Kinematics of Conformal Field Theory and Diagrams in AdS Space

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Publications

This doctoral dissertation contains the results of the following works in collaboration with Professor Heng-Yu Chen and Mr. En-Jui Kuo.

[1] "Anatomy of Geodesic Witten Diagrams"

H. Y. Chen, E. J. Kuo and H. Kyono,
JHEP **1705**, 070 (2017), arXiv:1702.08818 [hep-th].

[2] "Towards Spinning Mellin Amplitudes"

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Absrtact

In this thesis, we discuss kinematic aspects of conformal field theory (CFT) and diagrams in AdS space. The four-point function in CFT is expanded in terms of conformal blocks which are eigenfunctions of the conformal Casimir equation. As an orthogonal basis for the eigenfunctions, the conformal partial wave (CPW) is introduced. Thanks to the orthogonality, the CPW gives us a systematic way to obtain the conformal block expansion of any four-point functions. This procedure is packaged as the so-called OPE inversion formula. We introduce this formula and give a simple example to see how the method works. This formula can also be applied to four-point diagrams in $(d+1)$-dimensional AdS space. Then CPW is lifted to AdS space and can be interpreted as a bulk diagram. There, the AdS harmonic function plays an essential role, and we can see that the orthogonality of CPW comes from properties of the harmonic function. This application to AdS diagrams is useful not only to investigate the AdS/CFT correspondence, but also as a technique to calculate CFT four-point functions. We also discuss the so-called geodesic diagram in AdS space which is proposed as the bulk dual of a conformal block. In the geodesic diagrams, the interaction points are integrated over geodesics connecting two boundary points. Through the split representation of the propagator, we will see why this diagram corresponds to a conformal block itself. The external fields in correlation functions or AdS diagrams can be easily generalized to tensor fields by using differential operators, and we discuss the relation between bulk interactions and CFT tensor structures. As another interesting concept related to CFT correlation functions, we also discuss the Mellin representation. From the Mellin representation of CPW, we derive the expansion form of the d-dimensional conformal block. We also propose an extension of the Mellin representation of CPW involving external tensor fields. Finally, the crossing kernel which is the inner product between CPWs in different channels is discussed. Through the crossing kernel, for example, t-channel exchange diagrams and conformal blocks can be decomposed into s-channel CPWs. We discuss the actual calculation of the crossing kernel and possible applications to bootstrap approaches.

Notation

Here, the notation is summarized we use in the main text. For the space time dimension d, h is defined as half of d:

$$h = \frac{d}{2}$$

Combinations of scaling dimension:

$$\Delta_{ij} = \Delta_i - \Delta_j\,, \quad \Delta_{ij}^+ = \Delta_i + \Delta_j\,, \quad a = -\frac{\Delta_{12}}{2}\,, \quad b = \frac{\Delta_{34}}{2}$$

Coordinates in the embedding space:

$$(X^+, X^-, X^\mu) = \frac{1}{z}(1, x^2 + z^2, x^\mu)\,, \quad (P^+, P^-, P^\mu) = (1, x^2, x^\mu)$$

Inner products of coordinates in $\mathbb{R}^d$

$$P_{ij} = -2P_i \cdot P_j = x_{ij}^2\,, \qquad \text{where} \;\; x_{ij}^\mu = (x_i - x_j)^\mu$$

Cross ratios:

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

The one-dimensional conformal block:

$$k_\Delta(z) = z^\Delta {}_2F_1(\Delta + a, \Delta + b; 2\Delta; z)$$

The kinematic factor for you-point function:

$$\mathcal{F}(x_i) = \frac{1}{(x_{12}^2)^{\frac{1}{2}\Delta_{12}^+}(x_{34}^2)^{\frac{1}{2}\Delta_{34}^+}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b$$

Conformal blocks in two and four dimensions:

$$g^{(2d)}_{\Delta,l}(z,\bar{z}) = \frac{1}{1+\delta_{l,0}} \left[k_{\frac{\Delta+l}{2}}(z) k_{\frac{\Delta-l}{2}}(\bar{z}) + k_{\frac{\Delta-l}{2}}(z) k_{\frac{\Delta+l}{2}}(\bar{z}) \right] ,$$
$$g^{(4d)}_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}} \left[k_{\frac{\Delta+l}{2}}(z) k_{\frac{\Delta-l-2}{2}}(\bar{z}) - k_{\frac{\Delta-l-2}{2}}(z) k_{\frac{\Delta+l}{2}}(\bar{z}) \right]$$

The box tensor structures for three-point function:

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ J_1 & J_2 & J_3 \\ n_{23} & n_{13} & n_{12} \end{bmatrix} = \frac{V_{1,23}^{m_1} V_{2,31}^{m_2} V_{3,12}^{m_3} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{(P_{12})^{\frac{1}{2}(\tau_1+\tau_2-\tau_3)} (P_{13})^{\frac{1}{2}(\tau_1+\tau_3-\tau_2)} (P_{23})^{\frac{1}{2}(\tau_2+\tau_3-\tau_1)}}$$

The elements of tensor structure:

$$\begin{aligned} H_{ij} &= -2\left[(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(Z_j \cdot P_i)\right] = -\mathrm{Tr}(C_i \cdot C_j), \\ V_{i,jk} &= \frac{(P_j \cdot Z_i)(P_i \cdot P_k) - (P_j \cdot P_i)(Z_i \cdot P_k)}{(P_j \cdot P_k)} = \frac{(P_j \cdot C_i \cdot P_k)}{(P_j \cdot P_k)}, \\ C_i^{AB} &= Z_i^A P_i^B - Z_i^B P_i^A \quad i,j,k = 1,2,3 \end{aligned}$$

Mellin variables for four-point functions or diagrams:

$$t = -\delta_{23}\,, \qquad s = -a + \delta_{23} + \delta_{24}$$

The remaining variables are determined as

$$\delta_{12} = \frac{1}{2}\Delta_{12}^{+} - s\,, \quad \delta_{34} = \frac{1}{2}\Delta_{34}^{+} - s\,, \quad \delta_{13} = b + s + t\,, \quad \delta_{14} = -a - b - t\,.$$

The short-hand notation for a product of gamma functions:

$$\Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b)$$

Contents

Chapter 1

Introduction

Motivation to Conformal Field Theory

In theoretical physics, quantum field theory (QFT) is one of the most successful concepts which appears in various fields, elementary particle, condensed matter, statistical physics, etc. The behavior of QFT at a long or short distance is understood by a renormalization group flow, and at the fixed points of the flow, there are scaling invariant theories. It is known that in such theory, combining the Poincaré symmetry, the scaling symmetry is enhanced to the conformal symmetry which is described by $SO(2,d)$ group in d-dimension [3–5]. Such a theory is called conformal field theory (CFT), and we often encounter CFT in theoretical physics, world sheet of string theory, critical phenomena, etc. According to the enhanced symmetry, the dynamics of theories is restricted, and the so-called bootstrap approach is an attempt to classify the possible physical theories only by the constraints coming from the symmetry and some physical assumptions. Knowing that what kind of theories appear in the renormalization group flow is important and leads to understanding properties of QFT. In the below, we introduce some recent developments and topics relating CFT.

Conformal Block in Even Dimensions

As mentioned above, in CFT, the conformal symmetry gives some constraint on the dynamics. For example, correlation functions in CFT are strongly restricted, and two- and three-point functions are completely determined by the symmetry up to overall constants. The operator product expansion is also highly restricted, and it is turned out that CFT is characterized completely by the so-called CFT data which consists of three-point coefficients and the operator spectrum. The strategy of the conformal bootstrap is classifying the possible CFT data by using some physical assumptions, crossing symmetry, unitarity, etc., In two dimensional case, the conformal symmetry is enhanced further into the Virasoro symmetry, and thanks to this infinite symmetry, this strategy works well. As for theories including a finite number of operators, the possible CFT data are determined, which are known as minimal models [6]. In these decades, there are some developments also in d-dimensional CFT. Recently in [7, 8], it is found that conformal blocks which are building blocks for four-point function in CFT are characterized by a differential equation which is the so-called conformal Casimir equation, and especially in even dimensions, the closed forms of conformal block are written in terms of hypergeometric functions. In two or four dimensions, they have simple forms and this fact greatly contributes to the development of the numerical bootstrap approach established recently [9, 10].

Diagrams in AdS Space

The explicit form of conformal blocks in even dimension is also applied to investigate the AdS/CFT duality which is a conjecture claiming that a theory including gravity in the AdS_{d+1} space is dual to a CFT in the boundary of AdS, roughly speaking [11]. According to this conjecture, in a specific parameter region, correlation functions in CFT are computed as summations of diagrams in the dual AdS space [12, 13]. This conjecture has been under tests, especially taking the most famous example, the correspondence between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four-dimensional Minkowski space and the type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, some correlation functions in SYM are reproduced by diagrams of type IIB supergravity in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [14–16]. On the other hand, in [17], the correspondence of correlation function and AdS diagrams is investigated from a kinematic aspect. They have searched what kind of conformal block expansion is obtained from a particular diagram in AdS. In this analysis, the explicit form of conformal blocks also plays an important role, and it is recently generalized to other diagrams [18, 19]. Another interesting topic relating to AdS diagrams is the so-called geodesic diagram. As mentioned above, CFT correlation functions are dual to diagrams in AdS, however, the dual of a conformal block was not known. In [20], they proposed that a geodesic diagram in AdS whose bulk interaction is restricted on geodesic is dual to a CFT conformal block. This statement has been generalized to some cases including fields with spin [2, 21–25] .

Mellin Space

The correlation functions and diagram in AdS are usually represented as complicated functions, and sometimes this fact makes it harder to analyze. In [26–28], it is pointed out that in Mellin space, they can be described in relatively simple forms. Moreover, in the Mellin space, we can see an analogy with QFT amplitudes in flat space. As an application of the Mellin representation, recently, in [29, 30], they have revisited a bootstrap method proposed by Polyakov [31]. In this approach, four-point functions are decomposed into a summation of building blocks in a crossing symmetrical way. From the unitarity of theory, in [31], building blocks are determined, and these are mathematically equivalent tree exchange diagrams in AdS space. This decomposition naturally includes some redundant contributions and we have to impose some conditions on CFT data so that these contributions vanish. In the Mellin space, because the building blocks have relatively simple form, we can partially solve the constraints. This method has been investigated further in [32–34].

Conformal Partial Wave and Inversion Formula

Another important concept recently developed is the conformal partial wave (CPW) which is also an eigenfunction of the conformal Casimir equation. CPW is a linear combination of conformal blocks, and it has been turned out that CPW is an orthogonal function [35–37]. Through the orthogonality and relation to conformal blocks, we can write down a formula which gives the three-point coefficient from a given four-point function directly. It is called the OPE inversion formula. This formula gives us a systematic way to obtain the conformal block expansion from four-point functions, however, the actual calculation is difficult usually to obtain meaningful results. Recently, as a calculation technique, it is pointed out that during the computation, once we perform the wick rotation from the Euclidean space to the Lorentzian spacetime, the inner products with CPW and four-point function can be reduced to a simple form, and there the calculation is relatively easy [36, 37]. One of the most important applications of this formula is to derive the crossing kernel which is the inner product of s- and t-channel CPWs. The crossing kernel tells us what kind of contributions come from other channel CPWs or conformal blocks, and this question becomes important in the conformal bootstrap approach. In [38], in two and four dimensions, the crossing kernels are derived through the technique introduced above.

In this article, we discuss the kinematic aspects of d-dimensional CFT and diagrams in AdS_{d+1} space. In chapter 2, we give a brief review for d-dimensional conformal field theory and introduce some recent topics including the conformal Casimir equation and CPW. Firstly we introduce the embedding formalism where the coordinates of $\mathbb{R}^d$ are embedded in the $d+2$-dimensional Minkowski space. In the embedding space, the conformal symmetry can be regarded as the Lorentz symmetry, and when we consider correlation functions including spinning fields, the expressions are drastically simplified in the embedding space. The inversion formula is also discussed and then we will see how the inversion formula works using a simple theory, the so-called generalized free theory. In chapter 3, we consider diagrams in AdS space. In AdS space, CPW can be described as a bulk diagram using the AdS harmonic function. According to this fact, we are able to compute inner product with CPW and an AdS diagram as a bubble diagram in AdS space. We will give some example of this computation, and then we can see that the orthogonality of CPW is followed by the property of the AdS harmonic function. This part is based on the recent discussion with my collaborator [39]. In this chapter, we also discuss geodesic diagrams and its generalization including fields with spin, which is related to my work [1]. In chapter 4, Mellin representations of CPW and some diagrams are discussed. From the Mellin representation of CPW, we drive a closed form of conformal block in d-dimension. Its extensions including external spins are also concerned which is based on [2]. In chapter 5, we try to calculate the crossing kernel in d-dimension, and we will see that it can be also regarded as a bubble diagram in AdS. In chapter 6, we give a summary and discuss future directions.

Chapter 2

Conformal Field Theory in d-dimension

In this chapter, we will review generic ingredients of d-dimensional conformal field theory(CFT). In physics, we may encounter CFT in many places. It is known that CFTs appear at fixed points under the renormalization group for QFT. These fixed points are described by scaling invariant theories, and it is believed that the scaling symmetry is enhanced to the conformal symmetry [3,4]. Because of the larger symmetry, the kinematics of theories is strongly restricted. Knowing what kind of theories there are leads understanding of QFTs through the view of renormalization flow.

There are some attempts using the large symmetry to determine the dynamics of theories, which is called the bootstrap approach. In two-dimension, the conformal symmetry is enhanced as the Virasoro symmetry, and the bootstrap approach has been succeeded [6]. On the other hand, in higher dimensions, recently the analytical expression for the conformal blocks is discovered [7]. Using the expression, numerical bootstrap approaches are developed [9]. Besides numerical ones, some techniques like the so-called inversion formula [36, 37] are also developed recently. In the below, we will see such analytical methods to investigate d-dimensional CFTs.

In section 2.1, we review some basic concepts in d-dimensional CFT. In section 2.2, the embedding formalism is introduced, and in section 2.3, some correlation functions with tensor fields are discussed by using the formalism. In section 2.4, an important concept, conformal block is introduced. Section 2.5 is devoted for the review of the conformal Casimir equation which has conformal blocks as its eigenfunctions. In section 2.6, a key ingredient called the conformal partial wave which is an orthogonal basis in the space of four-point function is introduced. In section 2.7, as a simple example of how to use the orthogonal basis, we will consider the generalized free theory whose correlation functions are given as products of two-point function.

2.1 Conformal Symmetry and Correlation Function

In this section, we will review the basic aspects of the d-dimensional Euclidean conformal field theory mainly. For the reference of this section, there are some nice textbooks and review articles [40–44].

2.1.1 Conformal Symmetry

The conformal transformation is defined as the transformation which does not change the metric up to a position dependent scaling factor. Under a conformal transformation: $x \to x'$, the metric $g_{\mu\nu}(x)$ changes in he following manner

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x)\,, \tag{2.1.1}$$

where $\Omega(x)$ is a position dependent function called the scaling factor and the index μ runs from 1 to d. When $\Omega(x)$ equals to 1, such transformations are noting but the Lorentz transformation (or rotation in the Euclidean space) and translation. In this sense, the conformal transformation includes the transformations in the Poincaré group. To see the detail of transformation, let us consider an infinitesimal transformation: $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)\,.$ Under this transformation, we suppose that the scaling factor also has the expansion: $\Omega^2(x) = 1 - f(x) + \mathcal{O}(\epsilon^2)\,.$ Under a diffeomorphism, the metric is transformed as a tensor: $g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)\,.$ From (2.1.1), the constraint for $\epsilon^\mu(x)$ is obtained:

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = f(x)\delta_{\mu\nu}\,, \tag{2.1.2}$$

where we have set the metric $g_{\mu\nu}$ as the Euclidean metric $\delta_{\mu\nu}\,.$ [1] This condition is called the conformal Killing equation. As we expect, the rotation and translation are solutions of this equation:

$$\begin{aligned} \text{translation}: \quad & \epsilon_\mu(x) = a_\mu \qquad && (a_\mu = \text{const.}) \\ \text{rotation}: \quad & \epsilon_\mu(x) = m_{\mu\nu}\, x^\nu \qquad && (m_{\mu\nu} = -m_{\nu\mu} = \text{const.}) \end{aligned} \tag{2.1.3}$$

Aside from the Poincaré group, it contains the dilatation and special conformal transformation(SCT)[2]:

$$\begin{aligned} \text{dilatation}: \quad & \epsilon_\mu(x) = c\, x_\mu \qquad && (c = \text{const.}) \\ \text{SCT}: \quad & \epsilon_\mu(x) = 2(x\cdot b)x_\mu - x^2\, b_\mu \qquad && (b_\mu = \text{const.}) \end{aligned} \tag{2.1.4}$$

[1] In the Minkowski spacetime, the condition is obtained by replacing $\delta_{\mu\nu}$ with $\eta_{\mu\nu} = \mathrm{diag}(-,+,...,+)\,.$

[2] When $d = 2$, there are infinite independent transformations and the symmetry is enhanced to the Virasoro symmetry.

For the each transformation, we can find the corresponding generator as written below:

$$\begin{aligned}
\text{translation}: \quad P_\mu &= \partial_\mu \\
\text{rotation}: \quad M_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu \, x^\nu \\
\text{dilatation}: \quad D &= x_\mu \partial_\mu \\
\text{SCT}: \quad K_\mu &= 2x_\mu (x \cdot \partial) - x^2 \, \partial_\mu
\end{aligned} \tag{2.1.5}$$

These generators satisfies the following algebra called the conformal algebra:

$$\begin{aligned}
[K_\mu, P_\nu] &= 2(\delta_{\mu\nu} \, D - M_{\mu\nu}) \,, \quad [D, P_\mu] = P_\mu \,, \quad [D, K_\mu] = -K_\mu \,, \\
[P_\mu, M_{\nu\rho}] &= \delta_{\mu\nu} \, P_\rho - \delta_{\mu\rho} \, P_\nu \,, \quad [K_\mu, M_{\nu\rho}] = \delta_{\mu\nu} \, K_\rho - \delta_{\mu\rho} \, K_\nu \,, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \delta_{\mu\sigma} \, M_{\nu\rho} + \delta_{\nu\rho} \, M_{\mu\sigma} - \delta_{\mu\rho} \, M_{\nu\sigma} - \delta_{\nu\sigma} \, M_{\mu\rho} \,.
\end{aligned} \tag{2.1.6}$$

This algebra is isomorphic to the algebra of $SO(1,d+1)$.[3] To see this correspondence, we introduce L_{AB} $(A, B = -1, 0, 1, ..., d)$ as the following combination:

$$\begin{aligned}
&L_{-1\ \mu} = \frac{1}{2}\left(P_\mu - K_\mu\right) \,, \qquad L_{0\ \mu} = -\frac{1}{2}\left(P_\mu + K_\mu\right) \,, \\
&L_{\mu\nu} = M_{\mu\nu} \,, \qquad L_{-1\, 0} = D \,.
\end{aligned} \tag{2.1.7}$$

Then L is anti-symmetric and satisfies the commutation relation of $SO(1,d+1)$:

$$[L_{AB}, L_{CD}] = \eta_{AD} \, L_{BC} + \eta_{BC} \, L_{AD} - \eta_{AC} \, L_{BD} - \eta_{BD} \, L_{AC} \,, \tag{2.1.8}$$

where η_{AB} is a diagonal matrix whose elements are $\eta_{AB} = \text{diag}(-, +, ..., +)$.

So far we have seen the infinitesimal conformal transformations, and here we will exponentiate these infinitesimal transformations. When $\epsilon_\mu(x)$ satisfies the conformal Killing equation, the derivatives of the new coordinate is given by:

$$\begin{aligned}
\frac{\partial x'^\mu}{\partial x^\nu} &= \delta^\mu_\nu + \partial_\nu \epsilon^\mu \\
&= \left(1 + \frac{\partial \cdot \epsilon}{d}\right)\left(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu x^\mu - \partial^\mu x_\nu)\right) \,.
\end{aligned} \tag{2.1.9}$$

In the last line, the first factor is the infinitesimal form of the scaling factor $\Omega(x) = 1 - \frac{1}{2} f(x)$ [4], and the second factor is the infinitesimal form of rotation. Thus these factors are easily exponentiated,

[3] In the Lorenzian signiture, the conformal group is $SO(2,d)$

[4] Here we have used the fact that $f(x) = \frac{2}{d}\,(\partial \cdot \epsilon)$. This relation is derived by taking the trace of (2.1.2).

and we can rewrite it as a finite tranformation:

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x)^{-1} R^{\mu}{}_{\nu}(x)\,. \tag{2.1.10}$$

Here $R^{\mu}{}_{\nu}(x)$ is a position dependent rotation, and this equation is consistent with (2.1.1). By taking the determinant in the both side, the scaling factor is represented as the derivatives:

$$\Omega(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{1}{d}}\,. \tag{2.1.11}$$

In the generators in (2.1.5), the finite transformations of translation, rotation and dilatation can be easily derived. In order to obtain the exponentiation for SCT, it is convenient to introduce the inversion transformation I:

$$I : x^{\mu} \to \frac{x^{\mu}}{x^2} \tag{2.1.12}$$

The inversion is included in the conformal transformation, but it is not connected to the identity transformation continuously. Through the inversion, SCT can be regarded as $K_{\mu} = -I \cdot P_{\mu} \cdot I$. K_{μ} contains the inversion twice, and it is connected to the identity continuously. From this interpretation of K_{μ}, we can derive the finite form of SCT:

$$x^{\mu} \to x'^{\mu} = \frac{x^{\mu} - b^{\mu}\, x^2}{1 - 2(b \cdot x) + b^2\, x^2}\,. \tag{2.1.13}$$

2.1.2 Primary Operator

Here let us consider the action of the generators on operators. We denote an operator at position x in d-dimensional Euclidean or Minkowski space as $\mathcal{O}(x)$. Firstly, we focus on only an operator at the origin $\mathcal{O}(0)$. Using the action of the translation generator, we can see that operators at an arbitrary position x can be created by applying the translation generator to $\mathcal{O}(0)$.

$$\mathcal{O}_{\Delta,r}(x) = e^{x^{\mu} P_{\mu}}\, O_{\Delta,r}(0)\, e^{-x^{\mu} P_{\mu}}\,. \tag{2.1.14}$$

where the translation generator P_{μ} acts on an operator in the following way:

$$[P_{\mu}, \mathcal{O}_{\Delta,r}(x)] = \partial_{\mu}\mathcal{O}_{\Delta,r}(x)\,. \tag{2.1.15}$$

Since the dilatation D commutes with the rotation generators $M_{\mu\nu}$, it is natural to diagonalize $\mathcal{O}(0)$ as the eigenstate for the dilatation and rotation, and we will denote $\mathcal{O}(0)$ as $\mathcal{O}_{\Delta,r}(0)$. Here Δ

is the eigenvalue for the dilatation and r means an irreducible representation of the rotation group $SO(d)$. More explicitly, $\mathcal{O}_{\Delta,r}(0)$ obeys the following commutation relation:

$$\begin{aligned} [D, \mathcal{O}_{\Delta,r}(0)] &= \Delta\, \mathcal{O}_{\Delta,r}(0)\,, \\ \left[M_{\mu\nu}, \mathcal{O}^I_{\Delta,r}(0)\right] &= (\mathcal{R}_{\mu\nu})^I{}_J \mathcal{O}^J_{\Delta,r}(0)\,, \end{aligned} \tag{2.1.16}$$

where $\mathcal{R}_{\mu\nu}$ are generators of the representation r, and $I, J, \ldots$ are abstract indexes for the representation r. Now because of the conformal algebra, we can regard K_μ as a lowering operator for dimension,

$$\begin{aligned} [D, K_\mu \mathcal{O}_{\Delta,r}(0)] &= [D, K_\mu]\mathcal{O}_{\Delta,r}(0) + K_\mu[D, \mathcal{O}_{\Delta,r}(0)] \\ &= (\Delta - 1)K_\mu \mathcal{O}_{\Delta,r}(0)\,. \end{aligned} \tag{2.1.17}$$

Similarly, the translation is a raising operator as we can see as follows:

$$[D, P_\mu \mathcal{O}_{\Delta,r}(0)] = (\Delta + 1)P_\mu \mathcal{O}_{\Delta,r}(0)\,. \tag{2.1.18}$$

We can create any fields with arbitrary low dimension applying K_μ many times. However, in physics, we are interested in fields with real and bounded dimensions. Thus, main targets in below are operators which have the lowest states satisfying the vanishing commutation relation with K_μ:

$$[K_\mu, \mathcal{O}_{\Delta,r}(0)] = 0\,. \tag{2.1.19}$$

This kind of operators is called primary operators. Once we know a primary operator, we can create fields with arbitrary large dimensions applying the raising (translation) operator. Such operators are derivatives of a primary operator and called its descendants.

Using the Campbell-Baker-Hausdorff formula;

$$e^{-A}\, B\, e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \ldots\,, \tag{2.1.20}$$

the action of the other generators on an operator $\mathcal{O}_{\Delta,r}(x)$ can be obtained. For example, the action of rotation on $\mathcal{O}_{\Delta,r}(x)$ is computed as follows:

$$\begin{aligned} [M_{\mu\nu}, \mathcal{O}_{\Delta,r}(x)] &= e^{x\cdot P}[e^{-x\cdot P} M_{\mu\nu} e^{x\cdot P}, \mathcal{O}_{\Delta,r}(0)]e^{-x\cdot P} \\ &= e^{x\cdot P}[M_{\mu\nu} + x_\nu P_\mu - x_\mu P_\nu, \mathcal{O}_{\Delta,r}(0)]e^{-x\cdot P} \\ &= (x_\nu \partial_\mu - x_\mu \partial_\nu + \mathcal{R}_{\mu\nu})\, \mathcal{O}_{\Delta,r}(x)\,. \end{aligned} \tag{2.1.21}$$

where $\mathcal{O}_{\Delta,r}(0)$ is a primary operator. From the similar calculation, we can obtain the following transformation rules:

$$
\begin{aligned}
[D, \mathcal{O}_{\Delta,r}(x)] &= (\Delta + x^\mu \partial_\mu)\, \mathcal{O}_{\Delta,r}(x)\,, \\
[K_\mu, \mathcal{O}_{\Delta,r}(x)] &= \left(2\Delta x_\mu - 2x^\nu \mathcal{R}_{\mu\nu} + 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) \mathcal{O}_{\Delta,r}(x)\,.
\end{aligned}
\tag{2.1.22}
$$

We can summarize these commutation relation as for the generic generator Q_ϵ:

$$
Q_\epsilon = a^\mu P_\mu + \frac{1}{2} m^{\mu\nu} M_{\mu\nu} + c\, D + b^\mu\, K_\mu\,, \tag{2.1.23}
$$

where a^μ , $m^{\mu\nu}$, c and b^μ are infinitesimal parameters whose orders are $\mathcal{O}(\epsilon)$. According to (2.1.15), (2.1.21) and (2.1.22), Q_ϵ satisfies the following commutation relation:

$$
[Q_\epsilon, \mathcal{O}_{\Delta,r}(x)] = \left(\epsilon \cdot \partial + \frac{\Delta}{d}(\partial \cdot \epsilon) - \frac{1}{2}(\partial^\mu \epsilon^\nu)\mathcal{R}_{\mu\nu}\right) \mathcal{O}_{\Delta,r}(x)\,, \tag{2.1.24}
$$

where $\epsilon_\mu = a_\mu + m_{\mu\nu}\, x^\nu + c\, x_\mu + 2(x \cdot b)x_\mu - x^2 b_\mu$. Now we can exponentiate the generator Q_ϵ as $U = e^{Q_\epsilon}$ in the similar way as in (2.1.9) and (2.1.10);

$$
U \mathcal{O}^I_{\Delta,r}(x') U^{-1} = \Omega(x)^\Delta D(R(x))^I{}_J \mathcal{O}^J_{\Delta,r}(x)\,. \tag{2.1.25}
$$

Especially, a primary scalar transforms under the conformal transformation in the following manner;

$$
\mathcal{O}_\Delta(x) \to U \mathcal{O}_\Delta(x') U^{-1} = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \mathcal{O}_\Delta(x)\,. \tag{2.1.26}
$$

2.1.3 Correlation Function

In CFT, according to the conformal symmetry, the form of correlation function is strongly restricted. Here we will consider correlation functions with scalar fields for simplicity, and later discuss cases containing fields with spin. Firstly, let us consider a two-point function. According to the rotation and translation invariance, the two-point functions can depend on only the distance between two points:

$$
\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = f\left(|x_1 - x_2|\right)\,, \tag{2.1.27}
$$

where f is an arbitrary function. On the other side, because the vacuum is invariant under the conformal trans formation, the following VEV of commutator should vanish[5]:

$$\langle 0|\,[D, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)]\,|0\rangle = 0\,. \tag{2.1.29}$$

Using the commutation relation in (2.1.22), we can obtain the following differential equation for two-point function:

$$\left((x_1 - x_2)^\mu \partial_\mu + \Delta_1 + \Delta_2\right) \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = 0\,. \tag{2.1.30}$$

The solution under the ansatz in (2.1.27) has the following form with an overall constant:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\text{const.}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{2.1.31}$$

So far, we used translation, rotation and dilatation. Next, let us consider the full conformal transformation. According to (2.1.25) and the fact that the vacuum does not change under the conformal transformation, a two-point function transforms in the following way:

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle &= \langle U\mathcal{O}_{\Delta_1}(x_1)U^{-1}U\mathcal{O}_{\Delta_2}(x_2)U\rangle \\ &= \Omega(x_1')^{\Delta_1}\Omega(x_2')^{\Delta_2}\langle \mathcal{O}_{\Delta_1}(x_1')\mathcal{O}_{\Delta_2}(x_2')\rangle\end{aligned} \tag{2.1.32}$$

On the other hand, the distance $(x_1 - x_2)^2$ in (2.1.31) transforms as

$$(x_1 - x_2)^2 = \frac{(x_1' - x_2')^2}{\Omega(x_1')\Omega(x_2')}\,. \tag{2.1.33}$$

This means that under the general conformal transformation, the function in (2.1.31) transforms in the following way:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \Omega(x_1')^{\frac{\Delta_1+\Delta_2}{2}}\Omega(x_2')^{\frac{\Delta_1+\Delta_2}{2}}\frac{\text{const.}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} \tag{2.1.34}$$

By comparing (2.1.32) with (2.1.34), in order to get a consistent result, the conditions that $\Delta_1 = \Delta_2$ or const. $= 0$ are demanded. This implies that a two-point function has the following form:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{x_{12}^{2\Delta_1}}\,, \tag{2.1.35}$$

[5] Similarly, n-point correlation function satisfy the following equation reflecting the conformal invariance of the vacuum:

$$\langle 0|\,[L_{AB}, \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)]\,|0\rangle = 0\,, \tag{2.1.28}$$

where L_{AB} is the combination of conformal generators as in (2.1.7).

where we used the notation that $x_{ij} = |x_i - x_j|$ and the overall factor is absorbed by the definition of operators.

As for a three-point function, the functional form is also determined up to an overall constant by the conformal symmetry. By translation, rotation and dilatation, the possible from of three-point function is restricted as follows:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \sum_{a,b,c} \frac{\lambda_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c}\,, \tag{2.1.36}$$

where the summation is taken over all a, b and c which satisfy $a+b+c = \Delta_1+\Delta_2+\Delta_3$. Using the conformal transformation, we can fix a, b and c completely, and the the form of three-point function is determined up to a constant factor:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.1.37}$$

Because operators are already re-defined in order to normalize two-point functions, the overall factor in three-point function cannot be normalized, and it gives dynamical information.

