On the Casson-Walker invariant of 3-manifolds with genus one open book decompositions (summary)

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1 The Casson-Walker invariant of 3-manifolds with genus one open book decompositions via surgery presentations

1.1 Surgery presentations of 3-manifolds with genus one open book decompositions

Firstly, we recall the definition of a *genus one and one boundary component open book decomposition* of a 3-manifold. (We will call it a genus one open book decomposition in short in the following.) Let $\Sigma_{1,1}$ be an oriented compact surface with genus one and one boundary component and let φ be an orientation preserving homeomorphism of $\varphi : \Sigma_{1,1} \to \Sigma_{1,1}$ restricting to the identity on the boundary $\partial\Sigma_{1,1}$. For a pair $(\Sigma_{1,1}, \varphi)$, we define a 3-manifold M_φ as follows.

$$M_\varphi = ((\Sigma_{1,1} \times [0,1])/\sim) \underset{\psi}{\cup} (D^2 \times S^1),$$

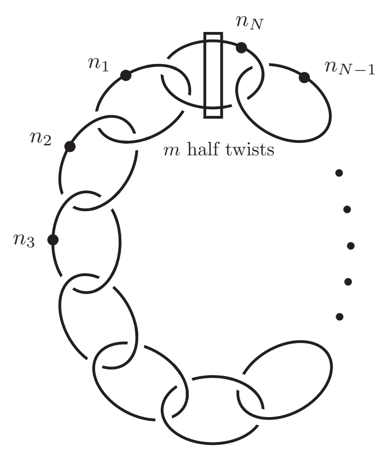
where "$\sim$" is defined by $(x,1) \sim (\varphi(x), 0)$, and ψ is the following homeomorphism

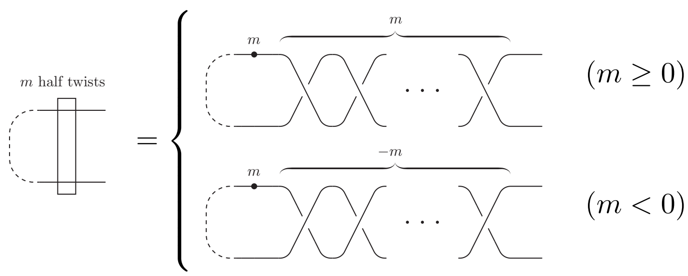
$$\psi : \partial(D^2 \times S^1) \longrightarrow \partial\left((\Sigma_{1,1} \times [0,1])/\sim\right) = S^1 \times S^1,$$

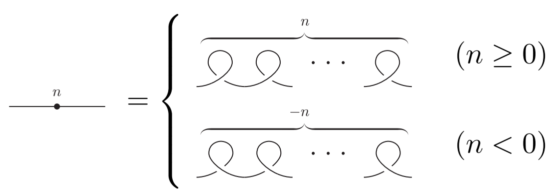
which maps a meridian $\partial(D^2 \times S^1)$ to {a point} $\times [0,1]$ on the boundary $\partial\left((\Sigma_{1,1} \times [0,1])/\sim\right)$. When a 3-manifold M is homeomorphic to M_φ, we call $(\Sigma_{1,1}, \varphi)$ a genus one open book decomposition of M.

When a 3-manifold M is obtained from S^3 by surgery along a framed link L, we call L a surgery presentation of M.

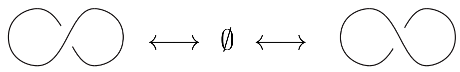
In the following, we express a framed link by a standard blackboard framing convention. For integers $n_1, \cdots, n_N, m$, let $L_{n_1,\cdots,n_N,m}$ be the framed link in the following figure, where $n_1, \cdots, n_N$ represents the framing of each component, N represents the number of the components of the link, where $N \geq 1$, and m represents the number of half twists. (If $m > 0$ or $m < 0$, $|m|$ represents the number of the positive or negative half twists, respectively.) Besides, we fix the shape of clasps between components as in the following left picture and we set a positive half twist as in the following right picture. Let $M_{n_1,\cdots,n_N;m}$ be the 3-manifold obtained from S^3 by surgery along $L_{n_1,\cdots,n_N;m}$.

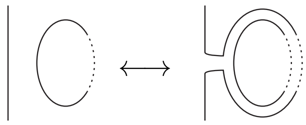


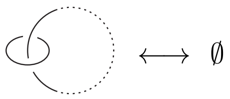




Next, we recall that the Kirby moves for framed links in a cube with 2-handles are given as in the following pictures.

the KI move : 

the KII move : 

the KIII move : 

For a compact connected orientable 3-manifold M and a framed link L in M, we denote by M_L the 3-manifold obtained from M by surgery along L. Let L and L' be framed links in M. It is known [10] that M_L and $M_{L'}$ are homeomorphic, if and only if L and L' are related by a sequence of isotopies and the moves KI, KII and KIII.

Lemma 1.1.
(1) $M_{n_1,\cdots,n_N;m}$ has a genus one open book decomposition.
(2) Any 3-manifold with a genus one open book decomposition is homeomorphic to $M_{n_1,\cdots,n_N;m}$ for some $n_1,\cdots,n_N,m$.

1.2 The Casson-Walker invariant of 3-manifolds with genus one open book decompositions

We show the values of the Casson-Walker invariant of a rational homology sphere which admits a genus one open book decomposition through its surgery presentation.

