is a closed subvariety of $Z^\bullet_\beta$ whose complement is $Z^\bullet_\beta|_{\mathbb{O}_{\le i-1}}$. Then we consider the following commutative diagram arising from (A.1):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\geq i+1}}) & = & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\geq i+1}}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\geq i}}) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\leq i-1}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{i}}) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\leq i}}) & \longrightarrow & \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\leq i-1}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.25}
$$

Arguing as in Lemma 3.3.10, we see that the left column and the lower row in (3.25) are exact. By downward induction on $i$ and diagram chases, we see that the middle row (and hence the middle column) in the diagram (3.25) is also exact.

Therefore we can regard $\mathsf{I}_i := \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\geq i+1}})$ for each $i \in \{0, \dots, r\}$ as a two-sided ideal of the algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$. What we have to do is to prove that the chain of ideals

$$
0 = \mathsf{I}_r \subsetneq \mathsf{I}_{r-1} \subsetneq \cdots \subsetneq \mathsf{I}_1 \subsetneq \mathsf{I}_0 = \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) \tag{3.26}
$$

gives an affine quasi-heredity chain. Observe that

$$
\mathsf{I}_{i-1}/\mathsf{I}_i \cong \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta|_{\mathbb{O}_i}) \cong \widehat{W}(\mathbf{m}_i) \otimes_{\widehat{R}(\mathbf{m}_i)} \widehat{W}(\mathbf{m}_i)^\sharp \cong \widehat{W}(\mathbf{m}_i)^{\oplus s_i}
$$

as a left $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$-module, where $s_i := \dim W(\mathbf{m}_i)$. By Theorem 3.3.6, the category of finite-dimensional modules over $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathsf{I}_i$ is identified with the full subcategory of $\mathscr{C}_{Q,\beta}$ consisting of modules $M$ whose $\ell$-weights belong to the set $\bigcup_{j \leq i}\{\boldsymbol{\mu} \in \mathscr{P} \mid \boldsymbol{\mu} \leq \mathbf{m}_j\}$. For such a module $M$, we have $\mathrm{Hom}_{\dot{U}_q(L\mathfrak{g})}(\widehat{W}(\mathbf{m}_i), M) \cong M_{\mathbf{m}_i}$ by the universal property of the deformed local Weyl module (Proposition 2.2.15 (1)). In particular, the functor $\mathrm{Hom}_{\dot{U}_q(L\mathfrak{g})}(\widehat{W}(\mathbf{m}_i), -)$ is exact on the category $(\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathsf{I}_i)\text{-mod}_{\mathrm{fd}}$. Since any module $M \in (\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta)/\mathsf{I}_i)\text{-mod}_{\mathrm{fg}}$ can be written as a projective limit of finite-dimensional modules (i.e. $M \cong \varprojlim M/\mathfrak{r}_{\boldsymbol{\lambda}}^N$), we see that the deformed local Weyl module $\widehat{W}(\mathbf{m}_i)$ is a projective module in the category $(\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(\mathbb{Z}_\beta)/\mathsf{I}_i)\text{-mod}_{\mathrm{fg}}$ with its simple head $L(\mathbf{m}_i)$. Moreover, we have

$$\mathrm{Hom}_{\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)}(\mathsf{I}_{i-1}/\mathsf{I}_i, \widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)/\mathsf{I}_{i-1}) = 0.$$

From these observations and Proposition 2.2.15, we conclude that the chain (3.26) is an affine quasi-heredity chain.

By Remark A.2.5, the algebra $\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z_\beta^\bullet)$ is affine quasi-hereditary for the poset $(\mathrm{KP}(\beta), \preceq)$, whose standard module (resp. proper standard module) associated with $\mathbf{m} \in \mathrm{KP}(\beta)$ (as a $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\boldsymbol{\lambda}_\beta)$-bimodule) is the deformed local Weyl module $\widehat{W}(\mathbf{m})$ (resp. local Weyl module $W(\mathbf{m})$). To prove the assertion for proper costandard modules, we have to show the Ext-orthogonality as in Theorem A.2.6. This is done in Proposition 3.3.18 below. □

To show the Ext-orthogonality between deformed local Weyl modules and dual local Weyl modules, we need to prepare some notation. Note that the dual local Weyl module $W^\vee(\mathbf{m})$ corresponding to $\mathbf{m} \in \mathrm{KP}(\beta)$ actually belongs to $\mathscr{C}_{Q,\beta}$ by Proposition 2.2.19 and Theorem 3.3.6.

We need to prepare some duality functors. We temporarily use the ambient category $\mathscr{B}$ of all $(\dot{U}_q(L\mathfrak{g}), \widehat{R}(\boldsymbol{\lambda}))$-bimodules. Note that each $M \in \mathscr{B}$ has the weight space decomposition $M = \bigoplus_{\mu \in \mathsf{P}} M_\mu$, where $M_\mu = a_\mu M$ and each weight space $M_\mu$ is preserved by the action of $\widehat{R}(\boldsymbol{\lambda}_\beta)$. We define the full dual and the topological dual of $M \in \mathscr{B}$ by $M^* := \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ and $D(M) := \bigcup_N \mathrm{Hom}_{\Bbbk}(M/\mathfrak{r}_\beta^N, \Bbbk)$ respectively. We define a left $\dot{U}_q(L\mathfrak{g})$-module structure on $M^*$ (resp. on $D(M)$) by twisting the natural right $\dot{U}_q(L\mathfrak{g})$-module structure by the antipode $S$ (resp. $S^{-1}$). Thus we obtain an exact contravariant endofunctor $(-)^* \colon \mathscr{B} \to \mathscr{B}$ and a right exact contravariant endofunctor $D \colon \mathscr{B} \to \mathscr{B}$. If $M \in \mathscr{B}$ is finitely generated over $\widehat{R}(\boldsymbol{\lambda}_\beta)$, we have $(D(M))^* \cong M$. If $M \in \mathscr{C}_{\mathfrak{g}}$ (with trivial $\widehat{R}(\boldsymbol{\lambda}_\beta)$-action), the dual $M^*$ (resp. $D(M)$) coincides with the left dual $M^*$ (resp. the right dual ${}^*M$) of $M$.

**Proposition 3.3.18.** Let $\mathbf{m}_1, \mathbf{m}_2 \in \mathrm{KP}(\beta)$. Then we have:

$$\mathrm{Ext}^i_{\widehat{\mathscr{C}}_{Q,\beta}}(\widehat{W}(\mathbf{m}_1), W^\vee(\mathbf{m}_2)) = \begin{cases} \Bbbk & i = 0, \mathbf{m}_1 = \mathbf{m}_2; \\ 0 & \text{otherwise}. \end{cases}$$

*Proof.* The case $i = 0$ follows from the fact that the module $W^\vee(\mathbf{m}_2)$ has a simple socle $L(\mathbf{m}_2)$ with $\dim W^\vee(\mathbf{m}_2)_{\mathbf{m}_2} = 1$ and the universality of the deformed Weyl module (Proposition 2.2.15 (1)).

For $i = 1$, we consider an extension in $\widehat{\mathscr{C}}_{Q,\beta}$:

$$0 \to W^\vee(\mathbf{m}_2) \to E \to \widehat{W}(\mathbf{m}_1) \to 0. \tag{3.27}$$

Put $\mu_j := \mathsf{cl}(\mathbf{m}_j)$ for $j = 1, 2$. If $\mu_1 \not< \mu_2$, then the $\ell$-weight $\mathbf{m}_1$ is maximal in $E/\mathfrak{r}_\beta^N$ for any $N \in \mathbb{Z}_{>0}$. By the universal property of the deformed local Weyl module $\widehat{W}(\mathbf{m}_1)$, we see that the sequence (3.27) must be split. If $\mu_1 < \mu_2$, we apply the topological duality functor $D$ to the sequence (3.27) to get the following exact sequence:

$$0 \to D(\widehat{W}(\mathbf{m}_1)) \to D(E) \to W(^*\mathbf{m}_2). \tag{3.28}$$

Since $\mu_1 < \mu_2$, we have

$$\dim D(E)_{-w_0\mu_2} = \dim E_{w_0\mu_2} = \dim E_{\mu_2} = 1 = \dim W(^*\mathbf{m}_2)_{-w_0\mu_2},$$

where the second equality is due to Weyl group symmetry coming from integrability. In particular, the image of the weight space $D(E)_{-w_0\mu_2}$ coincides with $W(^*\mathbf{m}_2)_{-w_0\mu_2}$, which generates $W(^*\mathbf{m}_2)$. Therefore the rightmost arrow in the sequence (3.28) is surjective. Moreover, by the universal property of the local Weyl module $W(^*\mathbf{m}_2)$, we see that the sequence (3.28) is split. By applying the full duality functor $(-)^*$, we find that the sequence (3.27) is also split. Therefore we have $\mathrm{Ext}^1_{\widehat{\mathscr{C}}_{Q,\beta}}(\widehat{W}(\mathbf{m}_1), W^\vee(\mathbf{m}_2)) = 0$.

The cases $i > 1$ follow from the case $i = 1$ by a standard argument in (affine) highest weight categories. $\square$

### 3.3.5 Comparison with a geometric extension algebra

Since the group $\mathbb{G}_\beta (= \mathbb{G}(\boldsymbol{\lambda}_\beta))$ is the centralizer of the torus $\mathbb{T}_\beta (= \mathbb{T}(\boldsymbol{\lambda}_\beta))$ inside $\mathbb{G}(\lambda_\beta)$ (where $\lambda_\beta := \mathsf{cl}(\boldsymbol{\lambda}_\beta)$ as before), the multiplication map induces an isomorphism

$$G_\beta \times \mathbb{T}_\beta \xrightarrow{\cong} \mathbb{G}_\beta \tag{3.29}$$

of algebraic groups. Note that this decomposition yields an isomorphism

$$K^{\mathbb{G}_\beta}(X) \otimes_A \Bbbk \cong K^{G_\beta}(X)_\Bbbk$$

for any $\mathbb{G}_\beta$-variety $X$ with a trivial $\mathbb{T}_\beta$-action. As a special case when $X = \mathrm{pt}$, we have an isomorphism

$$R(\boldsymbol{\lambda}_\beta) \overset{(3.12)}{=} R(\mathbb{G}(\boldsymbol{\lambda}_\beta)) \otimes_A \Bbbk \cong R(G_\beta)_\Bbbk$$

of $\Bbbk$-algebras, via which the maximal ideal $\mathfrak{r}_\beta \subset R(\boldsymbol{\lambda}_\beta)$ defined in Section 2.2.3 corresponds to the augmentation ideal $I \subset R(G_\beta)_\Bbbk$. Therefore we have an isomorphism

$$\left[K^{\mathbb{G}_\beta}(X) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_\beta} \cong \widehat{K}^{G_\beta}(X)_\Bbbk, \tag{3.30}$$

where $[-]^\wedge_{\mathfrak{r}_\beta}$ denotes the $\mathfrak{r}_\beta$-adic completion. See Appendix A.1.2 for the RHS. For $X = Z^\bullet_\beta$, we get an isomorphism

$$\widehat{\mathsf{K}}^{\mathbb{G}_\beta}(Z^\bullet_\beta) = \left[K^{\mathbb{G}_\beta}(Z^\bullet_\beta) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_\beta} \cong \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$$

of convolution algebras. Here the first equality follows because we know that $\mathsf{K}^{\mathbb{G}_\beta}(Z^\bullet_\beta)$ is a finitely generated $R(\boldsymbol{\lambda}_\beta)$-module thanks to the discussions in the previous section (see Lemmas 3.3.7, 3.3.9 and 3.3.10).

**Proposition 3.3.19.** The Riemann-Roch homomorphism gives an isomorphism of $\widehat{R}(G_\beta)_\Bbbk$-algebras:

$$\mathrm{RR}^{G_\beta} \colon \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk \xrightarrow{\cong} H^{G_\beta}_*(Z^\bullet_\beta, \Bbbk)^\wedge.$$

*Proof.* By Proposition A.1.3, the map $\mathrm{RR}^{G_\beta} \colon \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk \to H^{G_\beta}_*(\mathcal{Z}_\beta, \Bbbk)^\wedge$ is an algebra homomorphism. To prove that the map $\mathrm{RR}^{G_\beta} \colon \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk \to H^{G_\beta}_*(Z^\bullet_\beta, \Bbbk)^\wedge$ is an isomorphism, it suffices to check that the equivariant Chern character map $(\mathrm{ch}^{G_\beta})^{\mathfrak{M}^\bullet_\beta \times \mathfrak{M}^\bullet_\beta}_{Z^\bullet_\beta} \colon \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk \to H^{G_\beta}_*(Z^\bullet_\beta, \Bbbk)^\wedge$ gives an isomorphism of $\widehat{R}(G_\beta)_\Bbbk$-modules since $\mathrm{RR}^{G_\beta}$ is obtained from $(\mathrm{ch}^{G_\beta})^{\mathfrak{M}^\bullet_\beta \times \mathfrak{M}^\bullet_\beta}_{\mathcal{Z}_\beta}$ by multiplying the $G_\beta$-equivariant Todd class $p_1^* \mathrm{Td}^{G_\beta}_{\mathcal{F}_\beta}$, which is an invertible element.

Likewise as in the proof of Theorem 3.3.17, we fix a total ordering $\mathrm{KP}(\beta) = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_s\}$ refining the partial order $\preceq$, and set $\mathbb{O}_i := \mathbb{O}_{\mathbf{m}_i}$, $\mathbb{O}_{\geq i} := \bigsqcup_{j \geq i} \mathbb{O}_j$, $\mathbb{O}_{\leq i} := \bigsqcup_{j \leq i} \mathbb{O}_j$. By Lemma 3.1.4, Lemma 3.2.5 and the reduction, we have

$$K^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_k}) \cong K^{G(\mathbf{m}_k)}(\mathfrak{L}^\bullet(\mathbf{m}_k) \times \mathfrak{L}^\bullet(\mathbf{m}_k)),$$
$$H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_k}, \Bbbk) \cong H^{G(\mathbf{m}_k)}_*(\mathfrak{L}^\bullet(\mathbf{m}_k) \times \mathfrak{L}^\bullet(\mathbf{m}_k), \Bbbk)$$

for each $k$. Then, using [42, Theorem 7.4.1], we can prove that the equivariant Chern character map gives an isomorphism $\mathrm{ch}^{G_\beta}\colon \widehat{K}^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_k})_{\Bbbk} \xrightarrow{\cong} H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_k}, \Bbbk)^\wedge$ for each $k$. Moreover, we obtain the following commutative diagram with exact rows for each $k$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{K}^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\le k-1}})_{\Bbbk} & \longrightarrow & \widehat{K}^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\le k}})_{\Bbbk} & \longrightarrow & \widehat{K}^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_k})_{\Bbbk} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \\ 0 & \longrightarrow & H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_{\le k-1}}, \Bbbk)^\wedge & \longrightarrow & H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_{\le k}}, \Bbbk)^\wedge & \longrightarrow & H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_k}, \Bbbk)^\wedge & \longrightarrow & 0. \end{array}$$

By induction on $k$, the equivariant Chern character map gives an isomorphism $\mathrm{ch}^{G_\beta}\colon \widehat{K}^{G_\beta}(Z^\bullet_\beta|_{\mathbb{O}_{\le k}})_{\Bbbk} \xrightarrow{\cong} H^{G_\beta}_*(Z^\bullet_\beta|_{\mathbb{O}_{\le k}}, \Bbbk)^\wedge$ for all $k$. □

We consider the proper push-forward

$$\mathcal{L}^\bullet_\beta := (\pi_\beta)_* \underline{\Bbbk}$$

of the trivial local system $\underline{\Bbbk}$ on $\mathfrak{M}^\bullet_\beta$. By the decomposition theorem, we have

$$\mathcal{L}^\bullet_\beta \cong \bigoplus_{\mathbf{m}\in \mathrm{KP}(\beta)} L^\bullet_{\mathbf{m}} \otimes_{\Bbbk} \mathcal{IC}_{\mathbf{m}} = \bigoplus_{\mathbf{m}\in \mathrm{KP}(\beta)} \bigoplus_{k\in\mathbb{Z}} L^\bullet_{\mathbf{m},k} \otimes_{\Bbbk} \mathcal{IC}_{\mathbf{m}}[k],$$

where $\mathcal{IC}_{\mathbf{m}}$ denotes the intersection cohomology complex associated with the trivial local system on the orbit $\mathbb{O}_{\mathbf{m}}$ and $L^\bullet_{\mathbf{m}} = \bigoplus_k L^\bullet_{\mathbf{m},k}$ is a finite-dimensional graded $\Bbbk$-vector space, which is known to be non-zero for each $\mathbf{m}$ (see [42, Theorem 14.3.2]). We consider the Yoneda algebra

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^k_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)$$

in the derived category of $G_\beta$-equivariant constructible complexes on $E_\beta$. This is a $\mathbb{Z}$-graded $\Bbbk$-algebra whose grading is bounded from below.