When there are four points, there are two independent conformal invariant combinations:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \tag{2.1.38}$$

and these are called cross ratios. Using the conformal transformation, we can fix four points on a two-dimensional plane. For example, we can set them as $x_1 = (0,0)$, $x_2 = (x,y)$, $x_3 = (1,0)$ and $x_4 = \infty$. Here only x_2 cannot be fixed in the two-dimensional plane. This is why we have two independent scalar quantities which is invariant under the conformal transformation. In fact, u and v are written in terms of (x,y) through a complex coordinate $z \equiv x+iy$;

$$u = z\bar{z}\,, \qquad v = (1-z)(1-\bar{z})\,. \tag{2.1.39}$$

This relation is described in Figure 2.1. By the conformal symmetry, a four point function is determined as a function of cross ratios. We can check that the following form transforms properly

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{F}(x_i)\ \mathcal{G}(u,v) \tag{2.1.40}$$

where $\mathcal{G}(u,v)$ is an arbitrary function of cross ratios, and $\mathcal{F}(x_i)$ is defined as

$$\mathcal{F}(x_i) = \frac{1}{(x_{12}^2)^{\frac{1}{2}\Delta_{12}^+}(x_{34}^2)^{\frac{1}{2}\Delta_{34}^+}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b\,. \tag{2.1.41}$$

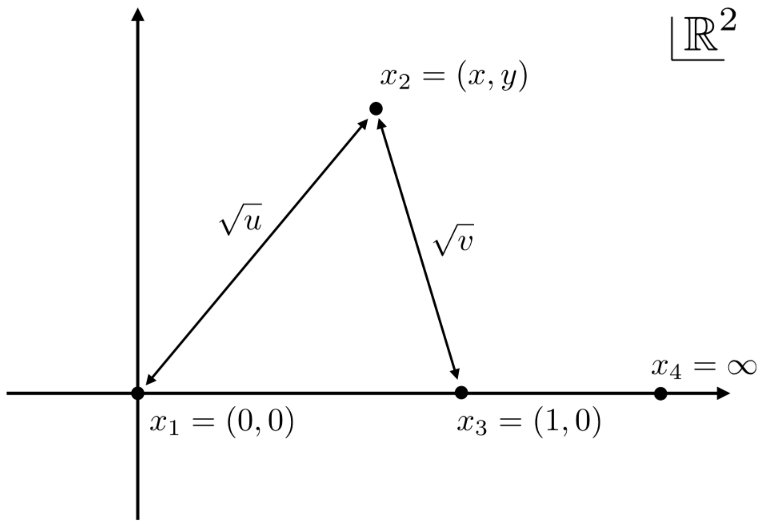


Figure 2.1: By the conformal transformation, except x_2, points are fixed in a two-dimensional plane as in the figure. The cross ratio u and v correspond to the distances between x_1 and x_2 or x_2 and x_3 .

Here we introduced short-hand notations:

$$\Delta_{ij} \equiv \Delta_i - \Delta_j \,, \qquad \Delta^+_{ij} \equiv \Delta_i + \Delta_j \,, \tag{2.1.42}$$

especially, a and b are:

$$a = -\frac{\Delta_{12}}{2} \,, \quad b = \frac{\Delta_{34}}{2} \,. \tag{2.1.43}$$

Four point functions are not fully determined through the conformal symmetry, and the function $\mathcal{G}(u, v)$ is a theory dependent function. In other words, $\mathcal{G}(u, v)$ contains dynamical information. It becomes our main target to consider below.

2.2 Embedding Formalism

In this section, we review the essential details about the embedding space formalism for encoding the tensors in the d-dimensional Euclidean space. It is introduced in [45], firstly. In this formalism, the d-dimensional Euclidean space is embedded on a light-cone in the $d+2$-dimensional Minkowski space which is called the embedding space, and the conformal group in the d-dimensional Euclidean space $SO(1, d+1)$ is interpreted as the Lorentz group in the embedding space. The essence of the embedding formalism is that the non-linear conformal transformation of the lower dimensional space corresponds to the linear Lorentz transformation which is much simpler. This formalism is especially convenient when dealing with tensors because tensor structures take simple forms by introducing polarization vectors in the embedding space. In Section 3.1, we will also introduce this formalism for Euclidean AdS_{d+1}, and there we can deal with coordinates of AdS_{d+1} and CFT_d on an equal footing. For the reference, there are some articles including review of this formalism [46–48].

2.2.1 Embedding Space

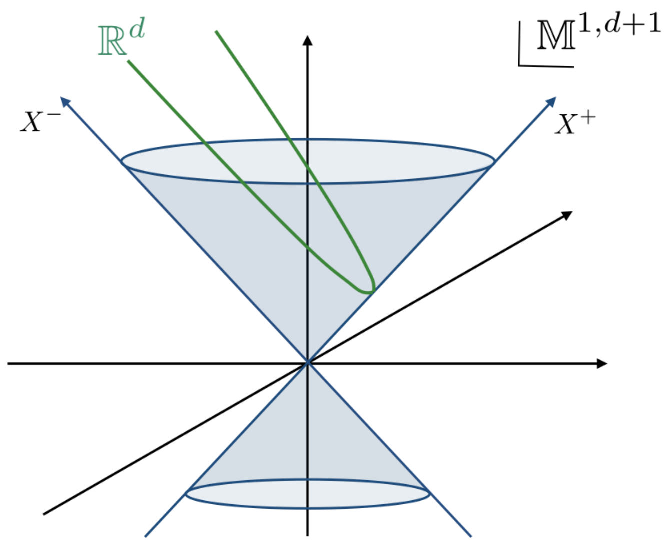


Figure 2.2: $\mathbb{R}^d$ is embedded in $d+2$-dimensional Minkowski space $\mathbb{M}^{1,d+1}$ as the green line on light cone.

Supposing $P^A = (P^+, P^-, P^\mu)$ is a point in $\mathbb{M}^{1,d+1}$ and x^μ is a point in $\mathbb{R}^d$, we consider the following map between P^A and x^μ:

$$(P^+, P^-, P^\mu) = \lambda\,(1, x^2, x^\mu) \qquad \lambda \in \mathbb{R}. \tag{2.2.1}$$

where λ is a non-zero constant. Note that in this parametrization, the square of P^μ is zero: $P \cdot P = 0$. This means that the point x^μ is mapped on the light cone in the Minkowski space. The parametrization in (2.2.1) covers the entire surface of light cone, and a section which is given for each fixed λ corresponds to $\mathbb{R}^d$. Conventionally, we consider a section with $\lambda = 1$. Under the Lorentz transformation in the embedding space, a point on light cone is mapped to another point on light cone. In order to interpret the Lorentz transformation as a map in $\mathbb{R}^d$, the new point should be rescaled. For example, under the Lorentz transformation, P^A transforms $P^A \to R^A{}_B P^B$. Then because the $+$ component is 1 in the section we consider, we rescale it as $R^A{}_B P^B/(R^A{}_B P^B)^+$. The map $P^A \to R^A{}_B P^B/(R^A{}_B P^B)^+$ moves a point on the section to another point on the same section, and this transformation is just the action of the conformal group on $\mathbb{R}^d$.

2.2.2 Scalar Field

Here we consider embedding a scalar field in $\mathbb{R}^d$ into the embedding space. Let us start with a scalar field in the embedding space which has the following homogeneous propaty:

$$\Phi(\lambda P) = \lambda^{-\Delta}\, \Phi(P)\,. \tag{2.2.2}$$

Under the Lorentz transformation in $\mathbb{M}^{1,d+1}$, $\Phi(P)$ transforms as a scalar $\Phi(P) \to \Phi(R \cdot P)$ where the matrix R_{AB} is an element of $SO(1, d+1)$. Now we define a scalar field $\phi(x)$ in $\mathbb{R}^d$ through $\Phi(P)$ by using the scaling property in (2.2.2):

$$\Phi(P) = \Phi\left(P^+ \frac{P}{P^+}\right) = (P^+)^{-\Delta}\Phi\left(\frac{P}{P^+}\right) = (P^+)^{-\Delta}\phi\left(\frac{P}{P^+}\right)\,, \tag{2.2.3}$$

and it transforms in the same way as in (2.1.26) ;

$$\phi\,(P) \to ((R \cdot P)^+)^{-\Delta}\phi\left(\frac{R \cdot P}{(R \cdot P)^+}\right)\,. \tag{2.2.4}$$

In the above expression, the transformation can be understood as the conformal transformation.

2.2.3 Symmetric Traceless Tensor Field

Next, we consider embedding physical tensor fields $\mathbb{R}^d$ into embedding space $\mathbb{M}^{1,d+1}$. Explicitly, given an arbitrary rank-J tensor field in $\mathbb{R}^d$, it is related to its embedding space counterpart through the pull-back operation:

$$\mathcal{F}^{(\mathbb{R})}_{\mu_1 \dots \mu_J}(y) = \frac{\partial P^{A_1}}{\partial x^{\mu_1}} \dots \frac{\partial P^{A_J}}{\partial x^{\mu_J}} F_{A_1 \dots A_J}(P). \tag{2.2.5}$$

In particular, the $\mathbb{R}^d$ Cartesian metric is given by:

$$\delta^{(\mathbb{R})}_{\mu\nu} = \frac{\partial P^A}{\partial x^\mu}\frac{\partial P^B}{\partial x^\nu}\eta_{AB}. \tag{2.2.6}$$

However the pull-back operations defined in (2.2.5) are surjective but not injective, in other words given a physical tensor in $\mathbb{R}^d$, they do not have a unique representative in the embedding space $\mathbb{M}^{d+1,1}$, but rather the embedding introduces redundant unphysical degrees of freedom. We can see this from the orthogonal conditions:

$$P_A \frac{\partial P^A}{\partial x^\mu}\Big|_{P \cdot P = 0} = 0, \tag{2.2.7}$$

we can see that any tensor components proportional to $P_{(A_1}H'_{A_2\dots A_J)}(P)$ contained in $F_{A_1\dots A_J}(P)$ vanish under the pull-back operation in (2.2.5), hence unphysical. Geometrically we can regard these additional components as being normal to the hypersurface (2.2.1). We can thus eliminate these unphysical redundant degrees of freedom in the embedding space tensors by further imposing the transverse condition:

$$P^{A_1}F_{A_1\dots A_J}(P)\,|_{P\cdot P=0}=0 \tag{2.2.8}$$

such that $F_{A_1\dots A_J}(P)$ only contains the components which are tangent to $\mathbb{R}^d$. This is the embedding representative of the $\mathbb{R}^d$ tensor field.

Moreover, we would like to consider symmetric traceless $\mathbb{R}^d$ tensor fields. To construct their representatives in embedding space $\mathbb{M}^{d+1,1}$ they need to be symmetric traceless also transverse (STT) from the discussion above, let us first introduce the following generating polynomial:

$$F(P,Z)=Z^{A_1}\dots Z^{A_J}F_{A_1\dots.A_J}(P), \qquad P\cdot Z=Z\cdot Z=0. \tag{2.2.9}$$

Here we have introduced the an auxiliary vector Z^A and the conditions $Z\cdot Z=0$ and $P\cdot Z=0$ imply $F_{A_1\dots A_J}(P)$ is defined up to equivalence $\sim P_{(A_1}H_{A_2\dots A_J)}(P)+\eta_{(A_1A_2}S_{A_3\dots A_J)}(P)$, the contraction with Z^As only picks up the symmetric, traceless and transverse components. It is worth noting that under the rescaling $F_{A_1\dots A_J}(\lambda P)=\lambda^{-\Delta}F_{A_1\dots A_J}(P),\ \lambda>0$, it is a homogenous polynomial of degree $-\Delta$.

To recover embedding space STT tensors representing symmetric traceless $\mathbb{R}^d$ tensors directly from (2.2.9), it is convenient to define the operator D_A which acts on the symmetric products Z^A as:

$$\frac{1}{J!\left(\frac{d-2}{2}\right)_J}D_{A_1}\dots D_{A_J}Z^{B_1}\dots Z^{B_J}=\Pi_{\mu_1\dots\mu_J}{}^{\nu_1\dots\nu_J}\frac{\partial P_{A_1}}{\partial y_{\mu_1}}\dots\frac{\partial P_{A_J}}{\partial y_{\mu_J}}\frac{\partial P^{B_1}}{\partial y^{\nu_1}}\dots\frac{\partial P^{B_J}}{\partial y^{\nu_J}} \tag{2.2.10}$$

where $\Pi_{a_1\dots a_J}{}^{b_1\dots b_J}$ is the following symmetric traceless tensor and the matrix $\frac{\partial P^A}{\partial y^\nu}$ is given as:

$$\Pi_{\mu_1\dots\mu_J}{}^{\nu_1\dots\nu_J}=\delta^{\nu_1}_{(\mu_1}\dots\delta^{\nu_J}_{\mu_J)}-\text{traces}, \qquad \frac{\partial P^A}{\partial y^\nu}=(0,2x_\nu,\delta^\mu_\nu). \tag{2.2.11}$$

In other words we obtain the manifestly symmetric, traceless and transverse tensorial projectors, and the resultant embedding space tensor

$$F_{\{A_1\dots A_J\}}(P)=\Pi_{\mu_1\dots\mu_J}{}^{\nu_1\dots\nu_J}\frac{\partial P_{A_1}}{\partial y_{\mu_1}}\dots\frac{\partial P_{A_J}}{\partial y_{\mu_J}}\frac{\partial P^{B_1}}{\partial y^{\nu_1}}\dots\frac{\partial P^{B_J}}{\partial y^{\nu_J}}F_{B_1\dots B_J}(P) \tag{2.2.12}$$

is the desired STT representative of $\mathbb{R}^d$ tensor in the embedding space $\mathbb{M}^{d+1,1}$. For completeness, the explicit expression for the operator D_A can be given in terms of the following differential operator:

$$D_A = \left(\frac{d-2}{2} + Z \cdot \frac{\partial}{\partial Z}\right) \frac{\partial}{\partial Z^A} - \frac{1}{2} Z_A \frac{\partial^2}{\partial Z \cdot \partial Z}. \tag{2.2.13}$$

2.3 Correlation Function with Spins

Here we will discuss correlation functions again using the embedding formalism. This formalism is a powerful tool to consider correlation function with tensor fields, where the tensor structures become simple forms. This part is based on the previous works [47, 49] .

Correlation Function in Embedding Space

Firstly, let us consider a two-point function again using the embedding formalism, and suppose that $\mathcal{O}_\Delta(P)$ is a scalar field in the embedding space with dimension Δ. According to the Lorentz invariance in the embedding space and the scaling property in (2.2.2), the possible functional form is fixed as

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = \frac{1}{P_{12}^{\Delta}} = \frac{1}{(x_{12}^2)^{\Delta}}\,, \tag{2.3.1}$$

where P_{12} is defined as $P_{12} \equiv -2P_1 \cdot P_2 = x_{12}^2$. The rightmost expression is the same as the conformal covariant two-point function in (2.1.35) . Similarly, the three-point function with scalar fields is written as below:

$$\langle \mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{\Delta_3}(P_2)\rangle = \frac{\lambda_{123}}{P_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{23}^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)} P_{31}^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}}\,. \tag{2.3.2}$$

In this way, using the Lorentz transformation and scaling in the embedding space, we can find the correct kinematical from of correlation functions in CFT.

Two-Point Function with Spin

Next, we consider a two-point function with vector fields $\mathcal{O}_A(P)$ in the embedding space. According to the Lorentz covariance and the scaling property of the vector fields, the possible form of correlation function is fixed again as below:

$$\langle \mathcal{O}_A(P_1)\mathcal{O}_B(P_2)\rangle = \frac{\eta_{AB} + c\,\frac{P_{2,A}P_{1,B}}{P_{12}}}{P_{12}^{\Delta}}\,, \tag{2.3.3}$$

where c is an undetermined constant Here the terms which contain $P_{1,A}$ or $P_{2,B}$ are discarded in advance because of the transverse condition, and the ambiguity from the overall constant is already fixed by rescaling of fields. Imposing the transverse conditions for $\mathcal{O}_A(P_1)$ and $\mathcal{O}_A(P_2)$, we can determine the coefficient c as $c=-1$. Eventually, a two-point function with two vector fields in the embedding space has the following form:

$$\langle \mathcal{O}_A(P_1)\mathcal{O}_B(P_2)\rangle = \frac{\eta_{AB}-\frac{P_{2,A}P_{1,B}}{P_{12}}}{P_{12}^{\Delta}}\,. \tag{2.3.4}$$

In the original physical space $\mathbb{R}^d$, through the pull-bulk relation, we obtain the well-known result:

$$\langle \mathcal{O}_\mu(x_1)\mathcal{O}_\nu(x_2)\rangle = \frac{I_{\mu\nu}(x_1-x_2)}{x_{12}^{2\Delta}}\,, \tag{2.3.5}$$

where the tensor in the numerator $I_{\mu\nu}(x_1-x_2)$ is defined as:

$$I_{\mu\nu}(x)=\delta_{\mu\nu}-\frac{2x^\mu x^\nu}{x^2}\,. \tag{2.3.6}$$

It is worthy noting that $I_{\mu\nu}(x)$ satisfies the following relation:

$$I_{\mu\nu}(x)I_{\nu\rho}(x)=\delta_{\mu\rho}\,,\quad I_{\mu\nu}(x)=\frac{\partial x'_\mu}{\partial x_\nu}\,,\quad I_{\mu\nu}(x'_{12})=I_{\mu\rho}(x_1)I_{\rho\sigma}(x_{12})I_{\sigma\nu}(x_2)\,, \tag{2.3.7}$$

where x' is the inversion transformed coordinate $x'^\mu=x^\mu/x^2$. Under the conformal transformation, a vector field transforms as

$$\mathcal{O}_\mu(x)\to\ \mathcal{O}'_\mu(x')=\Omega(x)^{-\Delta}\,\frac{\partial x'^\nu}{\partial x^\mu}\,\mathcal{O}_\nu(x)\,. \tag{2.3.8}$$

Using this fact, we can check that the right hand side of (2.3.5) is transformed in the correct way[6]. Furthermore, by using the polarization vector Z_A, the two-point function can be represented as for $\mathcal{O}(P,Z)$:

$$\langle \mathcal{O}^{\text{vec.}}_\Delta(P_1,Z_1)\mathcal{O}^{\text{vec.}}_\Delta(P_2,Z_2)\rangle = \frac{H_{12}}{P_{12}^{\Delta+1}}\,, \tag{2.3.9}$$

where $\mathcal{O}^{\text{vec.}}_\Delta(P,Z)$ is contracted with Z_μ: $\mathcal{O}^{\text{vec.}}_\Delta(P_1,Z_1)=Z^A\mathcal{O}^{\text{vec.}}_{\Delta,A}(P_1)$, and we defined H_{12} as the following combination in P_i and Z_i

$$\begin{aligned} H_{12} &= \mathrm{Tr}\,[C_1\cdot C_2]=2\,[(Z_1\cdot P_2)(Z_2\cdot P_1)-(Z_1\cdot Z_2)(P_1\cdot P_2)]\,,\\ C_i^{AB} &= Z_i^A P_i^B - Z_i^B P_i^A\,. \end{aligned} \tag{2.3.10}$$

[6] It is easy to check it for translation, rotation and dilatation. As for SCT, it is enough to see the inversion transformation, and then the properties (2.3.5) would be useful.

The original form of two-point function (2.3.4) can be recovered form (2.3.9) by removing the polarization vectors with the differential operator D_A in (2.2.13). Similarly, a two-point function with two tensor fields is also determined by the conformal symmetry. In general for symmetric traceless tensor fields with spin J, a two-point function is given by the following form:

$$\langle \mathcal{O}^{\mu_1 \ldots \mu_J}(x_1) \mathcal{O}_{\nu_1 \ldots \nu_J}(x_2) \rangle = \frac{I^{(\mu_1}{}_{\nu_1}(x_{12}) \ldots I^{\mu_J)}{}_{\nu_J}(x_{12})}{x_{12}^{2\Delta}} - (\text{traces}) , \tag{2.3.11}$$

where $(\mu_1 \ldots \mu_J)$ means the complete symmetrization for indexes, and it is subtracted by all possible terms proportional to $\delta_{\mu_i \mu_j}$ or $\delta_{\nu_i \nu_j}$ so that the result is symmetric and traceless in μ and ν respectively. Similarly to the vector case, using the polarization vector, this expression can be expressed simply as follows:

$$\langle \mathcal{O}^{\text{ten.}}_{\Delta,J}(P_1, Z_1) \mathcal{O}^{\text{ten.}}_{\Delta,J}(P_2, Z_2) \rangle = \frac{(H_{12})^J}{P_{12}^{\Delta+J}} , \tag{2.3.12}$$

where again the field $\mathcal{O}^{\text{ten.}}_{\Delta,J}(P_1, Z_1)$ is defined by contraction with Z_A as in (2.2.9). Note that if two fields belong to different representations, which means that the dimensions are different $\Delta_1 \neq \Delta_2$ or their spins are different $J_1 \neq J_2$, the two-point function vanishes according to the conformal symmetry.

Three-Point Function with Scalar-Scalar-Tensor

Next, we will consider three-point functions in the embedding space. The simplest generalization of the scalar three-point result is a correlation function with two scalar and one vector field. According to the scaling property, the possible form is restricted as

$$\langle \mathcal{O}_{\Delta_1}(P_1) \mathcal{O}_{\Delta_2}(P_2) \mathcal{O}^{\text{vec.}}_{\Delta_3}(P_3, Z_3) \rangle = \lambda_{123} \frac{(Z_3 \cdot P_3) + \alpha (Z_3 \cdot P_2)}{P_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3-J)} P_{23}^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1+J)} P_{31}^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2+J)}} , \tag{2.3.13}$$

where due to the relations $Z_3 \cdot Z_3 = Z_3 \cdot P_3 = 0$, these terms are dropped already and α is an arbitrary constant. Moreover, this function should be invariant under a shift of the polarization vector $Z_A \to Z_A + \beta P_A$ where $\beta \in \mathbb{R}$ which implies the transverse condition of the embedded field. By imposing that the right hand side is invariant under the shit, the coefficient α is determined as -1.

$$\langle \mathcal{O}_{\Delta_1}(P_1) \mathcal{O}_{\Delta_2}(P_2) \mathcal{O}^{\text{vec.}}_{\Delta_3}(P_3, Z_3) \rangle = \lambda_{123} \frac{V_{3,12}}{P_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3+J)} P_{23}^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1+J)} P_{31}^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2+J)}} , \tag{2.3.14}$$

where we introduced $V_{3,12}$ as the following combination:

$$V_{3,12} = \frac{P_1 \cdot C_3 \cdot P_2}{P_1 \cdot P_2} = \frac{(Z_3 \cdot P_1)(P_2 \cdot P_3) - (Z_3 \cdot P_2)(P_1 \cdot P_3)}{P_1 \cdot P_2} \,. \tag{2.3.15}$$

As similar as the two-point function case, the tensor structure constructed by the tensor C_{AB}. Projecting the result, in the physical space we obtain the well known form for a three-point function including a primary vector:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}^{\text{vec.},\mu}_{\Delta_3}(x_3)\rangle = \lambda_{123}\frac{x^{\mu}_{13}/x^2_{13} - x^{\mu}_{23}/x^2_{23}}{(x^2_{12})^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3-J)}(x^2_{23})^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1+J)}(x^2_{31})^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2+J)}} \,. \tag{2.3.16}$$

In general, through the similar argument, we can determine the functional form os three-point function with a symmetric traceless tensor whose spin is J as:

$$\langle \mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}^{\text{ten.}}_{\Delta_3,J}(P_3,Z_3)\rangle = \lambda_{123}\frac{[V_{3,12}]^J}{P_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3+J)}P_{23}^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1+J)}P_{31}^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2+J)}} \,. \tag{2.3.17}$$

In the physical space, it is written in the following form:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}^{\mu_1\ldots\mu_J}_{\Delta_3,J}(x_3)\rangle = \lambda_{123}\frac{Z^{(\mu_1}_{3,12}\ldots Z^{\mu_J)}_{3,12} - (\text{traces})}{(x^2_{12})^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3+J)}(x^2_{23})^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1+J)}(x^2_{31})^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2+J)}} \,,$$

where $Z^{\mu}_{3,12}$ is defined as:

$$Z^{\mu}_{3,12} = \frac{x^{\mu}_{13}}{x^2_{13}} - \frac{x^{\mu}_{23}}{x^2_{23}} \,. \tag{2.3.18}$$

In (2.3.18), as same as the two-point function case, for the tensor structure $Z^{\mu}_{3,12}$, the indexes are symmetrized, and the possible traces are subtracted.

General Three-Point Function

As a more general case, let us consider a three-point function with arbitrary three symmetric traceless primary tensors. The three-point correlation functions involving $\{\mathcal{O}_{\Delta_i,l_i}(P_i,Z_i)\}$ are crucial building blocks for higher point correlation functions, their form can also be completely fixed by conformal symmetries manifest in the embedding space. According to the scaling transformation, it should have the following form:

$$< \mathcal{O}_{\Delta_1,J_1}(P_1,Z_1)\mathcal{O}_{\Delta_2,J_2}(P_2,Z_2)\mathcal{O}_{\Delta_3,J_3}(P_3,Z_3) >= \frac{Q(\{P_i,Z_i\})}{P_{12}^{\frac{1}{2}(\tau_1+\tau_2-\tau_3)}P_{23}^{\frac{1}{2}(\tau_2+\tau_3-\tau_1)}P_{31}^{\frac{1}{2}(\tau_3+\tau_1-\tau_2)}} \tag{2.3.19}$$

where $\tau = \Delta_i + J_i$. Q is a polynomial of combinations P_i and Z_i, and its degree is J_i in both of Z_i and P_i. Additionally, Q should satisfy the transverse condition, and this means that Q is invariant under the shift $Z_i \to Z_i + \beta P_i$ where β is an arbitrary constant. As a summary, Q is a polynomial which satisfies:

$$Q(\{\alpha_i P_i, \gamma_i Z_i + \beta_i P_i\}) = Q(\{P_i, Z_i\}) \prod_{i=1}^{3} (\alpha_i \gamma_i)^{J_i} . \tag{2.3.20}$$

This kind of polynomials is constructed by a combinations of the tensor C_i^{AB} because of the transverse condition. However, some of the combinations among C_i^{AB} are trivial. For example $C_1 \cdot C_1 = 0$. Non-trivial combinations come from the contraction between different points like $C_1 \cdot C_2 \cdot C_1$, and these combination can be constructed by the following building blocks:

$$\begin{aligned} H_{ij} &= -2\left[(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(Z_j \cdot P_i)\right] = -\mathrm{Tr}(C_i \cdot C_j), \\ V_{i,jk} &= \frac{(P_j \cdot Z_i)\,(P_i \cdot P_k) - (P_j \cdot P_i)\,(Z_i \cdot P_k)}{(P_j \cdot P_k)} = \frac{(P_j \cdot C_i \cdot P_k)}{(P_j \cdot P_k)}, \quad i, j, k = 1, 2, 3. \end{aligned} \tag{2.3.21}$$

These are generalizations of the tensor structures (2.3.10) and (2.3.15) we introduced before. Let us show that the transverse polynomial $Q(P_i, Z_i)$ $(i = 1, 2, 3)$ be built only from H_{ij} and $V_{i,jk}$. Because this polynomial Q do not have the Lorentz indices, Q can only consist of three scalar products; $P_i \cdot P_j$, $Z_i \cdot P_j$ and $Z_i \cdot Z_j$. The combination $Z_i \cdot Z_j$ can be replaced to H_{ij} and other scalar product through (2.3.21). Then Q can be represented as;

$$Q(P_i, Z_i) = \sum_{(m_1,m_2,m_3)=(0,0,0)}^{(J_1,J_2,J_3)} R_{m_1,m_2,m_3}((P_i \cdot P_j)\, , (Z_i \cdot P_j),\, H_{ij})\, , \tag{2.3.22}$$

where R_{m_1,m_2,m_3} is a polynomial inculing $P_i \cdot P_j$, $Z_i \cdot P_j$ and H_{ij} and it is degree m_i in Z_i besides H_{ij}. We can decompose R_{m_1,m_2,m_3} further;

$$R_{m_1,m_2,m_3} = \sum_{n=0}^{m_1} c_{n,m_1-n}\,(Z_1 \cdot P_2)^n\,(Z_1 \cdot P_3)^{m_1-n} . \tag{2.3.23}$$

Here we focused on the Z_1 dependence. The coefficient c_{n,m_1-n} depends on Z_1 only through H_{12} or H_{31} . By demanding the transverse condition, the following equation should be satisfied

$$\frac{\partial}{\partial \beta}\left[\sum_{n=0}^{m_1} c_{n,m_1-n}(Z_1 \cdot P_2 + \beta P_1 \cdot P_2)^n (Z_1 \cdot P_3 + \beta P_1 \cdot P_3)^{m_1-n}\right] = 0 . \tag{2.3.24}$$

Because this condition should satisfied at each order of $(Z_1 \cdot P_2)$ or $(Z_1 \cdot P_2)$, we can obtain the following recursion equation;

$$(m_1 - n)\, c_{n,m_1-n}(P_1 \cdot P_3) + (n+1)\, c_{n+1,m_1-n-1}(P_1 \cdot P_2) = 0\,. \tag{2.3.25}$$

According to this relation, c_{n,m_1-n} is determined as

$$c_{n,m_1-n} = {}_{m_1}C_n \left(-\frac{P_1 \cdot P_3}{P_1 \cdot P_2}\right)^n c_{0,m_1}\,. \tag{2.3.26}$$

Then the decomposition in (2.3.23) is just a binomial expansion and R_{m_1,m_2,m_3} can be rewritten as;

$$R_{m_1,m_2,m_3} = c_{0,m_1} \left(-\frac{P_2 \cdot P_3}{P_1 \cdot P_2} V_{1,23}\right)^{m_1}\,. \tag{2.3.27}$$

The discussions for Z_2 and Z_3 go through similarly. Therefore transverse polynomials Q should depend on depends on Z_i only through H_{ij} and $V_{i,jk}$.

Note that H_{ij} and $V_{i,jk}$ has a symmetry in their indexes: $H_{ij} = H_{ji}$ and $V_{i,jk} = -V_{i,kj}$, and sometimes we will use a short notation for V as $V_1 = V_{1,23}$, $V_2 = V_{2,31}$ and $V_3 = V_{3,12}$. H_{ij} is degree 1 polynomial in P_i, P_j, Z_i and Z_j, and $V_{i,jk}$ is degree 1 in P_i and Z_i. The polynomial Q can be decomposed into a sum of combination of the building blocks:

$$Q(\{P_i, Z_i\}) = \lambda_{n_{12},n_{13},n_{23}} \prod_{i=1}^{3} V_i^{m_i} \prod_{i<j}^{3} H_{ij}^{n_{ij}}\,, \tag{2.3.28}$$

where the exponents satisfy the following relation reflecting the homogeneous property in (2.3.20)

$$m_1 = J_1 - n_{12} - n_{13} \geq 0, \quad m_2 = J_2 - n_{12} - n_{23} \geq 0 \quad m_3 = J_3 - n_{13} - n_{23} \geq 0. \tag{2.3.29}$$

$\lambda_{\{n_{ij}\}}$ are theory dependent constant factors, and now because there are some possible tensor structures, there are the same number of the three-point coupling constant as the number of tensor structures. Here we introduce the elementary structures of the three-point correlation function, which is called the box tensor basis is given by:

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ J_1 & J_2 & J_3 \\ n_{23} & n_{13} & n_{12} \end{bmatrix} = \frac{V_{1,23}^{m_1} V_{2,31}^{m_2} V_{3,12}^{m_3} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{(P_{12})^{\frac{1}{2}(\tau_1+\tau_2-\tau_3)} (P_{13})^{\frac{1}{2}(\tau_1+\tau_3-\tau_2)} (P_{23})^{\frac{1}{2}(\tau_2+\tau_3-\tau_1)}}. \tag{2.3.30}$$

Note that once J_i and n_{ij} are given, m_i are determined through the relation (2.3.29), therefore the boxes are labeled by J_i and n_{ij} . Finally, we obtained the general form of three-point function with symmetric traceless primary tensors:

$$< \mathcal{O}_{\Delta_1,l_1}(P_1,Z_1)\mathcal{O}_{\Delta_2,l_2}(P_2,Z_2)\mathcal{O}_{\Delta_3,l_3}(P_3,Z_3) >= \sum_{n_{12},n_{13},n_{23}\geq 0} \lambda_{n_{12},n_{13},n_{23}} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ l_1 & l_2 & l_3 \\ n_{23} & n_{13} & n_{12} \end{bmatrix} \tag{2.3.31}$$

This expression is written in terms of the coordinates in the embedding space, and applying the differential operators D_{Z_i}, we can obtain the the result in the original physical space $\mathbb{R}^d$. The number of the set of non-negative integers satisfying (2.3.29) is the possible elementary structures listed in (2.3.30), for $J_3 \geq J_2 \geq J_1$ and $p = \max(0, J_1 + J_2 - J_3)$, it is given by:

$$N(J_1,J_2,J_3) = \frac{(J_1+1)(J_1+2)(3J_2-J_1+3)}{6} - \frac{p(p+2)(2p+5)}{24} - \frac{1-(-1)^p}{16}. \tag{2.3.32}$$

Note here that in three dimension or (3+2) dimension in the embedding space, there are further non-trivial relations among tensor structures V and H because the polarization vectors and coordinates are not linearly independent. In fact, for example, a tensor structure $H_{12}H_{13}H_{23}$ can be written in terms of a combination of Vs[7] .