Theorem 1.2. Let $M_{n_1,\cdots,n_N;m}$ be the 3-manifold obtained from S^3 by surgery along the framed link $L_{n_1,\cdots,n_N;m}$, which has a genus one open book decomposition. We assume that $M_{n_1,\cdots,n_N;m}$ is a rational homology sphere. Then, the value of the Casson-Walker invariant of $M_{n_1,\cdots,n_N;m}$ is the following.

$$\lambda_W(M_{n_1,\cdots,n_N;m}) = -\frac{1}{24}\left(\sum_i n_i - 3\sigma\right) + \frac{(-1)^{m+\sigma_+}}{24|H_1|}\left(2\sum_i n_i + 6N + 12m\right),$$

where σ, σ_+ represents the signature, the number of the positive eigenvalues of the linking matrix of $L_{n_1,\cdots,n_N;m}$, respectively. $|H_1|$ represents the order of $H_1(M_{n_1,\cdots,n_N;m};\mathbb{Z})$.

Here, we set the linking matrices of circular chain links and straight chain links as follows.

$$A_{n_1,\cdots,n_N;m} = \begin{cases} \begin{pmatrix} n_1 & 1 & & & & (-1)^m \\ 1 & n_2 & 1 & & & \\ & 1 & n_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & n_{N-1} & 1 \\ (-1)^m & & & & 1 & n_N \end{pmatrix} & N > 2, \\ \begin{pmatrix} n_1 & 1+(-1)^m \\ 1+(-1)^m & n_2 \end{pmatrix} & N = 2, \\ \begin{pmatrix} n_1+(-1)^m 2 \end{pmatrix} & N = 1, \end{cases}$$

$$A(n_1,\cdots,n_N) = \begin{cases} \begin{pmatrix} n_1 & 1 & & & & \\ 1 & n_2 & 1 & & & \\ & 1 & n_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & n_{N-1} & 1 \\ & & & & 1 & n_N \end{pmatrix} & N > 2, \\ \begin{pmatrix} n_1 & 1 \\ 1 & n_2 \end{pmatrix} & N = 2, \\ \begin{pmatrix} n_1 \end{pmatrix} & N = 1. \end{cases}$$

Then, for $M_{n_1,\cdots,n_N;m}$, we have that

$$|H_1| = (-1)^{\sigma_-} \det A_{n_1,\cdots,n_N;m},$$

where σ_- denotes the number of negative eigenvalues of $A_{n_1,\cdots,n_N;m}$.

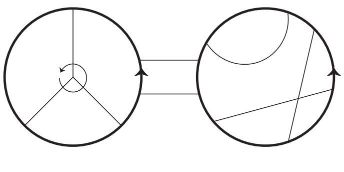
We give a proof of the theorem in Section 1.4.

1.3 The relation between the Casson-Walker invariant and the LMO invariant

In this section, as a preparation of a proof of Theorem 1.2 in Section 1.4, we review the LMO invariant, its relation to the Casson-Walker invariant, and some useful formulae for the computation of the LMO invariant.

1.3.1 The degree 1 part of the LMO invariant and the Casson-Walker invariant

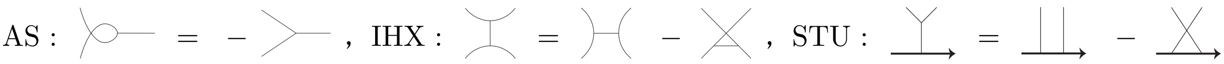
Firstly, we review the Kontsevich invariant. We use [8] as a basic reference for the theory of the LMO invariant, and will use the same notation as [8] in the following. A Jacobi diagram on $\sqcup^\ell S^1$ is a 1-manifold $\sqcup^\ell S^1$ with a graph which has univalent vertices and trivalent vertices. A univalent vertex is necessarily located on $\sqcup^\ell S^1$ and a trivalent vertex is oriented, that is, the set of the three adjacent edges is given a cyclic ordering. (We express graphs as thin lines and 1-manifolds as thick lines in the following.) The *degree* of a Jacobi diagram is half the number of both univalent and trivalent vertices. The picture in the following represents an example of a Jacobi diagram $S^1 \sqcup S^1$ of degree 7.



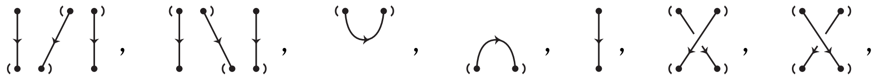
The space of Jacobi diagrams on $\sqcup^\ell S^1$ is defined as follows.

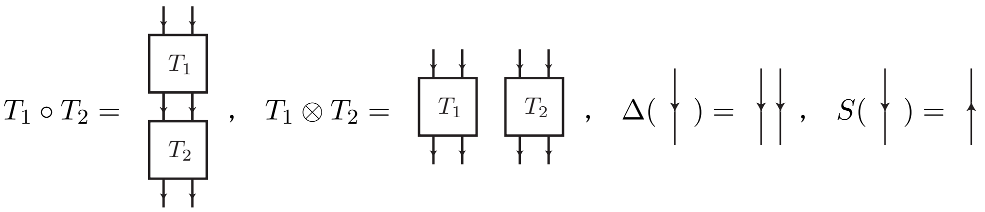
$$\mathcal{A}(\sqcup^\ell S^1) = \mathrm{span}_{\mathbb{C}} \{\text{Jacobi diagrams on } \sqcup^\ell S^1\}/\text{AS, IHX, STU},$$

where AS, IHX, STU relations are the following.