By a standard argument (see [11, Section 8.6]), we have an isomorphism of $\Bbbk$-algebras

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta) \cong H^{G_\beta}_*(Z^\bullet_\beta, \Bbbk).$$

Note that this is not compatible with the $\mathbb{Z}$-grading. This isomorphism induces an isomorphism between the completions:

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)^\wedge \cong H^{G_\beta}_*(Z^\bullet_\beta, \Bbbk)^\wedge. \tag{3.31}$$

As a conclusion, we obtain the following.

**Corollary 3.3.20.** We have the following isomorphisms of $\Bbbk$-algebras:

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)^\wedge \cong H_*^{G_\beta}(Z^\bullet_\beta, \Bbbk)^\wedge \cong \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk.$$

**Remark 3.3.21.** By a general theory by Kato [32], or also by McNamara [38], the Yoneda algebra $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)$ is a $\mathbb{Z}$-graded affine quasi-hereditary algebra. Therefore its completion $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}^\bullet_\beta, \mathcal{L}^\bullet_\beta)^\wedge \cong \widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk$ inherits a structure of affine quasi-hereditary algebra (see [19, Section 4] for the completion). This gives an alternative proof of the first statement of Theorem 3.3.17.

# Chapter 4

# Dynkin quiver type quantum affine Schur-Weyl duality

## 4.1 Quiver Hecke algebras

In this section, we explain the basics of the quiver Hecke algebras, also known as Khovanov-Lauda-Rouquier algebras, of finite $\mathsf{ADE}$ types (Section 4.1.1) and their geometric realization due to Varagnolo-Vasserot [49] (Section 4.1.2). We remark that Rouquier [48] also considered a similar geometric interpretation of the quiver Hecke algebras independently.

Let $\Bbbk$ be a field of characteristic zero in this section. Later we will set $\Bbbk = \overline{\mathbb{Q}(q)}$.

### 4.1.1 Definition and properties

Fix an element $\beta = \sum_{i \in I} d_i \alpha_i \in \mathsf{Q}^+$ and put $d := \sum_{i \in I} d_i = \operatorname{ht} \beta$. Let

$$I^\beta := \{\mathbf{i} = (i_1, \ldots, i_d) \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \beta\}.$$

The symmetric group $\mathfrak{S}_d$ of degree $d$ acts on the set $I^\beta$ from the right by $(i_1, \ldots, i_d) \cdot \sigma := (i_{\sigma(1)}, \ldots, i_{\sigma(d)})$. Let $s_k \in \mathfrak{S}_d$ denote the transposition of $k$ and $k+1$ for $1 \le k < d$.

**Definition 4.1.1** (Khovanov-Lauda [34], Rouquier [47])**.** The quiver Hecke algebra $H_Q(\beta)$ is defined to be a $\Bbbk$-algebra with the generating set

$$\{1_{\mathbf{i}} \mid \mathbf{i} \in I^\beta\} \cup \{x_1, \ldots, x_d\} \cup \{\tau_1, \ldots, \tau_{d-1}\},$$

satisfying the following relations:

$$1_{\mathbf{i}}1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'}1_{\mathbf{i}}, \quad \sum_{\mathbf{i}\in I^\beta} 1_{\mathbf{i}} = 1, \quad x_k x_l = x_l x_k, \quad x_k 1_{\mathbf{i}} = 1_{\mathbf{i}} x_k,$$

$$\tau_k 1_{\mathbf{i}} = 1_{\mathbf{i}\cdot s_k}\tau_k, \quad \tau_k\tau_l = \tau_l\tau_k \quad \text{if } |k-l|>1,$$

$$\tau_k^2 1_{\mathbf{i}} = \begin{cases} (x_k - x_{k+1})1_{\mathbf{i}}, & \text{if } i_k \leftarrow i_{k+1}, \\ (x_{k+1} - x_k)1_{\mathbf{i}}, & \text{if } i_k \rightarrow i_{k+1}, \\ 1_{\mathbf{i}} & \text{if } a_{i_k,i_{k+1}} = 0, \\ 0 & \text{if } i_k = i_{k+1}, \end{cases}$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)1_{\mathbf{i}} = \begin{cases} -1_{\mathbf{i}} & \text{if } l = k, i_k = i_{k+1}, \\ 1_{\mathbf{i}} & \text{if } l = k+1, i_k = i_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)1_{\mathbf{i}} = \begin{cases} 1_{\mathbf{i}} & \text{if } i_k = i_{k+2}, i_k \leftarrow i_{k+1}, \\ -1_{\mathbf{i}} & \text{if } i_k = i_{k+2}, i_k \rightarrow i_{k+1}, \\ 0 & \text{otherwise}. \end{cases}$$

We set

$$P_\beta := \bigoplus_{\mathbf{i}\in I^\beta} \Bbbk[x_1, \ldots, x_d]1_{\mathbf{i}}$$

with a commutative $\Bbbk[x_1, \ldots, x_d]$-algebra structure $1_{\mathbf{i}} \cdot 1_{\mathbf{i}'} = \delta_{\mathbf{i}\mathbf{i}'}1_{\mathbf{i}}$. By the defining relations, there is a $\Bbbk$-algebra homomorphism $P_\beta \to H_Q(\beta)$ sending the generators $1_{\mathbf{i}}, x_k \in P_\beta$ to $1_{\mathbf{i}}, x_k \in H_Q(\beta)$ respectively. For each $\sigma \in \mathfrak{S}_d$, we fix a reduced expression $\sigma = s_{k_1}\cdots s_{k_p}$. Then we define $\tau_\sigma := \tau_{k_1}\cdots\tau_{k_p} \in H_Q(\beta)$. Note that this element $\tau_\sigma$ depends on the choice of a reduced expression of $\sigma$ in general because we do not have the braid relation for $\tau_k$'s. Then the following fact is known.

**Proposition 4.1.2** (Khovanov-Lauda [34])**.** The quiver Hecke algebra $H_Q(\beta)$ is a left (or right) free module over the commutative algebra $P_\beta$ with a free basis $\{\tau_\sigma \mid \sigma \in \mathfrak{S}_d\}$.

*Proof.* See [34, Proposition 2.7]. □

The quiver Hecke algebra $H_Q(\beta)$ is equipped with a $\mathbb{Z}$-grading given by

$$\deg 1_{\mathbf{i}} = 0, \quad \deg x_k = 2, \quad \deg \tau_k 1_{\mathbf{i}} = -a_{i_k, i_{k+1}}.$$

Since the grading is bounded from below by Proposition 4.1.2, the completion $\widehat{H}_Q(\beta) := H_Q(\beta)^\wedge$ inherits a natural structure of $\Bbbk$-algebra. Explicitly, we have

$$\widehat{H}_Q(\beta) = \bigoplus_{\sigma \in \mathfrak{S}_d} \widehat{P}_\beta \tau_\sigma = \bigoplus_{\sigma \in \mathfrak{S}_d} \tau_\sigma \widehat{P}_\beta,$$

where $\widehat{P}_\beta := \bigoplus_{\mathbf{i}} \Bbbk[\![x_1, \ldots, x_d]\!] 1_{\mathbf{i}}$.

For $\beta, \beta' \in \mathsf{Q}^+$ with $\operatorname{ht} \beta = d$, $\operatorname{ht} \beta' = d'$, we have an embedding

$$H_Q(\beta) \otimes H_Q(\beta') \hookrightarrow H_Q(\beta + \beta')$$

given by $1_{\mathbf{i}} \otimes 1_{\mathbf{i}'} \mapsto 1_{\mathbf{i} \circ \mathbf{i}'}$, $x_k \otimes 1 \mapsto x_k$, $\tau_k \otimes 1 \mapsto \tau_k$, $1 \otimes x_k \mapsto x_{d+k}$, $1 \otimes \tau_k \mapsto \tau_{d+k}$, where we set $\mathbf{i} \circ \mathbf{i}' := (i_1, \ldots, i_d, i'_1, \ldots, i'_{d'}) \in I^{\beta+\beta'}$. Using this embedding, we define the *convolution product* $M \circ M'$ of a left $H_Q(\beta)$-module $M$ and a left $H_Q(\beta')$-module $M'$ by

$$M \circ M' := H_Q(\beta + \beta') \otimes_{H_Q(\beta) \boxtimes H_\beta(\beta')} (M \boxtimes M'),$$

which is a left $H_Q(\beta + \beta')$-module.

The quiver Hecke algebras categorify the quantum group in the following sense. Let $H_Q(\beta)$-$\mathsf{proj}$ be the additive category of finitely generated graded projective left $H_Q$-modules and $H_Q(\beta)$-$\mathsf{gmod}$ be the abelian category of finite-dimensional graded left $H_Q(\beta)$-modules. We equip the direct sum categories

$$H_Q\text{-}\mathsf{proj} := \bigoplus_{\beta \in \mathsf{Q}^+} H_Q(\beta)\text{-}\mathsf{proj}, \quad H_Q\text{-}\mathsf{gmod} := \bigoplus_{\beta \in \mathsf{Q}^+} H_Q(\beta)\text{-}\mathsf{gmod}$$

with structures of monoidal category using the convolution product. Since the convolution product is a bi-exact functor, the Grothendieck groups $K(H_Q\text{-}\mathsf{proj})$ and $K(H_Q\text{-}\mathsf{gmod})$ become $\mathsf{Q}^+$-graded $\mathbb{Z}[v, v^{-1}]$-algebras, where the multiplication with $v^{\pm 1}$ is given by the grading shifts. The following fundamental result is a consequence of Khovanov-Lauda [34], Rouquier [47, 48] and Varagnolo-Vasserot [49].

**Theorem 4.1.3** (Categorification Theorem)**.** Let $U_v^+(\mathfrak{g})_\mathbb{Z}$ be the $\mathbb{Z}[v, v^{-1}]$-form of the positive part of the quantized enveloping algebra $U_v(\mathfrak{g})$ (generated by the divided powers of the Chevalley generators) and $U_v^+(\mathfrak{g})_\mathbb{Z}^\vee$ be its (graded) dual. Then there is a $\mathsf{Q}^+$-graded $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism

$$K(H_Q\text{-}\mathsf{proj}) \cong U_v^+(\mathfrak{g})_\mathbb{Z}, \quad (\text{resp. } K(H_Q\text{-}\mathsf{gmod}) \cong U_v^+(\mathfrak{g})_\mathbb{Z}^\vee)$$

under which the classes of self-dual indecomposable projective modules (resp. self-dual simple modules) correspond to the canonical basis elements (resp. the dual canonical basis elements) bijectively.

For each $\beta \in \mathsf{Q}^+$, let us consider the category

$$\mathscr{M}_{Q,\beta} := \widehat{H}_Q(\beta)\text{-mod}_{\mathrm{fd}}$$

of finite-dimensional $\widehat{H}_Q(\beta)$-modules. By Proposition 4.1.2, we see that this category is identical to the category of finite-dimensional $H_Q(\beta)$-modules on which the elements $x_k$ act nilpotently. Note that here we do not consider the gradings of $H_Q(\beta)$ modules. By forgetting the gradings, we have an exact functor

$$H_Q(\beta)\text{-}\mathsf{gmod} \to \mathscr{M}_{Q,\beta}.$$

Again by Proposition 4.1.2, we can prove that every simple module in $\mathscr{M}_{Q,\beta}$ is gradable. Therefore the above functor induces an isomorphism

$$K(H_Q(\beta)\text{-}\mathsf{gmod})|_{v=1} \cong K(\mathscr{M}_{Q,\beta})$$

of the Grothendieck groups. The convolution product equips the direct sum

$$\mathscr{M}_Q := \bigoplus_{\beta \in \mathsf{Q}^+} \mathscr{M}_{Q,\beta}$$

with a structure of monoidal category. By forgetting the gradings, we obtain an isomorphism

$$K(H_Q\text{-}\mathsf{gmod})|_{v=1} \cong K(\mathscr{M}_Q)$$

of the Grothendieck rings.

Let $G$ be a complex affine algebraic group whose Lie algebra is $\mathfrak{g}$ and $N$ be a maximal unipotent subgroup of $G$ corresponding to the positive roots $\mathsf{R}^+$. It is well-known that the coordinate ring $\mathbb{C}[N]$ is isomorphic to the (graded) dual of the positive part of the enveloping algebra $U(\mathfrak{g})$. Combining the observation above with the categorification theorem (Theorem 4.1.3) we obtain the following:

**Corollary 4.1.4.** There is an isomorphism of $\mathbb{C}$-algebras

$$K(\mathscr{M}_Q)_{\mathbb{C}} \cong \mathbb{C}[N]$$

which sends the classes of simple modules to the elements of the dual canonical basis bijectively.

Finally we recall the faithful polynomial right representation of $H_Q(\beta)$ from [34, Section 2.3]. We define $f^w(x_1, \dots, x_d) := f(x_{w(1)}, \dots, x_{w(d)})$ for $f \in \Bbbk[x_1, \dots, x_d]$ and $w \in \mathfrak{S}_d$.