Differensial Operators

Another very useful basis for expressing the structures of three-point functions involve the following differential operators:

$$D_{11} = \big((P_1\cdot P_2)Z_1^A - (Z_1\cdot P_2)P_1^A\big)\frac{\partial}{\partial P_2^A} + \big((P_1\cdot Z_2)Z_1^A - (Z_1\cdot Z_2)P_1^A\big)\frac{\partial}{\partial Z_2^A}, \tag{2.3.33}$$
$$D_{12} = \big((P_1\cdot P_2)Z_1^A - (Z_1\cdot P_2)P_1^A\big)\frac{\partial}{\partial P_1^A} + \big((P_2\cdot Z_1)Z_1^A\big)\frac{\partial}{\partial Z_1^A}, \tag{2.3.34}$$
$$D_{22} = \big((P_1\cdot P_2)Z_2^A - (Z_2\cdot P_1)P_2^A\big)\frac{\partial}{\partial P_1^A} + \big((P_2\cdot Z_1)Z_2^A - (Z_1\cdot Z_2)P_2^A\big)\frac{\partial}{\partial Z_1^A}, \tag{2.3.35}$$
$$D_{21} = \big((P_1\cdot P_2)Z_2^A - (Z_2\cdot P_1)P_2^A\big)\frac{\partial}{\partial P_2^A} + \big((P_1\cdot Z_2)Z_2^A\big)\frac{\partial}{\partial Z_2^A}, \tag{2.3.36}$$

and they only have the following non-vanishing commutators:

$$[D_{11}, D_{22}] = \frac{H_{12}}{2}\left(Z_1\cdot\frac{\partial}{\partial Z_1} - Z_2\cdot\frac{\partial}{\partial Z_2} + P_1\cdot\frac{\partial}{\partial P_1} - P_2\cdot\frac{\partial}{\partial P_2}\right), \tag{2.3.37}$$
$$[D_{12}, D_{21}] = \frac{H_{12}}{2}\left(Z_1\cdot\frac{\partial}{\partial Z_1} - Z_2\cdot\frac{\partial}{\partial Z_2} - P_1\cdot\frac{\partial}{\partial P_1} + P_2\cdot\frac{\partial}{\partial P_2}\right), \tag{2.3.38}$$

[7] For the detail of this part please see, [47, 50].

while all other commutators vanish, including $[D_{ij}, H_{12}] = 0$. We shall express such differential basis using curly brackets, and they are defined through the following relations:

$$\begin{aligned}
\begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ J_1 & J_2 & J_3 \\ n_{23} & n_{13} & n_{12} \end{Bmatrix} &= H_{12}^{n_{12}} D_{12}^{n_{13}} D_{21}^{n_{23}} D_{11}^{m_1} D_{22}^{m_2} \Sigma^{l_1+n_{23}-n_{13}, l_2+n_{13}-n_{23}} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & 0 & J_3 \\ 0 & 0 & 0 \end{bmatrix}, \\
&= H_{12}^{n_{12}} D_{12}^{n_{13}} D_{21}^{n_{23}} D_{11}^{m_1} D_{22}^{m_2} \begin{bmatrix} \tilde{\tau}_1 & \tilde{\tau}_2 & \Delta_3 \\ 0 & 0 & J_3 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{2.3.39}$$

where the shift operators $\Sigma^{a,b}$ which shifts the scaling dimensions (Δ_1, Δ_2) to $(\Delta_1 + a, \Delta_2 + b)$, such that $\tilde{\tau}_1 = \tau_1 + (n_{23} - n_{13})$ and $\tilde{\tau}_2 = \tau_2 + (n_{13} - n_{23})$. Notice that for given integer spins $\{J_1, J_2, J_3\}$, (2.3.39) are also labeled by triplet of non-negative integers $\{n_{12}, n_{13}, n_{23}\}$ satisfying (2.3.29), we therefore have equal number $N(J_1, J_2, J_3)$ of differential basis (2.3.39) as in the original box basis (2.3.30), and they are related by linear transformation with constant coefficients. In contrast with the box basis (2.3.30), where we can cyclicly permute the three primary operators involved, in the differential basis we break this cyclicity such that the differential operators (2.3.33)-(2.3.36) only act on $(P_{1,2}, Z_{1,2})$.

Four Point Function with External Spins

In the previous section, we have discussed conformal blocks or four-point functions with four scalar operators. Here we also consider four point function with tensor fields as in below:

$$\langle \mathcal{O}_{\Delta_1,J_1}(P_1, Z_1) \mathcal{O}_{\Delta_2,J_2}(P_2, Z_2) \mathcal{O}_{\Delta_3,J_3}(P_3, Z_3) \mathcal{O}_{\Delta_4,J_4}(P_4, Z_4) \rangle \,, \tag{2.3.40}$$

for convenience, it is written in the embedding space. The Lorentz invariance and scaling transformation restrict the function to the following form:

$$\begin{aligned}
&\langle \mathcal{O}_{\Delta_1,J_1}(P_1, Z_1) \mathcal{O}_{\Delta_2,J_2}(P_2, Z_2) \mathcal{O}_{\Delta_3,J_3}(P_3, Z_3) \mathcal{O}_{\Delta_4,J_4}(P_4, Z_4) \rangle \\
&= \frac{1}{P_{12}^{\frac{1}{2}(\tau_1+\tau_2)} P_{34}^{\frac{1}{2}(\tau_3+\tau_4)}} \left(\frac{P_{14}}{P_{24}}\right)^{\frac{\tau_2-\tau_1}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\tau_3-\tau_4}{2}} \sum_k \mathcal{G}_k(u,v) Q^k(\{P_i, Z_i\}) \,,
\end{aligned} \tag{2.3.41}$$

where $\mathcal{G}_k(u, v)$ are arbitrary functions and $Q_k(\{P_i, Z_i\})$ are polynomials of P_i and Z_i which satisfy the transverse condition and scaling property as in like (2.3.42)

$$Q^k(\{\alpha_i P_i, \gamma_i Z_i + \beta_i P_i\}) = Q^k(\{P_i, Z_i\}) \prod_{i=1}^{4} (\alpha_i \gamma_i)^{J_i} \,. \tag{2.3.42}$$

Now there are some possible $Q_k(\{P_i, Z_i\})$ functions, and this means the tensor structure of four point function is not unique. Therefore, according to the number of tensor structure, the same number of arbitrary functions of the cross ratios are introduced. In general, a four-point function is a summation of such arbitrary functions with tensor structure. To construct the polynomial Q^k, again the elements of tensor structures H_{ij} and $V_{i,jk}$ are useful which are introduced in (2.3.21). Now the indexes sun over 1 to 4 because there are four points. Through a similar argument, we can show that the polynomial Q is expressed as a product of H_{ij} and $V_{i,jk}$. In this case, there are 6 H_{ij}s and 12 $V_{i,jk}$s naively because of their symmetry; $H_{ij} = H_{ji}$ and $V_{i,jk} = -V_{i,kj}$. However the $V_{i,jk}$s are not independent, in fact these are related as follows:

$$P_{23}P_{14}V_{1,23} + P_{24}P_{13}V_{1,42} + P_{34}P_{12}V_{1,34} = 0\,. \tag{2.3.43}$$

These are also similar relations for $V_{2,ij}$, $V_{3,ij}$ and $V_{4,ij}$, hence there are 6 H_{ij}s and 8 $V_{i,jk}$s listed in below:

$$\begin{aligned}
&H_{12}\,, \quad H_{13}\,, \quad H_{14}\,, \quad H_{23}\,, \quad H_{24}\,, \quad H_{34}\,, \\
&V_{1,23}\,, \quad V_{1,24}\,, \quad V_{2,13}\,, \quad V_{2,14}\,, \quad V_{3,41}\,, \quad V_{3,42}\,, \quad V_{4,31}\,, \quad V_{4,32}\,.
\end{aligned} \tag{2.3.44}$$

These are the building blocks to construct tensor structure of four-point functions.

2.4 Operator Product Expansion and Conformal Block

Operator Product Expansion

A key relation in CFT is the operator product expansion(OPE) which implies a two product of operator can be expanded as a series of primary and its descendant operator. This relation is justified through the radial quantization formalism which is not explained in this thesis (for the detail, for example, see the reference [20, 43]). OPE between two scalar primary operator $\mathcal{O}_i(x_i)$ are given as

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k C_{12k}(x_{12}, \partial_2)\, \mathcal{O}_k(x_2)\,. \tag{2.4.1}$$

Here $C_{12k}(x_{12}, \partial_2)$ is an undermined polynomial in x_{12} and $, \partial_2$ and the summation is taken over all possible primary fields $\mathcal{O}_k$. In below, we will see that the conformal symmetry strongly restricts the expansion. For simplicity, we assume that the operators are scalar. Firstly, by applying the dilation for both sides of the expansion, the expansion of C_{12k} is given as

$$C_{12k}(x,\partial) = \#|x|^{\Delta_k-\Delta_1-\Delta_2}\left(1+\#x^\mu\partial_\mu+\#x^\mu x^\nu\partial_\mu\partial_\nu+\#x^2\partial^2+...\right). \tag{2.4.2}$$

where since $C_{12k}(x_{12},\partial_2)$ is a scalar under the ration, the Lorentz indexes should be contracted, and all #s are undermined coefficients. To restrict the expansion further, let us take the correlation function with a operator $\mathcal{O}_k(x_3)$ ($|x_{12}|<|x_3|$) in the OPE (2.4.1). Then it becomes

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_k(x_3)\rangle = \sum_{k'} C_{12k'}(x_{12},\partial_2)\,\langle\mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle\,. \tag{2.4.3}$$

The two or three-point function is completely fixed by the conformal symmetry. If the two-point function is diagonalized so that it is proportional to the Kronecker's delta $\langle\mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle\sim\delta_{kk'}$, the summation in the RHS is collapsed and it becomes

$$\frac{\lambda_{12k}}{x_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_k)}x_{23}^{\frac{1}{2}(\Delta_2+\Delta_k-\Delta_1)}x_{31}^{\frac{1}{2}(\Delta_k+\Delta_1-\Delta_2)}} = C_{12k}(x_{12},\partial_2)|x_{23}|^{-2\Delta_k}\,. \tag{2.4.4}$$

This equation means that $C_{12k}(x_{12},\partial_2)$ is proportional to the three-point coefficient λ_{12k}. Furthermore, by using this equation, $C_{12k}(x_{12},\partial_2)$ is determined in principle. Firstly, we can rewrite C_{12k} as a derivatives in x_3: $C_{12k}(x_{12},-\partial_3)$, and set $x_2=0$. Now the RHS is regarded as a expansion in x_1 around $x_1=0$. Then comparing the expansion of the LHS, we can determine each coefficient in the series expansion (2.4.2).

So far we consider scalar operators only, however, for other representations the same argument holds. In general, some operators which have spins appear in RHS of (2.4.1), then the OPE is given as

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k C_{12k,\mathcal{I}}(x_{12},\partial_2)\,\mathcal{O}_k^{\mathcal{I}}(x_2)\,, \tag{2.4.5}$$

where $\mathcal{I}$ is an index for an irreducible representation of $SO(d)$. However, from a product of two scalar fields, there are only spin-J symmetric traceless tensor fields in the RHS of OPE.

Using OPE, we can reduce the number of operators in a correlation function. In (2.4.3), the three-point function is expressed as a summation of two-point functions. Similarly, n point function can be expressed as a summation of $n-1$ point functions due to the OPE. Through iterating this relation, an arbitrary n point function can be expressed as a summation of one point function which is zero unless the operator $\mathcal{O}$ is identity operator. In this way, we can compute any point function in principle.

Conformal Block

Especially, for a four point function $\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$, if $|x_{12}|$ and $|x_{34}|$ are suficiently small, we can take OPE for both of pairs $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ and $\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)$, then it becomes

$$\begin{aligned}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle &= \sum_{k,k'}\lambda_{12k}\lambda_{34k'}C_{12k,\mathcal{I}}(x_{12},\partial_2)C_{34k',\mathcal{J}}(x_{34},\partial_4)\langle\mathcal{O}_k^{\mathcal{I}}(x_2)\mathcal{O}_{k'}^{\mathcal{J}}(x_4)\rangle \\ &= \sum_{k}\lambda_{12k}\lambda_{34k}C_{12k,\mathcal{I}}(x_{12},\partial_2)C_{34k,\mathcal{J}}(x_{34},\partial_4)\frac{I^{\mathcal{I}\mathcal{J}}(x_{24})}{x_{24}^{2\Delta}} \\ &= \sum_{\Delta,J}\lambda_{12k}\lambda_{34k}\, g_{\Delta,J}(x_i)\,. \end{aligned} \tag{2.4.6}$$

In the last line we defined the function $g_{\Delta,J}(x_i)$ as

$$g_{\Delta,J}(x_i) = C_{12k,\mathcal{I}}(x_{12},\partial_2)C_{34k,\mathcal{J}}(x_{34},\partial_4)\frac{I^{\mathcal{I}\mathcal{J}}(x_{24})}{x_{24}^{2\Delta}}\,. \tag{2.4.7}$$

Here we rescaled the function $C_{12k,\mathcal{I}}$ with the three-point coefficient λ_{12k}, and $I^{\mathcal{I}\mathcal{J}}(x_{24})$ is the tensor structure of two-point function with two symmetric traceless tensors which is given in 2.3.11 . The dimension Δ is that of $\mathcal{O}_k$. The functions $g_{\Delta,J}(x_i)$ are called conformal blocks. It is a kinematical object which depends on the external dimensions Δ_i $(i = 1, ..4)$ and the internal dimension Δ and spin J . In the last line of the above equation, the summation is taken over Δ and J because symmetric traceless tensors are characterized by them.

In order to see the property, let us consider another expression for conformal blocks. The conformal block decomposition of a four point function can be understood as a projection to a conformal multiplet of a primary operator $\mathcal{O}$:

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\Delta,J}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{P}_{\Delta,J}|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\,, \tag{2.4.8}$$

where $\mathcal{P}_{\Delta,J}$ is a projection operator to the conformal family of $\mathcal{O}$ [8]:

$$\mathcal{P}_{\Delta,J} = \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},P^2\mathcal{O},\ldots}\frac{|\alpha\rangle\langle\beta|}{\langle\alpha|\beta\rangle}\,, \tag{2.4.9}$$

By the construction, conformal blocks is related to the projected four point function as follows:

$$\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g_{\Delta,J}(x_i) = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{P}_{\Delta,J}|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\,, \tag{2.4.10}$$

Because the projection $\mathcal{P}_{\Delta,J}$ is invariant under the conformal transformation, projected correlation

[8] Here the state $|\mathcal{O}\rangle$ is defined as $|\mathcal{O}\rangle \equiv \mathcal{O}(0)|0\rangle$, and also $\langle\mathcal{O}| \equiv \lim_{x\to\infty} x^{2\Delta}\langle 0|\mathcal{O}(x)$.

functions transform in the same way as four point functions. Therefore the conformal block can also be written as a function of cross ratio u and v:

$$g_{\Delta,J}(x_i) = \mathcal{F}(x_i)\, g_{\Delta,J}(u,v)\,, \tag{2.4.11}$$

where $\mathcal{F}(x_i)$ is the kinematic factor defined in (2.1.41) .

According to OPE, a conformal block is regarded as a product of two three-point functions, and as discussed in the previous section, a three-point function with three tensor fields is reproduced by applying differential operators. Following these observations, a conformal block with tensor operators is also constructed by applying differential operators to a scalar conformal block. We will argue the detail of this structure later after introducing the conformal partial wave.

2.5 Conformal Casimir Equation

In the previous section, we have seen that a conformal block is obtained as a projected four point function.

$$g_{\Delta,J}(x_i) \sim \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\mathcal{P}_{\Delta,J}|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\,, \tag{2.5.1}$$

where the projection $\mathcal{P}_{\Delta,J}$ is defined in (2.4.9) . From this expression, let us drive a differential equation which characterizes the conformal blocks, and it is called the conformal Casimir equation. To drive the equation, we insert the conformal Casimir operator $L^2 \equiv \frac{1}{2}L_{AB}L^{AB}$ in front of the projection operator, where L_{AB} are generators in $SO(1,d{+}1)$ in (2.1.8) . Now we have the following object:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)L^2|P^n\mathcal{O}\rangle\,. \tag{2.5.2}$$

Here the part depending on $x_{3,4}$ is dropped. Considering the cases when L^2 acts on the left and right, and equating them, we can obtain the conformal Casimir equation. Firstly let us consider the case that L^2 acts on the right. Because L^2 commutes with the conformal generators, it is enough to compute $L^2|\mathcal{O}\rangle$. After expanding L^2, it becomes

$$L^2|\mathcal{O}\rangle = -\frac{1}{2}\left\{M_{\mu\nu}M^{\mu\nu} + K_\mu P^\mu - 2D^2\right\}|\mathcal{O}\rangle\,, \tag{2.5.3}$$

Here some terms are dropped due to the property of property fields: $K_\mu|\mathcal{O}\rangle = 0$. After some calculation, the result becomes:

$$L^2|\mathcal{O}\rangle = C_{\Delta,J}|\mathcal{O}\rangle\,, \qquad C_{\Delta,J} \equiv \Delta(\Delta-d)+J(J+d-2)\,, \tag{2.5.4}$$

On the other hand, in the case that L^2 acts on the left, according to the relation $[L_{AB},\mathcal{O}_1(x_1)] = L_{1,AB}\mathcal{O}_1(x_1)$ discussed in section 2.1, the conformal generator can be written as a differential operator acting in the coordinate as follows:

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)L_{AB}...\rangle = -\left(L_{1,AB}+L_{2,AB}\right)\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\rangle \tag{2.5.5}$$

By using this relation twice and equating the previous result in (2.5.4), we obtain the following equation:

$$\left(L_{1,AB}+L_{2,AB}\right)\left(L_1^{AB}+L_2^{AB}\right)g_{\Delta,J}(x_i) = C_{\Delta,J}g_{\Delta,J}(x_i) \tag{2.5.6}$$

This is called the conformal Casimir equation. In the above expression, the differential operator is acting on the coordinates x_i, and using the property in (2.4.11), we can rewrite the equation as a differential equation of the cross ratios as below:

$$\mathcal{D}\,g_{\Delta,J}(z,\bar{z}) = C_{\Delta,J}\,g_{\Delta,J}(z,\bar{z}) \tag{2.5.7}$$

where $\mathcal{D}$ is a differential operator acting on the cross ratios z and $\bar{z}$which are related to u and v through $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$:

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}} + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}\left[(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right]\,, \tag{2.5.8}$$

and $\mathcal{D}_z$ and $\mathcal{D}_{\bar{z}}$ are defined as below:

$$\begin{aligned}\mathcal{D}_z &= z^2(1-z)\partial_z^2 - (a+b+1)z^2\partial_z - abz\,,\\ \mathcal{D}_{\bar{z}} &= \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - (a+b+1)\bar{z}^2\partial_{\bar{z}} - ab\bar{z}\,.\end{aligned} \tag{2.5.9}$$

By solving the differential equation under the proper boundary condition, in even dimensions, in principle we can have the solution as a closed form. For example, the two-dimensional conformal block is given by:

$$g^{(2d)}_{\Delta,l}(z,\bar{z}) = \frac{1}{1+\delta_{l,0}}\left[k_{\frac{\Delta+l}{2}}(z)k_{\frac{\Delta-l}{2}}(\bar{z}) + k_{\frac{\Delta-l}{2}}(z)k_{\frac{\Delta+l}{2}}(\bar{z})\right]\,, \tag{2.5.10}$$

and in four dimension, it is given by:

$$g^{(4d)}_{\Delta,l}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left[k_{\frac{\Delta+l}{2}}(z)k_{\frac{\Delta-l-2}{2}}(\bar{z}) - k_{\frac{\Delta-l-2}{2}}(z)k_{\frac{\Delta+l}{2}}(\bar{z})\right]. \tag{2.5.11}$$

Here we define the function k_Δ as :

$$k_\Delta(z) = z^\Delta {}_2F_1(\Delta+a, \Delta+b; 2\Delta; z)\,, \tag{2.5.12}$$

and a and b are the combination of the external dimensions as in (2.1.43). Sometimes this function is called the one-dimensional conformal block because it is an eigenfunction of the following one-dimensional conformal Casimir equation:

$$\mathcal{D}_z\, k_\Delta(z) = \Delta(\Delta-1)\, k_\Delta(z)\,, \tag{2.5.13}$$

where the differential operator $\mathcal{D}_z$ is the same one defined in (2.5.9). In one dimension, there is only one independent cross ratio z which take a real value.

In the end, we will show some useful property of conformal block using the conformal Casimir equation. In terms of cross ratios u and v, the differential operator $\mathcal{D}$ takes the following form:

$$\begin{aligned}\mathcal{D} = (1-u-v)\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}+a+b\right) + u\frac{\partial}{\partial u}\left(2u\frac{\partial}{\partial u}-d\right) \\ -(1+u-v)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+a\right)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+b\right).\end{aligned} \tag{2.5.14}$$

Here w e take a limit $u \to 0$ and after the limit of u, take $v \to 1$. In this limit, if we assume that the conformal block behaves as $g_{\Delta,J} \sim u^p(1-v)^q$, the conformal Casimir equation (2.5.7) becomes:

$$\mathcal{D}\, u^p(1-v)^q = \left[2p^2 - dp + q^2 + 2pq - q\right] u^p(1-v)^q = C_{\Delta,J}\, u^p(1-v)^q \tag{2.5.15}$$

This relation means that the conformal block should have the following asymptotic behavior:

$$g_{\Delta,J} \sim u^{\frac{\Delta-J}{2}}(1-v)^J + \ldots\,, \qquad u \to 0 \text{ and } v \to 1\,, \tag{2.5.16}$$

where ... denotes the higher order terms in the expansion.

Next, we regard $z\bar{z}$ and $(z+\bar{z})/2\sqrt{z\bar{z}}$ as independent new variables, and then in the small $z\bar{z}$ limit, the conformal blocks may behaves as below:

$$g_{\Delta,J} \sim (z\bar{z})^{\frac{\Delta}{2}} f(\sigma) + \ldots\,, \qquad \text{with } \sigma = \frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\,. \tag{2.5.17}$$

Note that the power of the leading term is determined by the relation (2.5.16) . Substituting this form to the conformal Casimir equation, we can obtain an equation for the arbitrary function $f(\sigma)$:

$$\left[(1-\sigma^2)\,\partial_\sigma^2 - (2\epsilon+1)\sigma\,\partial_\sigma \,+ J(J+2\epsilon)\right] f(\sigma) = 0\,, \tag{2.5.18}$$

where we define ϵ as $\epsilon = h-1 = \frac{d-2}{2}$. The solution is given as the Gegenbaur polynomial $C_J^{(\epsilon)}(\sigma)$, we conclude that in the limit: $z\bar{z} \to 0$, the conformal block behaves as:

$$g_{\Delta,J} \sim (z\bar{z})^{\frac{\Delta}{2}} C_J^{(\epsilon)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right) + \dots\,. \tag{2.5.19}$$

2.6 Conformal Partial Wave

Here we introduce another nice basis to expand four-point functions which are called conformal partial wave(CPW). CPWs are defined as an integral of a pair of three-point functions as below:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = \int_{\mathbb{R}^d} d^d x_0\ \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{h+i\nu,J}(x_0)^{\mu_1\dots\mu_J}\rangle_1 \langle \tilde{\mathcal{O}}_{h-i\nu,J}(x_0)^{\mu_1\dots\mu_J}\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle_1 \tag{2.6.1}$$

In the three-point functions, there are operators whose dimensions are complex value. These operators are not really included in the theory, and these three-point functions are defined through the kinematic form of three-point function in (2.3.18). Here the three-point coefficient is 1, therefore we denote a subscript 1 in this three-point function as $\langle ... \rangle_1$. In this sense, the CPW is a purely kinematical quantity which does not depend on three-point coefficient.

$\Psi^{\Delta_i}_{h+i\nu,J}$ is transformed in the same manner as the scalar four-point function with scalar fields $\mathcal{O}_{\Delta_i}$. In the conformal transformation including the integrated point P_0, the three-point functions in the integrand in (2.6.1) transforms as:

$$\begin{aligned}&\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{h+i\nu,J}(x_0)^{\mu_1\dots\mu_J}\rangle_1 \\ &\qquad \to \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_1+\Delta_2+h+i\nu}{d}} D(R(x))^{\mu_1\dots\mu_J}{}_{\nu_1\dots\nu_J}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{h+i\nu,J}(x_0)^{\nu_1\dots\nu_J}\rangle_1\,.\end{aligned} \tag{2.6.2}$$

Here the rotation factor $D(R(x))$ is canceled with the same factor coming form another three-point function, and the Jacobian coming form the integration measure is canceled with the determinant factors coming from operators $\mathcal{O}_{h+i\nu,J}$ and $\tilde{\mathcal{O}}_{h-i\nu,J}$. The remaining determinant factors are the same as what a four-point function produces under the conformal transformation. Therefore by extracting the kinematic factor $\mathcal{F}(x_i)$, Ψ can be written as a function of cross ratios:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = \mathcal{F}(x_i)\,\Psi^{\Delta_i}_{h+i\nu,J}(u,v) = \mathcal{F}(x_i)\,\Psi^{\Delta_i}_{h+i\nu,J}(z,\bar{z})\,. \tag{2.6.3}$$

Here $\mathcal{F}(x_i)$ is defined in (2.1.41) .

Another important property of CPW is that the CPWs are also eigenfunctions of the conformal Casimir equation. We can show this fact easily in the same way as the case of conformal block by inserting the conformal Casimir operator L^2 before $\mathcal{O}_{h+i\nu,J}$ in the first three-point function. Eventually, we can confirm that $\Psi^{\Delta_i}_{h+i\nu,J}$ satisfies the following equation:

$$\mathcal{D}\,\Psi^{\Delta_i}_{h+i\nu,J}(z,\bar{z}) = \left[h^2+\nu^2+J(J+d-2)\right]\Psi^{\Delta_i}_{h+i\nu,J}(z,\bar{z})\,. \tag{2.6.4}$$

From the fact that CPW is also an eigenfunction of conformal Casimir equation, it is expected that a CPW is written as a linear combination of conformal blocks, and in fact, it is true. Comparing the asymptotic behavior of CPW and conformal blocks in the limits $x_{12}\to 0$ or $x_{34}\to 0$, it is shown that CPW is a linear combination of two conformal blocks:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = K^{\Delta_3,\Delta_4}_{h-i\nu,J}\,g^{\Delta_i}_{h+i\nu,J}(x_i) + K^{\Delta_1,\Delta_2}_{h+i\nu,J}\,g^{\Delta_i}_{h-i\nu,J}(x_i)\,. \tag{2.6.5}$$

where the coefficient K is defined as:

$$K^{\Delta_3,\Delta_4}_{h-i\nu,J} = \frac{1}{(-2)^J}\frac{\pi^h(h-i\nu-1)_J\Gamma(-i\nu)}{\Gamma(h+i\nu+J)}\frac{\Gamma\left(\frac{h+i\nu+J\pm\Delta_{34}}{2}\right)}{\Gamma\left(\frac{h-i\nu+J\pm\Delta_{34}}{2}\right)} \tag{2.6.6}$$

Note here this $K^{\Delta_1,\Delta_2}_{\Delta,J}$ satisfies the following identities $K^{d-\Delta_1,d-\Delta_2}_{h+i\nu,J} = K^{\Delta_1,\Delta_2}_{h+i\nu,J}$ and:

$$K^{\Delta_1,\Delta_2}_{h+i\nu,J}K^{\Delta_1,\Delta_2}_{h-i\nu,J} = K^{\Delta_3,\Delta_4}_{h+i\nu,J}K^{\Delta_3,\Delta_4}_{h-i\nu,J} = \frac{\pi^{2h}}{2^{2J}}\frac{\Gamma(\pm i\nu)\,(h\pm i\nu-1)_J}{\Gamma(h\pm i\nu+J)}\,, \tag{2.6.7}$$

and through this relation, it is shown that $\Psi^{\Delta_i}_{h+i\nu,J}$ is almost symmetric under $\nu\leftrightarrow-\nu$:

$$\Psi^{\Delta_i}_{h-i\nu,J} = \frac{K^{\Delta_1,\Delta_2}_{h-i\nu}}{K^{\Delta_3,\Delta_4}_{h-i\nu}}\Psi^{\Delta_i}_{h+i\nu,J}\,. \tag{2.6.8}$$

The most useful fact is that the CPWs satisfy the orthogonal relation:

$$\left(\Psi^{\Delta_i}_{h+i\nu,J},\Psi^{d-\Delta_i}_{h-i\nu',J'}\right) = \frac{1}{2}\,n_{\nu,J}\,\delta_{J,J'}\,\left[\delta(\nu-\nu')+\frac{K^{\Delta_1,\Delta_2}_{h+i\nu,J}}{K^{\Delta_3,\Delta_4}_{h+i\nu,J}}\,\delta(\nu+\nu')\right] \tag{2.6.9}$$

where $n_{\nu,J}$ is a normalization factor which is determined later in section 3.5. The inner product $(..,...)$ is defined as:

$$(f^{\Delta_i}, g^{d-\Delta_i}) = \int_{\mathbb{R}^d} \frac{\prod_{i=1}^4 d^d x_i}{\mathrm{vol}(SO(1,d+1))}\ f^{\Delta_i}(x_i) g^{d-\Delta_i}(x_i)\,. \tag{2.6.10}$$

Here we assume that under the conformal transformation, the function $f^{\Delta_i}(x_i)$ changes in the same way as a four-point function of scalar fields whose dimensions are Δ_i , and $g^{d-\Delta_i}(x_i)$ also has the same covariance as a four-point function with scalars whose dimensions are $d-\Delta_i$. An operator with dimension $d-\Delta$ is called the shadow operator of the original physical operator whose dimension is Δ . In the definition of the inner product, the function in the second argument should transform as a four-point function with shadow operators, so that the inner product is conformally invariant. Under the conformal transformation, the determinant factors which come from f^{Δ_i} and $g^{d-\Delta_i}$ are canceled with the factor coming from the integration measure in the inner product. The inner product is divided by the volume of the conformal group $\mathrm{vol}(SO(1,d+1))$, so that this integration has a finite value. According to the conformal symmetry, this integration has a redundant degree of freedom, and to fix it, this volume factor is needed. A detail discussion of this inner product is in [37] . In section 3.5, we will consider the bulk interpretation of CPW and this inner product, see that the orthogonal relation for CPW follows from the orthogonality of the so-called AdS harmonic function.

According to the orthogonality and completeness of CPW, we can easily obtain the conformal block expansion of an arbitrary four-point function through the following formula:

$$\begin{aligned}&\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle \\ &= \sum_{J=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{n_{\nu,J}}\ \Big(\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle, \Psi^{d-\Delta_i}_{h-i\nu,J}\Big)\ \Psi^{\Delta_i}_{h+i\nu,J}(x_i)\,.\end{aligned} \tag{2.6.11}$$

The inner product (..,...) is the same as defined in (2.6.10) . After calculating this inner product, the resultant function in ν has poles in the ν-plane. Basically, these poles correspond to the dimension of operators which appear in the conformal block expansion, and the residues give OPE coefficients.

Finally, for convenience, we give the definition of CPW in the embedding space:

$$\begin{aligned}&\Psi^{\Delta_i}_{h+i\nu,J}(P_i) \\ &= \frac{1}{J!(h-1)_J}\int_{\mathbb{R}^d} d^d P_0\ \langle \mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{h+i\nu,J}(P_0,\mathcal{D}_{Z_0})\rangle_1 \langle \tilde{\mathcal{O}}_{h-i\nu,J}(P_0,Z_0)\mathcal{O}_{\Delta_3}(P_3)\mathcal{O}_{\Delta_4}(P_4)\rangle_1 \\ &= \frac{1}{P_{12}^{\gamma_{12}}P_{34}^{\gamma_{34}}}\int_{\mathbb{R}^d} d^d P_0\ \frac{1}{J!(h-1)_J}\frac{(-2P_2\cdot C_0^{\mathcal{D}}\cdot P_1)^J(-2P_4\cdot C_0^{Z}\cdot P_3)^J}{\Pi_{i=1}^4 P_{0i}^{\gamma_{0i}}}\,.\end{aligned} \tag{2.6.12}$$

Here we substitute the kinematical form of three-point functions in the integrand. The powers γ_{ij} are defined as follows:

$$\begin{aligned}
\gamma_{12} &= \frac{\Delta_{12}^{+} - h - i\nu + J}{2}\,, & \gamma_{01} &= \frac{h + i\nu + J}{2} - a\,, & \gamma_{02} &= \frac{h + i\nu + J}{2} + a\,, \\
\gamma_{34} &= \frac{\Delta_{34}^{+} - h + i\nu + J}{2}\,, & \gamma_{03} &= \frac{h - i\nu + J}{2} + b\,, & \gamma_{04} &= \frac{h - i\nu + J}{2} - b\,.
\end{aligned} \tag{2.6.13}$$

In $C_0^{\mathcal{D}}$, the polarization vector Z_0 is replaced with $\mathcal{D}_{Z_0}$ defined in (2.2.13) to take contractions, and C_0^Z is the usual one $C_0^Z = C_0$.