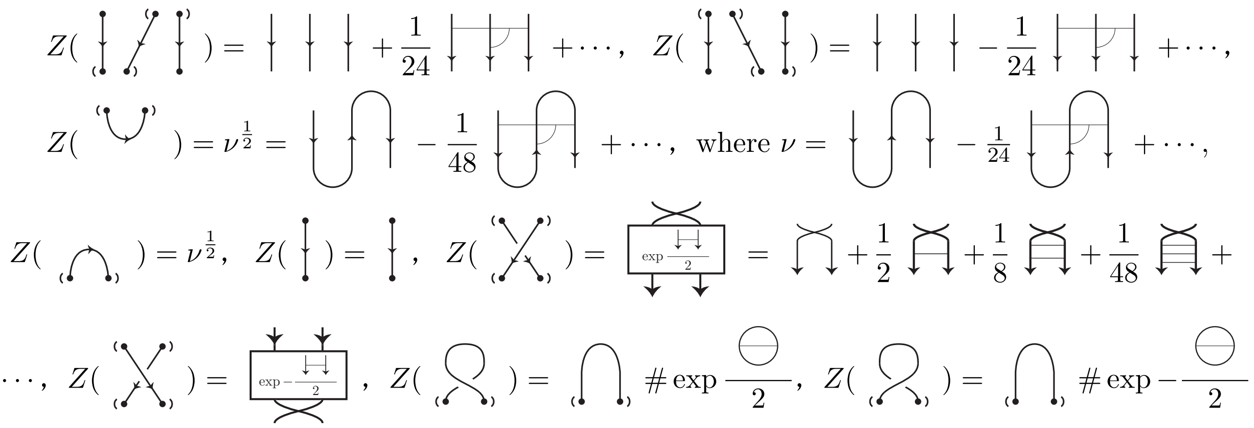


Any oriented q-tangle is a tangle whose endpoints are parenthesised, generated by the following fundamental q-tangles by conducting several operations of composition $\circ$, tensor product $\otimes$, duplication Δ and antipode S.



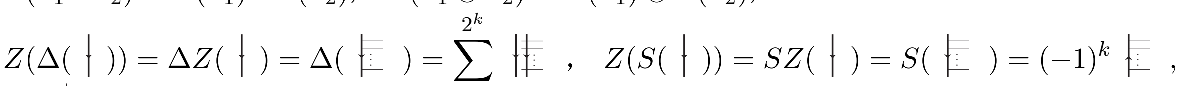


The Kontsevich invariant is an invariant of framed links that takes value in $\mathcal{A}(\sqcup^{\ell} S^1)$, where l is the number of components. The Kontsevich invariant can be extended to an invariant of q-tangles that takes value in certain spaces of Jacobi diagrams. (See [8] for details.) We list the Kontsevich invariant of the fundamental q-tangles below.



where # represents connected sum of 1-manifolds.
Besides, for composition $\circ$, tensor product $\otimes$, duplication Δ and antipode S, we define their values as follows.

$$Z(T_1 \circ T_2) = Z(T_1) \circ Z(T_2), \quad Z(T_1 \otimes T_2) = Z(T_1) \otimes Z(T_2),$$



where | represents a certain component of tangle, and k represents the number of the univalent vertices on the component. (As for the well-definedness of this definition, see for example [4].)

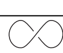
Next, we review the definition of the LMO invariant up to degree 1. (As for the LMO invariant with general degree, see for example [5].) We set

$$\check{Z}(L) = Z(L) \# \nu^{\otimes \ell}.$$

Here, # denotes the connected sum of Jacobi diagrams, and $Z(L)\#\nu^{\otimes \ell}$ means that we take a connected sum with ν along each component of $Z(L)$.

The LMO invariant up to degree 1 is defined as follows. ($\emptyset$ represents an empty diagram.)

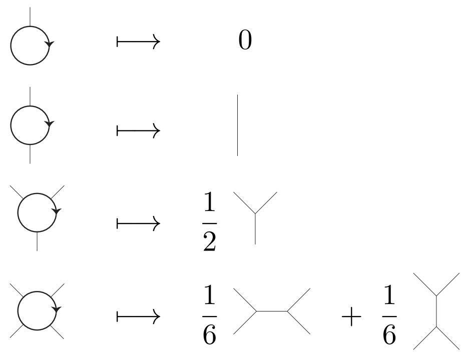
$$Z_1^{\mathrm{LMO}}(M) = \frac{\iota(\check{Z}(L))}{\iota(Z(\ \cdot\ ))^{\sigma_+}\iota(Z(\ \cdot\ ))^{\sigma_-}} \in \mathrm{span}_{\mathbb{C}}\{\emptyset,\ \cdot\ \}$$

where ι is the map

$$\iota : \mathcal{A}(\sqcup^{\ell} S^1) \longrightarrow \mathrm{span}_{\mathbb{C}}\{\emptyset,\ \cdot\ \}$$

defined as follows. For $D \in \mathcal{A}(\sqcup^{\ell} S^1)$, we remove each S^1 by the following correspondences.



S^1 with more than 5 univalent vertices $\longmapsto$ 0.