**Theorem 4.1.5** ([34] Proposition 2.3)**.** The following formulas give a faithful right $H_Q(\beta)$-module structure on the $\Bbbk$-vector space $P_\beta$:

$$
\begin{aligned}
a \cdot 1_{\mathbf{i}} &= a 1_{\mathbf{i}}, \\
a \cdot x_k &= a x_k, \\
(f 1_{\mathbf{i}}) \cdot \tau_k &= \begin{cases} \dfrac{f^{s_k} - f}{x_k - x_{k+1}} 1_{\mathbf{i}} & \text{if } i_k = i_{k+1}, \\ (x_{k+1} - x_k) f^{s_k} 1_{\mathbf{i} \cdot s_k} & \text{if } i_k \leftarrow i_{k+1}, \\ f^{s_k} 1_{\mathbf{i} \cdot s_k} & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $a \in P_\beta$ and $f 1_{\mathbf{i}} \in \Bbbk[x_1, \dots, x_d] 1_{\mathbf{i}}$.

Replacing the polynomial ring $\Bbbk[x_1, \dots, x_d]$ with the ring $\Bbbk[\![x_1, \dots, x_d]\!]$ of formal power series, we get the completion of the representation $P_\beta$:

$$
\widehat{P}_\beta = \bigoplus_{\mathbf{i} \in I^\beta} \Bbbk[\![x_1, \dots, x_d]\!] 1_{\mathbf{i}} = P_\beta \otimes_{H_Q(\beta)} \widehat{H}_Q(\beta). \tag{4.1}
$$

### 4.1.2 Varagnolo-Vasserot's realization

Fix an $I$-graded $\mathbb{C}$-vector space $D = \bigoplus_{i \in I} D_i$ with $\underline{\dim}\, D = \beta$, i.e. $\dim D_i = d_i$ as in Section 2.1.2. We consider the following two non-singular $G_\beta$-varieties:

$$
\begin{aligned}
&\mathcal{B}_\beta = \{F^\bullet = (D = F^0 \supsetneq F^1 \supsetneq \cdots \supsetneq F^d = 0) \mid F^k \text{ is an } I\text{-graded subspace of } D\}, \\
&\mathcal{F}_\beta = \{(F^\bullet, x) \in \mathcal{B}_\beta \times E_\beta \mid x(F^k) \subset F^k \text{ for any } 1 \le k \le d\}.
\end{aligned}
$$

The $G_\beta$-action on $\mathcal{F}_\beta$ is defined so that the projections $\mathrm{pr}_1 \colon \mathcal{F}_\beta \to \mathcal{B}_\beta$ and $\mu_\beta := \mathrm{pr}_2 \colon \mathcal{F}_\beta \to E_\beta$ are $G_\beta$-equivariant. They decompose into connected components as

$$
\mathcal{B}_\beta = \bigsqcup_{\mathbf{i} \in I^\beta} \mathcal{B}_{\mathbf{i}}, \qquad \mathcal{F}_\beta = \bigsqcup_{\mathbf{i} \in I^\beta} \mathcal{F}_{\mathbf{i}},
$$

where we put

$$
\mathcal{B}_{\mathbf{i}} := \{F^\bullet \in \mathcal{B}_\beta \mid \underline{\dim}\, F^{k-1} = \underline{\dim}\, F^k + \alpha_{i_k},\ \forall k\}, \quad \mathcal{F}_{\mathbf{i}} := (\mathrm{pr}_1)^{-1}(\mathcal{B}_{\mathbf{i}})
$$

for $\mathbf{i} = (i_1, \ldots, i_d) \in I^\beta$.

We fix a basis $\{v_k\}_{1\le k\le d}$ of the vector space $D$ so that the set $\{v_{i,j}\}_{1\le j\le d_i}$ forms a basis of the vector space $D_i$ for each $i \in I$, where we put $v_{i,j} := v_{d_1+\cdots+d_{i-1}+j}$. Let $H_i \subset GL(D_i)$ be the maximal torus fixing the lines $\{\mathbb{C}v_{i,j}\}_{1\le j\le d_i}$ for each $i \in I$. We set $H_\beta := \prod_{i\in I} H_i \subset G_\beta$.

Let $F_0^\bullet \in \mathcal{B}_\beta$ be the flag defined by $F_0^k := \bigoplus_{l>k} \mathbb{C}v_l$, which belongs to the component $\mathcal{B}_{\mathbf{i}_0}$ with $\mathbf{i}_0 := (1^{d_1}, 2^{d_2}, \ldots, n^{d_n}) \in I^\beta$. For each $\mathbf{i} \in I^\beta$, we fix an element $w_\mathbf{i} \in \mathfrak{S}_d$ such that $\mathbf{i} = \mathbf{i}_0 \cdot w_\mathbf{i}$. The set $\{w_\mathbf{i}\}_{\mathbf{i}\in I^\beta}$ forms a complete system of coset representatives for the quotient $\mathfrak{S}_\beta \backslash \mathfrak{S}_d$, where $\mathfrak{S}_\beta := \mathrm{Stab}_{\mathfrak{S}_d}(\mathbf{i}_0) = \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n}$. For each $w \in \mathfrak{S}_d$, we define the flag $F_w^\bullet$ by $F_w^k := \bigoplus_{l>k} \mathbb{C}v_{w(l)}$ which belongs to the component $\mathcal{B}_{\mathbf{i}_0\cdot w}$. Let $F_\mathbf{i}^\bullet := F_{w_\mathbf{i}}^\bullet \in \mathcal{B}_\mathbf{i}$ for $\mathbf{i} \in I^\beta$. Then we have $\mathcal{B}_\mathbf{i} \cong G_\beta / B_\mathbf{i}$ with $B_\mathbf{i} := \mathrm{Stab}_{G_\beta}(F_\mathbf{i}^\bullet) \subset G_\beta$ being the Borel subgroup fixing the flag $F_\mathbf{i}^\bullet$, which contains the maximal torus $H_\beta$. Then we have

$$H_*^{G_\beta}(\mathcal{B}_\mathbf{i}, \Bbbk) \cong H_*^{B_\mathbf{i}}(\mathrm{pt}, \Bbbk) \cong H_{H_\beta}^*(\mathrm{pt}, \Bbbk) \cong \Bbbk[x_1, \ldots, x_d]1_\mathbf{i}, \tag{4.2}$$

where the last isomorphism sends the 1st $H_\beta$-equivariant Chern class of the line $\mathbb{C}v_{w_\mathbf{i}(k)}$ to the element $x_k 1_\mathbf{i}$. Thus we get an isomorphism

$$H_*^{G_\beta}(\mathcal{B}_\beta, \Bbbk) = \bigoplus_{\mathbf{i}\in I^\beta} H_*^{G_\beta}(\mathcal{B}_\mathbf{i}, \Bbbk) \cong \bigoplus_{\mathbf{i}\in I^\beta} \Bbbk[x_1, \ldots, x_d]1_\mathbf{i} = P_\beta. \tag{4.3}$$

We consider the Steinberg type variety $\mathcal{Z}_\beta := \mathcal{F}_\beta \times_{E_\beta} \mathcal{F}_\beta$ associated with the morphism $\mu_\beta \colon \mathcal{F}_\beta \to E_\beta$. Its $G_\beta$-equivariant Borel-Moore homology group $H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)$ becomes a $\Bbbk$-algebra with respect to the convolution product relative to $\mathcal{F}_\beta \times \mathcal{F}_\beta \times \mathcal{F}_\beta$. We identify the variety $\mathcal{B}_\beta$ with the fiber product $\{0\} \times_{E_\beta} \mathcal{F}_\beta$. Then the convolution product relative to $\{0\} \times \mathcal{F}_\beta \times \mathcal{F}_\beta$ makes the space $H_*^{G_\beta}(\mathcal{B}_\beta, \Bbbk)$ into a right $H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)$-module.

Let $\mu_\mathbf{i}$ denote the restriction of the proper morphism $\mu_\beta \colon \mathcal{F}_\beta \to E_\beta$ to the component $\mathcal{F}_\mathbf{i}$ for $\mathbf{i} \in I^\beta$. We put

$$\mathcal{L}_\beta := \bigoplus_{\mathbf{i}\in I^\beta} (\mu_\mathbf{i})_* \underline{\Bbbk}[\dim \mathcal{F}_\mathbf{i}],$$

where $\underline{\Bbbk}[\dim \mathcal{F}_\mathbf{i}]$ is the trivial local system (i.e. the constant $\Bbbk$-sheaf of rank 1) on $\mathcal{F}_\mathbf{i}$ homologically shifted by $\dim \mathcal{F}_\mathbf{i}$. By the decomposition theorem, we have

$$\mathcal{L}_\beta \cong \bigoplus_{\mathbf{m}\in \mathrm{KP}(\beta)} L_\mathbf{m} \otimes_\Bbbk \mathcal{IC}_\mathbf{m} = \bigoplus_{\mathbf{m}\in \mathrm{KP}(\beta)} \bigoplus_{k\in\mathbb{Z}} L_{\mathbf{m},k} \otimes_\Bbbk \mathcal{IC}_\mathbf{m}[k],$$

where $\mathcal{IC}_{\mathbf{m}}$ denotes the intersection cohomology complex associated with the trivial local system on the orbit $\mathbb{O}_{\mathbf{m}}$ and $L_{\mathbf{m}} = \bigoplus_{k \in \mathbb{Z}} L_{\mathbf{m},k}[k]$ is a self-dual finite-dimensional graded $\Bbbk$-vector space for each $\mathbf{m} \in \mathrm{KP}(\beta)$. The vector space $L_{\mathbf{m}}$ is known to be non-zero for all $\mathbf{m} \in \mathrm{KP}(\beta)$ (see [31, Corollary 2.8]).

Similarly to Section 3.3.5, we have a standard isomorphism of $\Bbbk$-algebras

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta) \cong H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk). \tag{4.4}$$

Let $\mathcal{L}_{\mathbf{i}}(k)$ be the $G_\beta$-equivariant line bundle on $\mathcal{F}_{\mathbf{i}}$ whose fiber at the point $(F^\bullet, x) \in \mathcal{F}_{\mathbf{i}}$ is $F^{k-1}/F^k$ for $\mathbf{i} \in I^\beta$ and $1 \leq k \leq d$.

**Theorem 4.1.6** (Varagnolo-Vasserot [49])**.** There is a unique isomorphism of $\mathbb{Z}$-graded $\Bbbk$-algebras

$$H_Q(\beta) \xrightarrow{\cong} \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta) \tag{4.5}$$

which satisfies the following properties:

(1) The composition $H_Q(\beta) \xrightarrow{\cong} H^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)$ of the isomorphisms (4.5) and (4.4) sends the element $1_{\mathbf{i}}$ (resp. $x_k 1_{\mathbf{i}}$) to the push-forward of the fundamental class $[\mathcal{F}_{\mathbf{i}}]$ (resp. the 1st $G_\beta$-equivariant Chern class of the line bundle $\mathcal{L}_{\mathbf{i}}(k)$) with respect to the diagonal embedding $\mathcal{F}_{\mathbf{i}} \to \mathcal{F}_{\mathbf{i}} \times_{E_\beta} \mathcal{F}_{\mathbf{i}}$;

(2) We have the following commutative diagram:

$$\begin{array}{ccc} H_Q(\beta) & \xrightarrow{\cong} & H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk) \\ \downarrow & & \downarrow \\ \mathrm{End}\,(P_\beta)^{\mathrm{op}} & \xrightarrow{\cong} & \mathrm{End}\left(H_*^{G_\beta}(\mathcal{B}_\beta, \Bbbk)\right)^{\mathrm{op}}, \end{array}$$

where the lower horizontal arrow denotes the isomorphism induced from (4.3) and the vertical arrows denote the right module structures.

**Remark 4.1.7.** Because our convention of the flag variety $\mathcal{B}_\beta$ differs from Varagnolo-Vasserot's [49], we need a modification. Actually, our isomorphism (4.5) is obtained by twisting the original isomorphism $H_Q(\beta) \cong \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)$ in [49] by a $\Bbbk$-algebra involution on $H_Q(\beta)$ given by

$$1_{\mathbf{i}} \mapsto 1_{\mathbf{i}^{\mathrm{op}}}, \quad x_k \mapsto x_{d-k+1}, \quad \tau_k 1_{\mathbf{i}} \mapsto \begin{cases} -\tau_{d-k} 1_{\mathbf{i}^{\mathrm{op}}} & \text{if } i_k = i_{k+1}; \\ \tau_{d-k} 1_{\mathbf{i}^{\mathrm{op}}} & \text{if } i_k \neq i_{k+1}, \end{cases}$$

where $\mathbf{i}^{\mathrm{op}} := (i_d, \ldots, i_2, i_1)$ for $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in I^\beta$.

Similarly to the case of the $G_\beta$-equivariant Borel-Moore homologies, the $K$-group $K^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$ becomes an $R(G_\beta)_\Bbbk$-algebra and the $K$-group $K^{G_\beta}(\mathcal{B}_\beta)_\Bbbk$ becomes a right $K^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$-module with respect to the convolution products.

For each $\mathbf{i} \in I^\beta$, we have

$$K^{G_\beta}(\mathcal{B}_\mathbf{i})_\Bbbk \cong K^{B_\mathbf{i}}(\mathrm{pt})_\Bbbk \cong K^{H_\beta}(\mathrm{pt})_\Bbbk = R(H_\beta)_\Bbbk \cong \Bbbk[y_1^{\pm 1}, \dots, y_d^{\pm 1}]1_\mathbf{i}$$

where the last isomorphism sends the class $[\mathbb{C}v_{w_\mathbf{i}(k)}]$ of the 1-dimensional $H_\beta$-module $\mathbb{C}v_{w_\mathbf{i}(k)}$ to the element $y_k 1_\mathbf{i}$. The $G_\beta$-equivariant Chern character map $(\mathrm{ch}^{G_\beta})^{\mathcal{B}_\mathbf{i}}_{\mathcal{B}_\mathbf{i}}$ gives an isomorphism of $\Bbbk$-algebras

$$\widehat{K}^{G_\beta}(\mathcal{B}_\mathbf{i})_\Bbbk \cong \Bbbk[\![y_1 - 1, \dots, y_d - 1]\!]1_\mathbf{i} \xrightarrow{\cong} \Bbbk[\![x_1, \dots, x_d]\!]1_\mathbf{i} \cong H_*^{G_\beta}(\mathcal{B}_\mathbf{i}, \Bbbk)^\wedge,$$

where the middle arrow sends the element $y_k 1_\mathbf{i}$ to the exponential $e^{x_k}1_\mathbf{i}$ for $1 \leq k \leq d$. Applying the equivariant Riemann-Roch theorem (=Theorem A.1.1) to the inclusion $\mathcal{B}_\mathbf{i} \hookrightarrow \mathcal{F}_\mathbf{i}$, we have

$$(\mathrm{ch}^{G_\beta})^{\mathcal{F}_\mathbf{i}}_{\mathcal{B}_\mathbf{i}} = C_\mathbf{i} \cdot (\mathrm{ch}^{G_\beta})^{\mathcal{B}_\mathbf{i}}_{\mathcal{B}_\mathbf{i}}, \quad C_\mathbf{i} := (\mathrm{Td}^{G_\beta}_{\mathcal{F}_\mathbf{i}})^{-1}\mathrm{Td}^{G_\beta}_{\mathcal{B}_\mathbf{i}} \cdot 1_\mathbf{i} \in \Bbbk[\![x_1, \dots, x_d]\!]1_\mathbf{i} \tag{4.6}$$

and hence the map $(\mathrm{ch}^{G_\beta})^{\mathcal{F}_\mathbf{i}}_{\mathcal{B}_\mathbf{i}}$ is an isomorphism of $\widehat{R}(G_\beta)_\Bbbk$-modules. Summing up over $\mathbf{i} \in I^\beta$, we obtain an isomorphism of $\widehat{R}(G_\beta)_\Bbbk$-modules

$$(\mathrm{ch}^{G_\beta})^{\mathcal{F}_\beta}_{\mathcal{B}_\beta} \colon \widehat{K}^{G_\beta}(\mathcal{B}_\beta)_\Bbbk \xrightarrow{\cong} H_*^{G_\beta}(\mathcal{B}_\beta, \Bbbk)^\wedge. \tag{4.7}$$

**Proposition 4.1.8.** The Riemann-Roch homomorphism gives an isomorphism of $\widehat{R}(G_\beta)_\Bbbk$-algebras:

$$\mathrm{RR}^{G_\beta} \colon \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk \xrightarrow{\cong} H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)^\wedge,$$

which makes the following diagram commute:

$$\begin{array}{ccc} \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk & \xrightarrow{\cong} & H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)^\wedge \\ \downarrow & & \downarrow \\ \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathcal{B}_\beta)_\Bbbk\right)^{\mathrm{op}} & \xrightarrow{\cong} & \mathrm{End}\left(H_*^{G_\beta}(\mathcal{B}_\beta, \Bbbk)^\wedge\right)^{\mathrm{op}}, \end{array} \tag{4.8}$$

where the lower horizontal arrow denotes the isomorphism induced from (4.7) and the vertical arrows denote the right module structures.