Fields with Spin

In section 2.3, we saw that by applying differential operators defined in (2.3.33)-(2.3.35) to a three-point function with two scalars and one rank-J tensor field, the three-point functions with three tensor fields can be constructed. The CPWs are defined as a product of three-point functions with two scalars and one tensor field. Applying the differential operators to the left and right three-point functions, we can obtain CPWs with four external tensors and an internal tensor field. In this case, the internal operator can be a more general representation, however, here we consider symmetric traceless tensor only[9]. When restricting to only the exchange of symmetric traceless operators, we can construct it by fusing the differential basis for a pair of three-point correlation functions involving primary operators $\mathcal{O}_{\Delta_{1,2},J_{1,2}}, \mathcal{O}_{\Delta,J}$ and $\mathcal{O}_{\Delta_{3,4},J_{3,4}}, \tilde{\mathcal{O}}_{d-\Delta,J}$ and the resultant conformal partial wave schematically contains the following tensor structures:

$$\Psi_{\Delta,J}^{\{n_{10},n_{20},n_{12}\};\{n_{30},n_{40},n_{34}\}}(P_i, Z_i) = \mathcal{D}_{\text{Left}}^{n_{10},n_{20},n_{12}} \mathcal{D}_{\text{Right}}^{n_{30},n_{40},n_{34}} \Psi_{\Delta,J}(P_i). \tag{2.6.14}$$

Here the composite operators are given by:

$$\mathcal{D}_{\text{Left}}^{n_{10},n_{20},n_{12}} = H_{12}^{n_{12}} D_{12}^{n_{10}} D_{21}^{n_{20}} D_{11}^{m_1} D_{22}^{m_2} \Sigma^{l_1+n_{20}-n_{10},l_2-n_{20}+n_{10}}, \tag{2.6.15}$$

$$\mathcal{D}_{\text{Right}}^{n_{30},n_{40},n_{34}} = H_{34}^{n_{34}} D_{34}^{n_{30}} D_{43}^{n_{40}} D_{33}^{m_3} D_{44}^{m_4} \Sigma^{l_3+n_{40}-n_{30},l_4-n_{40}+n_{30}}. \tag{2.6.16}$$

They should form a basis for four-point function with four tensors like:

$$< \mathcal{O}_{\Delta_1,J_1}(P_1,Z_1)\mathcal{O}_{\Delta_2,J_2}(P_2,Z_2)\mathcal{O}_{\Delta_3,J_3}(P_3,Z_3)\mathcal{O}_{\Delta_4,J_4}(P_4,Z_4) > . \tag{2.6.17}$$

Unlike scalar case (2.6.1) whose conformal partial wave for a given exchanged operator $\mathcal{O}_{\Delta,J}$ can be packaged into a single scalar function of cross-ratios, the conformal partial wave for (2.6.17) for a given exchange operator consists of multiple terms each with independent tensor structures.

[9] The method to construct more general representations is given in [51].

The basis Ψ are now labeled by two sets of triplet of integers $\{n_{10}, n_{20}, n_{12}\}$ and $\{n_{30}, n_{40}, n_{34}\}$ satisfying (2.3.29), and we have denoted the exchanged operator as $\mathcal{O}_{\Delta_0,l_0} \equiv \mathcal{O}_{\Delta,J}$. Now there are $N(J_1, J_2, J_0) \times N(J_3, J_4, J_0)$ possible tensor structures listed in (2.6.14), and we can express each conformal partial wave as the same kinematical form as four-point functions in (2.3.41) .

2.7 Generalized Free Theory

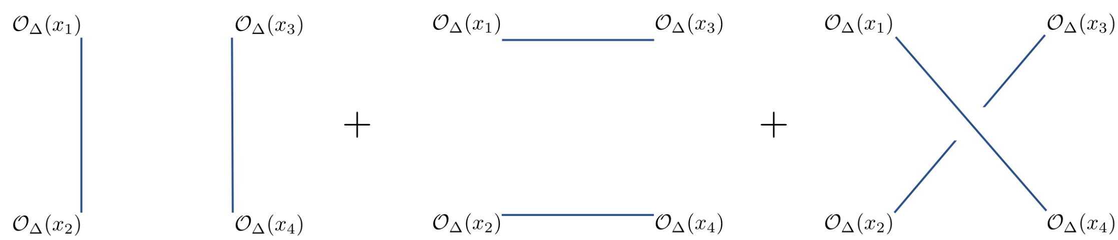


Figure 2.3: A four point function in the generalized free theory is given by the three possible combinations of two-point functions. Each line describes a two-point function.

In this section, as a simple example, we consider the so-called generalized free theory. The four point function of scalar primary is simply given by products of two-point functions.

$$\begin{aligned}\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_\Delta(x_4)\rangle_{\mathrm{GFT}} &= \frac{1}{(x_{12}^2)^\Delta (x_{34}^2)^\Delta} + \frac{1}{(x_{13}^2)^\Delta (x_{24}^2)^\Delta} + \frac{1}{(x_{14}^2)^\Delta (x_{23}^2)^\Delta} \\ &= \frac{1}{(x_{12}^2)^\Delta (x_{34}^2)^\Delta}\left(1 + (z\bar{z})^\Delta + \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^\Delta\right) . \end{aligned} \tag{2.7.1}$$

This theory is a free theory in AdS space, and in CFT, it appears as a leading contributions in the large N expansion. In below, we will consider the conformal block expansion of this four-point function as an example. Firstly we will see it in 2 and 4 dimensions using the explicit form of conformal blocks and later using the inversion formula, we will obtain the results for arbitrary dimensions.

2.7.1 Two-Dimension

Here we rewrite the four-point function in (2.7.1) into a summation of the two-dimensional conformal blocks. The first term in the last line of (2.7.1) corresponds to the conformal block for the identity operator, and we will rewrite the second and third terms. Firstly, there are useful identities to expand z and $\bar{z}$ in terms of k-function:

$$
\begin{aligned}
z^{\Delta} &= \sum_{n=0}^{\infty}(-1)^n C_n^{(\Delta)}\, k_{\Delta+n}(z)\,, \qquad (2.7.2)\\
\left(\frac{z}{1-z}\right)^{\Delta} &= \sum_{n=0}^{\infty} C_n^{(\Delta)}\, k_{\Delta+n}(z)\,. \qquad \left(C_n^{(\Delta)} \equiv \frac{(\Delta)_n^2}{n!(2\Delta+n-1)_n}\right)
\end{aligned}
$$

We can explicitly show these identities through the Jacobi transformation discussed in appendix C.1. The first identity is given in C.1.14, and the second identity is also given as a special case of (C.1.17). Substituting these expansion, we can obtain the expansion of the correlation function $\mathcal{A}^{\mathrm{GFT}}(x_i) \equiv \langle \mathcal{O}_{\Delta}(x_1)\mathcal{O}_{\Delta}(x_2)\mathcal{O}_{\Delta}(x_3)\mathcal{O}_{\Delta}(x_4)\rangle_{\mathrm{GFT}}$:

$$
\mathcal{A}^{\mathrm{GFT}}(z,\bar{z}) = 1 + \sum_{n,m=0}^{\infty} C_n^{(\Delta)} C_m^{(\Delta)} \left[\,1+(-1)^{n+m}\,\right] k_{\Delta+n}(z) k_{\Delta+m}(\bar{z})\,. \qquad (2.7.3)
$$

Here we used cross ratios z and $\bar{z}$ extracting the kinematical factor $cF(x_i)$. In order to obtain two-dimensional conformal bock expansion, we split the n and m summation in the following way:

$$
\sum_{n,m=0}^{\infty}(...) = \sum_{n,l=0}^{\infty}\frac{1}{1+\delta_{l,0}}(...)\Bigg|_{m=n+l} + \sum_{m,l=0}^{\infty}\frac{1}{1+\delta_{l,0}}(...)\Bigg|_{n=m+l}. \qquad (2.7.4)
$$

After some calculation, this summation becomes the conformal block expansion of $\mathcal{A}^{\mathrm{GFT}}$:

$$
\begin{aligned}
\mathcal{A}^{\mathrm{GFT}}(u,v) &= 1 + \sum_{n,l=0}^{\infty}\frac{C_n^{(\Delta)}C_{n+l}^{(\Delta)}}{1+\delta_{l,0}}\left[1+(-1)^l\right] k_{\Delta+n}(z)k_{\Delta+n+l}(\bar{z}) \qquad (2.7.5)\\
&\qquad\qquad + \sum_{m,l=0}^{\infty}\frac{C_m^{(\Delta)}C_{m+l}^{(\Delta)}}{1+\delta_{l,0}}\left[1+(-1)^l\right] k_{\Delta+m+l}(z)k_{\Delta+m}(\bar{z})\\
&= 1 + \sum_{n,l=0}^{\infty}\left[1+(-1)^l\right] C_n^{(\Delta)}C_{n+l}^{(\Delta)}\; g^{(2d)}_{2\Delta+2n+l,l}(z,\bar{z})\,.
\end{aligned}
$$

In the last line, we used the definition of the two-dimensional conformal block given in (2.5.10). Supposing the usual notation of the following conformal block expansion, we can write this result as in:

$$
\mathcal{A}^{\mathrm{GFT}}(u,v) = 1 + \sum_{\mathcal{O}} p_0(\Delta_{\mathcal{O}}, l_{\mathcal{O}})\, g^{(s;2d)}_{\Delta_{\mathcal{O}},l_{\mathcal{O}}}(z,\bar{z})\,, \qquad (2.7.6)
$$

Now the summation is taken over the double trace operators which are parametrized integers n and l. The OPE coefficient $p_0(n,l)$ is given by

$$p_0(n,l) = \left[1 + (-1)^l \right] C_n^{(\Delta)} C_{n+l}^{(\Delta)} . \tag{2.7.7}$$

This result is the same as obtained in [17].

2.7.2 Four dimension

Next, we consider the expansion of (2.7.1) into four-dimensional conformal blocks. Firstly, let us try to expand the following factors $\tilde{\mathcal{A}}^{\mathrm{GFT}}$ into the k-function.

$$\tilde{\mathcal{A}}^{\mathrm{GFT}}(z,\bar{z}) \equiv \left(\frac{1}{\bar{z}} - \frac{1}{z} \right) \left[(z\bar{z})^{\Delta} + \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta} \right] . \tag{2.7.8}$$

Once we could expansion this polynomial in the k-function, multiplying the factor $(1/\bar{z} - 1/z)$ to the both sides, we may obtain the four-dimensional conformal block expansion. Here in addition to (2.7.2), the following formula is needed:

$$\frac{z^{\Delta-1}}{(1-z)^{\Delta}} = \sum_{n=0}^{\infty} \left[C_n^{(\Delta-1)} + C_{n-1}^{(\Delta)} \right] k_{\Delta+n-1}(z) . \tag{2.7.9}$$

This expansion can be also given as a special case of the general formula in (C.1.17). Substituting these formulae, $\tilde{\mathcal{A}}^{\mathrm{GFT}}$ is expanded as below:

$$\begin{aligned} \tilde{\mathcal{A}}^{\mathrm{GFT}} = \sum_{n,m=0}^{\infty} \Big[& c_{n+m}\, C_n^{(\Delta)} C_m^{(\Delta-1)}\, k_{\Delta+n}(z) k_{\Delta+m-1}(\bar{z}) - c_{n+m}\, C_n^{(\Delta-1)} C_m^{(\Delta)}\, k_{\Delta+n-1}(z) k_{\Delta+m}(\bar{z}) \\ & + C_n^{(\Delta)} C_{m-1}^{(\Delta)}\, k_{\Delta+n}(z) k_{\Delta+m-1}(\bar{z}) - C_{n-1}^{(\Delta)} C_m^{(\Delta)}\, k_{\Delta+n-1}(z) k_{\Delta+m}(\bar{z}) \Big] , \end{aligned} \tag{2.7.10}$$

where c_n is defined as:

$$c_n = 1 + (-1)^n . \tag{2.7.11}$$

After changing the way of summation as in the two-dimensional case, it is rewritten in the following summations:

$$\begin{aligned} \tilde{\mathcal{A}}^{\mathrm{GFT}} = & \sum_{n,l=0}^{\infty} \left[c_l\, C_{n+l}^{(\Delta)} C_n^{(\Delta-1)} \left\{ k_{\Delta+n+l}(z) k_{\Delta+n-1}(\bar{z}) - k_{\Delta+n-1}(z) k_{\Delta+n+l}(\bar{z}) \right\} \right] \\ & + \sum_{n=0,l=2}^{\infty} \left[c_l\, C_n^{(\Delta)} C_{n+l}^{(\Delta-1)} \left\{ k_{\Delta+n}(z) k_{\Delta+n+l-1}(\bar{z}) - k_{\Delta+n}(z) k_{\Delta+n+l-1}(\bar{z}) \right\} \right] \end{aligned} \tag{2.7.12}$$

Note here that in the second summation, it starts from $l = 2$. By the shift of integers: $n = n' - 1$ and $l = l' + 2$, the second summation rewritten as:

$$\text{(the 2nd line)} = -\sum_{n,l=0}^{\infty}\left[c_l\, C_{n-1}^{(\Delta)} C_{n+l+1}^{(\Delta-1)}\left\{k_{\Delta+n-1}(z)k_{\Delta+n+l}(\bar{z}) - k_{\Delta+n-1}(z)k_{\Delta+n+l}(\bar{z})\right\}\right] . \tag{2.7.13}$$

Now $\tilde{\mathcal{A}}^{\mathrm{GFT}}$ has a compact form:

$$\tilde{\mathcal{A}}^{\mathrm{GFT}}(z,\bar{z}) = \sum_{n,l=0}^{\infty} p_0(n,l)\left\{k_{\Delta+n-1}(z)k_{\Delta+n+l}(\bar{z}) - k_{\Delta+n-1}(z)k_{\Delta+n+l}(\bar{z})\right\}, \tag{2.7.14}$$

where the coefficient $p_0(n,l)$ is now defined as:

$$p_0(n,l) = c_l\left(C_{n+l}^{(\Delta)} C_n^{(\Delta-1)} - C_{n-1}^{(\Delta)} C_{n+l+1}^{(\Delta-1)}\right) \tag{2.7.15}$$

Multiplying the factor $(1/\bar{z} - 1/z)$ to the both sides of (2.7.14), we obtain the four-dimensional conformal block expansion:

$$\mathcal{A}^{\mathrm{GFT}}(z,\bar{z}) = \sum_{n,l=0}^{\infty} p_0(n,l)\, g_{2\Delta+2n+l,l}^{(4d)}(z,\bar{z}) \tag{2.7.16}$$

Again there are only the double trace operator in the spectrum, and the OPE coefficient is given in (2.7.15) . This result is also consistent with that obtained in [17].

2.7.3 General Dimension through the inversion formula

Here we will demonstrate how we can obtain the conformal block expansions through the inversion formula (2.6.11), using the simple example called generalized free theory. So far, the conformal block decompositions for two or four dimension are derived through the formulae associated with the Jacobi transformation. We will see that the inversion formula gives us a systematic method to drive decomposition which is available in arbitrary $d > 2$-dimensions.

Firstly, we focus on the t-channel diagram which is a product of two-point functions; $(x_{13}^2)^{\Delta}$ and $(x_{24}^2)^{\Delta}$. The inner product with the corresponding conformal partial wave is given as:

$$\begin{aligned}&\left(\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}}\,,\ \Psi_{h-i\nu,J}^{d-\Delta}\right) \\ &\qquad = \frac{1}{J!(h-1)_J}\int \frac{dP_0\,\Pi_{i=1}^4 dP_i}{\mathrm{vol}(SO(d+1,1))}\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}}\frac{(-2P_2\cdot C_0^{\mathcal{D}}\cdot P_1)^J(-2P_4\cdot C_0^{Z}\cdot P_3)^J}{P_{12}^{\gamma_{12}}P_{34}^{\gamma_{34}}\,\Pi_{i=1}^4 P_{0i}^{\gamma_{0i}}}\end{aligned} \tag{2.7.17}$$

Here we used the expression of CPW in the embedding space given in (2.6.12) . Now we consider identical scalars, namely the dimensions of external operators are the same: $\Delta_i = \Delta$. According to this reason, the external operators of CPW are also identical, and such CPWs can be regarded as the basis. The powers γ_{ij} are defined in (2.6.13), however, in this case Δ_i are replaced with $d-\Delta$, and the sign of ν is flipped $\nu \to -\nu$. Now the P_1 integration has the following form:

$$\mathcal{I}_1 \equiv \int dP_1 \frac{(-2P_2 \cdot C_0^{\mathcal{D}} \cdot P_1)^J}{P_{01}^{\frac{1}{2}(h-i\nu+J)} P_{12}^{d-\Delta+\frac{1}{2}(-h+i\nu+J)} P_{13}^{\Delta}} \tag{2.7.18}$$

According to the generalized Symanzik formula given in appendix B, this type of integration can be evaluated for arbitrary a_i as follows:

$$\int dP_1 \frac{(-2P_2 \cdot C_0 \cdot P_1)^J}{P_{01}^{a_0} P_{12}^{a_2} P_{13}^{a_3}} = \pi^h \frac{\Gamma(\delta_{02})\Gamma(\delta_{03})\Gamma(\delta_{23})}{\Gamma(a_0)\Gamma(a_2)\Gamma(a_3)} \frac{(2P_2 \cdot C_0^{\mathcal{D}} \cdot P_3)^J}{P_{02}^{\delta_{02}} P_{03}^{\delta_{03}} P_{23}^{\delta_{23}}} \tag{2.7.19}$$

where δ_{0i} are defined by:

$$\delta_{02} = \frac{1}{2}\left(a_0 + a_2 - a_3 - J\right), \ \delta_{03} = \frac{1}{2}\left(a_0 + a_3 - a_2 + J\right), \ \delta_{23} = \frac{1}{2}\left(a_2 + a_3 - a_0 + J\right).$$

Applying this formula we can evaluate the P_1-integration:

$$\mathcal{I}_1 = \mathcal{N}_1 \frac{(-2P_2 \cdot C_0^{\mathcal{D}} \cdot P_3)^J}{P_{23}^{\frac{1}{2}(h+i\nu+J)} P_{02}^{h-\Delta} P_{03}^{\frac{1}{2}(2\Delta-h-i\nu+J)}}, \quad \mathcal{N}_1 = \pi^h \frac{\Gamma\left(h-\Delta\right)\Gamma\left(\frac{h+i\nu+J}{2}\right)\Gamma\left(\frac{2\Delta-h-i\nu+J}{2}\right)}{\Gamma(\Delta)\Gamma\left(\frac{h-i\nu+J}{2}\right)\Gamma\left(\frac{3h+i\nu+J-2\Delta}{2}\right)}. \tag{2.7.20}$$

Now the inner product becomes:

$$\begin{aligned}\left(\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}}, \ \Psi^{d-\Delta}_{h-i\nu,J}\right) = \frac{(-1)^J \mathcal{N}_1}{J!(h-1)_J} \int \frac{dP_0 \prod_{i=2}^4 dP_i}{\mathrm{vol}(SO(d+1,1))} \\ \times \frac{(-2P_3 \cdot C_0^{\mathcal{D}} \cdot P_2)^J (-2P_4 \cdot C_0^{Z} \cdot P_3)^J}{P_{24}^{\Delta} P_{23}^{\frac{1}{2}(h+i\nu+J)} P_{02}^{\frac{1}{2}(3h-i\nu+J-2\Delta)} P_{34}^{\frac{1}{2}(3h-i\nu+J-2\Delta)} P_{03}^{\Delta+J} P_{04}^{\frac{1}{2}(h+i\nu+J)}}\end{aligned} \tag{2.7.21}$$

Next, the P_2 integration has the same form as the P_1 integration, and it is also easily evaluated through the formula again and the result is given by:

$$\begin{aligned}\mathcal{I}_2 &= \int dP_2 \frac{(-2P_3 \cdot C_0^{\mathcal{D}} \cdot P_2)^J}{P_{24}^{\Delta} P_{23}^{\frac{1}{2}(h+i\nu+J)} P_{02}^{\frac{1}{2}(3h-i\nu+J-2\Delta)}} \\ &= \pi^h \frac{\Gamma\left(h-\Delta\right)\Gamma\left(\frac{-h+i\nu+J+2\Delta}{2}\right)\Gamma\left(\frac{h-i\nu+J}{2}\right)}{\Gamma(\Delta)\Gamma\left(\frac{h+i\nu+J}{2}\right)\Gamma\left(\frac{3h-i\nu+J-2\Delta}{2}\right)} \frac{(-2P_3 \cdot C_0^{\mathcal{D}} \cdot P_4)^J}{P_{03}^{h-\Delta} P_{04}^{\frac{1}{2}(h-i\nu+J)} P_{34}^{\frac{1}{2}(-h+i\nu+J+2\Delta)}}\end{aligned} \tag{2.7.22}$$

After P_1 and P_2 integration, the inner product becomes the following form:

$$\begin{aligned}
&\left(\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}}\,,\ \Psi_{h-i\nu,J}^{d-\Delta}\right) &(2.7.23)\\
&= \frac{\pi^{2h}}{J!(h-1)_J}\frac{\Gamma(h-\Delta)^2\Gamma\left(\frac{2\Delta+J-h\pm i\nu}{2}\right)}{\Gamma(\Delta)^2\Gamma\left(\frac{2d-2\Delta+J-h\pm i\nu}{2}\right)}\int\frac{dP_0dP_3dP_4}{\mathrm{vol}(SO(d+1,1))}\frac{(-2P_4\cdot C_0^{\mathcal{D}}\cdot P_3)^J(-2P_4\cdot C_0^Z\cdot P_3)^J}{P_{34}^{h+J}P_{03}^{h+J}P_{04}^{h+J}}
\end{aligned}$$

The remaining integrations produce only constant factors because this part dose not depend on ν, and now we are interested in the pole structure in the ν plane. As for the contraction of C_0, it can be evaluated as the following way:

$$\begin{aligned}
\frac{1}{J!(h-1)_J}(-2P_4\cdot C_0^{\mathcal{D}}\cdot P_3)^J(-2P_4\cdot C_0^Z\cdot P_3)^J &= \sum_{r=0}^{[J/2]}(-1)^r\frac{J!(J+h-1)_{-r}}{2^{2r}r!(J-2r)!}(-4P_0\cdot\mathcal{P}_{34}\cdot\mathcal{P}_{34}\cdot P_0)^J\\
&= \frac{(2h-2)_J}{2^J(h-1)_J}(P_{03}P_{04}P_{34})^J\,. \qquad (2.7.24)
\end{aligned}$$

Now the integrals becomes:

$$\int\frac{dP_0dP_3dP_4}{\mathrm{vol}(SO(d+1,1))}\frac{1}{P_{34}^hP_{03}^hP_{04}^h}\,, \tag{2.7.25}$$

and it is evaluated by "gauge fixing". This integral is conformal invariant because we can regard it as an integration of products of three-point function and its shadow:

$$\int\frac{dP_0dP_3dP_4}{\mathrm{vol}(SO(d+1,1))}\langle\mathcal{O}_0(P_0)\mathcal{O}_3(P_3)\mathcal{O}_4(P_4)\rangle_1\langle\tilde{\mathcal{O}}_0(P_0)\tilde{\mathcal{O}}_3(P_3)\tilde{\mathcal{O}}_4(P_4)\rangle_1\,, \tag{2.7.26}$$

here the operators $\mathcal{O}_i$ have dimensions Δ_i and the shadow operators $\tilde{\mathcal{O}}_i$ have dimensions $d-\Delta_i$. In this expression, manifestly the integral is conformally invariant. Now we can fix the points at special values $(x_0,x_3,x_4)=(0,1,\infty)$, this integral is evaluated:

$$\int\frac{dP_0dP_3dP_4}{\mathrm{vol}(SO(d+1,1))}\frac{1}{P_{34}^hP_{03}^hP_{04}^h} = \frac{1}{\mathrm{vol}(SO(d-1))}\,. \tag{2.7.27}$$

Here $SO(d-1)$ is the stabilizer group of the fixed three-points. Therefore the integral in (2.7.24) is evaluated as follows:

$$\frac{1}{J!(h-1)_J}\int\frac{dP_0dP_3dP_4}{\mathrm{vol}(SO(d+1,1))}\frac{(-2P_4\cdot C_0^{\mathcal{D}}\cdot P_3)^J(-2P_4\cdot C_0^Z\cdot P_3)^J}{P_{34}^{h+J}P_{03}^{h+J}P_{04}^{h+J}} = \frac{(2h-2)_J}{2^J(h-1)_J}\frac{1}{\mathrm{vol}(SO(d-1))}\,. \tag{2.7.28}$$

Substituting the result into the inversion formula (2.6.11), we have the following expansion:

$$\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}} = \sum_{J=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{n_{\nu,J}}\frac{(2h-2)_J}{2^J(h-1)_J}\frac{\pi^h}{\mathrm{vol}(SO(d-1))}\frac{\Gamma(h-\Delta)^2\Gamma\left(\frac{2\Delta+J-h\pm i\nu}{2}\right)}{\Gamma(\Delta)^2\Gamma\left(\frac{2d-2\Delta+J-h\pm i\nu}{2}\right)}\Psi_{h+i\nu}^{\Delta}(P_i) \tag{2.7.29}$$

Note here that the normalization factor $n_{\nu,J}$ and the resultant inner product are invariant under $\nu \leftrightarrow -\nu$. By using this property and (2.6.5), the above equation is written as the following form:

$$\frac{1}{P_{13}^{\Delta}P_{24}^{\Delta}} = \frac{(2h-2)_J\Gamma(h-\Delta)^2}{2^J(h-1)_J\Gamma(\Delta)^2}\frac{\pi^h}{\mathrm{vol}(SO(d-1))}\sum_{J=0}^{\infty}\int_{-\infty}^{\infty}d\nu\frac{K_{h-i\nu}^{\Delta,\Delta}}{n_{\nu,J}}\frac{\Gamma\left(\frac{2\Delta+J-h\pm i\nu}{2}\right)}{\Gamma\left(\frac{2d-2\Delta+J-h\pm i\nu}{2}\right)}g_{h+i\nu}^{\Delta}(P_i)\,. \tag{2.7.30}$$

Here $g_{h+i\nu}^{\Delta}$ is the conformal block. Now the integration contour can be closed in the lower half plane, and then the integration picks up poles at $h+i\nu = 2\Delta + J + 2n$ $(n = 0, 1, 2, ...)$ which come from the gamma function in the numerator. This pole values precisely are the dimensions of the double trace operators we observed in two or four dimensions. And the residue becomes the coefficient of the conformal block expansion.

In the u-channel calculation, the corresponding two-point functions are $P_{14}^{-\Delta}P_{23}^{-\Delta}$. This diagram produces the same $\mathcal{I}_1$ and $\mathcal{I}_2$ with an exchange $3 \leftrightarrow 4$. Then we have $(-2P_3 \cdot C_0^{\mathcal{D}} \cdot P_4)^J$ in stead of $(-2P_4 \cdot C_0^{\mathcal{D}} \cdot P_3)^J$ in (2.7.24). This difference gives the additional factor $(-1)^J$.

In this way, through the inversion formula, we can obtain the conformal decomposition of four-point correlation function systematically.

Chapter 3

Diagrams in AdS Space

In this chapter, we will consider some Feynman diagrams in AdS space whose legs are on the boundary of the AdS space. It is known that such diagrams have the same symmetry as CFT correlation functions, for example, a three-point diagram is proportional to the kinematical form of CFT three-point function as we show in the below. Especially, in the context of the AdS/CFT correspondence, it is conjectured that a summation of such diagrams in a bulk theory gives a correlation function of the dual conformal field theory. In the previous chapter, we have discussed that a four-point function has a conformal block decomposition. Now the question we are interested in is what kind of conformal block appears from a given AdS diagram. The conformal partial wave we introduced in the previous chapter is a useful tool to answer this question. According to the correspondence between three-point diagrams and correlation functions, CPW naturally has its bulk interpretation. Then the inversion formula provides a systematic way to see the conformal block decomposition of bulk diagrams. We will see some simple examples of conformal block decomposition of tree diagrams. Not only to obtain the conformal block decomposition of AdS diagram, but the bulk interpretation is also useful as a calculation tool. The orthogonality of CPW can be seen clearly in bulk, thanks to properties of the AdS harmonic functions.

Another interesting concept we discuss in this chapter is the so-called geodesic diagram. In the geodesic diagram, the interaction points are restricted on geodesics which are connecting two boundary points. This diagram is proposed as the bulk dual of conformal block [20]. We will show that the three-point geodesic diagram has a different coefficient form the usual AdS three-point diagram, due to the difference, the four-point geodesic diagram is proportional to the conformal block while the usual diagrams are not. We will also discuss extensions to the case including external tensor fields. Parametrizing the bulk interaction properly, we can reproduce the CFT tensor structures discussed in the previous chapter from the bulk diagram with tensor fields.

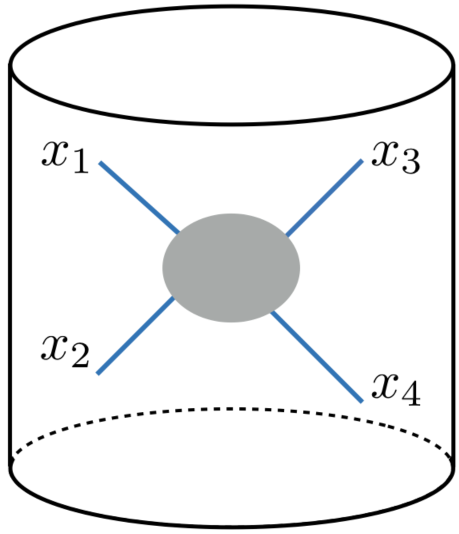


Figure 3.1: A diagram in AdS space. The external legs which is expressed as blue lines have the edges on the boundary of AdS space.

3.1 Embedding Formalism for AdS

In section 2.2, the embedding formalism for d-dimensional euclidean space is discussed. Here we will briefly review the embedding formalism for AdS space (for the detail, see [46, 52–54]). $d+1$-dimensional AdS is also embedded into $\mathbb{M}^{1,d+1}$ which is the same embedding space before. It is convenient to see the relation between AdS diagrams and CFT correlation functions because the coordinates of AdS and CFT are dealt with on an equal footing in the embedding space. The following argument is almost parallel to the previous Euclidean case.

In $d+2$ dimensional embedding space $\mathbb{M}^{d+1,1}$, the euclidean AdS_{d+1} space is defined by the set of future directed unit vectors satisfying:

$$X \cdot X = \eta_{AB} X^A X^B = -1, \qquad \eta_{AB} = \mathrm{diag}(-1, 1, \ldots, 1), \qquad X^0 > 1 \tag{3.1.1}$$

which can also be viewed as a $d+1$ dimensional hyperboloid, and we have set the radius of curvature to be 1. We can parametrize the solutions to (3.1.1) explicitly in the light cone coordinates:

$$(X^+, X^-, X^a) = \frac{1}{z}(1, x^2 + z^2, x^\mu), \quad X \cdot X = -X^+ X^- + \delta_{\mu\nu} X^\mu X^\nu, \quad \mu, \nu = 0, \ldots, d \tag{3.1.2}$$

in terms of the Poincare coordinates $y^a = (z, x^\mu)$ of AdS_{d+1} space. Towards the boundary AdS_{d+1}: $z \to 0$, the hyperboloid asymptotes to the light cone $X \cdot X = 0$ by dropping a divergent factor, i. e. the conformal boundary $\mathbb{R}^d$ is identified with the projective cone of light rays in the embedding

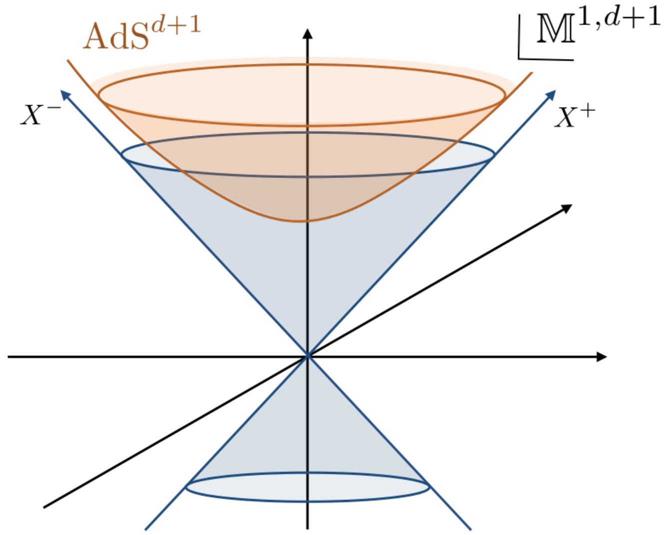


Figure 3.2: $d+1$ dimensional Euclidean AdS space is embedded in $\mathbb{M}^{1,d+1}$ as a hyperboloid.

space which corresponds to the $\mathbb{R}^d$ we discussed in section 2.2.