Besides, in the definition of ι, if there appears a circle  with a thin line after replacing S^1, then we remove such a circle by formally putting  $= -2$. Besides, we also remove a graph with more than 3 trivalent vertices by formally letting the graph be 0. We put the resulting diagram to be $\iota(D)$. In the following, we set $\theta =$ . The values of the ± 1-framed trivial knots are the following.

$$\iota(Z(\ \cdot\ )) = \left(-1 + \frac{1}{16}\theta\right),\ \iota(Z(\ \cdot\ )) = \left(1 + \frac{1}{16}\theta\right). \tag{1.1}$$

Next, we mention the relation between the Casson-Walker invariant and the coefficient of the degree 1 part of the LMO invariant. Let M be a rational homology sphere. When the LMO invariant up to degree 1 is described as

$$Z_1^{LMO}(M) = c_0(M) + c_1(M)\theta,$$

the relations to the order of the first homology and the Casson-Walker invariant are

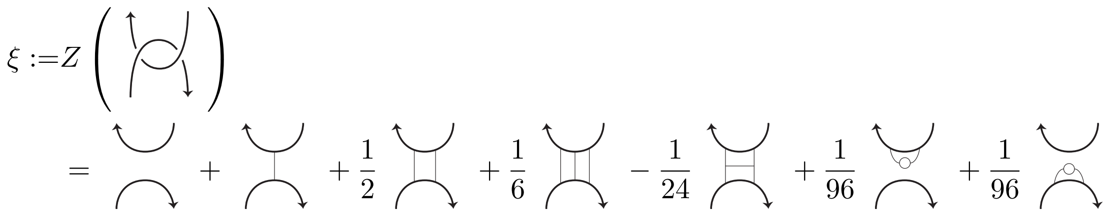
$$|H_1| = c_0(M),\ \lambda_W(M) = \frac{2c_1(M)}{|H_1|}, \tag{1.2}$$

if the first Betti number of M is equal to 0. (See [6].) For the coefficients $b_0(L)$, $b_1(L)$ of the degree 0 part and the degree 1 part of $\iota\check{Z}(L)$ of the surgery link L,

$$c_0(M) = (-1)^{\sigma_+} b_0(L),$$
$$c_1(M) = (-1)^{\sigma_+}\left(\frac{\sigma}{16} b_0(L) + b_1(L)\right).$$

1.3.2 A formula for the Kontsevich invariant of a clasp

Here, we show a formula for the calculation of the LMO invariant up to degree 1. It is known [9] that the value of the Kontsevich invariant of the clasp is the following.



(1.3)

+ (the terms with at least 3 trivalent vertices).

1.4 Proof of Theorem 1.2

Let $n_1, \cdots, n_N, m$ be integers. For the calculation of the Kontsevich invariant of the circular chain link $L_{n_1,\cdots,n_N;m}$, we define the *straight chain link* $L(n_1, \cdots, n_N)$ in the following picture, where $n_1, \cdots, n_N$ represents the framing of each component.

For $N \geq 3$, the coefficients of the degree 0 and 1 parts of $\iota\check{Z}(L(n_1, \cdots, n_N))$, $b_0\left(L(n_1, \cdots, n_N)\right)$ and
$b_1\left(L(n_1, \cdots, n_N)\right)$, are as follows. (We calculate them in Appendix A.)

$$b_0\left(L(n_1, \cdots, n_N)\right) = (-1)^N \det A(n_1, \cdots, n_N), \tag{1.4}$$

$$\begin{aligned} b_1(L(n_1, \cdots, n_N)) =& \frac{(-1)^{N-1}}{48}\left(\det A(n_1, \cdots, n_N)\sum_{i=1}^{N} n_i \right. \\ & \left. + \det A(n_1, \cdots, n_{N-1}) + \det A(n_2, \cdots, n_N)\right). \end{aligned} \tag{1.5}$$

From here, we will calculate the degree 1 part of the LMO invariant of 3-manifolds with genus one open book decompositions. Recall that, in Example 2.5, we have confirmed the validity of the formula of Theorem 1.2 for the case where $N = 1$. Although we can also compute the LMO invariant in the proof of the theorem for the case where $N = 1$ by similar methods, to make computations simpler in the following, we assume that $N \geq 2$.

1.4.1 The degree 1 part of the LMO invariant of $M_{n_1,\cdots,n_N;m}$

Firstly, we compute the coefficient of the degree 1 part of $\iota\check{Z}(L_{n_1,\cdots,n_N;m})$ in the following proposition.

Proposition 1.3. The coefficient $b_1(L_{n_1,\cdots,n_N;m})$ of the degree 1 part of $\iota\check{Z}(L_{n_1,\cdots,n_N;m})$ is given by

$$\begin{aligned} b_1(L_{n_1,\cdots,n_N;m}) = & -(-1)^N \frac{1}{48}\left(\det A_{n_1,\cdots,n_N;m} \sum_{i=1}^{N} n_i\right) \\ & + (-1)^m \frac{1}{48}\left(2\sum_{i=1}^{N} n_i + 6N + 12m\right). \end{aligned}$$

Proof of Theorem 1.2. By Proposition 1.3, the degree 1 coefficient of the LMO invariant of $M_{n_1,\cdots,n_N;m}$ is the following.

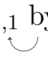
$$\begin{aligned} c_1\left(M_{n_1,\cdots,n_N;m}\right) = & -\frac{1}{48}(-1)^{N+\sigma_+} \det A_{n_1,\cdots,n_N;m}(\mathrm{tr}A_N - 3\sigma) \\ & + \frac{1}{48}(-1)^{m+\sigma_+}\left(2\mathrm{tr}A_{n_1,\cdots,n_N;m} + 6N + 12m\right). \end{aligned}$$

Considering the relation between $\lambda_W(M_{n_1,\cdots,n_N;m})$ and $c_1\left(M_{n_1,\cdots,n_N;m}\right)$, we obtain the theorem. □

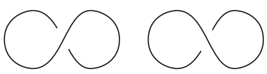
2 The Casson-Walker invariant of 3-manifolds with genus one open book decompositions via a representation of the mapping class group

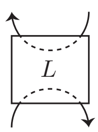
In this section, we present the Casson-Walker invariant in terms of a representation of a central extension of $\mathfrak{M}_{1,1}$ on the space of Jacobi diagrams. We give a central extension of $\mathfrak{M}_{1,1}$ as the group of equivalence classes of certain 2-tangles modulo a modification of the Kirby moves.