*Proof.* As in the proof of Proposition 3.3.19, it suffices to prove that the equivariant Chern character map $(\mathrm{ch}^{G_\beta})^{\mathcal{F}_\beta\times\mathcal{F}_\beta}_{\mathcal{Z}_\beta}\colon \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_{\Bbbk} \to H_*^{G_\beta}(\mathcal{Z}_\beta,\Bbbk)^\wedge$ is an isomorphism.

Because we have the connected component decomposition

$$\mathcal{Z}_\beta = \bigsqcup_{\mathbf{i},\mathbf{i}'\in I^\beta} \mathcal{Z}_{\mathbf{i},\mathbf{i}'}, \quad \mathcal{Z}_{\mathbf{i},\mathbf{i}'} := \mathcal{F}_{\mathbf{i}} \times_{E_\beta} \mathcal{F}_{\mathbf{i}'},$$

we focus on a connected component

$$\mathcal{Z}_{\mathbf{i},\mathbf{i}'} = \{(F^\bullet, F'^\bullet, x) \in \mathcal{B}_{\mathbf{i}} \times \mathcal{B}_{\mathbf{i}'} \times E_\beta \mid x(F^k) \subset F^k, x(F'^k) \subset F'^k,\ \forall k\}.$$

For each $w \in \mathfrak{S}_\beta w_{\mathbf{i}'}$, we define a locally closed $G_\beta$-subvariety

$$\mathcal{Z}^w_{\mathbf{i},\mathbf{i}'} = G_\beta \times^{B_{\mathbf{i}}} \{(F^\bullet_{\mathbf{i}}, F'^\bullet, x) \in \mathcal{Z}_{\mathbf{i},\mathbf{i}'} \mid F'^\bullet \in B_{\mathbf{i}} F^\bullet_w\}$$

which is a $G_\beta$-equivariant affine bundle over $\mathcal{B}_{\mathbf{i}}$. They give a $G_\beta$-stable stratification $\mathcal{Z}_{\mathbf{i},\mathbf{i}'} := \bigsqcup_{w\in\mathfrak{S}_\beta w_{\mathbf{i}'}} Z^w_{\mathbf{i},\mathbf{i}'}$. Fix a total ordering $\mathfrak{S}_\beta w_{\mathbf{i}'} = \{w_1, w_2, \ldots, w_m\}$ such that we have $w_k w_{\mathbf{i}}^{-1} < w_l w_{\mathbf{i}}^{-1}$ in the Bruhat ordering only if $k < l$. We simply write $\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'} := \mathcal{Z}^{w_k}_{\mathbf{i},\mathbf{i}'}$ and set $\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'} := \bigsqcup_{j\le k} \mathcal{Z}^j_{\mathbf{i},\mathbf{i}'}$. Then for each $k$, the variety $\mathcal{Z}^{\le k-1}_{\mathbf{i},\mathbf{i}'}$ is closed in $\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'}$ and its complement is $\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'}$. Since $\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'}$ is a $G_\beta$-equivariant affine bundle over $\mathcal{B}_{\mathbf{i}}$, its homology of odd degree vanishes: $H^{G_\beta}_{\mathrm{odd}}(\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'},\Bbbk) = 0$. Therefore an inductive argument with respect to $k$ yields $H^{G_\beta}_{\mathrm{odd}}(\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'},\Bbbk) = 0$. Using the cellular fibration lemma [11, 5.5.1] for equivariant $K$-groups and Proposition A.1.2, we obtain the following commutative diagram with exact rows for each $k$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{K}^{G_\beta}(\mathcal{Z}^{\le k-1}_{\mathbf{i},\mathbf{i}'})_{\Bbbk} & \longrightarrow & \widehat{K}^{G_\beta}(\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'})_{\Bbbk} & \longrightarrow & \widehat{K}^{G_\beta}(\mathcal{Z}^{k}_{\mathbf{i},\mathbf{i}'})_{\Bbbk} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \downarrow{\scriptstyle \mathrm{ch}^{G_\beta}} & & \\
0 & \longrightarrow & H_*^{G_\beta}(\mathcal{Z}^{\le k-1}_{\mathbf{i},\mathbf{i}'},\Bbbk)^\wedge & \longrightarrow & H_*^{G_\beta}(\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'},\Bbbk)^\wedge & \longrightarrow & H_*^{G_\beta}(\mathcal{Z}^{k}_{\mathbf{i},\mathbf{i}'},\Bbbk)^\wedge & \longrightarrow & 0.
\end{array}$$

Note that the map $\mathrm{ch}^{G_\beta}\colon \widehat{K}^{G_\beta}(\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'})_{\Bbbk} \to H_*^{G_\beta}(\mathcal{Z}^k_{\mathbf{i},\mathbf{i}'},\Bbbk)^\wedge$ is an isomorphism for any $k$ since again the variety $Z^k_{\mathbf{i},\mathbf{i}'}$ is an affine bundle over $\mathcal{B}_{\mathbf{i}}$. Hence, by induction on $k$, we conclude that $\mathrm{ch}^{G_\beta}\colon \widehat{K}^{G_\beta}(\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'})_{\Bbbk} \to H_*^{G_\beta}(\mathcal{Z}^{\le k}_{\mathbf{i},\mathbf{i}'},\Bbbk)^\wedge$ is an isomorphism for all $k$. □

Note that the isomorphism (4.4) induces an isomorphism between the completions:

$$\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)^\wedge \cong H^{G_\beta}_*(\mathcal{Z}_\beta, \Bbbk)^\wedge.$$

As a summary of this subsection, we have the following.

**Corollary 4.1.9.** We have the following isomorphisms of $\Bbbk$-algebras:

$$\widehat{H}_Q(\beta) \cong \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)^\wedge \cong H^{G_\beta}_*(\mathcal{Z}_\beta, \Bbbk)^\wedge \cong \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk.$$

## 4.2 Algebraic construction

In this section and the next section, we study the Dynkin quiver type quantum affine Schur-Weyl duality. In the present section, we give a concise explanation of the original algebraic construction due to Kang-Kashiwara-Kim [25, 26]. We remark that actually their construction is a special (but quite interesting) case of more general framework of their *generalized quantum affine Schur-Weyl duality* developed further in their joint works [27, 28] with Se-jin Oh.

In this section, we take the field $\Bbbk$ to be $\overline{\mathbb{Q}(q)}$ as before.

### 4.2.1 Kang-Kashiwara-Kim's bimodule

Recall that for each $i \in I$, the global Weyl module $\mathbb{W}(\varpi_i)$ of highest weight $\varpi_i$ is a $(U_q(L\mathfrak{g}), R(\varpi_i))$-bimodule (see Theorem 2.2.10), where we can write $R(\varpi_i) = \Bbbk[z^{\pm 1}_{\varpi_i}]$. We set $\lambda_i := \varpi_j$ and $a_i := q^p$ if $\phi(\alpha_i) = (j,p) \in \widehat{I}$. Then the deformed local Weyl module

$$\widehat{V}_i := \widehat{W}(\phi(\alpha_i))$$

is a $(U_q(L\mathfrak{g}), \widehat{R}(\phi(\alpha_i)))$-bimodule, where we can write $\widehat{R}(\phi(\alpha_i)) = \Bbbk[\![z_{\lambda_i} - a_i]\!]$.

For each $\mathbf{i} = (i_1, i_2, \dots, i_d) \in I^\beta$, we consider a left $U_q(L\mathfrak{g})$-module

$$\widehat{V}^{\otimes \mathbf{i}} := \widehat{V}_{i_1} \hat{\otimes} \widehat{V}_{i_2} \hat{\otimes} \cdots \hat{\otimes} \widehat{V}_{i_d}.$$

We define $X_k := z_{\lambda_{i_k}}$, which is the equivariant parameter acting on the $k$-th tensor factor $\widehat{V}_{i_k}$ of $\widehat{V}^{\otimes \mathbf{i}}$ for each $1 \le k \le d$. Under this notation, we have a commuting action of the algebra $\Bbbk[\![X_1 - a_{i_1}, \dots, X_d - a_{i_d}]\!]$ on the $U_q(L\mathfrak{g})$-module $\widehat{V}^{\otimes \mathbf{i}}$.

We want to construct on the left $U_q(L\mathfrak{g})$-module

$$\widehat{V}^{\otimes\beta} := \bigoplus_{\mathbf{i}\in I^\beta} \widehat{V}^{\otimes\mathbf{i}}$$

a commuting right action of the quiver Hecke algebra $H_Q(\beta)$ using the normalized $R$-matrices in Section 2.2.5. In order to do this, we need the following technical assumption. Recall that the normalized $R$-matrix is a homomorphism of $(U_q(L\mathfrak{g}), \Bbbk[z_{\varpi_i}^{\pm 1}, z_{\varpi_j}^{\pm 1}])$-bimodules,

$$R_{i,j}^{\mathrm{norm}} \colon \mathbb{W}(\varpi_i) \otimes \mathbb{W}(\varpi_j) \to \Bbbk(z_{\varpi_j}/z_{\varpi_i}) \otimes_{\Bbbk[(z_{\varpi_j}/z_{\varpi_i})^{\pm 1}]} (\mathbb{W}(\varpi_j) \otimes \mathbb{W}(\varpi_i)),$$

whose denominator is denoted by $d_{i,j}(z_{\varpi_j}/z_{\varpi_i})$.

**Assumption 4.2.1.** For any $i_1, i_2 \in I$, the order of zero of the denominator $d_{j_1,j_2}(u)$ at the point $u = q^{p_2-p_1}(= a_{i_2}/a_{i_1})$ is at most one, where $\phi(\alpha_{i_1}) = (j_1, p_1), \phi(\alpha_{i_2}) = (j_2, p_2)$.

**Theorem 4.2.2** (Kang-Kashiwara-Kim [26])**.** Under Assumption 4.2.1, the following formulas define a commuting right action of the completed quiver Hecke algebra $\widehat{H}_Q(\beta)$ on the left $U_q(L\mathfrak{g})$-module $\widehat{V}^{\otimes\beta}$:

$$v \cdot 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} v, \tag{4.9}$$

$$v \cdot x_k = (a_{i_k}^{-1} X_k - 1) v, \tag{4.10}$$

$$v \cdot \tau_k = \begin{cases} (a_{i_k}^{-1} X_k - a_{i_{k+1}}^{-1} X_{k+1})^{-1} (R_k^{\mathbf{i}}(v) - v) & \text{if } i_k = i_{k+1}, \\ (a_{i_k}^{-1} X_{k+1} - a_{i_{k+1}}^{-1} X_k) R_k^{\mathbf{i}}(v) & \text{if } i_k \leftarrow i_{k+1}, \\ R_k^{\mathbf{i}}(v) & \text{otherwise,} \end{cases} \tag{4.11}$$

for $v \in \widehat{V}^{\otimes\mathbf{i}}$ with $\mathbf{i} = (i_1, \ldots, i_d) \in I^\beta$. Here the operator $R_k^{\mathbf{i}} \colon \widehat{V}^{\mathbf{i}} \to \widehat{V}^{\mathbf{i}\cdot s_k}$ is induced from the normalized $R$-matrix from $\mathbb{W}(\lambda_{i_k}) \otimes \mathbb{W}(\lambda_{i_{k+1}})$ to a localization of $\mathbb{W}(\lambda_{i_{k+1}}) \otimes \mathbb{W}(\lambda_{i_k})$. In particular, this operator $R_k^{\mathbf{i}}$ is well-defined under Assumption 4.2.1.

**Conjecture 4.2.3** (Kang-Kashiwara-Kim [26] Conjecture 4.3.2)**.** Assumption 4.2.1 is true for any Dynkin quiver $Q$.

This conjecture is now a theorem, i.e. Assumption 4.2.1 has been verified by some explicit computations of the denominators of the normalized $R$-matrices. For type $\mathsf{AD}$ this is checked by [26]. For type $\mathsf{E}$, a recent preprint

[46] by Oh-Scrimshaw has verified the assumption by explicit computations with a computer.

Later in Corollary 4.3.7, we give a uniform proof of Conjecture 4.2.3 for any Dynkin quiver of type $\mathsf{ADE}$ via our geometric realization of the bimodule $\widehat{V}^{\otimes\beta}$.

**Remark 4.2.4.** When our quiver is of type $\mathsf{A}_n$ with a monotone orientation $Q = (1 \to 2 \to \cdots \to n)$, the corresponding complete quiver Hecke algebra $\widehat{H}_Q(\beta)$ is known to be isomorphic to a certain central completion of the affine Hecke algebra $H_d^{\mathrm{af}}(q^{-2})$ with $d = \mathrm{ht}\,\beta$ by Brundan-Kleshchev [4] and by Rouquier [47]. Under this isomorphism, we can obtain Kang-Kashiwara-Kim's bimodule $\widehat{V}^{\otimes\beta}$ for this case as the corresponding completion of the bimodule $\mathbb{V}^{\otimes d}$ in the usual quantum affine Schur-Weyl duality. For a geometric interpretation of this fact, see Example 3.2.4 above together with Remark 4.3.10 below.