As similar as the Euclid space case, we consider embedding tensor fields in AdS$_{d+1}$ into the embedding space $\mathbb{M}^{1,d+1}$. Again, an arbitrary rank-J tensor field in AdS$_{d+1}$ is related to its embedding space counterpart through the pull-back operation:

$$\mathcal{T}^{(\mathrm{AdS})}_{a_1\dots a_J}(y) = \frac{\partial X^{A_1}}{\partial y^{a_1}} \dots \frac{\partial X^{A_J}}{\partial y^{a_J}} T_{A_1\dots A_J}(X), \tag{3.1.3}$$

In particular, the AdS$_{d+1}$ metrics are also given by:

$$g^{(\mathrm{AdS})}_{ab} = \frac{\partial X^A}{\partial y^a}\frac{\partial X^B}{\partial y^b}\eta_{AB}. \tag{3.1.4}$$

However this pull-back operations defined is not injective as same as the previous $\mathbb{R}^d$ case, because the following matrix in the pull-back relation has a zero vector:

$$X_A \frac{\partial X^A}{\partial y^a}\bigg|_{X\cdot X=-1} = 0. \tag{3.1.5}$$

This means that there are redundant unphysical degrees of freedom which correspond to components proportional to $X_{(A_1}H_{A_2\dots A_J)}(X)$ contained in a tensor field $T_{A_1\dots A_J}(X)$. Geometrically, these additional components regarded as being normal to the hypersurface (3.1.1). To eliminate these unphysical redundant degrees of freedom in the embedding space tensors, the transverse condition is imposed:

$$X^{A_1}T_{A_1\ldots A_J}(X)\mid_{X\cdot X=-1}=0, \tag{3.1.6}$$

such that $T_{A_1\ldots A_J}(X)$ only contains the components which are tangent to AdS_{d+1}.

In the below, again we mainly consider symmetric traceless AdS_{d+1} tensor fields. In embedding space $\mathbb{M}^{d+1,1}$, they need to be symmetric traceless also transverse (STT). For the convenience, we introduce the polarization vectors W^A and consider the following generating polynomial:

$$T(X,W)=W^{A_1}\ldots W^{A_J}T_{A_1\ldots A_J}(X), \qquad X\cdot W=W\cdot W=0. \tag{3.1.7}$$

Here we have introduced the auxiliary vectors W^A, $X\cdot W=0$ and $W\cdot W=0$ imply $T_{A_1\ldots A_J}(X)$ is defined up to equivalence $\sim X_{(A_1}H_{A_2\ldots A_J)}(X)+\eta_{(A_1A_2}S_{A_3\ldots A_J)}(X)$, the contraction with W^As only picks up the symmetric, traceless and transverse components.

To recover embedding space STT tensors representing symmetric traceless AdS_{d+1} tensors directly from (3.1.7), it is convenient to define the operators K_A which act on the symmetric products of W^A as:

$$\frac{1}{J!\left(\frac{d-1}{2}\right)_J}K_{A_1}\ldots K_{A_J}W^{B_1}\ldots W^{B_J}=G^{B_1}_{\{A_1}\ldots G^{B_J}_{A_J\}}=G^{B_1}_{(A_1}\ldots G^{B_J}_{A_J)}-\text{traces}, \tag{3.1.8}$$

where $(\ldots)$ in the above implies total symmetrization of indices, and $G_{AB}\equiv\eta_{AB}+X_AX_B$ is the induced AdS metric. This differential operator is the AdS counterpart of operator D_A defined in 2.2.13. The explicit expression for the operator K_A can be given in terms of the following differential operator:

$$\begin{aligned}K_A=&\frac{d-1}{2}\left(\frac{\partial}{\partial W^A}+X_A\left(X\cdot\frac{\partial}{\partial W}\right)\right)+\left(W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^A}\\&+X_A\left(W\cdot\frac{\partial}{\partial W}\right)\left(X\cdot\frac{\partial}{\partial W}\right)-\frac{1}{2}\left(\frac{\partial^2}{\partial W\cdot\partial W}+\left(X\cdot\frac{\partial}{\partial W}\right)\left(X\cdot\frac{\partial}{\partial W}\right)\right),\end{aligned} \tag{3.1.9}$$

however we mostly will not use these somewhat lengthy expressions in the main text, only the formal operation (3.1.8) will be sufficient. When the contracted embedding space tensor in the generating polynomial is already traceless and transverse, the action of K_A simplifies to

$$K_A=\left(\frac{d-1}{2}+W\cdot\frac{\partial}{\partial W}\right)\frac{\partial}{\partial W^A} \tag{3.1.10}$$

Finally, we can consider AdS_{d+1} covariant derivatives in the embedding space, it acts on the embedding space tensor satisfying the transverse condition (3.1.6), and the resultant tensor should remain so after its action. The following differential operator in $\mathbb{M}^{1,d+1}$ satisfies such requirement:

$$\nabla_A = \frac{\partial}{\partial X^A} + X_A \left(X \cdot \frac{\partial}{\partial X} \right) + W_A \left(X \cdot \frac{\partial}{\partial W} \right) = G_A{}^B \frac{\partial}{\partial X^B} + W_A \left(X \cdot \frac{\partial}{\partial W} \right) \tag{3.1.11}$$

we can clearly see that $X^A \nabla_A = 0$, and moreover if the contracted tensor in (3.1.7) already satisfies the transverse condition, the action of the last term is trivial. We can express the action of ∇_A on such a tensor which is the representative of an AdS_{d+1} tensor as:

$$\nabla_B T_{A_1 \ldots A_J}(X) = G_B{}^C G_{A_1}{}^{C_1} \ldots G_{A_J}{}^{C_J} \frac{\partial}{\partial X^C} T_{C_1 \ldots C_J}(X). \tag{3.1.12}$$

In particular, it is worth noting that induced AdS_{d+1} metric G_{AB} itself also satisfies transverse condition $X^A G_{AB} = G_{AB} X^B = 0$, we have

$$\nabla_C G_{AB} = G_C{}^{C'} G_A{}^{A'} G_B{}^{B'} \frac{\partial}{\partial X^{C'}} G_{A'B'} = 0 \tag{3.1.13}$$

as required for ∇_A to be the metric covariant derivative in the embedding space.

3.2 Propagator in AdS Space

In this section, we review some basics of propagator in the $d+1$-dimensional AdS space. Firstly, we will discuss scalar fields and derive its bulk-to-bulk and bulk-to-boundary propagator, after that, consider propagators of tensor fields. We can see that the embedding formalism introduced the previous section is a powerful tool to deal with tensor fields again.

3.2.1 Scalar Field

Let us consider a massive scalar field in AdS. The action with a scalar source $\mathcal{J}(y)$ is given as below:

$$\int_{\mathrm{AdS}_{d+1}} d^{d+1}y \, \sqrt{g} \left[\frac{1}{2} g^{ab} \, \partial_a \phi \, \partial_b \phi + \frac{1}{2} m^2 \phi^2 - \phi \, \mathcal{J} \right], \tag{3.2.1}$$

where the metric $g_{\mu\nu}$ is written in terms of the usual Poincaré coordinate:

$$ds^2 = g_{ab} \, dy^a dy^b = \frac{1}{z^2} \left(dz^2 + \eta_{\mu\nu} \, dx^\mu dx^\nu \right) . \tag{3.2.2}$$

Here the AdS radius is set as $R = 1$, and we follow the notation for AdS coordinate $y_i^a = (z_i, x_i^\mu)$. Then the field $\phi(y)$ can be written as response to the source by using the Green function $\Pi_\Delta(y, y')$:

$$\phi(y) = \int_{\mathrm{AdS}_{d+1}} d^{d+1}y' \, \sqrt{g} \, \Pi_\Delta(y, y') \, \mathcal{J}(y') \,, \tag{3.2.3}$$

where $\Pi_\Delta(y, y')$ is the Green function for the Klein-Gordon operator:

$$\left(\Box_{\mathrm{AdS},1} - m^2\right) \Pi_\Delta(y_1, y_2) = -\frac{1}{\sqrt{g}} \delta(z_1 - z_2) \, \delta^{(d)}(x_1 - x_2) \,. \tag{3.2.4}$$

$\Pi_\Delta(y, y')$ is characterized a parameter Δ which is related to the mass m^2 through the relation $m^2 = \Delta(\Delta - d)$[1]. $\Box_{\mathrm{AdS},1}$ is the Laplacian operator acting on y_1, and it is given as:

$$\begin{aligned} \Box_{\mathrm{AdS}} &= \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g} \, g^{ab} \, \partial_b \, \phi \right) \\ &= z^2 \partial_z^2 - (d-1) \, z \, \partial_z + z^2 \, \eta^{\mu\nu} \partial_\mu \partial_\nu \,. \end{aligned} \tag{3.2.5}$$

The solution of (3.2.4) is known, and it is given by the following hypergeometric function:

$$\Pi_\Delta(y_1, y_2) = \tilde{\mathcal{C}}_\Delta \, \xi^\Delta \, {}_2F_1 \left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - h + 1; \xi^2 \right) \,, \tag{3.2.6}$$

where ξ is defined as:

$$\xi = \frac{2 z_1 z_2}{z_1^2 + z_2^2 + |x_1 - x_2|^2} \,, \tag{3.2.7}$$

and $\tilde{\mathcal{C}}_\Delta$ is a normalization factor:

$$\tilde{\mathcal{C}}_\Delta = \frac{\Gamma(\Delta)}{2^{\Delta+1} \, \pi^h \, \Gamma(\Delta - h + 1)} \,. \tag{3.2.8}$$

The propagator in (3.2.6) is called the bulk-to-bulk propagator which connecting two bulk points. The bulk-to-boundary propagator can be obtained by pulling one point in bulk to the boundary of AdS. Then the answer becomes simple because the hypergeometric function is reduced to 1 in the limit:

$$\Pi_\Delta(y_1, x_2) = \mathcal{C}_\Delta \, \left(\frac{z_1}{z_1^2 + (x_1 - x_2)^2} \right)^\Delta \tag{3.2.9}$$

and the normalization factor is given by:

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2 \, \pi^h \Gamma(\Delta - h + 1)} \,. \tag{3.2.10}$$

[1] This convention comes the dictionary for the AdS/CFT correspondence. There are two solutions for Δ, however, due to the unitary bound $\Delta > d/2$, we always pick up the grater solution.

3.2.2 Field with Spin

Next we will consider bulk-to-bulk propagators for fields with spin. It is a difficult problem to write a consistent action in the Euclidean AdS space for general massive tensor fields. For example, in the case of massive spin-2 tensor fields, the action in Euclidean AdS is given in [55] as the linearization of the Einstein-Hilbert action with a negative cosmological constant and the Fierz-Pauli mass term. The equations which the field satisfies are given by:

$$\left(\Box_{\text{AdS}} - m^2\right) h_{ab} = 0\,, \quad D_a\, h^a{}_b = 0\,, \quad h^a{}_a = 0 \tag{3.2.11}$$

In the case of general spins, it is hard to write the action in AdS space, however, the equations which the on-shell fields satisfy are given by

$$\left(\Box_{\text{AdS}} - m^2\right) \mathcal{T}_{a_1 \ldots a_J} = 0\,, \quad D^a \mathcal{T}_{a a_2 \ldots a_J} = 0\,, \quad \mathcal{T}^a{}_{a a_3 \ldots a_J} = 0\,. \tag{3.2.12}$$

According to these equations, the bulk-to-bulk propagator should satisfy the following equations:

$$\begin{aligned} (\Box_{\text{AdS},1} - m^2)\, \Pi_{\Delta,J}(X_1, X_2; W_1, W_2) \,&=\, (W_{12})^J \delta(X_1, X_2)\,, \\ (K_1 \cdot \nabla_1)\, \Pi(X_1, X_2; W_1, W_2) \,&=\, 0\,. \end{aligned} \tag{3.2.13}$$

where the mass is taken as $m^2 = \Delta(\Delta - d) - J$, so that Δ corresponds to the dimension of the dual operator. Some concrete examples of solution of these equations for $J = 1, 2$ are given in [54], and the general solution is also discussed which is written by using the AdS harmonic function we will introduce in the next section. We postpone giving a form of the bulk-to-bulk propagator with spin J to the next section. As for bulk-to-boundary propagators, they can be determined by the transverse condition and scaling property in below:

$$\Pi_{\Delta,J}(X, \alpha P; \beta W, \gamma Z + \delta P) = \alpha^{-\Delta} (\beta\, \gamma)^J\ \Pi_{\Delta,J}(X, P; W, Z) \tag{3.2.14}$$

The solution has the following form:

$$\Pi_{\Delta,J}(X, P; W, Z) = \mathcal{C}_{\Delta,J} \frac{(-2W \cdot C \cdot X)^J}{(-2P \cdot X)^{\Delta + J}}\,, \tag{3.2.15}$$

where $\mathcal{C}_{\Delta,J}$ is a constant which cannot be determined by symmetry. The coefficient is determined by taking the limit to bring one of the bulk points of bulk-to-bulk propagator to the boundary. It is given by [54]:

$$\mathcal{C}_{\Delta,J} = \frac{(J + \Delta - 1)\Gamma(\Delta)}{2\, \pi^h (\Delta - 1)\Gamma(\Delta - h + 1)}\,. \tag{3.2.16}$$

3.3 AdS Harmonic Function

Here we introduce the AdS harmonic function. This function has some nice properties, especially it has orthogonality, and it plays a crucial role in later calculations. The definition of AdS harmonic function is given by the boundary integration of two bulk-to-boundary propagators:

$$\Omega_{\nu,J}(X_1, X_2; W_1, W_2) \equiv \frac{\nu^2}{\pi\, J!(h-1)_J} \int_{\partial\mathrm{AdS}} d^dP_0\; \Pi_{h+i\nu,J}(X_1, P_0; W_1, \mathcal{D}_{Z_0})\Pi_{h-i\nu,J}(X_2, P_0; W_2, Z_0) \tag{3.3.1}$$

and diagrammatically it can be written as in Fig: 3.3. It is known that the AdS harmonic function

$$\Omega_{\nu,J}(X_1, X_2; W_1, W_2) \sim \int_{\partial\mathrm{AdS}} dP_0$$

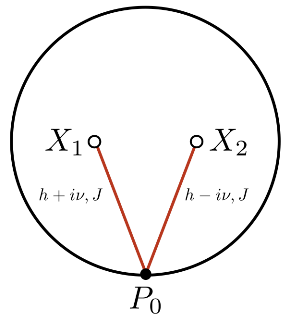


Figure 3.3: The definition of AdS harmonic function. It is given as a boundary integration with two bulk-to-boundary propagators which share a same boundary point. The mass (dimension) of the propagators are $h \pm i\nu$.

has a different expression as a linear combination of bulk-to-bulk propagators [54, 56]:

$$\Omega_{\nu,J}(X_1, X_2; W_1, W_2) = \frac{i\nu}{2\pi}\left(\Pi_{h+i\nu,J}(X_1, X_2; W_1, W_2) - \Pi_{h-i\nu,J}(X_1, X_2; W_1, W_2)\right) \tag{3.3.2}$$

According to this relation, we can show that the AdS harmonic function satisfies the following equations:

$$\begin{aligned} \left(\Box_{\mathrm{AdS},1} + h^2 + \nu^2 + J\right)\Omega_{\nu,J}(X_1, X_2; W_1, W_2) &= 0\,, \\ \left(\nabla_{\mathrm{AdS},1}\cdot K_1\right)\Omega_{\nu,J}(X_1, X_2; W_1, W_2) &= 0 \end{aligned} \tag{3.3.3}$$

Another important property of the harmonic function is the orthogonal relation. The AdS harmonic satisfies the following relation:

$$\begin{aligned} &\frac{1}{J!\left(h-\frac{1}{2}\right)_J} \int_{\mathrm{AdS}} dX_0\; \Omega_{\nu,J}(X_1, X_0; W_1, K_0)\, \Omega_{\nu',J'}(X_0, X_2; W_0, W_2) \\ &\qquad = \frac{1}{2}\,\delta_{J,J'}\,\left[\delta(\nu-\nu') + \delta(\nu+\nu')\right]\,\Omega_{\nu,J}(X_1, X_2; W_1, W_2) \end{aligned} \tag{3.3.4}$$

This relation means that two harmonic functions which are connected can be combined into one harmonic function with a delta function. A proof of this relation is given in [54]. The AdS harmonic function has also the completeness relation:

$$(W_{12})^J\,\delta(X_1,X_2) = \sum_{l=0}^{J}\int_{-\infty}^{+\infty} d\nu\, c_{J,l}(\nu)\,\left((W_1\cdot\nabla_1)(W_2\cdot\nabla_2)\right)^l\,\Omega_{\nu,J-l}(X_1,X_2;W_1,W_2) \quad (3.3.5)$$

where $c_{J,l}$ is given by:

$$c_{J,l}(\nu) = \frac{2^l(J-l+1)_l(h+J-l-\frac{1}{2})_l}{l!\,(2h+2J-2l-1)_l(h+J-l-i\nu)_l}\,. \quad (3.3.6)$$

This coefficients are determined recursively by using the equations (3.3.3) and setting $c_{J,0}(\nu)=1$. This completeness relation means that the AdS harmonic function forms a basis for symmetric traceless tensors in AdS space. By using the AdS harmonic function, the bulk-to-bulk propagator can be expanded as

$$\Pi_{\Delta,J}(X_1,X_2;W_1,W_2) = \sum_{l=0}^{J}\int_{-\infty}^{+\infty} d\nu\, a_{J,l}(\nu)\,\left((W_1\cdot\nabla_1)(W_2\cdot\nabla_2)\right)^l\,\Omega_{\nu,J-l}(X_1,X_2;W_1,W_2)\,. \quad (3.3.7)$$

The coefficients $a_{J,l}(\nu)$ are determined by the equations of motions in 3.2.13:

$$\begin{aligned} a_{J,J}(\nu) &= \frac{1}{\nu^2+(\Delta-h)^2}\,, \\ a_{J,l}(\nu) &= \sum_{q=1}^{J-l}\frac{(l+q)!}{l!q!}\frac{(-1)^{q+1}}{2^{q-1}(q-1)!(h+l)_{q-1}}\frac{a_{l+q}(i(h-1+l))}{\nu^2+(h+l+q-1)^2}. \end{aligned} \quad (3.3.8)$$

The details of (3.3.5) and (3.3.7) are also given in [54].

3.4 Tree Diagram

In this section, we will calculate some tree diagrams in AdS space.

3.4.1 Three-point scalar diagram

Firstly, we consider a three-point tree diagram with three scalar fields. In this case the bulk interaction is unique, and it is just the ϕ^3 interaction. This diagram is given by the following integration with three scalar bulk-to-boundary propagators:

$$\mathcal{A}^{\text{3-pt}}_{\text{scalar}}(P_i) = \int dX\;\Pi_{\Delta_1}(P_1,X)\,\Pi_{\Delta_2}(P_2,X)\,\Pi_{\Delta_3}(P_3,X)$$

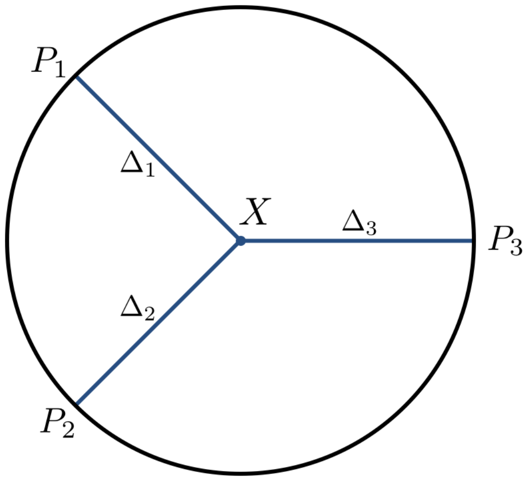


Figure 3.4: A three point scalar diagram with a ϕ^3 interaction in bulk. Each blue line is a bulk-to-boundary propagator.

$$= \int dX \frac{\mathcal{C}_{\Delta_1}}{(-2P_1 \cdot X)^{\Delta_1}} \frac{\mathcal{C}_{\Delta_2}}{(-2P_2 \cdot X)^{\Delta_2}} \frac{\mathcal{C}_{\Delta_3}}{(-2P_3 \cdot X)^{\Delta_3}} . \tag{3.4.1}$$

This integration can be done by applying the Symanzik formula given in appendix B for the $n = 3$ case. Usually, the Symanzik formula provides some Mellin integration. However, in the three-point case, the Mellin integration does not appear, and the result is proportional to a three-point correlation function in CFT as we will see in the below. Using the Schwinger parameterization,

$$\frac{1}{(-P_i \cdot X)^{\Delta_i}} = \frac{1}{\Gamma(\Delta_i)} \int_0^\infty \frac{dt_i}{t_i} \, t^{\Delta_i} e^{-(-2P_i \cdot X)t_i} , \tag{3.4.2}$$

we can rewrite the integration as

$$\mathcal{A}^{\text{3-pt}}_{\text{scalar}}(P_i) = \frac{1}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \int_0^\infty \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3} t_1^{\Delta_1} t_2^{\Delta_2} t_3^{\Delta_3} \int dX e^{2Q\cdot X} , \tag{3.4.3}$$

where Q is defined as $Q \equiv \sum_{i=1}^3 t_i P_i$. Because $Q \cdot X$ is a scalar under the Lorentz transformation in the embedding space $\mathbb{M}^{1,d+1}$, we can choose Q as $|Q|(1, 1, 0)$ where $|Q|^2 = \sum_{i>j} t_i t_j P_{ij}$. Now the coordinate X is parametrized as $(1, z^2 + y^2, y^\mu)/z$, we can evaluate the AdS integration

$$\begin{aligned} \int dX e^{2Q\cdot X} &= \int_0^\infty \frac{dz}{z} \int_{\mathbb{R}^d} \frac{d^d y}{z^d} e^{-\frac{|Q|}{z}(z^2+y^2+1)} \\ &= \pi^h \int_0^\infty \frac{dz}{z} \frac{1}{(z|Q|)^h} e^{-\frac{|Q|}{z}(z^2+1)} \\ &= \pi^h \int_0^\infty \frac{dz}{z} \frac{1}{z^h} e^{-\left(z+\frac{|Q|^2}{z}\right)} , \end{aligned} \tag{3.4.4}$$

in the last line, z is scaled as $z \to |Q|^{-1}z$. Scaling t_i as $t_i \to t_i\sqrt{z}$, we can perform the z integration

$$
\begin{aligned}
\mathcal{A}^{\text{3-pt}}_{\text{scalar}}(P_i) &= \pi^h \left(\prod_{i=1}^{3} \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right) \int_0^\infty \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} \int_0^\infty \frac{dz}{z} z^{\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}} e^{-z-|Q|^2} \\
&= \pi^h \left(\prod_{i=1}^{3} \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right) \Gamma\left(\frac{\sum_{i=1}^{3}\Delta_i - d}{2}\right) \int_0^\infty \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} t_1^{\Delta_1} t_2^{\Delta_2} t_3^{\Delta_3} e^{-\sum_{i>j} t_i t_j P_{ij}}
\end{aligned}
\tag{3.4.5}
$$

By utilizing the following parameterization:

$$
t_1 = \sqrt{\frac{m_1 m_3}{m_2}}\,, \qquad t_2 = \sqrt{\frac{m_1 m_2}{m_3}}\,, \qquad t_3 = \sqrt{\frac{m_2 m_3}{m_1}}\,, \tag{3.4.6}
$$

the t_i integration can be calculated as

$$
\begin{aligned}
&\int_0^\infty \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} t_1^{\Delta_1} t_2^{\Delta_2} t_3^{\Delta_3} e^{-\sum_{i>j} t_i t_j P_{ij}} \\
&\qquad = \frac{1}{2} \int_0^\infty \frac{dm_1}{m_1}\frac{dm_2}{m_2}\frac{dm_3}{m_3} m_1^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} m_2^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}} m_3^{\frac{\Delta_3+\Delta_1-\Delta_2}{2}} e^{-m_1 P_{12} - m_2 P_{23} - m_3 P_{31}} \\
&\qquad = \frac{1}{2}\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+\Delta_3}{2}\right) \\
&\qquad\quad \times P_{12}^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{23}^{-\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)} P_{31}^{-\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}\,.
\end{aligned}
\tag{3.4.7}
$$

Therefore the three point scalar diagram (3.4.1) can be evaluated as

$$
\mathcal{A}^{\text{3-pt}}_{\text{scalar}}(P_i) = \mathcal{B}^{\text{3-pt}}_{\text{scalar}} \frac{1}{P_{12}^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{23}^{\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)} P_{31}^{\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}}\,, \tag{3.4.8}
$$

where the coefficient $\mathcal{B}^{\text{3-pt}}_{\text{scalar}}$ is

$$
\begin{aligned}
\mathcal{B}^{\text{3-pt}}_{\text{scalar}} \equiv \pi^h &\left(\prod_{i=1}^{3} \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right) \Gamma\left(\frac{\sum_{i=1}^{3}\Delta_i - d}{2}\right) \\
&\times \Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+\Delta_3}{2}\right)\,.
\end{aligned}
\tag{3.4.9}
$$

The result in (3.4.8) is precisely proportional to the CFT three-point function with scalar primary operators whose dimensions are Δ_i.

3.4.2 Three-Point Diagram with Two Scalars and One Tensor

Next, for a later purpose, we will consider a three-point function with two scalars and one symmetric tensor field. This diagram plays an important role when considering the bulk interpretation of the conformal partial wave. In this case, the bulk interaction contains covariant derivatives because the

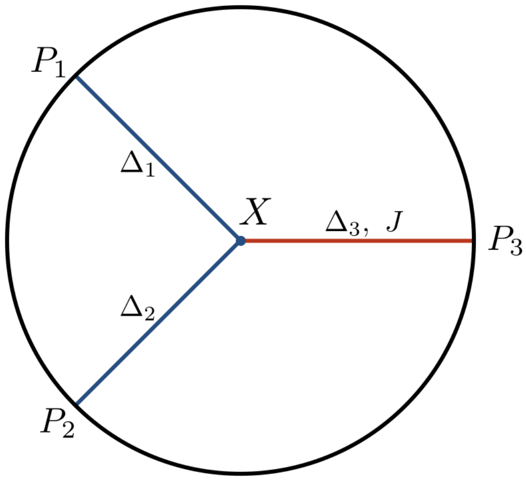


Figure 3.5: A three point diagram with two scalars and one spin-J tensor. The bulk interaction is given as $\phi_1 \nabla^J_{a_1 \dots a_J} \phi_2 \, \mathcal{T}^{a_1 .. a_J}$.

third field has spin indexes and it is needed to contract with some differential operators to obtain a Lorentz invariant vertex. The diagram is given by the following integral:

$$\mathcal{A}^{\text{3-pt}}_{0,0,J}(P_i) \equiv \frac{1}{J! \left(h - \frac{1}{2}\right)_J} \int_{\text{AdS}} dX \; \Pi_{\Delta_1,0}(P_1, X) \, (K \cdot \nabla)^J \; \Pi_{\Delta_2,0}(P_2, X) \, \Pi_{\Delta_3,J}(P_3, X; Z_3, W) \tag{3.4.10}$$

The factor $1/J! \left(h - \frac{1}{2}\right)_J$ in front of the integral is the same factor in (3.1.8), it is put to contract the Lorentz indexes in the embedding space. Now the covariant derivatives are acting on the second propagator, and this diagram gives the same result from the diagram with covariant derivatives acting on the first field. These diagrams are equivalent up to the integral by part. Note here that if a covariant derivative acts on the third field, it vanishes because of its equations of motion. This diagram is evaluated as follows:

$$\mathcal{A}^{\text{3-pt}}_{0,0,J}(P_i) \tag{3.4.11}$$
$$= \mathcal{C}_{\Delta_1} \mathcal{C}_{\Delta_2} \mathcal{C}_{\Delta_3,J} \int_{\text{AdS}} dX \; \frac{1}{(-2P_1 \cdot X)^{\Delta_1}} \, \nabla^{A_1 \dots A_J} \frac{1}{(-2P_2 \cdot X)^{\Delta_2}} \, \frac{(-2C_3 \cdot X)^J_{A_1 \dots A_J}}{(-2P_3 \cdot X)^{\Delta_3 + J}} \, . \tag{3.4.12}$$

Here $\nabla^{A_1 \dots A_J}$ means $\nabla^{A_1} \dots \nabla^{A_J}$ and $(-2C_3 \cdot X)^J_{A_1 \dots A_J}$ is also defined in the same manner. We have used the relation (3.1.8) to take the contraction and the trace subtraction can be dropped because $X \cdot C_3 \cdot C_3 \cdot X = 0$. The contraction part is computed as follows:

$$\nabla^{A_1 \dots A_J} \frac{1}{(-2P_2 \cdot X)^{\Delta_2}} \, \frac{(-2C_3 \cdot X)^J_{A_1 \dots A_J}}{(-2P_3 \cdot X)^{\Delta_3 + J}} = 2^J (\Delta_2)_J \frac{(P_2 \cdot G)_{A_1 \dots A_J}}{(-2P_2 \cdot X)^{\Delta_2 + J}} \, \frac{(-2C_3 \cdot X)^J_{A_1 \dots A_J}}{(-2P_3 \cdot X)^{\Delta_3 + J}} \tag{3.4.13}$$

$$= 2^J (\Delta_2)_J \; \frac{1}{(-2P_2\cdot X)^{\Delta_2+J}} \frac{(-2P_2\cdot C_3\cdot X)^J}{(-2P_3\cdot X)^{\Delta_3+J}} \,.$$

In the last line, because $X\cdot C_3\cdot X=0$, the contraction becomes simple: $P_2\cdot G\cdot C_3\cdot X = P_2\cdot C_3\cdot X$. Now the integral can be computed as the scalar diagram using a differential operator introduced below:

$$\mathcal{A}^{\text{3-pt}}_{0,0,J}(P_i) = \mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta_3,J}\frac{2^J(\Delta_2)_J}{(\Delta_3)_J}(\mathcal{D}_{32})^J \int_{\text{AdS}} dX \; \frac{1}{(-2P_1\cdot X)^{\Delta_1}} \frac{1}{(-2P_2\cdot X)^{\Delta_2+J}} \frac{1}{(-2P_3\cdot X)^{\Delta_3}} \,,$$

where the differential operator $\mathcal{D}_{32}$ is introduced as a combination of derivative for P_3 and Z_3:

$$\mathcal{D}_{32} \equiv (Z_3\cdot P_2)\left(Z_3\cdot\frac{\partial}{\partial Z_3} - P_3\cdot\frac{\partial}{\partial P_3}\right) + (P_3\cdot P_2)\left(Z_3\cdot\frac{\partial}{\partial P_3}\right) \,. \tag{3.4.14}$$

As a basis property of $\mathcal{D}_{32}$, when it acts on the scalar bulk-to-boundary operator, it produces the tensor C_3 in the numerator:

$$(\mathcal{D}_{32})^J \frac{1}{(-2P_3\cdot Y)^{\Delta}} = (\Delta)_J \frac{(-2P_2\cdot C_3\cdot Y)^J}{(-2P_3\cdot Y)^{\Delta+J}} \,, \tag{3.4.15}$$

where Y is an arbitrary vector in the embedding space. Using this property of $\mathcal{D}_{32}$, the diagram is reduced to the scalar diagram as in the last line of (3.4.11), where note that $\mathcal{D}_{32}$ acts on the boundary point, and it commutes with the bulk integral. Substituting the result in (3.4.8), and using (3.4.15) again, we can confirm that the diagram is proportional to the CFT three-point function with two scalar and one spin-J tensor field. Note that here $\mathcal{D}_{32}$ acts on P_{13} only because $\mathcal{D}_{32}P_{23}=0$. The result can be packaged in the following expression:

$$\mathcal{A}^{\text{3-pt}}_{0,0,J}(P_i) = \mathcal{B}^{\Delta_1,\Delta_2,\Delta_3}_{0,0,J} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & 0 & J \\ 0 & 0 & 0 \end{bmatrix} \,, \tag{3.4.16}$$

where the coefficient $\mathcal{B}^{\Delta_1,\Delta_2,\Delta_3}_{0,0,J}$ is defined as:

$$\begin{aligned} \mathcal{B}^{\Delta_1,\Delta_2,\Delta_3}_{0,0,J} &= \frac{\pi^h}{2}(-2)^J \mathcal{C}_{\Delta_1}\mathcal{C}_{\Delta_2}\mathcal{C}_{\Delta_3,J} \\ &\quad \times \frac{\Gamma\left(\frac{\sum_{i=1}^3\Delta_i + J - d}{2}\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3+J)} \Gamma\left(\frac{\Delta_3 \pm \Delta_{12} + J}{2}\right)\Gamma\left(\frac{\Delta_{12}^+ - \Delta_3 + J}{2}\right) . \end{aligned} \tag{3.4.17}$$

Here, we used the following short-hand notation:

$$\Gamma(a\pm b) \equiv \Gamma(a+b)\Gamma(a-b) \,, \tag{3.4.18}$$

and the box notation to represent the spinning conformal correlation function which is introduced in section 2.3.