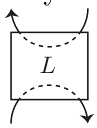
2.1 A central extension of the mapping class group $\mathfrak{M}_{1,1}$

Let $\mathfrak{M}_{1,1}$ be the mapping class group of $\Sigma_{1,1}$, a compact surface with genus one and one boundary component. It is known that every central extension of $\mathfrak{M}_{1,1}$ is trivial (see for example [3]), but, in order to avoid the complication of the calculation of the invariant, we consider a central extension $\widetilde{\mathfrak{M}_{1,1}}$ by signature, and construct the representation of $\widetilde{\mathfrak{M}_{1,1}}$ on the 2-tangle Jacobi diagram space $\hat{\mathcal{A}}($  $)$.

As for the Kirby moves for framed links in a compact 3-manifold (possibly with boundary), in order to make the representation well-defined with regard to the signature, we introduce the KI′ move, as follows.

the KI′ move :  $\longleftrightarrow \emptyset$

We regard a 2-tangle 

as in a cube. We associate this 2-tangle with the 3-cobordism obtained from the cube by removing tubular neighbourhood of the top and bottom components of the 2-tangle and by surgery along the closed components L. We define an *admissible 2-tangle* to be a 2-tangle 

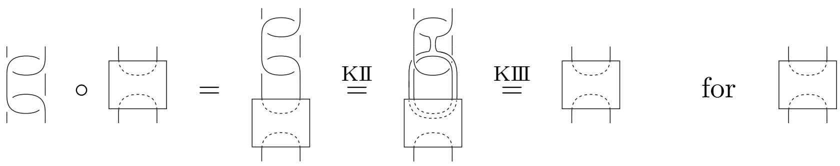
such that the associated 3-cobordism is homeomorphic to a mapping cylinder. We denote by $\mathcal{T}_2$ the set of admissible 2-tangles.

We regard the Kirby moves in $\mathcal{T}_2$ as in the following way. For L, we use the KI, KII, KIII moves of a link in (cube $- N(C_1 \cup C_2)$). For C_1 or C_2, we use the KII move of a tangle, that is, the handle slide of C_1 or C_2 along a component of L.

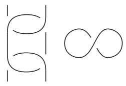
Lemma 2.1 (a particular case of a theorem in [7])**.** $\mathcal{T}_2$/KI, KII, KIII forms a group, and is isomorphic to $\mathfrak{M}_{1,1}$.

We put $\widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/\mathrm{KI}'$, KII , KIII. We give the product of $\widetilde{\mathfrak{M}_{1,1}}$ by the composition of 2-tangles. This product is naturally associative.

We can show that the unit element in $\widetilde{\mathfrak{M}_{1,1}}$ is given by  in the following formula.



$\in \widetilde{\mathfrak{M}_{1,1}}$.

Since $\mathcal{T}_2$/KI, KII, KIII is a group by Lemma 2.1, for any admissible 2-tangle T, there is a 2-tangle T' such that $T \circ T' =$  in $\mathcal{T}_2$/KI, KII, KIII. When we use the KI move in the deformation from $T \circ T'$ to  , we exchange the KI move for the KI$'$ move by adding  or  . Thus, we get the inverse of T in $\widetilde{\mathfrak{M}_{1,1}}$ as the union of T' and some copies of  or  . Therefore, any element in $\widetilde{\mathfrak{M}_{1,1}}$ has its inverse in $\widetilde{\mathfrak{M}_{1,1}}$. Hence, $\widetilde{\mathfrak{M}_{1,1}}$ forms a group. Moreover, since $\mathcal{T}_2$/KI, KII , KIII $= \mathfrak{M}_{1,1}$ is generated by admissible 2-tangles α, β below, this shows that $\widetilde{\mathfrak{M}_{1,1}}$ is generated by α, β and $\mu =$  .

Lemma 2.2. $\widetilde{\mathfrak{M}_{1,1}}$ is a central extension of $\mathfrak{M}_{1,1}$.

We define a map

$$\sigma : \widetilde{\mathfrak{M}_{1,1}} \longrightarrow \mathbb{Z} \tag{2.1}$$

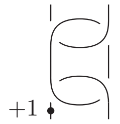
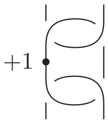
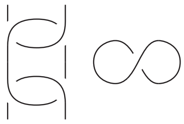
by putting $\sigma(T)$ to be the signature of the linking matrix of the closure of $T \in \widetilde{\mathfrak{M}_{1,1}}$. Here, the closure of T means the link obtained by connecting the upper ends and lower ends of T respectively. Then, $\sigma(T)$ is invariant under the KI$'$, KII, KIII moves. As for our central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_1} \widetilde{\mathfrak{M}_{1,1}} \xrightarrow{f_2} \mathfrak{M}_{1,1} \longrightarrow 1,$$

$\sigma \circ f_1$ is the identity on $\mathbb{Z}$.

2.2 The Casson-Walker invariant of 3-manifolds with genus one open book decompositions

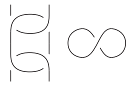
We choose generators of $\widetilde{\mathfrak{M}_{1,1}}$ as follows.