### 4.2.2 Properties of the induced functor

For each $\beta \in \mathsf{Q}^+$, we define the functor

$$\mathscr{F}_{Q,\beta} : \mathscr{M}_{Q,\beta} \to \mathscr{C}_{\mathfrak{g}}; \quad M \mapsto \widehat{V}^{\otimes\beta} \otimes_{\widehat{H}_Q(\beta)} M$$

induced from the bimodule $\widehat{V}^{\otimes\beta}$ between finite-dimensional modules. We also consider their direct sum over $\beta \in \mathsf{Q}^+$, i.e. we define the functor

$$\mathscr{F}_Q := \bigoplus_{\beta\in\mathsf{Q}^+} \mathscr{F}_{Q,\beta} : \quad \mathscr{M}_Q = \bigoplus_{\beta\in\mathsf{Q}^+} \mathscr{M}_{Q,\beta} \to \mathscr{C}_{\mathfrak{g}}$$

between monoidal categories. In the paper [26], Kang-Kashiwara-Kim studied some properties of this functor, which are summarized as follows.

**Theorem 4.2.5** (Kang-Kashiwara-Kim [26])**.** The functor $\mathscr{F}_Q$ is exact and monoidal, i.e. there is a natural isomorphism

$$\mathscr{F}_{Q,\beta+\beta'}(M \circ M') \cong \mathscr{F}_{Q,\beta}(M) \otimes \mathscr{F}_{Q,\beta'}(M')$$

for any $\beta, \beta' \in \mathsf{Q}^+$ and $M \in \mathscr{M}_{Q,\beta}, M' \in \mathscr{M}_{Q,\beta'}$. Moreover, the functor lands into the monoidal full subcategory $\mathscr{C}_Q \subset \mathscr{C}_{\mathfrak{g}}$ and induces an isomorphism between Grothendieck rings

$$[\mathscr{F}_Q] : K(\mathscr{M}_Q) \xrightarrow{\cong} K(\mathscr{C}_Q)$$

which is compatible with the isomorphisms in Theorem 2.3.4 and Corollary 4.1.4. In other words, the functor $\mathscr{F}_{Q,\beta}$ sends simple modules in $\mathscr{M}_{Q,\beta}$ to the simple modules in $\mathscr{C}_{Q,\beta}$ bijectively for each $\beta \in \mathsf{Q}^+$.

**Conjecture 4.2.6** (Kang-Kashiwara-Kim-Oh [28])**.** The functor $\mathscr{F}_Q$ gives an equivalence

$$\mathscr{F}_Q \colon \mathscr{M}_Q \xrightarrow{\cong} \mathscr{C}_Q$$

of monoidal categories.

We give a proof of Conjecture 4.2.6 in the next section via our geometric realization of the bimodule $\widehat{V}^{\otimes\beta}$.

**Remark 4.2.7.** Actually, Conjecture 4.2.6 is only a half of Kang-Kashiwara-Kim-Oh's [28, Conjecture 5.7]. For the other half, they consider Dynkin quiver type Schur-Weyl duality for twisted quantum loop algebras and conjecture that it is also an equivalence of monoidal categories. For the twisted cases, we do not have any geometric technique at the moment. This should be an interesting subject of a future research.

## 4.3 Geometric realization

In this section, we identify the convolution bimodule of the completed $K$-group of the fiber product $\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta$ with Kang-Kashiwara-Kim's bimodule $\widehat{V}^{\otimes\beta}$. After all, we give proofs of Conjecture 4.2.3 and Conjecture 4.2.6.

We keep the notation so far. In particular, $\Bbbk = \overline{\mathbb{Q}(q)}$.

### 4.3.1 Intermediary fiber product

We fix an element $\beta = \sum_{i \in I} d_i \alpha_i \in \mathsf{Q}^+$ and put $\lambda_\beta := \mathsf{cl}(\boldsymbol{\lambda}_\beta) \in \mathsf{P}^+$. From the two $G_\beta$-equivariant proper morphisms $\pi_\beta \colon \mathfrak{M}^\bullet_\beta \to E_\beta$ and $\mu_\beta \colon \mathcal{F}_\beta \to E_\beta$, we form the fiber product $\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta$. The convolution products make its completed $G_\beta$-equivariant $K$-group $\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk$ into a $(\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk, \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk)$-bimodule. More precisely, the convolution products give $\Bbbk$-algebra homomorphisms

$$\widehat{K}^{G_\beta}(Z^\bullet_\beta)_\Bbbk \to \operatorname{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk\right) \leftarrow \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)^{\mathrm{op}}_\Bbbk,$$

whose images commute with each other. In the rest of this subsection, we prove that this bimodule induces a Morita equivalence.

For a moment, we focus on a component $\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i}$ for a fixed $\mathbf{i} \in I^\beta$. Using the isomorphism $\mathcal{B}_\mathbf{i} \cong G_\beta / B_\mathbf{i}$ with $B_\mathbf{i} = \mathrm{Stab}_{G_\beta}(F^\bullet_\mathbf{i})$, we have

$$\begin{aligned}\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i} &\cong \mathfrak{M}^\bullet_\beta \times_{E_\beta} \left(G_\beta \times^{B_\mathbf{i}} \mathrm{pr}_1^{-1}(F^\bullet_\mathbf{i})\right) \\ &\cong G_\beta \times^{B_\mathbf{i}} \left(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathrm{pr}_1^{-1}(F^\bullet_\mathbf{i})\right), \end{aligned} \tag{4.12}$$

where $\mathrm{pr}_1$ is the projection $\mathcal{F}_\mathbf{i} \ni (F^\bullet, x) \mapsto F^\bullet \in \mathcal{B}_\mathbf{i}$. We define a 1-parameter subgroup $\rho_\mathbf{i} \colon \mathbb{C}^\times \to H_\beta$ by $\rho_\mathbf{i}(t) v_{w_\mathbf{i}(k)} := t^k v_{w_\mathbf{i}(k)}$ for $t \in \mathbb{C}^\times$. Note that this depends on the choice of $w_\mathbf{i} \in \mathfrak{S}_d$ fixed in Section 4.1.2. We observe that

$$\mathrm{pr}_1^{-1}(F^\bullet_\mathbf{i}) \cong \{x \in E_\beta \mid x(F^k_\mathbf{i}) \subset F^k_\mathbf{i}, \forall k\} = \left\{x \in E_\beta \;\middle|\; \lim_{t\to 0} \rho_\mathbf{i}(t) x = 0\right\}.$$

Therefore we get

$$\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathrm{pr}_1^{-1}(F^\bullet_\mathbf{i}) \cong \left\{x \in \mathfrak{M}^\bullet_\beta \;\middle|\; \lim_{t\to 0} \rho_\mathbf{i}(t) \pi_\beta(x) = 0\right\}.$$

Since the morphism $\pi_\beta \colon \mathfrak{M}^\bullet_\beta \to E_\beta$ is the $\mathbb{T}_\beta$-fixed part of $\pi \colon \mathfrak{M}(\lambda_\beta) \to \mathfrak{M}_0(\lambda_\beta)$, it is natural to consider the following subvariety of $\mathfrak{M}(\lambda_\beta)$:

$$\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i}) := \left\{x \in \mathfrak{M}(\lambda_\beta) \;\middle|\; \lim_{t\to 0} \rho_\mathbf{i}(t) \pi(x) = 0 \in \mathfrak{M}_0(\lambda_\beta)\right\},$$

which turns out to be the tensor product variety introduced by Nakajima [42]. Since the subgroups $\mathbb{T}_\beta$ and $\rho_\mathbf{i}(\mathbb{C}^\times)$ commute with each other, we have

$$\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathrm{pr}_1^{-1}(F^\bullet_\mathbf{i}) \cong \widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})^{\mathbb{T}_\beta}. \tag{4.13}$$

Using (4.12), (4.13) and the reduction, we obtain

$$K^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i}) \cong K^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})^{\mathbb{T}_\beta}), \tag{4.14}$$

$$H_*^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i}, \Bbbk) \cong H_*^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})^{\mathbb{T}_\beta}, \Bbbk). \tag{4.15}$$

**Proposition 4.3.1.** The $G_\beta$-equivariant Chern character map gives an isomorphism:

$$\mathrm{ch}^{G_\beta} \colon \widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i})_\Bbbk \xrightarrow{\cong} H_*^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i}, \Bbbk)^\wedge.$$

*Proof.* Thanks to (4.14) and (4.15), it is enough to show that the $H_\beta$-equivariant Chern character map

$$\mathrm{ch}^{H_\beta} \colon \widehat{K}^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_{\mathbf{i}})^{\mathbb{T}_\beta})_{\Bbbk} \to H_*^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta, w_{\mathbf{i}})^{\mathbb{T}_\beta}, \Bbbk)^\wedge$$

is an isomorphism. This latter assertion follows from a $\mathbb{T}_\beta$-fixed part analogue of [42, Theorem 3.10. (1)]. □

The $G_\beta$-equivariant Borel-Moore homology $H_*^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta, \Bbbk)$ becomes a $(H_*^{G_\beta}(Z_\beta^\bullet, \Bbbk), H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk))$-bimodule by the convolution products, similarly to the case of $K$-groups. On the other hand, the Ext-group $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta)$ becomes a $(\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta^\bullet), \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta))$-bimodule by the Yoneda products. This bimodule $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta)$ gives a Morita equivalence between $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta^\bullet)$ and $\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)$ because $\mathcal{IC}_{\mathbf{m}}$ appears as a non-zero direct summand of both $\mathcal{L}_\beta$ and $\mathcal{L}_\beta^\bullet$ for each $\mathbf{m} \in \mathrm{KP}(\beta)$. Moreover, we have a standard isomorphism

$$H_*^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta, \Bbbk) \cong \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta) \tag{4.16}$$

**Theorem 4.3.2.** We have the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{K}^{G_\beta}(Z_\beta^\bullet)_{\Bbbk} & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta)_{\Bbbk}\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_{\Bbbk}^{\mathrm{op}} \\
{\scriptstyle \mathrm{RR}^{G_\beta}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{RR}^{G_\beta}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{RR}^{G_\beta}}\downarrow{\scriptstyle \cong} \\
H_*^{G_\beta}(Z_\beta^\bullet, \Bbbk)^\wedge & \longrightarrow & \mathrm{End}\left(H_*^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta, \Bbbk)^\wedge\right) & \longleftarrow & H_*^{G_\beta}(\mathcal{Z}_\beta, \Bbbk)^{\wedge\mathrm{op}} \\
{\scriptstyle (3.31)}\downarrow{\scriptstyle \cong} & & {\scriptstyle (4.16)}\downarrow{\scriptstyle \cong} & & {\scriptstyle (4.4)}\downarrow{\scriptstyle \cong} \\
\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta^\bullet)^\wedge & \longrightarrow & \mathrm{End}\left(\mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta^\bullet, \mathcal{L}_\beta)^\wedge\right) & \longleftarrow & \mathrm{Ext}^*_{G_\beta}(\mathcal{L}_\beta, \mathcal{L}_\beta)^{\wedge\mathrm{op}},
\end{array}$$

where each row denotes the bimodule structure defined above. In particular, the bimodule $\widehat{K}^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta)_{\Bbbk}$ gives a Morita equivalence between two convolution algebras $\widehat{K}^{G_\beta}(Z_\beta^\bullet)_{\Bbbk}$ and $\widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_{\Bbbk}$.

*Proof.* The commutativity of the upper half (resp. lower half) of the diagram follows from Proposition A.1.3 (resp. an equivariant version of [11, Theorem 8.6.7]). □

### 4.3.2 The left action of $U_q(L\mathfrak{g})$

In this subsection, we fix $\mathbf{i} = (i_1, \dots, i_d) \in I^\beta$ and investigate the $U_q(L\mathfrak{g})$-module structure of the pull-back $\widehat{\Phi}^*_\beta(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_\mathbf{i})_\Bbbk)$.

We recall the notation introduced in Section 4.2.1. For each $i \in I$, we define $\lambda_i := \mathsf{cl}(\phi(\alpha_i)) = \varpi_j$ and $a_i := q^p$ if $\phi(\alpha_i) = (j, p) \in \widehat{I}$. We identify

$$\mathrm{End}_{U_q}(\mathbb{W}(\lambda_i)) \cong R(\mathbb{G}(\lambda_i)) \otimes_A \Bbbk = R(G(\lambda_i))_\Bbbk \cong \Bbbk[z_{\lambda_i}^{\pm 1}], \tag{4.17}$$

where $z_{\lambda_i}$ is identified with the class of the 1-dimensional representation of $G(\lambda_i) = \mathbb{C}^\times$ of weight 1.

We recall some properties of the tensor product variety $\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})$. Let

$$\mathbb{H}_\beta := H_\beta \times \mathbb{C}^\times \subset G_\beta \times \mathbb{C}^\times = \mathbb{G}_\beta \subset \mathbb{G}(\lambda_\beta)$$

be a maximal torus. By construction, the subvariety $\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i}) \subset \mathfrak{M}(\lambda_\beta)$ is stable under the action of $\mathbb{H}_\beta$. The convolution product makes the $\mathbb{H}_\beta$-equivariant $K$-group $K^{\mathbb{H}_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i}))$ into a left $K^{\mathbb{H}_\beta}(Z(\lambda_\beta))$-module. Via the composition of the homomorphisms

$$U_q(L\mathfrak{g}) \xrightarrow{\Phi_{\lambda_\beta}} K^{\mathbb{G}(\lambda_\beta)}(Z(\lambda_\beta)) \otimes_A \Bbbk \to K^{\mathbb{H}_\beta}(Z(\lambda_\beta)) \otimes_A \Bbbk,$$

where the latter one is the restriction to $\mathbb{H}_\beta \subset \mathbb{G}(\lambda_\beta)$, we regard the $\mathbb{H}_\beta$-equivariant $K$-group $K^{\mathbb{H}_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})) \otimes_A \Bbbk$ as a $U_q(L\mathfrak{g})$-module.

**Theorem 4.3.3** (Nakajima [42])**.** There is a $U_q(L\mathfrak{g})$-module isomorphism

$$K^{\mathbb{H}_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_\mathbf{i})) \otimes_A \Bbbk \cong \mathbb{V}^{\otimes \mathbf{i}} := \mathbb{W}(\lambda_{i_1}) \otimes \cdots \otimes \mathbb{W}(\lambda_{i_d}),$$

where the action of $R(\mathbb{H}_\beta) \otimes_A \Bbbk$ on the LHS is translated into the action on the RHS via the isomorphism

$$\begin{aligned} R(\mathbb{H}_\beta) \otimes_A \Bbbk \xrightarrow{\cong} \mathcal{O}_\mathbf{i} &:= \Bbbk[X_1^{\pm 1}, \dots, X_d^{\pm 1}] \subset \mathrm{End}_{U_q}(\mathbb{V}^{\otimes \mathbf{i}}); \\ [\mathbb{C} v_{w_\mathbf{i}(k)}] &\mapsto X_k, \end{aligned} \tag{4.18}$$

where we set $X_k := z_{\lambda_{i_k}}$ using the notation in (4.17).