3.4.3 Four-point Contact Diagram

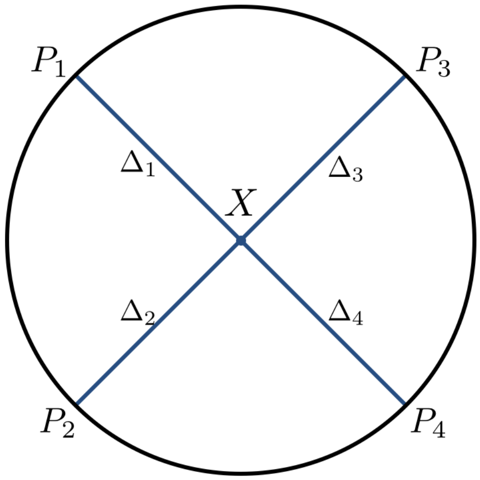


Figure 3.6: Four-point contact diagram with the ϕ^4 interaction.

Here we consider a tree four-point diagram with ϕ^4 interaction which is given in Fig. 3.6 . This diagram is known as the so-called D-function $D_{\Delta_i}(x_i)$ which depends on four external points x_i and dimensions Δ_i. The diagram is given by the following integral:

$$\mathcal{A}^{\phi^4}(P_i) = D_{\Delta_i}(x_i) = \int dX \, \prod_{i=1}^{4} \Pi_{\Delta_i}(P_i, X) \tag{3.4.19}$$

Applying the Symanzik formula, we can easily obtain the Mellin representation for this diagram:

$$\begin{aligned}\mathcal{A}^{\phi^4}(P_i) = \mathcal{N}^{\phi^4} \int_{\mathcal{C}_M} \frac{dsdt}{(2\pi i)^2} \, u^s \, v^t \, \Gamma\left(\frac{\Delta_{12}^+}{2} - s\right) \Gamma\left(\frac{\Delta_{34}^+}{2} - s\right) \\ \times\Gamma\left(-t\right)\Gamma\left(-a-b-t\right)\Gamma\left(a+s+t\right)\Gamma\left(b+s+t\right) \, .\end{aligned} \tag{3.4.20}$$

Here we define the Mellin variables s and t through the following relation:

$$\delta_{23} = -t, , \qquad \delta_{24} = a+s+t \, . \tag{3.4.21}$$

The other Mellin variables are determined by the condition $\sum_{j(\neq i)} \delta_{ij} = \Delta_i$. The coefficient $\mathcal{N}^{\phi^4}$ is given by:

$$\mathcal{N}^{\phi^4} = \frac{\pi^h}{2} \Gamma\left(\frac{\sum_{i=1}^4 \Delta_i - d}{2}\right) \prod_{i=1}^{4} \frac{1}{\Gamma(\Delta_i)} \, . \tag{3.4.22}$$

The integration contour is taken as usual as Mellin integration like in Fig. A.1. By picking up poles in gamma functions, we can perform the Mellin integrations and obtain the expansion form of the D-function. In the following section, we will consider the conformal block expansion of this diagram, and in the next chapter, more details about the Mellin representation are discussed. In 3.5.3, using the orthogonality of CPW, we will discuss the conformal block decomposition of this diagram.

3.4.4 Four-point Exchange Diagram

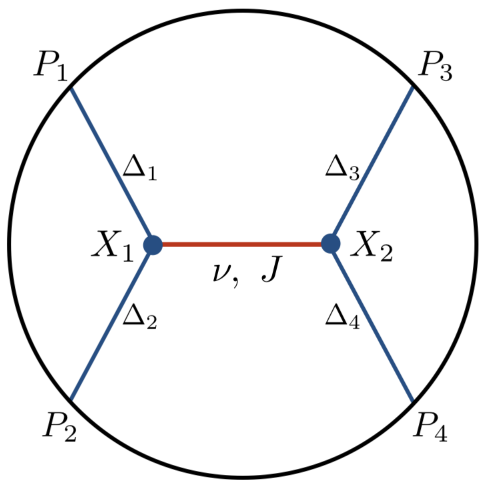


Figure 3.7: Four-point spin J tensor exchange diagram with two three point bulk interactions $\phi\nabla^J_{\mu_1\ldots\mu_J}\phi\mathcal{T}^{\mu_1\ldots\mu_J}$.

Next as another four-point tree diagram, we consider exchange diagram as in Fig.3.7. The internal line is a spin J symmetric traceless tensor and external fields are scalars. In this case, the three point interactions are uniquely determined up to integration by part as $\phi\nabla^J_{\mu_1\ldots\mu_J}\phi\mathcal{T}^{\mu_1\ldots\mu_J}$. In order to calculate this diagram, the expression of spin J bulk-to-bulk propagator is needed, which is expanded in terms of AdS harmonic functions. Substituting the expansion form of the propagator, we can evaluate the exchange diagram in principle, however, it becomes so complicated. In the next section, we will introduce the bulk interpretation of CPWs, and see it is an analog of the AdS harmonic function. According to this fact, we can immediately obtain the expansion of exchange diagram in terms of CPW through the split representation. In 3.5.4 and 4.2.2, we will discuss more exchange diagrams and obtain its conformal block expansion.

3.5 Conformal Block Decomposition of AdS Diagrams

Here we discuss how bulk four-point diagram can be decomposed into conformal blocks. The basic idea is to use the conformal partial wave as a basis for four-point diagrams. Through the relationship between the three-point diagram in AdS and three-point function in CFT, the conformal partial wave is naturally lifted up as a bulk diagram. Then according to the orthogonality of conformal partial wave, we can obtain the conformal block expansion of the desire four-point diagram.

3.5.1 Bulk Interpretation of Conformal Partial Wave

$$\Psi^{\Delta_i}_{h+i\nu,J}(P_i) \sim \int dX_1\, dX_2$$

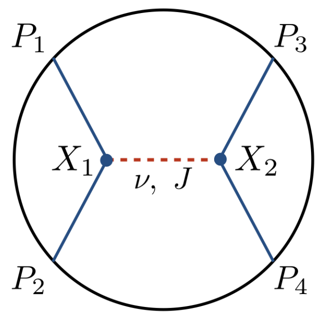


Figure 3.8: CPW can be expressed as a bulk exchange diagram, where the internal dashed line is the AdS harmonic function.

Firstly, we will see that a CPW is regarded as a product of three-point diagrams. In the definition of CPW (2.6.1), there are two three-point functions. Using the previous result, we can replace the three-point functions with three-point diagrams. Then CPW can be described as the following integral:

$$\begin{aligned}\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = &\frac{1}{\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu,J}\mathcal{B}^{\Delta_3,\Delta_4}_{h-i\nu,J}}\frac{1}{J!(h-1)_J}\left[\frac{1}{J!(h-\frac{1}{2})_J}\right]^2\int_{\mathbb{R}^d} dP_0 \\ &\times \int_{\mathrm{AdS}_{d+1}} dX_1\ \Pi_{\Delta_1}(X_1,P_1)\,(K_1\cdot\nabla_1)^J\Pi_{\Delta_2}(X_1,P_2)\ \Pi_{h+i\nu,J}(X_1,P_0;W_1,\mathcal{D}_{Z_0}) \\ &\times \int_{\mathrm{AdS}_{d+1}} dX_2\ \Pi_{\Delta_3}(X_2,P_3)\,(K_2\cdot\nabla_2)^J\Pi_{\Delta_4}(X_2,P_4)\ \Pi_{h-i\nu,J}(X_2,P_0;W_2,Z_0)\,.\end{aligned} \tag{3.5.1}$$

Here the coefficient $\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu,J}\equiv\mathcal{B}^{\Delta_1,\Delta_2,h+i\nu}_{0,0,J}$ is given in (3.4.17). In this expression, the P_0 integration is nothing but the definition of the AdS harmonic function. By substituting the definition (3.3.1), we obtain the bulk representation of CPW:

$$\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = \frac{\pi}{\nu^2\ \mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu,J}\mathcal{B}^{\Delta_3,\Delta_4}_{h-i\nu,J}}\left[\frac{1}{J!(h-\frac{1}{2})_J}\right]^2\int_{\mathrm{AdS}_{d+1}} dX_1\,dX_2\ \Pi_{\Delta_1}(X_1,P_1)\ (K_1\cdot\nabla_1)^J\Pi_{\Delta_2}(X_1,P_2)$$

$$\times\Pi_{\Delta_3}(X_2,P_3)\ (K_2\cdot\nabla_2)^J\Pi_{\Delta_4}(X_2,P_4)\ \Omega_{\nu,J}(X_1,X_2;W_1,W_2)\,. \tag{3.5.2}$$

Diagrammatically, this expression can be described as in Fig. 3.8 .

In the following sections, using this expression, we will consider conformal block decompositions of some AdS diagrams. By applying the inversion formula in (2.6.11), an arbitrary four-point AdS diagram $\mathcal{A}(P_i)$ is also decomposed by CPWs as follows:

$$\mathcal{A}(P_i)=\sum_{J=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{n_{\nu,J}}\ \left(\mathcal{A},\Psi^{d-\Delta_i}_{h-i\nu,J}\right)\ \Psi^{\Delta_i}_{h+i\nu,J}(P_i)\,, \tag{3.5.3}$$

where the parenthesis in the integrand denotes the inner product given in (2.6.10). Computing the inner product of a diagram $\mathcal{A}$ and Ψ, we can obtain the spectrum function for the diagram $\mathcal{A}$. After performing the ν-integral by picking up poles in the spectral function , we can obtain the conformal block decomposition of the diagram $\mathcal{A}$.

3.5.2 Orthogonality of Conformal Partial Wave

Inner Product as a Bubble Diagram

Before considering conformal block decompositions, we will demonstrate how the orthogonality of CPW works in bulk. In fact, we will see that the orthogonality relation can be identified as the orthogonality of the AdS harmonic function. Let us consider, the inner product of two Ψs which appears in the LHS of the orthogonality relation:

$$\left(\Psi^{\Delta_i}_{h+i\nu,J},\Psi^{d-\Delta_i}_{h-i\nu',J'}\right)=\int_{\mathbb{R}^d}\frac{d^dP_1...d^dP_4}{\mathrm{vol}\,(SO(1,d+1))}\ \Psi^{\Delta_i}_{h+i\nu,J}(P_i)\,\Psi^{d-\Delta_i}_{h-i\nu',J'}(P_i), \tag{3.5.4}$$

Here for two Ψs, by substituting the bulk integral form (3.5.2), the inner product (3.5.4) can be represented as follows:

$$\begin{aligned}
&\left(\Psi^{\Delta_i}_{h+i\nu,J},\Psi^{d-\Delta_i}_{h-i\nu',J'}\right)=\int_{\mathbb{R}^d}\frac{d^dP_1...d^dP_4}{\mathrm{vol}\,(SO(1,d+1))}\frac{1}{\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu,J}\mathcal{B}^{\Delta_3,\Delta_4}_{h-i\nu,J}\,\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu',J'}\mathcal{B}^{d-\Delta_3,d-\Delta_4}_{h+i\nu',J'}}\\
&\times\frac{\pi^2}{\nu^2\ \nu'^2}\left[\frac{1}{J!\,(h-\frac{1}{2})_J}\right]^4\int_{\mathrm{AdS}_{d+1}}dX^{12}dX^{34}d\tilde{X}^{12}d\tilde{X}^{34}\\
&\times\Pi_{\Delta_1}(P_1,X^{12})\,\Pi_{d-\Delta_1}(P_1,\tilde{X}^{12})\,(K^{12}\cdot\nabla_{12})^J\Pi_{\Delta_2}(P_2,X^{12})\,(\tilde{K}^{12}\cdot\tilde{\nabla}_{12})^J\Pi_{d-\Delta_2}(P_2,\tilde{X}^{12})\\
&\times\Pi_{\Delta_3}(P_3,X^{34})\,\Pi_{d-\Delta_3}(P_3,\tilde{X}^{34})\,(K^{34}\cdot\nabla_{34})^J\Pi_{\Delta_4}(P_4,X^{34})\,(\tilde{K}^{34}\cdot\tilde{\nabla}_{34})^J\Pi_{d-\Delta_4}(P_4,\tilde{X}^{34})\\
&\times\Omega_{\nu,J}(X^{12},\tilde{X}^{12};W^{12},\tilde{W}^{12})\ \Omega_{\nu,J}(X^{34},\tilde{X}^{34};W^{34},\tilde{W}^{34})
\end{aligned} \tag{3.5.5}$$

Here X^{12}, X^{34}, $\tilde{X}^{12}$ and $\tilde{X}^{34}$ are bulk points to be integrated over AdS. In each bulk point,

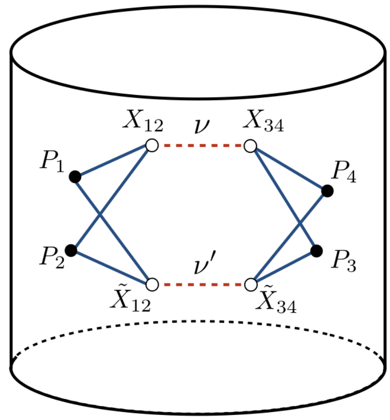


Figure 3.9: The diagrammatic expression of (3.5.5). The white circles are bulk points and the black circles are boundary points.

we have chosen particular interactions with covariant derivatives. Although we can choose another type of integrations, the final result would not be changed. In the above expression, each boundary integral has the same form of the definition of the AdS harmonic function (3.3.1) again, and the pairs of bulk-to-boundary propagator are combined into AdS harmonic functions. Finally, the inner product becomes the following bulk integral:

$$\begin{aligned}\left(\Psi^{\Delta_i}_{h+i\nu,J}, \Psi^{d-\Delta_i}_{h-i\nu',J'}\right) = \mathcal{N}^{\Delta_i}_{\nu,\nu';J,J'} & \left[\frac{1}{J!\,(h-\frac{1}{2})_J}\right]^4 \int \frac{dX^{12}dX^{34}d\tilde{X}^{12}d\tilde{X}^{34}}{\mathrm{vol}\,(SO(1,d+1))} \\ & \times\Omega_{\alpha_1}(X^{12},\tilde{X}^{12})(\tilde{K}^{12}\cdot\tilde{\nabla}_{12})^J(K^{12}\cdot\nabla_{12})^J\Omega_{\alpha_2}(X^{12},\tilde{X}^{12}) \\ & \times\Omega_{\alpha_3}(X^{34},\tilde{X}^{34})(\tilde{K}^{34}\cdot\tilde{\nabla}_{34})^J(K^{34}\cdot\nabla_{34})^J\Omega_{\alpha_4}(X^{34},\tilde{X}^{34}) \\ & \times\Omega_{\nu,J}(X^{12},X^{34};W^{12},W^{34})\Omega_{\nu',J}(\tilde{X}^{12},\tilde{X}^{34};\tilde{W}^{12},\tilde{W}^{34})\end{aligned} \tag{3.5.6}$$

Here in the indexes of harmonic functions, we have introduced α_i through the relation $\Delta_i = h+i\alpha_i$. The coefficients are combined as $\mathcal{N}^{\Delta_i}_{\nu,\nu';J,J'}$ which is given as:

$$\mathcal{N}^{\Delta_i}_{\nu,\nu';J,J'} = \frac{1}{\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu,J}\mathcal{B}^{\Delta_3,\Delta_4}_{h-i\nu,J}\,\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu',J'}\mathcal{B}^{d-\Delta_3,d-\Delta_4}_{h+i\nu',J'}}\left(\prod_{i=1}^{4}\frac{\pi}{\alpha_i^2}\right)\frac{\pi^2}{\nu^2\,\nu'^2}\,. \tag{3.5.7}$$

Diagrammatically, the inner product can be expressed as in Fig.3.10. Each dashed line in the bulk is an AdS harmonic function, not usual propagator. The blue dashed lines are the scalar functions and the red dashed lines are the functions with spin.

To compute the diagram in Fig.3.10, we need to evaluate the following bulk integrals with three

$$\left(\Psi^{\Delta_i}_{h+i\nu,J},\ \Psi^{d-\Delta_i}_{h-i\nu',J'}\right) =$$

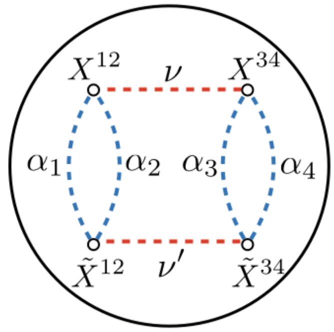


Figure 3.10: The inner product of two Ψs can interpreted as a bubble diagram in bulk. The bulk points are integrated over AdS.

harmonic functions.

$$\begin{aligned}
&\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) = \\
&\frac{1}{J!(h-\frac{1}{2})_J}\int_{\mathrm{AdS}_{d+1}} dY\ \Omega_{\alpha_1}(X_1,Y)\,(W_1\cdot\nabla_1)^J(K_Y\cdot\nabla_Y)^J\Omega_{\alpha_2}(X_1,Y)\,\Omega_{\nu,J}(Y,X_2;W_Y,W_2)\,.
\end{aligned} \tag{3.5.8}$$

Each bulk point in Fig. 3.10 has the same form as Ξ, and it is the building block of the bubble diagram.

Computation of Ξ

Here we will show the detail of computation of Ξ introduced in (3.5.8), and the result is simple as given in the following:

$$\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) = F(\alpha_1,\alpha_2,\nu)\,\Omega_{\nu,J}(X_1,X_2;W_1,W_2)\,, \tag{3.5.9}$$

where the coefficient $F(\alpha_1,\alpha_2,\nu)$ is given as:

$$F(\alpha_1,\alpha_2,\nu) = \frac{J!\,\pi^h}{2^{J-1}\,\Gamma(h+J)}\left(\prod_{i=1}^{2}\frac{\alpha_i^2}{\pi}\right)\frac{\nu^2}{\pi}\frac{1}{\mathcal{C}_{h\pm i\nu,J}}\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu;J}\,\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu;J}\,. \tag{3.5.10}$$

Because of the completeness of the AdS harmonic function, the function Ξ which depends on two bulk points can also be expanded in terms of the harmonic functions. This equation (3.5.9) means that Ξ is actually just proportional to one harmonic function.

Basically Ξ contains one bulk integral and three boundary integral which comes from the definition of the AdS harmonic function. Using the definition of AdS harmonic function, Ξ is expanded as the following integration:

$$\begin{aligned}
\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) \,&=\, \frac{\mathcal{N}_0}{J!(h-\frac{1}{2})_J}\int dP_0dP_1dP_2\int dY\frac{1}{(-2X_1\cdot P_1)^{h+i\alpha_1}}\frac{1}{(-2Y\cdot P_1)^{h-i\alpha_1}}\\
&\quad\times(W_1\cdot\nabla_1)^J\frac{1}{(-2X_1\cdot P_2)^{h+i\alpha_2}}\,(K_Y\cdot\nabla_Y)^J\frac{1}{(-2Y\cdot P_2)^{h-i\alpha_2}}\\
&\quad\times\frac{1}{J!(h-1)_J}\frac{(-2W_2\cdot C_0^{\mathcal{D}}\cdot X_2)^J}{(-2X_2\cdot P_0)^{h+i\nu+J}}\frac{(-2W_Y\cdot C_0^Z\cdot Y)^J}{(-2Y\cdot P_0)^{h-i\nu+J}}\,. && (3.5.11)\\
\mathcal{N}_0 \,&=\, \frac{\alpha_1^2\,\alpha_2^2\,\nu^2}{\pi^3}\,\mathcal{C}_{h\pm i\alpha_1}\mathcal{C}_{h\pm i\alpha_2}\mathcal{C}_{h\pm i\nu,J}\,, && (3.5.12)
\end{aligned}$$

where in $C_0^{\mathcal{D}}$, Z_0 is replaced with the differential operator $\mathcal{D}_{Z_0}$ to take a contraction. Firstly, we focus on the bulk integral.

$$\mathcal{I}_Y \equiv \frac{1}{J!(h-\frac{1}{2})_J}\int dY\,\frac{1}{(-2P_1\cdot Y)^{h-i\alpha_1}}(K_Y\cdot\nabla_Y)^J\,\frac{1}{(-2P_2\cdot Y)^{h-i\alpha_2}}\,\frac{(-2W_Y\cdot C_0^Z\cdot Y)^J}{(-2P_0\cdot Y)^{h-i\nu+J}} \qquad (3.5.13)$$

This integral is a usual three-point diagram with two scalars and one tensor, and easily evaluated in the same way as in 3.4.2, and the result is given by:

$$\begin{aligned}
\mathcal{I}_Y \,&=\, \mathcal{N}_Y\,\frac{(-2P_1\cdot C_0\cdot P_2)^J}{P_{01}^{\gamma^{-+-}}P_{02}^{\gamma^{+--}}P_{12}^{\gamma^{--+}}}\,, && (3.5.14)\\
\mathcal{N}_Y \,&=\, \frac{(-2)^J\pi^h\Gamma\left(\gamma^{--+}\right)\Gamma\left(\gamma^{-+-}\right)\Gamma\left(\gamma^{+--}\right)\Gamma\left(\gamma^{---}\right)}{2\,\Gamma(h-i\alpha_1)\Gamma(h-i\alpha_2)\Gamma(h-i\nu+J)}\,,
\end{aligned}$$

where $\gamma^{--+},\ldots$ are defined as

$$\gamma^{\sigma_1\sigma_2\sigma_0} \equiv \frac{1}{2}(h+J+i(\sigma_1\,\alpha_1+\sigma_2\,\alpha_2+\sigma_0\,\nu))\,. \qquad (3.5.15)$$

Here each σ_i is a signature which takes $+$ or $-$. Next, we focus on the boundary P_1 integral:

$$\mathcal{I}_1 = \int dP_1\,\frac{1}{(-2P_1\cdot X_1)^{h+i\alpha_1}}\frac{(-2P_1\cdot C_0^Z\cdot P_2)^J}{(-2P_0\cdot P_1)^{\gamma^{-+-}}(-2P_2\cdot P_1)^{\gamma^{--+}}}\,. \qquad (3.5.16)$$

Using the generalized Symanzik formula which is given in appendix B, and introducing a Mellin integral for t, it can be evaluated as:

$$\begin{aligned}
\mathcal{I}_1 \,&=\, \mathcal{N}_1\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\,\mu(t)\,\frac{(2P_2\cdot C_0^Z\cdot X_1)^J}{(-2P_0\cdot X_1)^{\gamma^{++-}+t}(-2P_2\cdot X_1)^{\gamma^{+-+}+t}(-2P_0\cdot P_2)^{-t-i\alpha_1}}\\
\mathcal{N}_1 \,&=\, \frac{\pi^h}{\Gamma(h+i\alpha_1)\Gamma(\gamma^{-+-})\Gamma(\gamma^{--+})}\,, \qquad \mu(t)=\Gamma(-t)\Gamma(-i\alpha_1-t)\Gamma(\gamma^{++-}+t)\Gamma(\gamma^{+-+}+t)\,.
\end{aligned}$$

Now the original bulk integration has the following form:

$$
\begin{aligned}
&\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) \\
&= \mathcal{N}_0\,\mathcal{N}_Y\,\mathcal{N}_1 \int dP_0\,dP_2\;\left[(W_1\cdot\nabla_1)^J\,\frac{1}{(-2X_1\cdot P_2)^{h+i\alpha_2}}\right]\frac{(-2W_2\cdot C_0^{\mathcal{D}}\cdot X_2)^J}{(-2X_2\cdot P_0)^{h+i\nu+J}} \\
&\times\frac{1}{J!(h-1)_J}\int\frac{dt}{2\pi i}\mu(t)\frac{(2P_2\cdot C_0^{Z}\cdot X_1)^J}{(-2P_0\cdot X_1)^{\gamma^{++-}+t}(-2P_0\cdot P_2)^{\gamma^{+--}-i\alpha_1-t}(-2P_2\cdot X_1)^{\gamma^{+-+}+t}}
\end{aligned}
\tag{3.5.17}
$$

The remaining P_2 integration is also evaluated by the Symanzik formula:

$$
\begin{aligned}
\mathcal{I}_2 &= 2^J\,(h+i\alpha_2)_J\int dP_2\;\frac{(W_1\cdot P_2)^J(2X_1\cdot C_0\cdot P_2)^J}{(-2X_1\cdot P_2)^{h+J+\gamma^{+++}+t}(-2P_0\cdot P_2)^{\gamma^{---}-t}} \\
&= \mathcal{N}_2(t)\frac{(-2W_1\cdot C_0\cdot X_1)^J}{(-2P_0\cdot X_1)^{\gamma^{---}-t}} \\
\mathcal{N}_2(t) &= (-1)^J(h+i\alpha_2)_J\frac{J!\,\pi^h\;\Gamma(\gamma^{+++}+t)}{\Gamma(h+J+\gamma^{+++}+t)}\,.
\end{aligned}
\tag{3.5.18}
$$

Eventually, Ξ becomes a boundary P_0 integral and a Mellin integration, however, the boundary integration is the same integral in the definition of the AdS harmonic function with spin J. Thanks to this fact, the integral is replaced with a harmonic function, and Ξ becomes

$$
\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) = \mathcal{N}_0\,\mathcal{N}_Y\,\mathcal{N}_1\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\mu(t)\,\mathcal{N}_2(t)\;\frac{\pi}{\nu^2\mathcal{C}_{h\pm i\nu,J}}\Omega_{\nu,J}(X_1,X_2;W_1,W_2)\,.
\tag{3.5.19}
$$

The remaining t integration gives the following gamma functions through the Barnes's second formula (see appendix A):

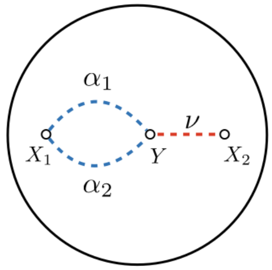
$$
\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\mu(t)\;\frac{\Gamma(\gamma^{+++}+t)}{\Gamma(h+J+\gamma^{+++}+t)} = \frac{\Gamma(\gamma^{+++})\Gamma(\gamma^{++-})\Gamma(\gamma^{+-+})\Gamma(\gamma^{-+-})\Gamma(\gamma^{--+})\Gamma(\gamma^{-++})}{\Gamma(h+J)\Gamma(h+i\alpha_2+J)\Gamma(h+i\nu+J)}
\tag{3.5.20}
$$

Finally we can conclude that the boundary integration Ξ is proportional to a AdS harmonic function and the coefficient is given as the following expression:

$$
\begin{aligned}
\Xi^{\alpha_1,\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) &= F(\alpha_1,\alpha_2,\nu)\;\Omega_{\nu,J}(X_1,X_2;W_1,W_2) \\
F(\alpha_1,\alpha_2,\nu) &= \frac{J!\,\pi^h}{2^{J-1}\,\Gamma(h+J)}\left(\prod_{i=1}^{2}\frac{\alpha_i^2}{\pi}\right)\frac{1}{\mathcal{C}_{h\pm i\nu,J}}\mathcal{B}^{\Delta_1,\Delta_2}_{h+i\nu;J}\,\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu;J}\,.
\end{aligned}
\tag{3.5.21}
$$

Note here the AdS function is a even function in ν, which means it is invariant under $\nu\to-\nu$, and the function $F(\alpha_1,\alpha_2,\alpha_3)$ is also even in each α_i. The result of this calculation is summarized in Fig. 3.11

$$\Xi^{\alpha_1;\alpha_2}_{\nu,J}(X_1,X_2;W_1,W_2) = \int dY$$



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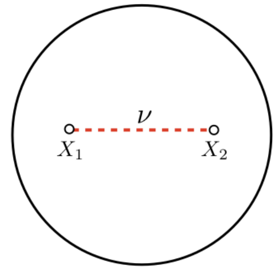


Figure 3.11: The summary of the calculation of Ξ. According to the completeness of the AdS harmonic functions, a loop of harmonic functions can be expanded as a series of harmonic functions and due to the orthogonality, finally, it is proportional to a single harmonic function.

Orthogonality from the AdS harmonic function

Applying the above result to X^{12} and $\tilde{X}^{34}$ integral in 3.5.6, the inner product can be simplified as:

$$\left(\Psi^{\Delta_i}_{h+i\nu,J},\Psi^{d-\Delta_i}_{h-i\nu',J}\right) = \mathcal{N}^{\Delta_i}_{\nu,\nu';J}\left[\frac{1}{J!\,(h-\frac{1}{2})_J}\right]^2\int_{\mathrm{AdS}_{d+1}}\frac{dX^{34}d\tilde{X}^{12}}{\mathrm{vol}\,(SO(1,d+1))} \tag{3.5.22}$$
$$\times F(\alpha_1,\alpha_2,\nu)\,\Omega_{\nu,J}(\tilde{X}^{12},X^{34};\tilde{K}^{12},W^{34})\,F(\alpha_3,\alpha_4,\nu')\,\Omega_{\nu',J}(\tilde{X}^{12},X^{34};\tilde{W}^{12},\tilde{K}^{34})\,.$$

Now we can use the orthogonality of the AdS harmonic function which is given in (3.3.4) for one of bulk integrals. Finally, we can conclude that the inner product of two Ψs is given in the following form:

$$\begin{aligned}\left(\Psi^{\Delta_i}_{h+i\nu,J},\Psi^{d-\Delta_i}_{h-i\nu',J}\right) &= \mathcal{N}^{\Delta_i}_{\nu,\nu';J}\,F(\alpha_1,\alpha_2,\nu)F(\alpha_3,\alpha_4,\nu')\frac{1}{2}\left[\delta(\nu-\nu')+\delta(\nu+\nu')\right]\\ &\quad\times\frac{1}{J!\,(h-\frac{1}{2})_J}\Omega_{\nu,J}(X,X;K,W)\int_{\mathrm{AdS}_{d+1}}\frac{dX}{\mathrm{vol}\,(SO(1,d+1))}\\ &= \frac{1}{2}\,n_{\nu,J}\left[\delta(\nu-\nu')+\frac{K^{\Delta_1,\Delta_2}_{h+i\nu,J}}{K^{\Delta_3,\Delta_4}_{h+i\nu,J}}\,\delta(\nu+\nu')\right]\end{aligned} \tag{3.5.23}$$

where the normalization factor $n_{\nu,J}$ is given as:

$$\begin{aligned}n_{\nu,J} &= \mathcal{N}^{\Delta_i}_{\nu,\nu;J}\,F(\alpha_1,\alpha_2,\nu)F(\alpha_3,\alpha_4,\nu)\\ &\quad\times\frac{1}{J!\,(h-\frac{1}{2})_J}\Omega_{\nu,J}(X,X;K,W)\int_{\mathrm{AdS}_{d+1}}\frac{dX}{\mathrm{vol}\,(SO(1,d+1))}\\ &= \left(\frac{\pi}{\nu^2}\frac{\pi^h\Gamma(J+1)}{2^{J-1}\,\Gamma(h+J)}\right)^2\frac{1}{J!\,(h-\frac{1}{2})_J}\frac{\Omega_{\nu,J}(X,X;K,W)}{(\mathcal{C}_{h\pm i\nu,J})^2}\int_{\mathrm{AdS}_{d+1}}\frac{dX}{\mathrm{vol}\,(SO(1,d+1))}\end{aligned} \tag{3.5.24}$$

Here we can evaluate the normalization factor of AdS harmonic function as:

$$\frac{1}{J!\,(h-\frac{1}{2})_J}\frac{\Omega_{\nu,J}(X,X;K,W)}{(\mathcal{C}_{h\pm i\nu,J})^2}=\frac{\pi^h\Gamma(2h+J)\Gamma(h)}{\Gamma(J+1)\Gamma(2h)^2}\frac{\nu^2}{\pi}\frac{1}{\mathcal{C}_{h\pm i\nu,J}}\,, \tag{3.5.25}$$

and the bulk integration is evaluated as well:

$$\int_{\mathrm{AdS}_{d+1}}dX=\mathrm{vol}(\mathrm{AdS}_{d+1})=\frac{\mathrm{vol}\,(SO(1,d+1))}{\mathrm{vol}\,(SO(d+1))}\,. \tag{3.5.26}$$

The volume of $SO(1,d+1)$ is infinite because it is a non-compact group, however, this factor precisely cancelled with the regularization factor in the definition of the inner product.