$\alpha =$  $, \ \beta =$  $, \ \mu =$  . (2.2)

Using the LMO invariant, we will define liner representations $\hat{\rho}_1, \hat{\rho}_2 : \widetilde{\mathfrak{M}_{1,1}} \longrightarrow \mathrm{GL}(4, \mathbb{C})$. As we will see in (2.5), (2.6) and (2.7), we have that

$$\hat{\rho}_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{96} & -\frac{1}{24} & \frac{1}{2} & 1 \end{pmatrix}, \ \hat{\rho}_1(\beta) = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{48} & \frac{1}{24} & 1 & -2 \\ 0 & -\frac{1}{48} & 0 & 1 \end{pmatrix}, \ \hat{\rho}_1(\mu) = -\sqrt{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{16} & 0 & 1 & 0 \\ 0 & -\frac{1}{16} & 0 & 1 \end{pmatrix},$$

$$\hat{\rho}_2(\alpha) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{24} & 1 \end{pmatrix}, \ \hat{\rho}_2(\beta) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{24} & 1 \end{pmatrix}, \ \hat{\rho}_2(\mu) = -\sqrt{-1} \begin{pmatrix} 1 & 0 \\ -\frac{1}{16} & 1 \end{pmatrix}.$$

It is convenient to use an element $h = (\alpha\beta)^3\mu^{-2}$ represented by the 2-tangle  . The tangle h represents a lift of the right-handed Dehn twist along $\partial\Sigma_{1,1}$. The image of h under ρ_1 and ρ_2 are given by

$$\hat{\rho}_1(h) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \hat{\rho}_2(h) = \begin{pmatrix} -1 & 0 \\ \frac{1}{8} & -1 \end{pmatrix}.$$

By taking the trace of this representation of the monodromy, we can calculate the Casson-Walker invariant of a 3-manifold admitting genus one and one boundary open book decompositions.

Theorem 2.3. Suppose that M_φ is a rational homology sphere. Taking an element $\tilde{\varphi}$ of the central extension corresponding to the monodromy φ, we can calculate the Casson-Walker invariant of M_φ as follows,

$$\lambda(M_\varphi) = \frac{2(\sqrt{-1})^{\sigma(\tilde{\varphi})}}{|H_1|} \left(\mathrm{tr}(Q_1\hat{\rho}_1(\tilde{\varphi})) - 2\mathrm{tr}(Q_2\hat{\rho}_2(\tilde{\varphi}))\right) + \frac{1}{8}\sigma(\tilde{\varphi}), \tag{2.3}$$

where σ is defined in (2.1), and Q_1, Q_2 are the following matrices,

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Q_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

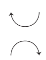
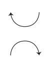
We give a proof of the theorem in Section 2.4.

Then, we show some examples of the concrete calculation. We will continue to assume that M_φ is a rational homology sphere.

2.3 Preparations for Proof of Theorem 2.3

2.3.1 The space of the Jacobi diagrams on two intervals

In order to construct a representation of the mapping class group of the surface $\Sigma_{1,1}$, we recall that the LMO invariant up to degree 1 of mapping cylinders can be defined by using the following map,

$\iota\check{Z}$:{surgery links with  }/K-moves, isotopy $\longrightarrow \hat{\mathcal{A}}($  $)$.

Here, we define the space $\hat{\mathcal{A}}(\;)$ by

$$\hat{\mathcal{A}}(\;) = \{\text{Jacobi diagrams on 2-tangles up to AS, IHX, STU}\}/P_2,\ O_1,\ I_{>2},$$

where P_2, O_1 and $I_{>2}$ are the equivalence relations generated by the following relations.

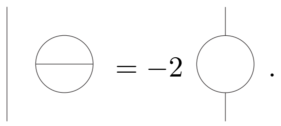
P_2 :  ~ 0

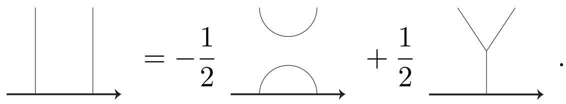
O_1 :  ~ -2

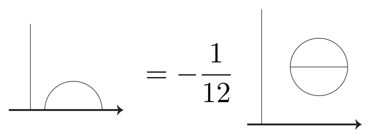
$I_{>2}$: the Jacobi diagram with more than 2 trivalent vertices ~ 0

In the rest of Section 2.3, we give a basis of $\hat{\mathcal{A}}(\;)$ in Lemmas 2.6 and 2.7. In order to prove these lemmas, we show the following two lemmas.

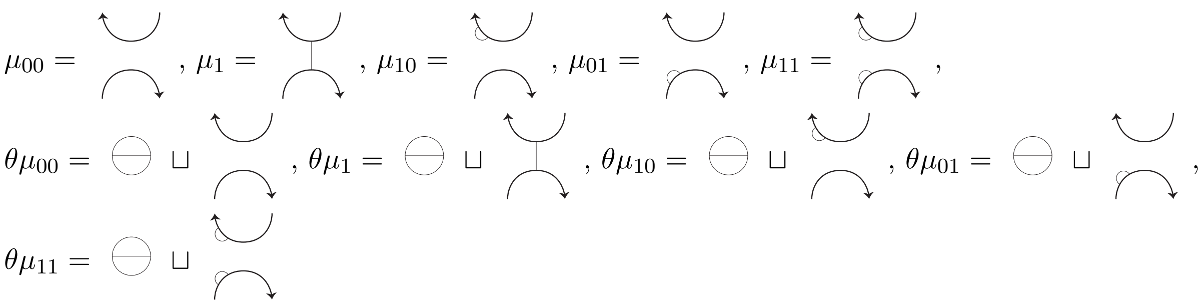
Lemma 2.4. ([8]) In the space $\hat{\mathcal{A}}(\;)$, the following relations hold.