The decomposition (3.29) $\mathbb{G}_\beta \cong G_\beta \times \mathbb{T}_\beta$ induces the decomposition $\mathbb{H}_\beta \cong H_\beta \times \mathbb{T}_\beta$ of the maximal torus $\mathbb{H}_\beta$. Similarly to the case of $\mathbb{G}_\beta$-equivariant $K$-groups in Section 3.3.1, this decomposition yields a natural isomorphism

$$K^{\mathbb{H}_\beta}(X) \otimes_A \Bbbk \cong K^{H_\beta}(X)_\Bbbk$$

for any $\mathbb{H}_\beta$-variety $X$ with a trivial $\mathbb{T}_\beta$-action. When $X = \mathrm{pt}$, we have the following commutative diagram:

$$\begin{array}{ccc} R(\mathbb{H}_\beta) \otimes_A \Bbbk & \xrightarrow{\cong} & R(H_\beta)_\Bbbk \\ {\scriptstyle (4.18)}\downarrow\cong & & \downarrow\cong \\ \mathcal{O}_{\mathbf{i}} = \Bbbk[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] & \xrightarrow{\cong} & \Bbbk[y_1^{\pm 1}, \ldots, y_d^{\pm 1}]1_{\mathbf{i}}, \end{array} \tag{4.19}$$

where the bottom horizontal arrow sends the element $a_{i_k}^{-1}X_k$ to $y_k 1_{\mathbf{i}}$ for $1 \leq k \leq d$. Under this isomorphism, the maximal ideal $\mathfrak{r}'_\beta \subset R(\mathbb{H}_\beta) \otimes_A \Bbbk$ defined as the kernel of the restriction $R(\mathbb{H}_\beta) \otimes_A \Bbbk \to R(\mathbb{T}_\beta) \otimes_A \Bbbk = \Bbbk$ corresponds to the augmentation ideal of $R(H_\beta)_\Bbbk$. Therefore we have a natural isomorphism

$$\left[K^{\mathbb{H}_\beta}(X) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}'_\beta} \cong \widehat{K}^{H_\beta}(X)_\Bbbk, \tag{4.20}$$

where $[-]^\wedge_{\mathfrak{r}'_\beta}$ denotes the $\mathfrak{r}'_\beta$-adic completion. In particular, completing the diagram (4.19), we get

$$\begin{array}{ccc} [R(\mathbb{H}_\beta) \otimes_A \Bbbk]^\wedge_{\mathfrak{r}'_\beta} & \xrightarrow{\cong} & \widehat{R}(H_\beta)_\Bbbk \\ \downarrow\cong & & \downarrow\cong \\ \widehat{\mathcal{O}}_{\mathbf{i}} := \Bbbk[\![X_1 - a_{i_1}, \ldots, X_d - a_{i_d}]\!] & \xrightarrow{\cong} & \Bbbk[\![y_1 - 1, \ldots, y_d - 1]\!]1_{\mathbf{i}}. \end{array}$$

**Theorem 4.3.4.** We have the following isomorphism of $U_q(L\mathfrak{g})$-modules:

$$\widehat{\Phi}^*_\beta\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_\Bbbk\right) \cong \mathbb{V}^{\otimes \mathbf{i}} \otimes_{\mathcal{O}_{\mathbf{i}}} \widehat{\mathcal{O}}_{\mathbf{i}} = \widehat{V}^{\otimes \mathbf{i}}.$$

*Proof.* Actually, there is the following isomorphism:

$$\begin{aligned} \widehat{V}^{\otimes \mathbf{i}} &\cong \left[K^{\mathbb{H}_\beta}(\widetilde{\mathfrak{Z}}(\lambda; w_{\mathbf{i}})) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}'_\beta} && \text{(Theorem 4.3.3)} \\ &\cong \left[K^{\mathbb{H}_\beta}(\widetilde{\mathfrak{Z}}(\lambda; w_{\mathbf{i}})^{\mathbb{T}_\beta}) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}'_\beta} && \text{(localization theorem)} \\ &\cong \widehat{K}^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda; w_{\mathbf{i}})^{\mathbb{T}_\beta})_\Bbbk && \text{(isomorphism (4.20))} \\ &\cong \widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_\Bbbk. && \text{(isomorphism (4.14))} \end{aligned}$$

We need to show that this is a $U_q(L\mathfrak{g})$-homomorphism. By construction, the following diagram of $\Bbbk$-algebras commutes:

$$\begin{array}{ccccc}
K^{\mathbb{G}(\lambda_\beta)}(Z(\lambda_\beta)) \otimes_A \Bbbk & \longrightarrow & \left[K^{\mathbb{G}_\beta}(Z_\beta^\bullet) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_\beta} & \xrightarrow[(3.30)]{\cong} & \widehat{K}^{G_\beta}(Z_\beta^\bullet)_\Bbbk \\
\downarrow & & \downarrow & & \downarrow \\
K^{\mathbb{H}_\beta}(Z(\lambda_\beta)) \otimes_A \Bbbk & \longrightarrow & \left[K^{\mathbb{H}_\beta}(Z_\beta^\bullet) \otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}'_\beta} & \xrightarrow[(4.20)]{\cong} & \widehat{K}^{H_\beta}(Z_\beta^\bullet)_\Bbbk,
\end{array}$$

where the vertical arrows denote the restrictions to the maximal tori. Moreover, by using an $H_\beta$-equivariant version of [41, Proposition 8.2.3], we can see that the following diagram also commutes:

$$\begin{array}{ccc}
K^{G_\beta}(Z_\beta^\bullet)_\Bbbk \otimes K^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_\Bbbk & \xrightarrow{\ *\ } & K^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_\Bbbk \\
{\scriptstyle (\text{restriction to } H_\beta) \otimes (4.14)} \downarrow & & {\scriptstyle (4.14)} \downarrow \cong \\
K^{H_\beta}(Z_\beta^\bullet)_\Bbbk \otimes K^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_{\mathbf{i}})^{\mathbb{T}_\beta})_\Bbbk & \xrightarrow{\ *\ } & K^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda_\beta; w_{\mathbf{i}})^{\mathbb{T}_\beta})_\Bbbk,
\end{array}$$

where the horizontal arrows denote the convolution products. From these commutative diagrams, combined with the definition of $\widehat{\Phi}_\beta$ and Theorem 4.3.3, we obtain the conclusion. □

### 4.3.3 The right action of $\widehat{H}_Q(\beta)$

Summarizing the discussion so far, we have obtained a $(U_q(L\mathfrak{g}), \widehat{H}_Q(\beta))$-bimodule structure on the left $U_q(L\mathfrak{g})$-module

$$\widehat{V}^{\otimes\beta} = \bigoplus_{\mathbf{i} \in I^\beta} \widehat{V}^{\otimes\mathbf{i}}$$

such that the following diagram commutes:

$$\begin{array}{ccccc}
U_q(L\mathfrak{g}) & \longrightarrow & \mathrm{End}(\widehat{V}^{\otimes\beta}) & \xleftarrow{\ \exists\psi\ } & \widehat{H}_Q(\beta)^{\mathrm{op}} \\
\downarrow {\scriptstyle \widehat{\Phi}_\beta} & & \downarrow \cong & & \downarrow \cong \\
\widehat{K}^{G_\beta}(Z_\beta^\bullet)_\Bbbk & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk^{\mathrm{op}}.
\end{array}$$

In this subsection, we describe the right action $\psi \colon \widehat{H}_Q(\beta) \to \mathrm{End}_{U_q}(\widehat{V}^{\otimes\beta})^{\mathrm{op}}$ of the quiver Hecke algebra $\widehat{H}_Q(\beta)$ on the space $\widehat{V}^{\otimes\beta}$.

For each $\mathbf{i} = (i_1, \ldots, i_d) \in I^\beta$, we set

$$v_{\mathbf{i}} := (w_{\lambda_{i_1}} \otimes \cdots \otimes w_{\lambda_{i_d}}) \otimes 1 \in \widehat{V}^{\otimes \mathbf{i}} = (\mathbb{W}(\lambda_{i_1}) \otimes \cdots \otimes \mathbb{W}(\lambda_{i_d})) \otimes_{\mathcal{O}_{\mathbf{i}}} \widehat{\mathcal{O}}_{\mathbf{i}}.$$

**Proposition 4.3.5.** The highest weight space $\bigoplus_{\mathbf{i} \in I^\beta} \widehat{\mathcal{O}}_{\mathbf{i}} v_{\mathbf{i}} \subset \widehat{V}^{\otimes \beta}$ of weight $\lambda$ is stable under the right action of $\widehat{H}_Q(\beta)$. Moreover it is isomorphic to the completed polynomial representation $\widehat{P}_\beta$ defined in (4.1).

*Proof.* Note that the connected component of the graded quiver variety $\mathfrak{M}^\bullet_\beta = \mathfrak{M}(\lambda_\beta)^{\mathbb{T}_\beta}$ corresponding to the highest weight space is $\mathfrak{M}(0, \lambda_\beta)^{\mathbb{T}_\beta} = \mathrm{pt}$ and hence $\mathfrak{M}(0, \lambda_\beta)^{\mathbb{T}_\beta} \times_{E_\beta} \mathcal{F}_\beta = \mathcal{B}_\beta$. Therefore we have

$$\bigoplus_{\mathbf{i} \in I^\beta} \widehat{\mathcal{O}}_{\mathbf{i}} v_{\mathbf{i}} \cong \widehat{K}^{G_\beta}(\mathfrak{M}(0, \lambda_\beta)^{\mathbb{T}_\beta} \times_{E_\beta} \mathcal{F}_\beta)_{\Bbbk} \cong \widehat{K}^{G_\beta}(\mathcal{B}_\beta)_{\Bbbk} \cong \widehat{P}_\beta$$

as $\widehat{H}_Q(\beta)$-module, where the last isomorphism comes from (4.3) and (4.7). □

Henceforth, we normalize the isomorphism $\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_{\Bbbk} \cong \widehat{V}^{\otimes \mathbf{i}}$ of $U_q(L\mathfrak{g})$-modules in Theorem 4.3.4 by multiplying the element of $\widehat{\mathcal{O}}_{\mathbf{i}}$ corresponding to the ratio $C_{\mathbf{i}}^{-1}$ of Todd classes defined in (4.6) for each $\mathbf{i} \in I^\beta$ so that the isomorphism

$$\bigoplus_{\mathbf{i} \in I^\beta} \widehat{\mathcal{O}}_{\mathbf{i}} v_{\mathbf{i}} = \bigoplus_{\mathbf{i} \in I^\beta} \Bbbk[\![X_1 - a_{i_1}, \ldots, X_d - a_{i_d}]\!] v_{\mathbf{i}} \xrightarrow{\cong} \widehat{P}_\beta = \bigoplus_{\mathbf{i} \in I^\beta} \Bbbk[\![x_1, \ldots, x_d]\!] 1_{\mathbf{i}}$$

in Proposition 4.3.5 above sends the element $v_{\mathbf{i}}$ to $1_{\mathbf{i}}$.

Let $\mathbb{K}_{\mathbf{i}}$ be the fraction field of the ring $\widehat{\mathcal{O}}_{\mathbf{i}}$ for each $\mathbf{i} \in I^\beta$. It is known that the $U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}$-module

$$\widehat{V}^{\otimes \mathbf{i}}_{\mathbb{K}} := \mathbb{V}^{\otimes \mathbf{i}} \otimes_{\mathcal{O}_{\mathbf{i}}} \mathbb{K}_{\mathbf{i}} = \widehat{V}^{\otimes \mathbf{i}} \otimes_{\widehat{\mathcal{O}}_{\mathbf{i}}} \mathbb{K}_{\mathbf{i}}$$

is irreducible (see e.g. [30, Proposition 9.5]). For each $w \in \mathfrak{S}_d$, the $\Bbbk$-algebra isomorphism

$$\varphi_w \colon \widehat{\mathcal{O}}_{\mathbf{i}} \xrightarrow{\cong} \widehat{\mathcal{O}}_{\mathbf{i} \cdot w};\ f(X_1, \ldots, X_d) \mapsto f^w(X_1, \ldots, X_d) := f(X_{w(1)}, \ldots, X_{w(d)})$$

induces an isomorphism $\mathbb{K}_{\mathbf{i}} \xrightarrow{\cong} \mathbb{K}_{\mathbf{i} \cdot w}$ of the fraction fields, which we denote by the same symbol $\varphi_w$. The pull-back $\varphi_w^* \widehat{V}^{\otimes \mathbf{i} \cdot w}_{\mathbb{K}}$ is an irreducible $U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}$-module.

For each $\mathbf{i} \in I^\beta$ and $1 \le k < d$, we define the following non-zero $U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}$-homomorphism

$$R_k^{\mathbf{i}} := \left(1^{\otimes(k-1)} \otimes R^{\mathrm{norm}} \otimes 1^{\otimes(d-k-1)}\right) \otimes \varphi_{s_k} \colon \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}} \to \varphi_{s_k}^* \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i} \cdot s_k},$$

where $R^{\mathrm{norm}}$ is the normalized $R$-matrix from $\mathbb{W}(\lambda_{i_k}) \otimes \mathbb{W}(\lambda_{i_{k+1}})$ to a localization of $\mathbb{W}(\lambda_{i_{k+1}}) \otimes \mathbb{W}(\lambda_{i_k})$. By the irreducibility, this is an isomorphism and we have

$$\mathrm{Hom}_{U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}} \left(\widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}}, \varphi_{s_k}^* \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i} \cdot s_k}\right) = \mathbb{K}_{\mathbf{i}} \cdot R_k^{\mathbf{i}}. \tag{4.21}$$

Let $\widehat{V}_{\mathbb{K}}^{\otimes \beta} := \bigoplus_{\mathbf{i} \in I^\beta} \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}}$. We regard $\widehat{V}^{\otimes \beta} \subset \widehat{V}_{\mathbb{K}}^{\otimes \beta}$ naturally.

**Theorem 4.3.6.** The right action of the quiver Hecke algebra $\widehat{H}_Q(\beta)$ on the space $\widehat{V}^{\otimes \beta}$ is given by the following formulas:

$$v \cdot 1_{\mathbf{i}'} = \delta_{\mathbf{i}, \mathbf{i}'} v \tag{4.22}$$

$$v \cdot x_k = \log(a_{i_k}^{-1} X_k) v \tag{4.23}$$

$$v \cdot \tau_k = \begin{cases} (\log(a_{i_k}^{-1} X_k) - \log(a_{i_{k+1}}^{-1} X_{k+1}))^{-1} (R_k^{\mathbf{i}}(v) - v) & \text{if } i_k = i_{k+1}, \\ (\log(a_{i_k}^{-1} X_{k+1}) - \log(a_{i_{k+1}}^{-1} X_k)) R_k^{\mathbf{i}}(v) & \text{if } i_k \leftarrow i_{k+1}, \\ R_k^{\mathbf{i}}(v) & \text{otherwise,} \end{cases} \tag{4.24}$$

where $v \in \widehat{V}^{\otimes \mathbf{i}}$ with $\mathbf{i} = (i_1, \ldots, i_d) \in I^\beta$ and

$$\log(X) := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(X-1)^m}{m}.$$

*Proof.* The first formula (4.22) is clear from Theorem 4.1.6 (1) and the construction.