3.5.3 Contact Diagram

Next we consider the conformal block decomposition of a contact diagram with ϕ^4 interaction:

$$\mathcal{A}^{\phi^4}(P_i)=\int_{\mathrm{AdS}}dX\,\prod_{i=1}^{4}\frac{\mathcal{C}_{\Delta_i}}{(-2P_i\cdot X)^{\Delta_i}}\,. \tag{3.5.27}$$

The spectral integral for this diagram is given through the inversion formula:

$$\mathcal{A}^{\phi^4}(P_i)=\sum_{J=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{n_{\nu,J}}\,\left(\mathcal{A}^{\phi^4},\Psi^{d-\Delta_i}_{h-i\nu,J}\right)\,\Psi^{\Delta_i}_{h+i\nu,J}(P_i)\,. \tag{3.5.28}$$

In the following, we will compute the inner product in the above integration. After the computation, the poles in the spectral function tell us what kind operators are contained in the contact diagram. Using the bulk representation of Ψ and gluing the bulk-to-boundary propagators through the AdS harmonic function, the inner product is evaluated as the following bulk diagram:

$$\begin{aligned}\left(\mathcal{A}_1^{\phi^4},\Psi^{d-\Delta_i}_{h-i\nu,J}\right)=&\left(\prod_{i=1}^{4}\frac{\pi}{\alpha_i^2}\right)\frac{\pi}{\nu^2}\frac{1}{\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu,J}\mathcal{B}^{d-\Delta_3,d-\Delta_4}_{h+i\nu,J}}\left[\frac{1}{J!(h-\frac{1}{2})_J}\right]^2\int_{\mathrm{AdS}}\frac{dXdX_LdX_R}{\mathrm{vol}\,(SO(1,d+1))}\\&\times\Omega_{\alpha_1}(X,X_L)\,(K_L\cdot\nabla_L)^J\Omega_{\alpha_2}(X,X_L)\Omega_{\alpha_3}(X,X_R)\,(K_R\cdot\nabla_R)^J\Omega_{\alpha_4}(X,X_R)\\&\times\Omega_{\nu,J}(X_L,X_R;W_L,W_R)\,.\end{aligned} \tag{3.5.29}$$

Next, we will focus on the X_L integration. This integration has the almost same structure as $\Xi^{\alpha_1,\alpha_2}_{\nu,J}$ which is defined in (3.5.8) except for differentiations at the bulk point X. Due to luck of differentiations, only $\Xi^{\alpha_1,\alpha_2}_{\nu,J}$ with $J=0$ can have non-zero value, and it is easily evaluated as:

$$\begin{aligned}\Xi^{\alpha_1,\alpha_2}_{\nu,0}&=\int dX_L\,\Omega_{\alpha_1}(X,X_L)\Omega_{\alpha_2}(X,X_L)\Omega_\nu(X_L,X_R)\\&=F(\alpha_1,\alpha_2,\nu)\Omega_\nu(X,X_R)\,.\end{aligned} \tag{3.5.30}$$

$$\left(\mathcal{A}^{\phi^4}, \Psi^{d-\Delta_i}_{h-i\nu,J}\right) =$$

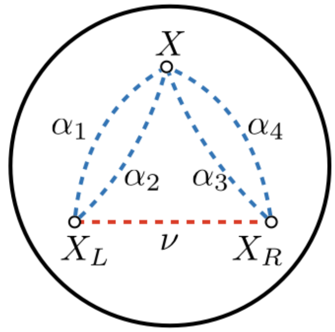


Figure 3.12: The inner product $\left(\mathcal{A}^{\phi^4}, \Psi^{d-\Delta_i}_{h-i\nu,J}\right)$ as a bulk diagram.

Now the X_R integration is also easily computed using the previous formula, and we obtain the following result:

$$\begin{aligned}\left(\mathcal{A}^{\phi^4}, \Psi^{d-\Delta_i}_{h-i\nu,J}\right) = \left(\prod_{i=1}^{4}\frac{\pi}{\alpha_i^2}\right)\frac{\pi}{\nu^2}\frac{F(\alpha_1,\alpha_2,\nu)F(\alpha_3,\alpha_4,\nu)}{\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu,J}\mathcal{B}^{d-\Delta_3,d-\Delta_4}_{h+i\nu,J}} \\ \times\delta_{J,0}\ \Omega_\nu(X,X)\int_{\mathrm{AdS}}\frac{dX}{\mathrm{vol}\,(SO(1,d+1))}\,.\end{aligned} \tag{3.5.31}$$

The result is proportional to a Kronecker's delta $\delta_{J,0}$ according to the reason argued above. Substituting this result into (3.5.28), we obtain the spectral representation for the contact diagram:

$$\begin{aligned}\mathcal{A}^{\phi^4}(P_i) &= N^{\phi^4}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\ \omega_0^{\phi^4}(\nu)\ \Psi^{\Delta_i}_{h+i\nu,J}(x_i)\,, \\ \omega_0^{\phi^4}(\nu) &= \Gamma\left(\frac{\Delta_{12}^+ - h \pm i\nu}{2}\right)\Gamma\left(\frac{\Delta_{34}^+ - h \pm i\nu}{2}\right)\frac{\Gamma\left(\frac{h+i\nu\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{h-i\nu\pm\Delta_{34}}{2}\right)}{8\ \Gamma(\pm i\nu)}\,.\end{aligned} \tag{3.5.32}$$

Now the function omega is called the spectral function because this function is regarded as the integration kernel and its pole structure determines the spectrum of operators in the conformal block decomposition. It is obvious that the spectral function contains the double trace poles at $h \pm i\nu = \Delta_{12}^+ + n$ $(n = 0, 1, 2, \ldots)$. The remaining poles are unphysical, and these are canceled with the coefficient $K^{\Delta_3,\Delta_4}_{h-i\nu,0}$ when we use the relation (2.6.5). Through the same procedure discussed in section 2.7, we can perform the ν-integration and obtain the conformal block decomposition of the contact diagram. This result is consistent with the fact that the contact diagram can be decomposed into conformal blocks of scalar double trace operator.

Even when the interaction contains differentiations, we can discuss the decomposition the diagram, applying the same method in the previous calculation. In such case, the diagram is decomposed into conformal blocks with spinning internal operator.

3.5.4 Exchange Diagram

As the next example, we consider exchange diagrams. According to the fact that the bulk-to-bulk propagator is expressed using the AdS harmonic function, and through the bulk interpretation of the CPW, the exchange diagram is expressed by using CPWs. Substituting the split representation (3.3.7), an exchange diagram is expressed as:

$$\begin{aligned}\mathcal{A}^{\text{exch.}}_{\Delta,J}(P_i) =& \sum_{l=0}^{J}\int_{-\infty}^{+\infty} d\nu\, a_{J,l}(\nu)\int_{\text{AdS}} dX_1 dX_2\ \Pi_{\Delta_1}(P_1,X_1)(K_1\cdot\nabla_1)^J\Pi_{\Delta_2}(P_2,X_1) \\ &\times\Pi_{\Delta_3}(P_3,X_2)(K_4\cdot\nabla_4)^J\Pi_{\Delta_4}(P_4,X_2)\left((W_1\cdot\nabla_1)(W_2\cdot\nabla_2)\right)^l\ \Omega_{\nu,J-l}(X_1,X_2;W_1,W_2)\,.\end{aligned} \tag{3.5.33}$$

Here the bulk integral is nothing but the bulk interpretation of CPW[2]. For example, for the highest contribution $l=J$ in the summation of bulk-to-bulk propagator, it has the following form:

$$\mathcal{A}^{\text{exch}}_{\Delta,J}(P_i) \ni N^{\text{exch}}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\ \omega^{\text{exch}}_J(\nu)\ \Psi^{\Delta_i}_{h+i\nu,J}(P_i) \tag{3.5.34}$$

Here ω^{exch}_J and N^{exch} are defined as:

$$\begin{aligned}\omega^{\text{exch}}_J(\nu) &= \frac{\Gamma\left(\frac{\Delta^+_{12}+J-h\pm i\nu}{2}\right)\Gamma\left(\frac{\Delta^+_{34}+J-h\pm i\nu}{2}\right)}{(\Delta-h\pm i\nu)}\frac{\Gamma\left(\frac{h+i\nu+J\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{h-i\nu+J\pm\Delta_{34}}{2}\right)}{8\,\Gamma(\pm i\nu)(h\pm i\nu-1)_J}\,, \\ N^{\text{exch}} &= 2^{2J}\left(\prod_{i=1}^{4}\frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)}\right)\,.\end{aligned} \tag{3.5.35}$$

Now it is already written as the spectral integration form. We can also obtain the conformal block expansion of exchange diagram in the same way as the contact diagram case. In this case, the spectrum function contains not only the double trace poles but also the so-called the single trace pole at $h\pm i\nu = \Delta$. Here Δ is the dimension associated with the internal field of this exchange diagram. From this expression, we can conclude that exchange diagram is decomposed into conformal blocks for the single trace operator and series of the double trace operators. This result is consistent with the analysis in [20].

3.6 Geodesic Diagram

Recently, in the previous work [20], the so-called four-point geodesic Witten diagram is proposed as a bulk dual of conformal block. In this diagram, the integration points are integrated along geodesics, not over the entire AdS space. We will calculate a three-point geodesic diagram firstly, and see that this diagram is also proportional to a CFT three-point function. The point is that a

[2] It is explicitly seen when $l=J$ and as for other contributions, we can check it after integration by part.

three-point geodesic diagram has a different coefficient from the case of a usual three-point AdS diagram case. According to the difference, through the inversion formula, we can see that the four-point geodesic diagrams have a different spectral functions from exchange diagrams and there is no pole corresponding the double-trace operators.

3.6.1 Three-Point Geodesic Diagram

Here we consider a 3-point geodesic diagram given in Fig.3.13. The interaction point is restricted on the geodesic line whose legs on the boundary points P_1 and P_2. Firstly, we consider a diagram with three scalar fields, and next, using the differential operators we compute a diagram with two scalar and one tensor as similar to the previous usual diagram case.

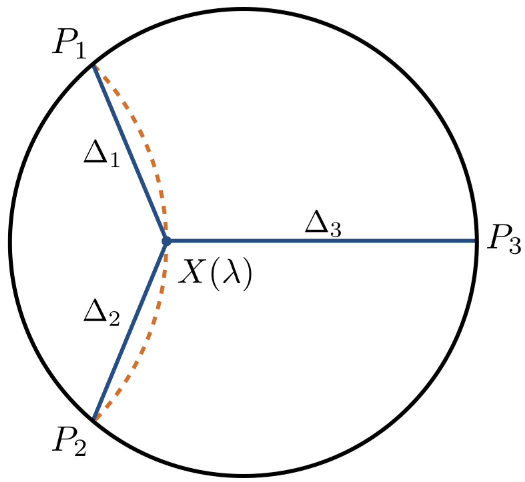


Figure 3.13: Geodesic 3-point diagram. The orange dashed line is the geodesic connecting boundary points P_1 and P_2. The bulk interaction point $X(\lambda)$ is integrated along the geodesic.

Three-Point Scalar Diagram

The diagram $\mathcal{A}^{\text{3-pt, geo}}_{\text{scalar}}$ with three scalar fields is calculated by the following integral:

$$\mathcal{A}^{\text{3-pt, geo}}_{\text{scalar}}(P_i) \equiv \int_{-\infty}^{\infty} d\lambda \, \frac{\mathcal{C}_{\Delta_1}}{(-2P_1 \cdot X(\lambda))^{\Delta_1}} \frac{\mathcal{C}_{\Delta_2}}{(-2P_2 \cdot X(\lambda))^{\Delta_2}} \frac{\mathcal{C}_{\Delta_3}}{(-2P_3 \cdot X(\lambda))^{\Delta_3}} \, . \tag{3.6.1}$$

Here the geodesic is taken between points P_1 and P_2 [3], and the interaction point X depends on λ which is the parameter of the geodesic. In the Poincare coordinate $y^a = (z, x^\mu)$, the geodesic is parametrized as below:

$$z(\lambda) = \frac{|x_1 - x_2|}{2\cosh\lambda} \, , \tag{3.6.2}$$

[3]There are three choices to take the geodesic, and the associated diagrams give a same result up to an overall coefficients.

$$x^\mu(\lambda) = \frac{x_1^\mu + x_2^\mu}{2} - \frac{x_1^\mu - x_2^\mu}{2}\tanh\lambda\,, \qquad \text{where } \lambda \in (-\infty,\infty)\,.$$

Here when $\lambda = -\infty$, it corresponds to the boundary point x_1, and when $\lambda = \infty$, it corresponds to the boundary point x_2. In the embedding space, the geodesic is described as a compact form as follows:

$$X(\lambda) = \frac{e^{-\lambda}P_1 + e^{\lambda}P_2}{P_{12}}\,, \qquad \text{where } \lambda \in (-\infty,\infty) \tag{3.6.3}$$

Substituting this expression, the integral $\mathcal{A}^{\text{3-pt, geo}}_{\text{scalar}}$ can be evaluated as below:

$$\begin{aligned}\mathcal{A}^{\text{3-pt, geo}}_{\text{scalar}}(P_i) &= \left(\prod_{i=1}^{3}\mathcal{C}_{\Delta_i}\right) P_{12}^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{13}^{-\Delta_3}\int_{-\infty}^{\infty} d\lambda\; e^{(-\Delta_1+\Delta_2+\Delta_3)\lambda}\left(\frac{P_{23}}{P_{13}}\,e^{2\lambda}+1\right)^{-\Delta_3} \\ &= \left(\prod_{i=1}^{3}\mathcal{C}_{\Delta_i}\right) P_{12}^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{13}^{-\Delta_3}\left(\frac{P_{13}}{P_{23}}\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_3)}\int_0^{\infty}\frac{d\tilde{t}}{2\tilde{t}}\;\tilde{t}^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_3)}\left(\tilde{t}+1\right)^{-\Delta_3}\end{aligned} \tag{3.6.4}$$

In the second line in (3.6.4), the integration variable λ is replaced with t as below:

$$e^{2\lambda} \longrightarrow \frac{P_{13}}{P_{23}}\,t\,. \tag{3.6.5}$$

The integration in the second line is just the beta function:

$$B(x,y) = \int_0^{\infty} ds\; s^{x-1}(s+1)^{-(x+y)} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \tag{3.6.6}$$

Through this relation, the scalar geodesic diagram becomes

$$\mathcal{A}^{\text{3-pt, geo}}_{\text{scalar}}(P_i) = \mathcal{B}^{\text{3pt, geo}}_{\text{scalar}}\; P_{12}^{-\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)} P_{23}^{-\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)} P_{31}^{-\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)}\,, \tag{3.6.7}$$

where the coefficient $\mathcal{B}^{\text{3pt, geo}}_{\text{scalar}}$ is given as

$$\mathcal{B}^{\text{3pt, geo}}_{\text{scalar}} = \left(\prod_{i=1}^{3}\mathcal{C}_{\Delta_i}\right)\frac{\Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3}{2}\right)}{2\Gamma(\Delta_3)} \tag{3.6.8}$$

Note that as in the result in (3.6.4), the diagram is proportional to the CFT kinematical form.

Three-Point Diagram with a Tensor Field

Next, as similar to the case of a standard diagram, we will consider a geodesic diagram with a symmetric traceless field. In this case, the bulk interaction is not unique, now we consider a certain interaction like:$\phi_1 \nabla_{a_1 \ldots a_J} \phi_2 \mathcal{T}^{a_1 \ldots a_J}$. The diagram is evaluated by the following integral:

$$\mathcal{A}^{\text{3-pt, geo}}_{0,0,J}(P_i) \equiv \frac{\mathcal{C}_{\Delta_1}}{J\left(h-\frac{1}{2}\right)_J} \int_{-\infty}^{\infty} d\lambda \frac{\mathcal{C}_{\Delta_2}}{(-2P_1 \cdot X(\lambda))^{\Delta_1}} (K \cdot \nabla)^J \frac{1}{(-2P_2 \cdot X(\lambda))^{\Delta_2}} \mathcal{C}_{\Delta_3,J} \frac{(-2W \cdot C_3 \cdot X(\lambda))}{(-2P_3 \cdot \tilde{X}(\lambda)^J)^{\Delta_3}} . \tag{3.6.9}$$

In this case, we do not have to use the differential operator to reduced this diagram into scalar one, because of the following useful relation on $X(\lambda)$:

$$-P_2 \cdot C_3 \cdot X(\lambda) = -(-P_2 \cdot X(\lambda))\, V_{3,12} , \tag{3.6.10}$$

where $V_{3,12}$ is an element of tensor structures introduced in (2.3.15) . Thanks to the above relation, the diagram is already reduced to a scalar diagram:

$$\begin{aligned} \mathcal{A}^{\text{3-pt, geo}}_{0,0,J}(P_i) &= (-2)^J (\Delta_2)_J \mathcal{C}_{\Delta_1} \mathcal{C}_{\Delta_2} \mathcal{C}_{\Delta_3,J} V^J_{3,12} \\ &\times \int_{-\infty}^{\infty} d\lambda \, \frac{1}{(-2P_1 \cdot X(\lambda))^{\Delta_1}} \frac{1}{(-2P_2 \cdot X(\lambda))^{\Delta_2}} \frac{1}{(-2P_3 \cdot X(\lambda))^{\Delta_3+J}} \end{aligned} \tag{3.6.11}$$

Using the result of the previous calculation, we obtain the following result:

$$\mathcal{A}^{\text{3-pt, geo}}_{0,0,J}(P_i) = \mathcal{B}^{\Delta_1,\Delta_2,\Delta_3}_{\text{geo};0,0,J} \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 0 & 0 & J \\ 0 & 0 & 0 \end{bmatrix} , \tag{3.6.12}$$

where the coefficient is given as:

$$\mathcal{B}^{\Delta_1,\Delta_2,\Delta_3}_{\text{geo};0,0,J} = (-2)^J (\Delta_2)_J \mathcal{C}_{\Delta_1} \mathcal{C}_{\Delta_2} \mathcal{C}_{\Delta_3,J} \frac{\Gamma\left(\frac{\pm\Delta_{12}+\Delta_3+J}{2}\right)}{2\Gamma(\Delta_3+J)} \tag{3.6.13}$$

Note that by comparing the case of normal diagram, the geodesic diagram gives a different coefficient, and it becomes important when considering four-point diagram.

3.6.2 Four-Point Geodesic Diagram

In the section 3.5, we lift up CPWs using the relation of three-point diagram and correlation function in (3.4.16). However we can use the relationship for three-point geodesic diagram (3.6.12) instead of (3.4.16) . The point is that the coefficients are different between (3.4.16) and (3.6.12) . From (3.6.12), we obtain the following form:

$$\mathcal{A}^{\text{geo}}_{\Delta,J}(P_i) \ni N^{\text{geo}} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \omega^{\text{geo}}_J(\nu) \; \Psi^{\Delta_i}_{h+i\nu,J}(P_i) . \tag{3.6.14}$$

Here $\omega_J^{\rm geo}$ and $N^{\rm geo}$ are defined as:

$$\begin{aligned}\omega_J^{\rm geo}(\nu) &= \frac{1}{(\Delta-h\pm i\nu)}\frac{\Gamma\left(\frac{h+i\nu+J\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{h-i\nu+J\pm\Delta_{34}}{2}\right)}{8\,\Gamma(\pm i\nu)(h\pm i\nu-1)_J} && (3.6.15)\\ N^{\rm geo} &= \frac{2^{2J}(\Delta_2)_J(\Delta_4)_J}{\pi^{2h}}\left(\prod_{i=1}^{4}\mathcal{C}_{\Delta_i,0}\right). && (3.6.16)\end{aligned}$$

Again we focus on only the highest contribution when $l=J$. In this case, the diagram is already written in terms of the Ψ basis. By substituting (2.6.5), it becomes

$$\begin{aligned}\mathcal{A}_1^{\rm geo}(P_i) = N^{\rm geo}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}&\left[\frac{1}{(\Delta-h\pm i\nu)}\frac{\pi^h}{(-2)^J}\frac{\Gamma\left(\frac{h+i\nu+J\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{h+i\nu+J\pm\Delta_{34}}{2}\right)}{8\,\Gamma(i\nu)\Gamma(h+i\nu+J)(h+i\nu-1)_J}\frac{1}{c_J}G^{\Delta_i}_{h+i\nu}(P_i)\right.\\ &\left.+\frac{1}{(\Delta-h\pm i\nu)}\frac{\pi^h}{(-2)^J}\frac{\Gamma\left(\frac{h-i\nu+J\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{h-i\nu+J\pm\Delta_{34}}{2}\right)}{8\,\Gamma(-i\nu)\Gamma(h-i\nu+J)(h-i\nu-1)_J}\frac{1}{c_J}G^{\Delta_i}_{h-i\nu}(P_i)\right]. \qquad (3.6.17)\end{aligned}$$

For the first term, the integral contour should be closed in the lower half plane, and for the second term, it should be closed on the upper half plane. Only the pole at $\nu=\mp i(\Delta-h)$ can contribute to the first (second) term, the result becomes:

$$\mathcal{A}_1^{\rm geo}(P_i) = \frac{(-2)^J}{8\pi^h}(\Delta_2)_J(\Delta_4)_J\left(\prod_{i=1}^{4}\mathcal{C}_{\Delta_i,0}\right)\frac{\Gamma\left(\frac{\Delta+J\pm\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J\pm\Delta_{34}}{2}\right)}{\Gamma(\Delta-h+1)\Gamma(\Delta+J)(\Delta-1)_J}\frac{1}{c_J}G^{\Delta_i}_{\Delta}(P_i)\,. \quad (3.6.18)$$

This relation implies the highest contribution of a geodesic diagram is proportional to a conformal block[4].

3.7 AdS Diagram with External Spinning Fields

In this section, we will consider more general diagrams which contain some spinning fields. In such case, in general, the bulk interactions are not unique, namely, when an interaction contains tensor fields, there are several ways to contract the Lorentz indexes. In the below, firstly we focus on three-point diagrams. If the interaction vertexes are characterized properly, we can see that the one-to-one correspondence between the three-point interaction vertex and the resulting the three-point tensor structures. In other wards, there is an invertible matrix between the interaction basis and the basis for CFT tensor structures [57–59]. As for geodesic three-point diagrams, basically through the same method, we can see the relation to the CFT three-point function [1,22,23]. Gluing three-point diagrams, we can also discuss the conformal partial wave and four-point diagrams with

[4] In other contributions $l<J$, there are the so-called spurious poles which come from the coefficient $a_{J,l}(\nu)$ in (3.3.8).

spinning fields.

3.7.1 Three-Point Diagram

Normal Diagram

Firstly, let us consider normal (not geodesic) three-point diagrams with three tensor fields. In general the bulk interaction is not unique, and it can be parametrized in the following way:

$$\mathcal{I}^{n_1,n_2,n_3}_{J_1,J_2,J_3}(X) = \mathcal{Y}_1^{J_1-n_2-n_3}\mathcal{Y}_2^{J_2-n_3-n_1}\mathcal{Y}_3^{J_3-n_1-n_2}\,\mathcal{H}_1^{n_1}\mathcal{H}_2^{n_2}\mathcal{H}_3^{n_3}\,\mathcal{T}_1(X_1,W_1)\mathcal{T}_2(X_2,W_2)\mathcal{T}_3(X_3,W_3)\Big|_{X_i=X}\,, \tag{3.7.1}$$

where $\mathcal{T}_i$ are symmetric traceless tensor fields with spin J_i which depends on a bulk point X_i and a polarization vector W_i, and the differential operators $\mathcal{Y}_i$ and $\mathcal{H}_i$ are defined as follows:

$$\begin{aligned} &\mathcal{Y}_1 = \partial_{W_1}\cdot\partial_{X_2}, \quad \mathcal{Y}_2 = \partial_{W_2}\cdot\partial_{X_3}, \quad \mathcal{Y}_3 = \partial_{W_3}\cdot\partial_{X_1}, \\ &\mathcal{H}_1 = \partial_{W_2}\cdot\partial_{W_3}, \quad \mathcal{H}_2 = \partial_{W_3}\cdot\partial_{W_1}, \quad \mathcal{H}_3 = \partial_{W_1}\cdot\partial_{W_2}. \end{aligned} \tag{3.7.2}$$

Basically each $\mathcal{Y}_i$ corresponds to a contraction with a differentiation and a tensor field like $\mathcal{T}_i^a\partial_a\cdots$, and $\mathcal{H}_i$ creates a contraction between two tensors like $\mathcal{T}_{i-1}^a\mathcal{T}_{i+1,a}$. In (3.7.1), after acting the differential operators, the points X_i are taken as $X_i\to X$ in order to separate the action of $\mathcal{Y}_i$. This parameterization covers the possible interactions by changing the parameter $\{n_1,n_2,n_3\}$ which satisfies the following relation[5] :

$$J_1-n_2-n_3\geq 0, \quad J_2-n_1-n_3\geq 0, \quad J_3-n_1-n_2\geq 0. \tag{3.7.3}$$

This condition is needed to keep the powers of $\mathcal{Y}_i$ are positive. An arbitrary three-point interaction vertex can be written as a linear combination of the vertex in (3.7.1): The interactions which are equivalent through integration by part are not included in the parametrization, in this sense, these are independent. In [59], the three-point diagram with the general interaction (3.7.1) is computed and it is shown that the result is proportional to a CFT tensor structure. The resulting tensor structures are also parametrized by n_i, and in this way, we can see the one-to-one correspondence between the bulk interactions and the CFT tensor structures. As a brief explanation why the result becomes again the CFT tensor structures, after the contractions, the interaction the the three-point

[5] Notice that while $\{n_1,n_2,n_0\}$ satisfy the same conditions of $\{n_{10},n_{20},n_{12}\}$ as in (2.3.29).

diagram becomes the following form:

$$\int dX \; \frac{Q(\{P_i, Z_i, X\})}{(P_1 \cdot X)^{\#}(P_2 \cdot X)^{\#}(P_3 \cdot X)^{\#}} \,, \tag{3.7.4}$$

where Q is a polynomial which scales properly under rescaling P_i and Z_i and it is invariant under the shift $Z_i \to Z_i + \alpha P_i$. As discussed in section 2.3, this type of polynomial depends on Z_i only through C_i defined in (2.3.10), and even after the integration, due to this property, the result can be written by using only $V_{i,jk}$ and H_{ij} defined in (2.3.21). Because the actual computations are a bit complicated, we skip the detail (please see [58, 59]), however, we conclude that the general three-point diagram is related to the box tensor structures:

$$\mathcal{A}^{\mathbf{k}}_{J_1,J_2,J_3}(x_i) = \sum_{\mathbf{n}} \mathbf{b}(\mathbf{k},\mathbf{n}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ J_1 & J_2 & J_3 \\ n_{23} & n_{31} & n_{12} \end{bmatrix} . \tag{3.7.5}$$

Here $\mathbf{k} = (n_1, n_2, n_3)$ is the parameter for the three-point interaction, and $\mathbf{n} = (n_{23}, n_{31}, n_{12})$ is the parameter for the tensor structures. $\mathbf{b}(\mathbf{k},\mathbf{n})$ is the coefficient which determined through the bulk integration. In the next subsection, we will see the geodesic diagram case.

Geodesic Diagram

As for three-point geodesic diagrams, we can parametrize the general interaction on a geodesic and compute a diagram with general tensor fields. The complete three point interaction vertices involving three symmetric traceless fields on a geodesic in AdS_{d+1} can be succinctly written in the following form:

$$\mathcal{V}_{J_1,J_2,J_3} = \sum_{0 \le n_i \le l_i} g^{n_1,n_2,n_3}_{J_1,J_2,J_3} \mathcal{J}^{n_1,n_2,n_3}_{J_1,J_2,J_3}(X(\lambda)), \quad i = 1,2,3. \tag{3.7.6}$$

Here $\{g^{n_1,n_2,n_3}_{J_1,J_2,J_3}\}$ are the theory dependent bulk coupling constants which can be eventually related the CFT OPE coefficients, and the integers $\{n_1, n_2, n_3\}$ need to satisfy the same conditions in (3.7.3). While the interaction vertices $\mathcal{J}^{n_1,n_2,n_3}_{J_1,J_2,J_3}(X(\lambda))$ along the geodesic between points P_1 and P_2 which is denoted as γ_{12} are parameterized by:

$$\mathcal{J}^{n_1,n_2,n_3}_{J_1,J_2,J_3}(X(\lambda)) = \tilde{\mathcal{Y}}_1^{J_1-n_2-n_3} \mathcal{Y}_2^{J_2-n_3-n_1} \mathcal{Y}_3^{J_3-n_1-n_2} \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \mathcal{T}_1(X_1,W_1)\mathcal{T}_2(X_2,W_2)\mathcal{T}_3(X_3,W_3)\,|_{X_i=X(\lambda)} \,. \tag{3.7.7}$$

where $\mathcal{T}_i^{\{A_1 \ldots A_{l_r}\}}(X)$ is a STT embedding space tensor field which is projected to symmetric traceless tensor field in AdS_{d+1}. Here we have almost adopted the general parameterizations found in (3.7.2) with an essential modification on the choice of operator $\tilde{\mathcal{Y}}_1$, which is changed from $\mathcal{Y}_1 = \partial_{W_1} \cdot \partial_{X_2} \to \tilde{\mathcal{Y}}_1 = \partial_{W_1} \cdot \partial_{X_3}$, we shall now explain the need for this modification. Notice that in original parameterization, which integrates over the entire AdS space, such a change is equivalent up to the equation of motion and a boundary term which we can safely discard. However restricting along the geodesic γ_{12}, we have made an explicit choice of external legs, i.e. the curves connecting $X(\lambda)$ and $P_{1,2}$ and the third leg connecting $X(\lambda)$ and P_3 which is not relevant to the geodesic, such a cyclic symmetry permuting the three tensor fields is explicitly broken. If we use the original parameterization, certain tensor structures appearing in the corresponding CFT three-point function become missing.

Let us work out a simple example of spin-scalar-scalar $(J, 0, 0)$ case to illustrate this. First we consider the parameterization in (3.7.2)

$$\mathcal{I}^{0,0,0}_{J,0,0} = \left(\partial_{W_1} \cdot \partial_{X_2}\right)^J \mathcal{T}_1\left(X_1, W_1\right) \mathcal{T}_2\left(X_2, W_2\right) \mathcal{T}_3\left(X_3, W_3\right)|_{X_i = X}, \tag{3.7.8}$$

and when we apply this vertex to integrate over the entire AdS-space, we have:

$$\begin{aligned}&\int_{\text{AdS}} dX \frac{\left(2P_2 \cdot C_1 \cdot X\right)^J}{\left(-2P_1 \cdot X\right)^{\Delta_1 + J}} \frac{1}{\left(-2P_2 \cdot X\right)^{\Delta_2 + J}} \frac{1}{\left(-2P_3 \cdot X\right)^{\Delta_3}} \\ &\propto \left(P_2 \cdot D_{P_1}\right)^J \mathcal{A}_3^{\Delta_1, \Delta_2, \Delta_3}\left(P_1, P_2, P_3\right) \propto \left[V_{1,23}\right]^J \mathcal{A}_3^{\Delta_1 + J, \Delta_2, \Delta_3}\left(P_1, P_2, P_3\right).\end{aligned} \tag{3.7.9}$$

Here $D^A_{P_i}$ is given by:

$$D^A_{P_i} = Z^A_i \left(Z_i \cdot \frac{\partial}{\partial Z_i} - P_i \cdot \frac{\partial}{\partial P_i}\right) + P^A_i \left(Z_i \cdot \frac{\partial}{\partial P_i}\right) \tag{3.7.10}$$

and $\mathcal{A}_3^{\Delta_1, \Delta_2, \Delta_3}$ is given by the scalar integral discussed in (3.4.1). This vertex (3.7.8) precisely reproduces the only and correct corresponding tensor structure in CFT side as we expected. However if we use the same interaction vertex as before but now restricted along geodesic γ_{12}:

$$\mathcal{I}^{0,0,0}_{J,0,0} = \left(\partial_{W_1} \cdot \partial_{X_2}\right)^J \mathcal{T}_1\left(X_1, W_1\right) \mathcal{T}_2\left(X_2, W_2\right) \mathcal{T}_3\left(X_0, W_3\right)|_{X_i = X(\lambda)}, \tag{3.7.11}$$

we now have

$$\int_{-\infty}^{+\infty} d\lambda \frac{\left(2P_2 \cdot C_1 \cdot X(\lambda)\right)^J}{\left(-2P_1 \cdot X(\lambda)\right)^{\Delta_1 + J}} \frac{1}{\left(-2P_2 \cdot X(\lambda)\right)^{\Delta_2 + J}} \frac{1}{\left(-2P_3 \cdot X(\lambda)\right)^{\Delta_3}} = 0 \tag{3.7.12}$$

due to the accidental orthogonality condition $2P_2 \cdot C_1 \cdot X(\lambda) = 0$ which only occurs along γ_{12} [6]. Now

[6] Similar cancelation was also noted in [22].

if use the new parametrization given in (3.7.7) instead, again we only have one type of interaction given by:

$$\mathcal{J}^{0,0,0}_{J,0,0} = \left(\partial_{W_1} \cdot \partial_{X_3}\right)^J \mathcal{T}_1\left(X_1, W_1\right) \mathcal{T}_2\left(X_2, W_2\right) \mathcal{T}_3\left(X_3, W_3\right) |_{X_i = X(\lambda)} \,. \tag{3.7.13}$$

The corresponding computation along the geodesic γ_{12} is given by (up overall constant):

$$\int_{-\infty}^{+\infty} d\lambda \frac{\left(2P_3 \cdot C_1 \cdot X(\lambda)\right)^J}{\left(-2P_1 \cdot X(\lambda)\right)^{\Delta_1 + J}} \frac{1}{\left(-2P_2 \cdot X(\lambda)\right)^{\Delta_2}} \frac{1}{\left(-2P_3 \cdot X(\lambda)\right)^{\Delta_3 + J}} \propto [V_{1,23}]^J \mathcal{A}_3^{\Delta_1 + J \Delta_2 \Delta_3} \tag{3.7.14}$$

where we have used $2P_3 \cdot C_1 \cdot X(\lambda) = P_{23}\, V_{1,23}/(-2P_2 \cdot X(\lambda))$. We have now seen that the modified parameterization instead gives the desired CFT tensor structure.