(1) 

(2) 

Lemma 2.5. In the space $\hat{\mathcal{A}}(\;)$, the following relation holds. 

Lemma 2.6. $\hat{\mathcal{A}}(\;)$ is spanned by the following 10 diagrams.



Lemma 2.7. The 10 diagrams of Lemma 2.6 give a basis of $\hat{\mathcal{A}}(\;)$.

2.3.2 The invariance under the Kirby moves

In this subsection, we construct the representation of the central extension of $\mathfrak{M}_{1,1}$ on the space of Jacobi diagrams on two intervals. (A similar representation and the relation with the Casson invariant appeared in [2].)

Let $\tilde{\mathcal{T}}$ be the set of the 2-tangles and we regard it as a monoid with the product as a composition. Instead of Z, we use a map $\check{Z} : \tilde{\mathcal{T}} \longrightarrow \hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown})$ such that $\hat{Z}(T)$ of a 2-tangle T is obtained from $Z(T)$ by connect-summing ν into each of the closed components $\sqcup^{\ell} S^1$ and connect-summing $\nu^{\frac{1}{2}}$ into each of the open components $\genfrac{}{}{0pt}{}{\smile}{\frown}$. When we consider $\iota : \hat{\mathcal{A}}(\sqcup^{\ell} S^1) \to \hat{\mathcal{A}}(\emptyset)$, we define $\hat{\iota} : \hat{\mathcal{A}}(\sqcup^{\ell} S^1) \to \hat{\mathcal{A}}(\emptyset)$ by $\hat{\iota} = \sqrt{-1}^{\ell}\iota$. Similarly, when we consider $\iota : \hat{\mathcal{A}}(\sqcup^{\ell} S^1 \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown}) \to \hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown})$, we define $\hat{\iota} : \hat{\mathcal{A}}(\sqcup^{\ell} S^1 \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown}) \to \hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown})$ by $\hat{\iota} = \sqrt{-1}^{\ell}\iota$. As for the map $\hat{\iota}\check{Z}$ from $\tilde{\mathcal{T}}$ to $\hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown})$, we have the following proposition.

Proposition 2.8. A monoid homomorphism

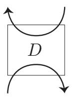
$$\hat{\iota}\check{Z} : \tilde{\mathcal{T}} \xrightarrow{\check{Z}} \hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown}) \xrightarrow{\hat{\iota}} \hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown})$$

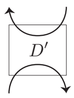
is invariant under the KI$'$, KII, KIII moves.

2.3.3 A representation of the central extension of the mapping class group on the space of the Jacobi diagrams on two intervals

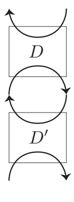
We set the composition

$$\circ : \hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown}) \otimes \hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown}) \longrightarrow \hat{\mathcal{A}}(S^1 \sqcup (\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown})$$

on the space $\hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown})$ as follows. For diagrams $\eta =$ 

, $\eta' =$ 

in $\hat{\mathcal{A}}((\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown})$, we define the composition $\eta \circ \eta'$ to be the diagram obtained from the union of these two diagrams by attaching the endpoints of the lower interval of η to the endpoints of the upper interval of η' as the orientations of intervals agree with each other, as in the following diagram.

$\eta \circ \eta' =$ 

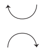
$\in \hat{\mathcal{A}}(S^1 \sqcup (\sqcup^{\ell} S^1) \sqcup \genfrac{}{}{0pt}{}{\smile}{\frown})$.

We define ρ to be the following map.

$$\rho : \widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/\mathrm{KI}', \mathrm{KII}\ , \mathrm{KIII} \longrightarrow \mathrm{End}(\hat{\mathcal{A}}(\genfrac{}{}{0pt}{}{\smile}{\frown}))$$
$$T \longmapsto \big(\eta \longmapsto \hat{\iota}(\check{Z}(T) \circ \eta)\big),$$

where $\check{Z}(T) \circ \eta$ represents the composition of Jacobi diagrams on two intervals

Proposition 2.9.

$\rho : \widetilde{\mathfrak{M}_{1,1}} = \mathcal{T}_2/\mathrm{KI}', \mathrm{KII}\ , \mathrm{KIII} \longrightarrow \mathrm{End}(\hat{\mathcal{A}}($  $))$

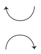
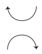
is a representation of $\widetilde{\mathfrak{M}_{1,1}}$.

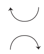
Then, we have the following proposition.

Proposition 2.10. Let $\varphi \in \mathfrak{M}_{1,1}$, and let M_φ be the 3-manifold with a genus one open book decomposition whose monodromy is φ. Suppose that M_φ is a rational homology sphere. Let $T \in \widetilde{\mathfrak{M}_{1,1}}$ be a lift of φ. Then, the Casson-Walker invariant of M_φ is presented by

$$\lambda(M_\varphi) = \frac{2(\sqrt{-1})^{\sigma(T)}}{|H_1|}\mathrm{tr}(\psi \circ \rho(T)) + \frac{1}{8}\sigma(T), \tag{2.4}$$

where $|H_1|$ denotes the order of $H_1(M_\varphi; \mathbb{Z})$, and we define

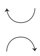
$\psi : \hat{\mathcal{A}}($  $) \longrightarrow \hat{\mathcal{A}}($  $)$

to be the linear map whose values for the basis vectors of $\hat{\mathcal{A}}($  $)$ are given by

$$\psi(\mu_{00}) = 0, \psi(\mu_1) = 0, \psi(\mu_{10}) = 0, \psi(\mu_{01}) = 0, \psi(\mu_{11}) = 0,$$
$$\psi(\theta\mu_{00}) = 0, \psi(\theta\mu_1) = -2\mu_1, \psi(\theta\mu_{10}) = \mu_{10}, \psi(\theta\mu_{01}) = \mu_{01}, \psi(\theta\mu_{11}) = 0.$$

2.4 Proof of Theorem 2.3

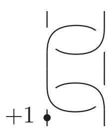
2.4.1 A matrix representation of the mapping class group

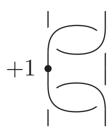
In this section, we calculate a matrix presentation of the representation of $\rho : \widetilde{\mathfrak{M}_{1,1}} \longrightarrow \mathrm{End}(\hat{\mathcal{A}}($  $))$ and the values of the invariant concretely for the basis of $\hat{\mathcal{A}}_1$ which we have taken in Lemma 2.6.