To prove the second formula (4.23), we assume that the vector $v \in \widehat{V}^{\otimes \mathbf{i}}$ corresponds to an element $\zeta \in \widehat{K}^{G_\beta}(\mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_{\Bbbk}$ under the isomorphism in Theorem 4.3.4. By Theorem 4.1.6 (1), the right action of $e^{x_k} \in \widehat{H}_Q(\beta)$ on $\widehat{V}^{\otimes \mathbf{i}}$ corresponds to the convolution with the class $\Delta_*[\mathcal{L}_{\mathbf{i}}(k)] \in \widehat{K}^{G_\beta}(\mathcal{F}_{\mathbf{i}} \times_{E_\beta} \mathcal{F}_{\mathbf{i}})_{\Bbbk}$ from the right, where $\mathcal{L}_{\mathbf{i}}(k)$ is the line bundle on $\mathcal{F}_{\mathbf{i}}$ defined in Section 4.1.2 and $\Delta \colon \mathcal{F}_{\mathbf{i}} \to \mathcal{F}_{\mathbf{i}} \times_{E_\beta} \mathcal{F}_{\mathbf{i}}$ is the diagonal embedding. By [41, Lemma 8.1.1], we have $\zeta * (\Delta_*[\mathcal{L}_{\mathbf{i}}(k)]) = \zeta \otimes p_2^*[\mathcal{L}_{\mathbf{i}}(k)]$, where $p_2 \colon \mathfrak{M}_\beta^\bullet \times_{E_\beta} \mathcal{F}_{\mathbf{i}} \to \mathcal{F}_{\mathbf{i}}$ is the second projection. The isomorphism (4.14) translates the operation $- \otimes p_2^*[\mathcal{L}_{\mathbf{i}}(k)]$

on $K^{G_\beta}(\mathfrak{M}^\bullet_\beta \times_{E_\beta} \mathcal{F}_{\mathbf{i}})$ into the multiplication of the element $y_k 1_{\mathbf{i}} \in R(H_\beta)$ on $K^{H_\beta}(\widetilde{\mathfrak{Z}}(\lambda; w_{\mathbf{i}})^{T_\beta})$. Thus we have $v \cdot e^{x_k} = (a_{i_k}^{-1} X_k) v$ (see (4.19)).

Let us verify the third formula (4.24). Let $\psi \colon \widehat{H}_Q(\beta) \to \mathrm{End}_{U_q(L\mathfrak{g})}(\widehat{V}^{\otimes \beta})^{\mathrm{op}}$ be the structure morphism. First, we consider the case $i_k = i_{k+1}$. From the commutation relation between $1_{\mathbf{i}}\tau_k$ and $x_l$ in $H_Q(\beta)$, and the formula (4.23) for $\psi(x_l)$ which we have proved in the previous paragraph, we see that

$$(\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1) f = f^{s_k} (\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1)$$

holds in $\mathrm{End}_{U_q(L\mathfrak{g})}(\widehat{V}^{\otimes \mathbf{i}})$ for any $f \in \widehat{\mathcal{O}}_{\mathbf{i}}$, where we put $\mathcal{D} := \log(a_{i_k}^{-1} X_k) - \log(a_{i_{k+1}}^{-1} X_{k+1})$. In other words, the operator $\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1$ belongs to the space $\mathrm{Hom}_{U_q(L\mathfrak{g}) \otimes \widehat{\mathcal{O}}_{\mathbf{i}}}(\widehat{V}^{\otimes \mathbf{i}}, \varphi_{s_k}^* \widehat{V}^{\otimes \mathbf{i}})$. Therefore it extends to an operator on the localizations. Namely, we can regard

$$\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1 \in \mathrm{Hom}_{U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}} \left( \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}}, \varphi_{s_k}^* \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}} \right) \cong \mathbb{K}_{\mathbf{i}} \cdot R_k^{\mathbf{i}},$$

where the last isomorphism is (4.21). By Proposition 4.3.5 and the formulas in Theorem 4.1.5, we see that $(\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1) v_{\mathbf{i}} = v_{\mathbf{i}} = R_k^{\mathbf{i}}(v_{\mathbf{i}})$. Therefore we obtain $\mathcal{D}\psi(1_{\mathbf{i}}\tau_k) + 1 = R_k^{\mathbf{i}}$ as an operator on $\widehat{V}^{\otimes \mathbf{i}}$.

The case $i_k \neq i_{k+1}$ is easier. In this case, the commutation relation in $H_Q(\beta)$ and the formula (4.23) for $\psi(x_l)$ show that the operator $\psi(1_{\mathbf{i}}\tau_k)$ already belongs to $\mathrm{End}_{U_q(L\mathfrak{g}) \otimes \widehat{\mathcal{O}}_{\mathbf{i}}}(\widehat{V}^{\otimes \mathbf{i}}, \varphi_{s_k}^* \widehat{V}^{\otimes \mathbf{i}})$. Therefore it extends to an element in $\mathrm{Hom}_{U_q(L\mathfrak{g}) \otimes \mathbb{K}_{\mathbf{i}}}(\widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}}, \varphi_{s_k}^* \widehat{V}_{\mathbb{K}}^{\otimes \mathbf{i}})$. Then we proceed just as in the previous paragraph to obtain the desired formula (4.24), taking Proposition 4.3.5, the formulas in Theorem 4.1.5 and (4.21) into consideration. □

**Corollary 4.3.7.** Conjecture 4.2.3 is true. Namely, for any $i_1, i_2 \in I$, the order of zero of the denominator $d_{j_1, j_2}(u)$ at the point $u = a_{i_2}/a_{i_1}$ is at most one, where $\phi(\alpha_{i_1}) = (j_1, p_1), \phi(\alpha_{i_2}) = (j_2, p_2)$ and $a_{i_1} = q^{p_1}, a_{i_2} = q^{p_2}$.

*Proof.* Since we know (2.9), we may assume that $i_1 \neq i_2$. We consider a sequence $\mathbf{i} = (i_1, i_2) \in I^\beta$ with $\beta = \alpha_{i_1} + \alpha_{i_2}$. When $i_1 \leftarrow i_2$, the formula (4.24) tells us that the operator $(\log(a_{i_1}^{-1} z_1) - \log(a_{i_2}^{-1} z_2)) R_1^{\mathbf{i}}$ belongs to $\mathrm{Hom}_{U_q(L\mathfrak{g})}(\widehat{V}^{\otimes \mathbf{i}}, \widehat{V}^{\otimes \mathbf{i} \cdot s_1})$, where we put $z_k = z_{\lambda_{i_k}}$ for $k = 1, 2$ as before. Notice that

$$\log(a_{i_1}^{-1} z_1) - \log(a_{i_2}^{-1} z_2) \in (z_2/z_1 - a_{i_2}/a_{i_1}) \cdot \widehat{\mathcal{O}}_{\mathbf{i}}^{\times}.$$

Therefore we find that the order of zero of $d_{j_1, j_2}(u)$ at $u = a_{i_2}/a_{i_1}$ is at most one. For the other case $i_k \not\leftarrow i_{k+1}$, by the formula (4.24), the operator $R_1^{\mathbf{i}}$

already belongs to $\mathrm{Hom}_{U_q(L\mathfrak{g})}(\widehat{V}^{\otimes \mathbf{i}}, \widehat{V}^{\otimes \mathbf{i}\cdot s_1})$. Therefore the order of zero of $d_{j_1,j_2}(u)$ at $u = a_{i_2}/a_{i_1}$ is zero. □

**Remark 4.3.8.** For each $\mathbf{i} \in I^\beta$, we define a topological $\Bbbk$-algebra automorphism $\sigma_{\mathbf{i}}$ of $\widehat{\mathcal{O}}_{\mathbf{i}}$ by setting

$$\sigma_{\mathbf{i}}(\log(a_{i_k}^{-1} X_k)) := a_{i_k}^{-1} X_k - 1$$

for all $k$. This induces a $U_q(L\mathfrak{g})$-module automorphism $\sigma := \bigoplus_{\mathbf{i}\in I^\beta}(1 \otimes \sigma_{\mathbf{i}})$ on the module $\widehat{V}^{\otimes\beta}$. If we twist our right $\widehat{H}_Q(\beta)$-action by this automorphism $\sigma$ (i.e. we replace the structure map $\psi$ with $\sigma\psi(-)\sigma^{-1}$), we get a new right $\widehat{H}_Q(\beta)$-action This new action is same as Kang-Kashiwara-Kim's action given by the formulas (4.9), (4.10) and (4.11) in Section 4.2.

**Theorem 4.3.9.** The formulas (4.22), (4.23) and (4.24) (or the formulas (4.9), (4.10) and (4.11)) define a structure of a $(U_q(L\mathfrak{g}), \widehat{H}_Q(\beta))$-bimodule on the left $U_q(L\mathfrak{g})$-module $\widehat{V}^{\otimes\beta}$. The functor gives an equivalence of categories:

$$\mathscr{F}_{Q,\beta} \colon \mathscr{M}_{Q,\beta} \xrightarrow{\cong} \mathscr{C}_{Q,\beta}.$$

Therefore, summing up over $\beta \in \mathsf{Q}^+$, we obtain the equivalence of monoidal categories:

$$\mathscr{F}_Q \colon \mathscr{M}_Q \xrightarrow{\cong} \mathscr{C}_Q.$$

Hence Conjecture 4.2.6 is true.

*Proof.* This follows from the discussions in this section, Theorem 3.3.6 and Theorem 4.3.2. □

**Remark 4.3.10.** Let us consider the case when our quiver $Q$ is of type $\mathsf{A}_n$ with a monotone orientation $Q = (1 \to 2 \to \cdots \to n)$. We shall remark that our geometric realization in this case can be obtained from Ginzburg-Reshetikhin-Vasserot's geometric realization of the usual quantum affine Schur-Weyl duality in Section 1.1.2. We freely use the notation in Section 1.1.2. As we have seen in Example 3.2.4, we have $\lambda_\beta = d\varpi_1$ in this case. The corresponding $\mathbb{G}(\lambda_\beta)$-equivariant morphism $\mathfrak{M}(\lambda_\beta) \to \mathfrak{M}_0(\lambda_\beta)$ between the quiver varieties is nothing but the $\mathbb{G}_d$-equivariant morphism $\mathfrak{M}_d \to \mathcal{N}_d$. Recall that we have $\mathcal{N}_d^{\mathbb{T}_\beta} = E_\beta$ in Example 3.2.4. Taking $\mathbb{T}_\beta$-fixed parts, we get the $G_\beta$-equivariant morphism $\pi_\beta \colon \mathfrak{M}_\beta^\bullet \to E_\beta$ for this case. On the other hand, the $G_\beta$-equivariant morphism $\mathcal{F}_\beta \to E_\beta$ is also obtained as the

$\mathbb{T}_\beta$-fixed part of the Springer resolution $\mathcal{F}_d \to \mathcal{N}_d$ for $GL_d(\mathbb{C})$. Completing the Ginzburg-Reshetikhin-Vasserot's diagram (1.1) with respect to the ideal $\mathfrak{m}_{\boldsymbol{\lambda}_\beta} \subset R(\lambda_\beta) = R(\mathbb{G}_d) \otimes_A \mathbb{k}$, we obtain our diagram in Theorem 1.2.1 by the localization theorem.

# Appendix A

# Preliminaries

## A.1 Equivariant $K$-theory

In this section, we collect some well-known facts around the equivariant $K$-theory in order to fix the notation. For the materials in this section, we refer to [11, Chapter 5], [14] and [41, Section 6].

### A.1.1 Notation

Let $G$ be a complex linear algebraic group. A $G$-variety $X$ is a quasi-projective complex algebraic variety equipped with an algebraic action of the group $G$. We set $\mathrm{pt} := \operatorname{Spec}\mathbb{C}$ with the trivial $G$-action. The equivariant $K$-group $K^G(X)$ is defined to be the Grothendieck group of the abelian category of $G$-equivariant coherent sheaves on $X$. For a $G$-equivariant coherent sheaf $\mathcal{F}$ on $X$, we denote by $[\mathcal{F}]$ the corresponding element in $K^G(X)$. We denote the structure sheaf of $X$ by $\mathcal{O}_X$. For a $G$-equivariant vector bundle $\mathcal{E}$ on $X$, the map $K^G(X) \ni [\mathcal{F}] \mapsto [\mathcal{E}] \cdot [\mathcal{F}] := [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}] \in K^G(X)$ is well-defined. Thus the equivariant $K$-group $K^G(X)$ becomes a module over the representation ring $R(G) = K^G(\mathrm{pt}) = K(\operatorname{Rep} G)$ of the group $G$.

For a $G$-equivariant vector bundle $\mathcal{E}$ on $X$, we also define $\bigwedge_u[\mathcal{E}] := \sum_{i=0}^{\operatorname{rank}\mathcal{E}} u^i[\bigwedge^i \mathcal{E}] \in [\mathcal{O}_X] + uK^G(X)[u]$. If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is an exact sequence of $G$-equivariant vector bundles on $X$, we have $\bigwedge_u[\mathcal{E}] = \bigwedge_u[\mathcal{E}_1] \cdot \bigwedge_u[\mathcal{E}_2]$. Therefore we set $\bigwedge_u([\mathcal{E}_1] + [\mathcal{E}_2]) := \bigwedge_u[\mathcal{E}_1] \cdot \bigwedge_u[\mathcal{E}_2]$ and $\bigwedge_u(-[\mathcal{E}]) := (\bigwedge_u[\mathcal{E}])^{-1} \in [\mathcal{O}_X] + uK^G(X)[\![u]\!]$.

We also use the equivariant topological $K$-homologies denoted by $K^G_{i,\mathrm{top}}(X)$

($i = 0, 1$). There is a canonical comparison map $K^G(X) \to K^G_{0,\mathrm{top}}(X)$ (see [11, Section 5.5.5]).

Let $Y$ be a $G$-invariant closed subvariety of $X$ and $U = X \setminus Y$ be the complement of $Y$. Then the inclusions $Y \xrightarrow{i} X \xleftarrow{j} U$ induce the followings:

(1) an exact sequence:

$$K^G(Y) \xrightarrow{i_*} K^G(X) \xrightarrow{j^*} K^G(U) \longrightarrow 0, \tag{A.1}$$

(2) an exact hexagon:

$$\begin{array}{ccccc} K^G_{0,\mathrm{top}}(Y) & \xrightarrow{i_*} & K^G_{0,\mathrm{top}}(X) & \xrightarrow{j^*} & K^G_{0,\mathrm{top}}(U) \\ \uparrow & & & & \downarrow \\ K^G_{1,\mathrm{top}}(U) & \xleftarrow{j^*} & K^G_{1,\mathrm{top}}(X) & \xleftarrow{i_*} & K^G_{1,\mathrm{top}}(Y). \end{array} \tag{A.2}$$

### A.1.2 Completion and equivariant Chern character

Let $\Bbbk$ be a field of characteristic zero. We put

$$K^G(X)_{\Bbbk} := K^G(X) \otimes_{\mathbb{Z}} \Bbbk, \quad R(G)_{\Bbbk} := R(G) \otimes_{\mathbb{Z}} \Bbbk.$$

Let $I \subset R(G)_{\Bbbk}$ be the augmentation ideal, i.e. the ideal generated by virtual representations of dimension 0. We define the $I$-adic completions by

$$\widehat{K}^G(X)_{\Bbbk} := \varprojlim_k K^G(X)_{\Bbbk}/I^k K^G(X)_{\Bbbk}, \quad \widehat{R}(G)_{\Bbbk} := \varprojlim_k R(G)_{\Bbbk}/I^k.$$

The completed $K$-group $\widehat{K}^G(X)_{\Bbbk}$ is a module over the algebra $\widehat{R}(G)_{\Bbbk}$.