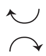
Proposition 2.11. We put

$\hat{\mathcal{A}}($  $) = \hat{\mathcal{A}}_1 \oplus \hat{\mathcal{A}}_1' \oplus \hat{\mathcal{A}}_2$
$$= \mathrm{span}_{\mathbb{C}}\{\mu_{00}, \mu_{10}, \theta\mu_{00}, \theta\mu_{10}\} \oplus \mathrm{span}_{\mathbb{C}}\{\mu_{01}, \mu_{11}, \theta\mu_{01}, \theta\mu_{11}\} \oplus \mathrm{span}_{\mathbb{C}}\{\mu_1, \theta\mu_1\}.$$

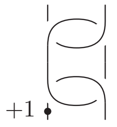
Then, the representation ρ is decomposed into $\rho_1 \oplus {\rho'}_1 \oplus \rho_2$, where ρ_i represents the restriction $\rho|_{\hat{\mathcal{A}}_i}$.

Next, we show matrix presentations of certain elements 

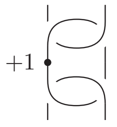
, 

and  of $\widetilde{\mathfrak{M}_{1,1}}$ through the map $\hat{\imath}\check{Z}$. We regard $\hat{\mathcal{A}}($  $)$ as a vector space $\mathbb{C}^{10}$ with regard to the

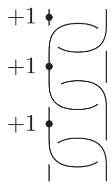
10 elements of a basis. We denote by $\hat{\rho}$ the representation $\hat{\rho} : \widetilde{\mathfrak{M}_{1,1}} \longrightarrow \mathrm{End}(\mathbb{C}^{10})$ such that $\hat{\rho}(T)$ is a 10 times 10 matrix for $T \in \widetilde{\mathfrak{M}_{1,1}}$. From Proposition 2.11, we decompose $\hat{\rho}(T)$ as $\hat{\rho}(T) = \hat{\rho}_1(T) \oplus \hat{\rho}_1'(T) \oplus \hat{\rho}_2(T)$.
We have the matrix presentations of the generators as

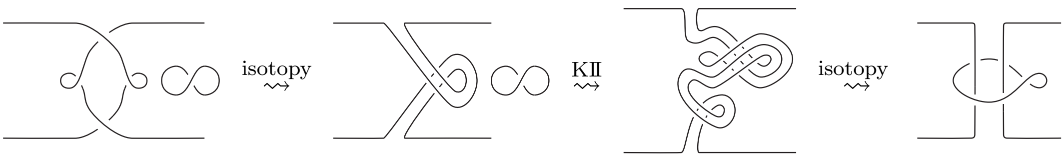
$\hat{\rho}\Big($  $\Big)$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{96} & -\frac{1}{24} & \frac{1}{2} & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{96} & -\frac{1}{24} & \frac{1}{2} & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ -\frac{1}{24} & 1 \end{pmatrix}, \tag{2.5}$$

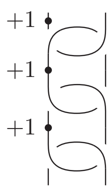
$\hat{\rho}\Big($  $\Big)$

$$= \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{48} & \frac{1}{24} & 1 & -2 \\ 0 & -\frac{1}{48} & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{48} & \frac{1}{24} & 1 & -2 \\ 0 & -\frac{1}{48} & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ -\frac{1}{24} & 1 \end{pmatrix}. \tag{2.6}$$

Finally, μ is represented by another form as  from the following transformation.



Then, by direct computation, we have that

$\hat{\rho}\Bigg($  $\Bigg)$

$$= (-\sqrt{-1}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{16} & 0 & 1 & 0 \\ 0 & -\frac{1}{16} & 0 & 1 \end{pmatrix} \oplus (-\sqrt{-1}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{16} & 0 & 1 & 0 \\ 0 & -\frac{1}{16} & 0 & 1 \end{pmatrix} \oplus (-\sqrt{-1}) \begin{pmatrix} 1 & 0 \\ -\frac{1}{16} & 1 \end{pmatrix}. \tag{2.7}$$

Proof of Thorem 2.3. We set

$$Q_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Q_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the proof of Proposition 2.10,

$$\begin{aligned} \mathrm{tr}(\psi \circ \rho(T)) &= \mathrm{tr}(Q\hat{\rho}(T)) \\ &= \mathrm{tr}(Q_1^{(1)}\hat{\rho}_1(\tilde{\varphi})) + \mathrm{tr}(Q_1^{(2)}\hat{\rho}_1{}'(\tilde{\varphi})) - 2\mathrm{tr}(Q_2\hat{\rho}_2(\tilde{\varphi})) \\ &= \mathrm{tr}(Q_1\hat{\rho}_1(\tilde{\varphi})) - 2\mathrm{tr}(Q_2\hat{\rho}_2(\tilde{\varphi})). \end{aligned}$$

From (2.4), we have got (2.3). □

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