Likewise, the $G$-equivariant Borel-Moore homology with $\Bbbk$-coefficients

$$H^G_*(X, \Bbbk) = \bigoplus_{k \in \mathbb{Z}} H^G_k(X, \Bbbk),$$

is a module over the $G$-equivariant cohomology ring $H^*_G(\mathrm{pt}, \Bbbk)$ of pt (with the cup product). Let us define the completion of a $\mathbb{Z}$-graded $\Bbbk$-vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$ by $V^\wedge := \prod_{k \in \mathbb{Z}} V_k$. The completion $H^*_G(\mathrm{pt}, \Bbbk)^\wedge$ naturally becomes a $\Bbbk$-algebra and the completion $H^G_*(X, \Bbbk)^\wedge$ becomes a module over $H^*_G(\mathrm{pt}, \Bbbk)^\wedge$.

Assume that our $G$-variety $X$ is a $G$-stable closed subvariety of a non-singular ambient $G$-variety $M$. Then we have the $G$-equivariant local Chern character map

$$(\mathrm{ch}^G)^M_X : \widehat{K}^G(X)_{\Bbbk} \to H^G_*(X, \Bbbk)^\wedge.$$

relative to $M$. We simply write $\mathrm{ch}^G$ instead of $(\mathrm{ch}^G)^M_X$ if the pair $(M, X)$ is obvious from the context. When $X = M = \mathrm{pt}$, the corresponding Chern character map induces an isomorphism of $\Bbbk$-algebras

$$\widehat{R}(G)_{\Bbbk} = \widehat{K}^G(\mathrm{pt})_{\Bbbk} \cong H^G_*(\mathrm{pt}, \Bbbk)^\wedge = H^*_G(\mathrm{pt}, \Bbbk)^\wedge.$$

We identify $H^G_*(\mathrm{pt}, \Bbbk)^\wedge$ with $\widehat{R}(G)_{\Bbbk}$ via this isomorphism. Then $(\mathrm{ch}^G)^M_X$ is regarded as an $\widehat{R}(G)_{\Bbbk}$-homomorphism.

For a $G$-equivariant vector bundle $E$ on a non-singular $M$, let $\mathrm{Td}^G(E) \in H^*_G(M, \Bbbk)^\wedge$ be the $G$-equivariant Todd class. This is an invertible element with respect to the cup product. For the tangent bundle $T_M$ of $M$, we put $\mathrm{Td}^G_M := \mathrm{Td}^G(T_M)$.

**Theorem A.1.1** (Equivariant Riemann-Roch [14])**.** For $i = 1, 2$, let $X_i$ be a $G$-variety which is a $G$-stable closed subvariety of a non-singular ambient $G$-variety $M_i$. Assume that a $G$-equivariant morphism $\tilde{f} \colon M_1 \to M_2$ restricts to a proper morphism $f \colon X_1 \to X_2$. Then we have

$$f_* \left( \mathrm{Td}^G_{M_1} \cdot (\mathrm{ch}^G)^{M_1}_{X_1}(\zeta) \right) = \mathrm{Td}^G_{M_2} \cdot (\mathrm{ch}^G)^{M_2}_{X_2}(f_* \zeta), \quad \zeta \in \widehat{K}^G(X_1)_{\Bbbk}.$$

The following proposition is standard.

**Proposition A.1.2.** Let $M$ be a non-singular $G$-variety. Let $Y \subset X \subset M$ be $G$-stable closed subvarieties, and $i \colon Y \hookrightarrow X$, $j \colon X \setminus Y \hookrightarrow X$ be inclusions. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{K}^G(Y)_{\Bbbk} & \xrightarrow{i_*} & \widehat{K}^G(X)_{\Bbbk} & \xrightarrow{j^*} & \widehat{K}^G(X \setminus Y)_{\Bbbk} \\
\downarrow {\scriptstyle (\mathrm{ch}^G)^M_Y} & & \downarrow {\scriptstyle (\mathrm{ch}^G)^M_X} & & \downarrow {\scriptstyle (\mathrm{ch}^G)^{M \setminus Y}_{X \setminus Y}} \\
H^G_*(Y, \Bbbk)^\wedge & \xrightarrow{i_*} & H^G_*(X, \Bbbk)^\wedge & \xrightarrow{j^*} & H^G_*(X \setminus Y, \Bbbk)^\wedge.
\end{array}$$

### A.1.3 Convolution product

Next we consider the convolution products. Let $M_i$ be non-singular $G$-varieties for $i = 1, 2, 3$. We denote by $p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ the projection to the $(i, j)$-factors for $(i, j) = (1, 2), (2, 3), (1, 3)$. Let $Z_{12} \subset M_1 \times M_2$

and $Z_{23} \subset M_2 \times M_3$ be $G$-stable closed subvarieties such that the morphism

$$p_{13} \colon p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$$

is proper. Then we define the convolution product $* \colon K^G(Z_{12}) \otimes_{R(G)} K^G(Z_{23}) \to K^G(Z_{13})$ relative to $M_1 \times M_2 \times M_3$ by

$$\zeta * \eta := p_{13*}(p_{12}^* \zeta \otimes^{\mathbb{L}}_{M_1 \times M_2 \times M_3} p_{23}^* \eta), \quad \zeta \in K^G(Z_{12}), \eta \in K^G(Z_{23}).$$

This naturally induces the convolution product on the completed $G$-equivariant $K$-groups $\widehat{K}^G(Z_{12})_{\Bbbk} \otimes_{\widehat{R}(G)_{\Bbbk}} \widehat{K}^G(Z_{23})_{\Bbbk} \to \widehat{K}^G(Z_{13})_{\Bbbk}$.

Similarly, we have the convolution product on the $G$-equivariant Borel-Moore homologies $* \colon H_*^G(Z_{12}, \Bbbk) \otimes_{H_G^*(\mathrm{pt}, \Bbbk)} H_*^G(Z_{23}, \Bbbk) \to H_*^G(Z_{13}, \Bbbk)$ relative to $M_1 \times M_2 \times M_3$ and its completed version $H_*^G(Z_{12}, \Bbbk)^\wedge \otimes_{\widehat{R}(G)_{\Bbbk}} H_*^G(Z_{23}, \Bbbk)^\wedge \to H_*^G(Z_{13}, \Bbbk)^\wedge$.

Under the situation in the previous paragraphs, for each $(i,j) = (1,2), (2,3), (1,3)$, we also define the $G$-equivariant Riemann-Roch homomorphism $\mathrm{RR}^G \colon \widehat{K}^G(Z_{ij})_{\Bbbk} \to H_*^G(Z_{ij}, \Bbbk)^\wedge$ relative to $M_i \times M_j$ by

$$\mathrm{RR}^G(\zeta) := (p_i^* \mathrm{Td}_{M_i}^G) \cdot (\mathrm{ch}^G)_{Z_{ij}}^{M_i \times M_j}(\zeta), \quad \zeta \in \widehat{K}^G(Z_{ij})_{\Bbbk},$$

where $p_i \colon M_i \times M_j \to M_i$ is the projection. By a completely similar discussion as in [11, 5.11.11], we can prove the following.

**Proposition A.1.3.** The $G$-equivariant Riemann-Roch homomorphisms are compatible with the convolution product, i.e. we have

$$\mathrm{RR}^G(\zeta * \eta) = \mathrm{RR}^G(\zeta) * \mathrm{RR}^G(\eta), \quad \zeta \in \widehat{K}^G(Z_{12})_{\Bbbk}, \eta \in \widehat{K}^G(Z_{23})_{\Bbbk}.$$

## A.2 Affine highest weight categories

In this section, we recall the definitions and some properties of (topologically complete) affine quasi-hereditary algebras and affine highest weight categories.

Let $A$ be a left Noetherian algebra over an algebraically closed field $\Bbbk$ and $J \subset A$ be the Jacobson radical of $A$. Throughout this section, we assume that $\dim(A/J) < \infty$ and $A$ is complete with respect to the $J$-adic topology, i.e. $\varprojlim A/J^n \cong A$. Let $\mathscr{C} := A\text{-mod}_{\mathrm{fg}}$ be the $\Bbbk$-linear abelian category of all finitely generated left $A$-modules. Our assumption guarantees that any simple module of $\mathscr{C}$ is finite-dimensional and the number of isomorphism classes of simple modules in $\mathscr{C}$ is finite. We parametrize the set $\mathsf{Irr}\,\mathscr{C}$ of simple isomorphism classes in $\mathscr{C}$ by a finite set $\Pi$ as $\mathsf{Irr}\,\mathscr{C} = \{L(\pi) \in \mathscr{C} \mid \pi \in \Pi\}$. For each $\pi \in \Pi$, we fix a projective cover $P(\pi)$ of the simple module $L(\pi)$.

**Definition A.2.1.** A two-sided ideal $I \subset A$ is called *affine heredity* if the following three conditions are satisfied:

(1) We have $\mathrm{Hom}_{\mathscr{C}}(I, A/I) = 0$;

(2) As a left $A$-module, we have $I \cong P(\pi)^{\oplus m}$ for some $\pi \in \Pi$ and $m \in \mathbb{Z}_{>0}$;

(3) The endomorphism $\Bbbk$-algebra $\mathrm{End}_A(P(\pi))$ is isomorphic to a ring of formal power series $\Bbbk[\![z_1, \ldots, z_n]\!]$ for some $n \in \mathbb{Z}_{\geq 0}$, and $P(\pi)$ is free of finite rank over $\mathrm{End}_A(P(\pi))$.

**Definition A.2.2.** We say that the algebra $A$ is *affine quasi-hereditary* if there is a chain of ideals:

$$0 = I_l \subsetneq I_{l-1} \subsetneq \cdots \subsetneq I_1 \subsetneq I_0 = A \tag{A.3}$$

such that, for each $i \in \{1, 2, \ldots, l\}$, the ideal $I_{i-1}/I_i$ is an affine heredity ideal of the algebra $A/I_i$. We refer to such a chain (A.3) as an *affine heredity chain*.

Let $\leq$ be a partial order of $\Pi$.

**Definition A.2.3.** The category $\mathscr{C} = A\text{-mod}_{\mathrm{fg}}$ is called an affine highest weight category for the poset $(\Pi, \leq)$ if, for each $\pi \in \Pi$, there exists an indecomposable module $\Delta(\pi)$ which is a nonzero quotient of $P(\pi)$ (i.e. $P(\pi) \twoheadrightarrow \Delta(\pi) \twoheadrightarrow L(\pi)$) satisfying the following three conditions:

(1) The endomorphism $\Bbbk$-algebra $B_\pi := \mathrm{End}_{\mathscr{C}}(\Delta(\pi))$ is isomorphic to a ring of formal power series $\Bbbk[\![z_1, \ldots, z_{n_\pi}]\!]$ for some $n_\pi \in \mathbb{Z}_{\geq 0}$, and $\Delta(\pi)$ is free of finite rank over $B_\pi$;

(2) Define $\bar{\Delta}(\pi) := \Delta(\pi)/\operatorname{rad} B_\pi$, where $\operatorname{rad} B_\pi$ denotes the maximal ideal of $B_\pi$. Then each composition factor of the kernel of the natural quotient map $\bar{\Delta}(\pi) \twoheadrightarrow L(\pi)$ is isomorphic to $L(\sigma)$ for some $\sigma < \pi$;

(3) The kernel of natural quotient map $P(\pi) \twoheadrightarrow \Delta(\pi)$ is filtered by various $\Delta(\sigma)$'s with $\sigma > \pi$.

We refer to the module $\Delta(\pi)$ (resp. $\bar{\Delta}(\pi)$) as the *standard module* (resp. *proper standard module*) corresponding to the parameter $\pi \in \Pi$.

**Theorem A.2.4** (Cline-Parshall-Scott [12], Kleshchev [35]). The category $A\text{-mod}_{\mathrm{fg}}$ is an affine highest weight category if and only if the algebra $A$ is an affine quasi-hereditary algebra.

*Proof.* See [35, Theorem 6.7]. □

**Remark A.2.5.** Let $A$ be an affine quasi-hereditary algebra with $\mathsf{Irr}\,\mathscr{C} = \{L(\pi) \in \pi \in \Pi\}$. Then standard modules of the affine highest weight category $\mathscr{C}$ are obtained as indecomposable direct summands of subquotients $I_{i-1}/I_i$ of an affine heredity chain (A.3). More precisely, an affine heredity chain (A.3) gives a total order $\{\pi_1, \pi_2, \ldots, \pi_l\}$ of the parameter set $\Pi$ by $I_{i-1}/I_i \cong \Delta(\pi_i)^{\oplus m_i}$. Using this notation, we define a partial order $\leq$ on the set $\Pi$ by the following condition:

(∗) For $\sigma, \tau \in \Pi$, we have $\sigma < \tau$ if and only if for any affine heredity chain we have $\sigma = \pi_i, \tau = \pi_j$ for some $i, j$ such that $1 \leq i < j \leq l$.

Then we can prove that the category $\mathscr{C}$ is an affine highest weight category for this partial order $\leq$ on $\Pi$.

The following theorem is the Ext-version of BGG type reciprocity.

**Theorem A.2.6** (Kleshchev [35]). Let $\mathscr{C}$ be an affine highest weight category for a poset $(\Pi, \leq)$. Then, for each $\pi \in \Pi$, there is an indecomposable module $\bar{\nabla}(\pi) \in \mathscr{C}$ characterized by the following Ext-orthogonality:

$$\mathrm{Ext}^i_{\mathscr{C}}(\Delta(\sigma), \bar{\nabla}(\pi)) = \begin{cases} \Bbbk & i = 0, \sigma = \pi; \\ 0 & \text{else.} \end{cases}$$

*Proof.* See [35, Lemma 7.2 and 7.4]. □

We refer to the module $\bar{\nabla}(\pi)$ as the *proper costandard module* corresponding to the parameter $\pi \in \Pi$. The following criterion is proved by using a theory of tilting modules in affine highest weight categories.

**Theorem A.2.7** ([17] Theorem 3.9). For $i = 1, 2$, let $\mathscr{C}_i = A_i\text{-mod}_{\mathrm{fg}}$ be an affine highest weight category for a poset $(\Pi_i, \leq_i)$. Assume that we have an exact functor $F\colon \mathscr{C}_1 \to \mathscr{C}_2$ and the following conditions are satisfied:

(1) The algebra $A_i$ is a finitely generated module over its center ($i = 1, 2$);

(2) There exists a bijection $f\colon \Pi_1 \xrightarrow{\cong} \Pi_2$ preserving partial orders and we have the following isomorphisms for each $\pi \in \Pi_1$:

$$F(\Delta(\pi)) \cong \Delta(f(\pi)), \qquad F(\bar{\nabla}(\pi)) \cong \bar{\nabla}(f(\pi)).$$

Then the functor $F$ gives an equivalence of categories $F\colon \mathscr{C}_1 \xrightarrow{\simeq} \mathscr{C}_2$.

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