We shall adopt the minimally modified parameterization (3.7.7) in our computation of the three-point geodesic Witten diagrams for symmetric traceless tensor fields. One important feature here is that for given spins (J_1, J_2, J_3), the allowed range of the non-negative integers $\{n_1, n_2, n_3\}$ imply that we have the same number (2.3.32) of independent interaction vertices as the independent box tensor structures given in (2.3.30), this implies that we should be able to express the resultant three-point geodesic diagrams as linear combinations of these box tensor structures, echoing our general argument at the beginning of this section. Moreover as shown in [58], the three-point Witten diagrams produced by the original parameterization of three-point vertices can also be expressed in terms of the same set of box tensor structures, this implies that we should also be able to expand the ordinary three-point diagrams in terms of three-point geodesic diagrams. We will explicitly do so in an example that follows. One further remark is that while the we have chosen $\mathcal{Y}_3 = \partial_{W_3} \cdot \partial_{X_1}$ in (3.7.7), the possible choice is $\mathcal{Y}_3 = \partial_{W_3} \cdot \partial_{X_2}$. But this choice is equivalent to starting with cyclically permuted three-point vertices in [58], then make a similar modification of the differential operator to switch the partial derivative to act on X_3. We believe for this other choice, and the story should go through the same.

The $(J_1, J_2, 0)$ case

Let us first consider the case with two external symmetric tensor fields with spins $J_{1,2}$ and one internal scalar field. We have the counting:

$$J_3 = 0, \quad J_1 - n_3 \geq 0, \quad J_2 - n_3 \geq 0, \quad n_1 = n_2 = 0. \tag{3.7.15}$$

The corresponding interaction vertices in this case are:

$$\mathcal{J}^{0,0,n_3}_{J_1,J_2,0} = (\partial_{W_1} \cdot \partial_{X_3})^{J_1 - n_3} (\partial_{W_2} \cdot \partial_{X_3})^{J_2 - n_3} (\partial_{W_1} \cdot \partial_{W_2})^{n_3} \mathcal{T}_1(X_1, W_1) \mathcal{T}_2(X_2, W_2) \mathcal{T}_3(X_3, W_3) |_{X_i = X(\lambda)} \tag{3.7.16}$$

which yield the following integral:

$$\mathcal{A}_{J_1,J_2,0} = \mathbb{C}\int_{\gamma_{12}} \eta^{A_1B_1}\dots\eta^{A_{n_3}B_{n_3}}\frac{(2X\cdot C_1)_{A_1\dots A_{l_1}}}{(-2P_1\cdot X)^{\tau_1}}\frac{(2X\cdot C_2)_{B_1\dots B_{l_2}}}{(-2P_2\cdot X)^{\tau_2}} \\ \times\left(\frac{\partial}{\partial X}\right)^{A_{n_3+1}\dots A_{J_1}}\left(\frac{\partial}{\partial X}\right)^{B_{n_3+1}\dots B_{J_2}}\frac{1}{(-2P_3\cdot X)^{\Delta_3}}\,, \tag{3.7.17}$$

where $\mathbb{C}=\prod_{r=1}^{3}\mathcal{C}_{\Delta_r,l_r}$. This integral can be done, and the result is written in the box basis:

$$\mathcal{A}_{J_1,J_2,0} = \mathbb{C}\;2^{J_1+J_2-2n_3}(-1)^{J_1-n_3}\beta_{\tau_{12},\Delta_3}\left(\frac{\tau_{12}+\Delta_3}{2}\right)_{J_2-n_3}\left(\frac{\Delta_3-\tau_{12}}{2}\right)_{J_1-n_3}\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ J_1 & J_2 & 0\\ 0 & 0 & n_3\end{bmatrix}$$

In this case, happily we found exact one box tensor structure for each interaction vertex.

The $(1,1,2)$ **case**

In the most general case involving three symmetric traceless fields with spins $J_{1,2}$ and J_3, as noted in [57,58], the corresponding three point ordinary Witten diagrams can only be expressed in terms of linear combination of box tensor basis (2.3.30). The same thing happens for the geodesic vertices in (3.7.7) and the resultant three-point geodesic Witten diagrams, they can only be expressed in terms of a linear combination of box basis.

As an illustrative example, we consider the case where $(J_1,J_2,J_3)=(1,1,2)$. First from the corresponding CFT three-point correlation function, we expect there are five box tensor structures arising, they are:

$$[I_1]:=\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & 2\\ 0 & 0 & 0\end{bmatrix},[I_2]:=\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & 2\\ 1 & 0 & 0\end{bmatrix},[I_3]:=\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & 2\\ 0 & 1 & 0\end{bmatrix},[I_4]:=\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & 2\\ 1 & 1 & 0\end{bmatrix},[I_5]:=\begin{bmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ 1 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}.$$

From the vertex parameterization (3.7.6), we now also have five independent interaction vertices. Let us denote the integral for the resultant three point geodesic Witten diagram for each vertex by $\left[J^{n_1,n_2,n_3}_{1,1,2}\right]$. The order of $\{n_1,n_2,n_3\}$ we pick is

$$[J_1]:=\left[J^{0,0,0}_{1,1,2}\right],[J_2]:=\left[J^{1,0,0}_{1,1,2}\right],[J_3]:=\left[J^{0,1,0}_{1,1,2}\right],[J_4]:=\left[J^{1,1,0}_{1,1,2}\right],[J_5]:=\left[J^{0,0,1}_{1,1,2}\right]. \tag{3.7.18}$$

The actual calculations producing them are complicated but somehow mechanical, however we can keep using the recursive relations of for the anti-symmetric tensor C_{iAB} to show that they can all be expressed in terms of box tensor structures given in (3.7.18).

We can express the final results through the following matrix multiplication: $[J_a] = \mathbb{T}_{ab}[I_b]$, $a, b = 1, \dots, 5$ where the mixing matrices $\mathbb{T}_{ab}$ for simplified case $\Delta_2 = \Delta_1, \Delta_3 = \Delta$ is given by:

$$\mathbb{T}_{ab} = 4\left(1+\Delta_1\right)\beta_{0,\Delta+2}\mathbb{C}\times$$

$$\begin{pmatrix} -\left(-4+\Delta^2\right)\left(2+\Delta_1\right) & \frac{2(2+\Delta)(1+\Delta+\Delta_1)}{\Delta} & 2(2+\Delta)\left(2+\Delta_1\right) & \frac{2(2+\Delta)(1+\Delta+\Delta_1)}{\Delta} & 0 \\ -\Delta & -1-\Delta & -\frac{\Delta+\Delta^2+2\Delta_1}{\Delta+\Delta\Delta_1} & -\frac{(1+\Delta)(\Delta+\Delta_1)}{\Delta(1+\Delta_1)} & 0 \\ -2+\Delta & -2 & -1-\frac{2}{\Delta}+\Delta & -\frac{1+\Delta}{\Delta} & 0 \\ \frac{1}{1+\Delta_1} & \frac{1+\Delta}{\Delta+\Delta\Delta_1} & \frac{1+\Delta}{\Delta+\Delta\Delta_1} & \frac{1+\Delta}{\Delta+\Delta\Delta_1} & 0 \\ 0 & 0 & 0 & 0 & \Delta_1 \end{pmatrix}. \quad (3.7.19)$$

In particular, one can check that $\mathbb{T}$ is invertible such that:

$$\mathrm{Det}[\mathbb{T}_{ab}] \propto \frac{(-1+\Delta)^3(2+\Delta)^2\Delta_1^2\left(1+\Delta_1\right)^3\left(2(1+\Delta)^2+\left(2+2\Delta+\Delta^2\right)\Delta_1\right)}{\Delta^3} \neq 0, \quad (3.7.20)$$

This implies that we can equivalently express each three point function tensor structures listed in (3.7.18) in terms of linear combination of three point GWDs for various vertices in (3.7.18). This clearly illustrates that the holographic dual of three-point function for primary operators with spins, as expressed in the box tensor basis, generally requires more than one type of interaction vertices, and to find the ideal basis for two sets of quantities which give one to one correspondence, this essentially becomes a matrix diagonalization problem [7]. Moreover, recalling that we further can connect the box tensor basis appearing in (3.7.18) with their corresponding differential tensor basis (2.3.39):

$$\{D_1\} := \begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{Bmatrix}, \{D_2\} := \begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{Bmatrix}, \{D_3\} := \begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{Bmatrix},$$
$$\{D_4\} := \begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}, \{D_5\} := \begin{Bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{Bmatrix}. \quad (3.7.21)$$

[7] Here we should, however, mention here that in [23], using the new CFT tensor basis constructed from linear combination of (2.3.21), and suitably constructed AdS space differential operators, the progress for direct identifications between CFT tensor structures and AdS interaction vertices has been made.

Again for $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta$, their mixing matrix is given by:

$$\mathbb{A}_{ab} = \begin{pmatrix} 1-\frac{1}{4}\Delta(4+\Delta) & -\frac{\Delta}{2} & -\frac{\Delta}{2} & -\frac{1}{2} & \frac{2-\Delta}{4} \\ -\frac{1}{4}(-2+\Delta)\Delta & \frac{\Delta}{2} & 1-\frac{\Delta}{2} & \frac{1}{2} & -\frac{\Delta}{4} \\ -\frac{1}{4}(-2+\Delta)\Delta & 1-\frac{\Delta}{2} & \frac{\Delta}{2} & \frac{1}{2} & -\frac{\Delta}{4} \\ -\frac{1}{4}(-2+\Delta)^2 & \frac{1}{2}(-2+\Delta) & \frac{1}{2}(-2+\Delta) & -\frac{1}{2} & \frac{2-\Delta}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.7.22}$$

such that $\{D_a\} = \mathbb{A}_{ab}[I_b]$, one can show that $\mathbb{A}^{-1}_{ab}$ is again invertible and agrees with Example 3.3.3 in [49] for $l = 2$. It should now be clear that, through two successive matrix multiplications, we can directly relate the differential tensor basis, which are somewhat more natural for constructing the integral representation of spinning conformal partial waves as explained in the previous section, to the three-point GWDs for different interaction vertices. We can succinctly summarize it as:

$$\{D_a\} = (\mathbb{A}\mathbb{T}^{-1})_{ab}[J_b], \tag{3.7.23}$$

again it would be very interesting to find the new combination of interaction vertices which diagonalizes the matrix $\mathbb{A}\mathbb{T}^{-1}$, such that we can have the simple one to one correspondence with the CFT differential tensor basis.

Changing the parametrization for the three-point interactions, we can conclude that even in the geodesic diagram case, the general three-point functions forms a basis for the three-point tensor structures, and we have similar relation as in (3.7.5) with different coefficients $\tilde{\mathbf{b}}(\mathbf{k}, \mathbf{n})$:

$$\mathcal{A}^{\text{geo.}\mathbf{k}}_{J_1,J_2,J_3}(x_i) = \sum_{\mathbf{n}} \tilde{\mathbf{b}}(\mathbf{k}, \mathbf{n}) \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ J_1 & J_2 & J_3 \\ n_{23} & n_{31} & n_{12} \end{bmatrix}. \tag{3.7.24}$$

3.7.2 Four-Point Diagram

So far, we have considered three-point normal and geodesic diagrams with a certain interaction. As for four-point exchange diagrams, even if there are spinning fields in the external legs, the intermediate bulk-to-bulk propagator is decomposed into the AdS harmonic functions through the relation (3.3.7), and it is further decomposed into two bulk-to-boundary propagators . Finally, exchange diagrams with external spinning fields can be represented as a sum of products of two three-point functions with three external spins (l_1, l_2, l) and arbitrary interactions:

$$\mathcal{A}^{\text{4-pt}}_{l_1,l_2,l_3,l_4}(x_i) \sim \sum_{l=0}^{J} \int d\nu \int dx_0 \; \mathcal{A}^{\text{3-pt},\mathbf{k}_L}_{l_1,l_2,l}(x_1, x_2, x_0) \mathcal{A}^{\text{3-pt},\mathbf{k}_R}_{l_3,l_4,l}(x_3, x_4, x_0) \tag{3.7.25}$$

To use the gluing identity (3.3.7), the dimension Δ_3 is taken as $h+i\nu$.[8] Note here that $\mathbf{k}_L$ are the integers (n_1,n_2,n_0) parametrizing the interactions, and $\mathbf{k}_R$ is also defined in a similar way. In the previous section, we saw that the general three-point function can be expanded in terms of three-point tensor structures. By using the differential basis, the three-point tensor structures are produced by a three-point function with external spins $(0,0,J)$ and differential operators. According to these facts, changing the order of integral and differentiation, the four point diagram $\mathcal{A}^{\text{4-pt}}$ is decomposed into a summation of CPW with differential operators:

$$\mathcal{A}^{\text{4-pt}}_{l_1,l_2,l_3,l_4}(x_i) \sim \sum_{l=0}^{J} \int d\nu \sum_{\mathbf{n}_L,\mathbf{n}_R} \mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)\,\mathbf{b}(\mathbf{k}_R,\mathbf{n}_R)\,\mathcal{D}^{\mathbf{n}_L}_{\text{Left}}\mathcal{D}^{\mathbf{n}_R}_{\text{Right}}\,\Psi^{\Delta_i}_{h+i\nu,l}(x_i)\,. \tag{3.7.26}$$

Here $\mathcal{D}^{\mathbf{n}_L}_{\text{Left}}\mathcal{D}^{\mathbf{n}_R}_{\text{Right}}\,\Psi^{\Delta_i}_{h+i\nu,l}(x_i)$ is the CPW with external spins we discussed in section 2.6 , and we can see that the exchange diagrams with symmetric and traceless tensors are decomposed into this generalized CPWs. In the next chapter, we will consider the Mellin representation of this generalized CPWs.

[8] For the right side diagram, it is taken as $h-i\nu$.

Chapter 4

Mellin Representation

In this chapter, we will discuss the Mellin representation of some four-point AdS diagram. In the computation of AdS diagrams, through the Symanzik formula [60], Mellin integrations naturally appear. Four-point diagrams or four-point functions are typically written as complicated functions in coordinate space, however, in the Mellin space, these can be described as a combination of gamma functions which are more acceptable. In principle, taking the contour properly, the Mellin integration can be evaluated as a pole integration. Not only as a calculation tool but recently some applications of the Mellin representation are proposed which are discussed in the below. In section 4.1, we explore the motivation and application more. In section 4.2, we will see some example of Mellin representations for AdS diagrams. In 4.3, the Mellin representation of the CPW are discussed, and as a byproduct, the expansion form conformal blocks in d-dimension is obtained in 4.4. In 4.5, extensions including external spin fields are discussed.

4.1 Motivation and Application

Relation to QFT Amplitudes

Recently, in [28], it is pointed out that Mellin representations of AdS diagrams have a similar structure as usual QFT diagrams. More precisely, for a n-point diagram $\mathcal{A}$, we define the Mellin amplitude $\mathcal{M}$ as the integrand of Mellin representation:

$$\mathcal{A}(x_i) = \int_{\mathcal{C}_M} [d\delta_{ij}]\, \mathcal{M}(\delta_{ij}) \prod_{i<j} \Gamma(\delta_{ij}) x_{ij}^{-\delta_{ij}} \tag{4.1.1}$$

where δ_{ij} satisfy the conditions: $\sum_{j(\neq i)} \delta_{ij} = \Delta_i$, and there are $n(n-3)/2$ independent variables. The integration measure $[d\delta_{ij}]$ is defined only for independent ones including a factor $(2\pi i)^{-1}$, and the integration contour $\mathcal{C}_M$ is taken as usual as the Mellin integral in Fig.A.1. Especially, in the case of four-point diagram, it can be written as:

$$\mathcal{A}(x_i) = \mathcal{F}(x_i) \int_{\mathcal{C}_M} \frac{dsdt}{(2\pi i)^2} \mathcal{M}(s,t)\, u^s\, v^t\, \Gamma\left(\frac{\Delta_{12}^+}{2} - s\right) \Gamma\left(\frac{\Delta_{34}^+}{2} - s\right)$$
$$\times \Gamma(-t)\,\Gamma(-a-b-t)\,\Gamma(a+s+t)\,\Gamma(b+s+t)\ , \tag{4.1.2}$$

where $\mathcal{F}(x_i)$ is the four-point kinematical factor and u and v are cross ratios. In this case, there are two independent variables, and the Mellin amplitude depends on them. Here the factors other than $\mathcal{M}$ are regarded as kinematical ones, and only $\mathcal{M}$ contains the dynamical information. Comparing the result for a contact diagram in (3.4), it is clear that the Mellin amplitudes for contact diagrams are just constants. In the case of exchange diagrams, the Mellin amplitude becomes a non-trivial function as we will see in the below. More explicitly, the Mellin amplitude for an exchange diagram with scalar field ϕ_Δ is written as

$$\mathcal{M}(s,t) \sim \sum_{m=0}^{\infty} \frac{R_m}{s - \Delta - 2m}\ , \tag{4.1.3}$$

where R_m is a constant factor, and this form looks like an QFT scattering amplitude in the flat space. To obtain the precise correspondence, we have to take the flat space limit: $R \to \infty$ where R is the AdS radius. According to the relation between the mass of AdS fields and the dimension of CFT operators: $R^2 m^2 = \Delta(\Delta - d)$, in this limit, Δ is also taken infinity large, while the mass m is kept finite. In some papers [61–64], this properly is investigated for the cases including tensor fields or more general diagrams.

In [65], this correspondence is applied to the so-called "S-matrix bootstrap" in QFT, which is an attempt to restrict the QFT dynamics by some kinematical constraints. Roughly speaking, the basic idea is that through the correspondence, a four-point QFT amplitude can be mapped to a four-point AdS diagram, and it gives a sort of CFT correlation function. Then we can apply the numerical bootstrap technique developed in [9, 10, 66] to the correlation function and obtain constraints on coupling constants. These constraints can be interpreted as constraints on QFT couplings, thought the correspondence.

The conformal partial wave we introduced in 2.6 also has the Mellin representation [26,27], and again it is expressed as a polynomial in the Mellin space. In section 4.5, we consider the Mellin representation of conformal partial waves with external spinning fields. In such cases, it contains tensor structures, namely, combinations of polarization vectors, in addition to coordinates. We proposed a natural form of the Mellin representation inching such tensor structures [2].

Application to Bootstrap in CFT

One of the applications of the Mellin representation is the conformal bootstrap which is developed in [29,30] . In the usual conformal bootstrap, a four-point function is expanded in conformal blocks for a particular channel, and taking OPE in a different channel, another expansion in conformal blocks for another channel. Then demanding the consistency of these expansions, it gives non-trivial constraints on the dynamical data of CFT, or three-point coefficients. On the other hand, there is another strategy to obtain non-trivial constraints on the CFT data, which is proposed in [31]. In this approach, a four-point function is expanded in a crossing symmetrical way in advance, while in the previous case, the conformal block expansion is not crossing symmetric explicitly and demanding the symmetry later. In [31], form a discussion about unitarity, the building blocks are determined, which are mathematically equivalent to exchange Witten diagrams. This expansion is described in Fig. 4.1. After the expansion, because the building blocks contain redundant contributions which correspond to the double trace operator coming from exchange diagrams, the conditions that the unnecessary contributions disappear is imposed. These conditions are the bootstrap constraints to restrict the dynamics. In [29, 30], they have revisited this approach, and because the exchange diagrams have a relatively simple form in the Mellin space, they considered solving the bootstrap condition in the Mellin space and applied this method to the Wilson Fisher fixed point. This approach is discussed more in [32, 33] and also in [34] .

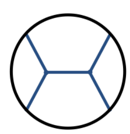
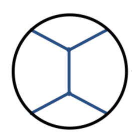
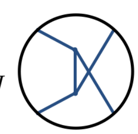
$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\Delta,J} c^{(s)}_{\Delta,J}$  $+\, c^{(t)}_{\Delta,J}$  $+\, c^{(u)}_{\Delta,J}$ 

Figure 4.1: A four point function expanded by AdS exchange diagram. In this expansion, the RHS is clearly crossing symmetric, however, in general it contains redundant contributions.

As a Calculation Technique

As mentioned above, through the Symanzik star formula, the boundary or bulk integrations produce Mellin integrations, and typically the integrands are just combinations of gamma functions. Basically, the integration can be performed by taking the contour properly and picking up the relevant poles, and then the result is obtained as infinite summations, which are typically hypergeometric series. In some special cases, thanks to mathematical identities, the infinite summations which come from the pole integration can be resumed and the results become relatively simple forms. In section 4.3, we will derive an expansion form of d-dimensional conformal block through the Mellin representation of the conformal partial wave, and in this case, the infinite summation can be packaged as Appell functions. This is a generalization of expansion for the scalar conformal blocks [67] .

As another interesting application of Mellin representations, in [68–70] , the expression of four-point functions for the $\mathcal{N} = 4$ super Yang-Mills theory(SYM) is conjectured. In the $\mathcal{N} = 4$ SYM, there is the R-symmetry $SO(6)$ in addition to the conformal symmetry, and thanks to the symmetry, the kinematical form of correlation functions are highly restricted, and under some assumptions, it is uniquely determined.

4.2 Diagrams in AdS

Here we will discuss Mellin representations of some basis AdS diagrams, and see that the Mellin integrations give the conformal block expansions in 2-dimension (in principle, in even dimensions).

4.2.1 Contact Diagram

As discussed in 3.4.3, a four-point contact diagram with four scalar fields has the following Mellin representation:

$$\begin{aligned}\mathcal{A}^{\phi^4}(u,v) = \mathcal{N}^{\phi^4} \int_{-i\infty}^{i\infty} \frac{dsdt}{(2\pi i)^2}\, u^s\, v^t\, \Gamma\left(\frac{\Delta^+_{12}}{2} - s\right)\Gamma\left(\frac{\Delta^+_{34}}{2} - s\right) \\ \times\Gamma\left(-t\right)\Gamma\left(-a-b-t\right)\Gamma\left(a+s+t\right)\Gamma\left(b+s+t\right) \ ,\end{aligned} \tag{4.2.1}$$

where the coefficient $\mathcal{N}^{\phi^4}$ is defined as

$$\mathcal{N}^{\phi^4} = \frac{\pi^h}{2}\Gamma\left(\frac{\sum_{i=1}^4 \Delta_i - d}{2}\right)\prod_{i=1}^4 \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)} \ . \tag{4.2.2}$$

The Mellin representation of a n-point contact diagram with n scalars is also immediately derived from the Symanzik-star formula as reviewed in appendix B:

$$\begin{aligned}\mathcal{A}^{\phi^n}(P_i) &= \int dX \prod_{i=1}^n \Pi_{\Delta_i}(P_i, X) \\ &= \mathcal{N}^{\phi^n} \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i<j} \Gamma(\delta_{ij}) P_{ij}^{\delta_{ij}} \ ,\end{aligned} \tag{4.2.3}$$

where the coefficient $\mathcal{N}^{\phi^n}$ is given by:

$$\mathcal{N}^{\phi^n} = \frac{\pi^h}{2}\Gamma\left(\frac{\sum_{i=1}^n \Delta_i - d}{2}\right)\left(\prod_{i=1}^n \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right) \ . \tag{4.2.4}$$

Here the Mellin variables δ_{ij} satisfy the condition:

$$\sum_{j(\neq i)} \delta_{ij} = \Delta_i \,. \tag{4.2.5}$$

The integration in (4.2.3) is taken for $\frac{n(n-3)}{2}$ independent Mellin variables, and the integration contour is taken in the same way as usual Mellin integral. Note that the number $\frac{n(n-3)}{2}$ is the same as the number of independent cross ratios for n-point function.

Here we will discuss the conformal block expansion of a four-point contact diagram via the Mellin representation. In the s-integration in (4.2.1), there are gamma functions which have poles at $s = \Delta_{12}^{+} + 2m$ and $s = \Delta_{34}^{+} + 2m$. These correspond to the dimensions of double trace operators which we discussed in 3.5 . Here we demonstrate that the s-integration actually gives the two-dimensional conformal block decomposition of the contact diagram, and as a result, it can be expanded as a summation of contributions from the double trace operators. In order to get the conformal block expansion, we utilize the formula in appendix C , to the cross ratios in the Mellin representation (4.2.1):

$$z^s(1-z)^t = \sum_{n=0}^{\infty} \frac{i^n}{(2s+n-1)_n}\, p_n(\, it\,;\, 0, -a-b, s+a, s+b)\, k_{s+n}(z)\,. \tag{4.2.6}$$

Here we use the notation z and $\bar{z}$ for cross ratios, and this formula is also applied to $\bar{z}$-part . Here $p_n(\, it\,;\, 0, -a-b, s+a, s+b)$ is the continuous Hahn polynomial, and the definition is given in appendix C. After applying the expansion formula to z and $\bar{z}$, the t-integration becomes the orthogonality relation of the Hahn polynomial:

$$\begin{aligned} &\int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \Gamma\,(-t)\,\Gamma\,(-a-b-t)\,\Gamma\,(a+s+t)\,\Gamma\,(b+s+t) \\ &\qquad \times p_n(\, it\,;\, 0, -a-b, s+a, s+b)\, p_m(\, it\,;\, 0, -a-b, s+a, s+b) \\ &\qquad = \frac{\Gamma(s+n\pm a)\Gamma(s+n\pm b)}{n!(2s+2n-1)\Gamma(2s+n-1)}\delta_{n,m} \end{aligned} \tag{4.2.7}$$

According to this nice propery, we can perform the t-integration and obtain a summation of k-functions:

$$\begin{aligned} &\int_{-i\infty}^{i\infty} \frac{dt}{2\pi i}\, u^s\, v^t\, \Gamma\,(-t)\,\Gamma\,(-a-b-t)\,\Gamma\,(a+s+t)\,\Gamma\,(b+s+t) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(s+n\pm a)\Gamma(s+n\pm b)}{\Gamma(2s+2n)\,(2s+n-1)_n}\, g^{(2d)}_{2s+2n,0}(z,\bar{z})\,. \end{aligned} \tag{4.2.8}$$

Here we identified the product of $k_{s+n}(z)$ and $k_{s+n}(\bar{z})$ as the two-dimensional conformal block:

$$k_{s+n}(z)k_{s+n}(\bar{z}) = g^{(2d)}_{2s+2n,0}(z,\bar{z}) \tag{4.2.9}$$

Using the above results, the original Mellin representation becomes:

$$\begin{aligned}\mathcal{A}^{\phi^4}(z,\bar{z}) = \mathcal{N}^{\phi^4}\sum_{n=0}^{\infty}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\,\frac{(-1)^n}{n!}\,\frac{\Gamma(s+n\pm a)\Gamma(s+n\pm b)}{\Gamma(2s+2n)\,(2s+n-1)_n} \\ \times\Gamma\left(\frac{\Delta^+_{12}}{2}-s\right)\Gamma\left(\frac{\Delta^+_{34}}{2}-s\right)g^{(2d)}_{2s+2n,0}(z,\bar{z})\end{aligned} \tag{4.2.10}$$

Now, the contour of s-integration has to be closed in the right half place due to the convergence property of $g^{(2d)}_{2s+2n,0}(z,\bar{z})$. There are two types of poles in the right half plane which correspond to the double trace contributions:

$$s = \frac{\Delta^+_{12}}{2}+m\,, \quad s = \frac{\Delta^+_{34}}{2}+m\,, \qquad \text{where} \;\; m\in\mathbb{N} \tag{4.2.11}$$

After performing the s-integral, the contact diagram is expressed by the following double summation:

$$\begin{aligned}\mathcal{A}^{\phi^4}(z,\bar{z}) = \mathcal{N}^{\phi^4}\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}}{n!\,m!} \\ \times\left[\frac{\Gamma(\frac{\Delta^+_{12}}{2}+m+n\pm a)\Gamma(\frac{\Delta^+_{12}}{2}+m+n\pm b)}{\Gamma(\Delta^+_{12}+2m+2n)\,(\Delta^+_{12}+2m+n-1)_n}\Gamma\left(\frac{\Delta^+_{34}-\Delta^+_{12}}{2}-m\right)g^{(s;2d)}_{\Delta^+_{12}+2n+2m,0}(z,\bar{z})\right. \\ \left. + \quad (\ \Delta^+_{12} \iff \Delta^+_{34}\ ) \quad \right].\end{aligned} \tag{4.2.12}$$

This expression means that the contact diagram can be expanded by the two-dimensional conformal blocks of double trace operator whose dimension is $\Delta^+_{12}+2n'$ or $\Delta^+_{34}+2n'$ $(n'\in\mathbb{N})$. In order to simplify the expression, we define an integer $N=n+m$. Then the double summation can be factorized, for example, the first term in the square parentheses in (4.2.12) becomes:

$$\begin{aligned}(\text{The first term}) = \mathcal{N}^{\phi^4}\sum_{N=0}^{\infty}\frac{(-1)^N}{N!}\left(\sum_{n=0}^{N}\frac{N!}{n!\,(N-n)!}\frac{\Gamma\left(\frac{\Delta^+_{34}-\Delta^+_{12}}{2}-N+n\right)}{(\Delta^+_{12}+2N-n-1)_n}\right) \\ \times\frac{\Gamma(\frac{\Delta^+_{12}}{2}+N\pm a)\Gamma(\frac{\Delta^+_{12}}{2}+N\pm b)}{\Gamma(\Delta^+_{12}+2N)}g^{(s;2d)}_{\Delta^+_{12}+2N,0}(z,\bar{z})\,.\end{aligned} \tag{4.2.13}$$

Now the range of n-summation is finite, and it is evaluated explicitly as:

$$\sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!} \frac{\Gamma\left(\frac{\Delta_{34}^{+}-\Delta_{12}^{+}}{2}-N+n\right)}{(\Delta_{12}^{+}+2N-n-1)_n} = \frac{\left(\frac{\Delta_{12}^{+}+\Delta_{34}^{+}}{2}-1\right)_N}{(\Delta_{12}^{+}+N-1)_N}\Gamma\left(\frac{\Delta_{34}^{+}-\Delta_{12}^{+}}{2}-N\right) \tag{4.2.14}$$

Then we can obtain the following expansion:

$$\begin{aligned}
&\mathcal{A}^{\phi^4}(z,\bar{z}) \\
&= \mathcal{N}^{\phi^4} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \frac{\left(\frac{\Delta_{12}^{+}+\Delta_{34}^{+}}{2}-1\right)_N \Gamma\left(\frac{\Delta_{34}^{+}-\Delta_{12}^{+}}{2}-N\right)\Gamma(\frac{\Delta_{12}^{+}}{2}+N\pm a)\Gamma(\frac{\Delta_{12}^{+}}{2}+N\pm b)}{(\Delta_{12}^{+}+N-1)_N\Gamma(\Delta_{12}^{+}+2N)} g^{(s;2d)}_{\Delta_{12}^{+}+2N,0}(z,\bar{z}) \\
&\quad + \quad (\ \Delta_{12}^{+} \iff \Delta_{34}^{+}\ ) \quad .
\end{aligned} \tag{4.2.15}$$

There are only conformal blocks for the double trace series and it is consistent with the previous result. The spectrum is determined by the pole structure of the s integration.

4.2.2 Exchange Diagram

As a next example, we consider a scalar exchange diagram. The Mellin representation is given by:

$$\begin{aligned}
\mathcal{A}_{\Delta}^{\text{exch}}(u,v) = \mathcal{N}^{\text{exch}} \int_{-i\infty}^{i\infty} \frac{ds dt}{(2\pi i)^2} \mathcal{M}_{\Delta}(s)\, u^s\, v^t\, \Gamma\left(\frac{\Delta_{12}^{+}}{2}-s\right)\Gamma\left(\frac{\Delta_{34}^{+}}{2}-s\right) \\
\times\Gamma\left(-t\right)\Gamma\left(-a-b-t\right)\Gamma\left(a+s+t\right)\Gamma\left(b+s+t\right)\ .
\end{aligned} \tag{4.2.16}$$

Here $\mathcal{N}^{\text{exch}}$ is a constant given by:

$$\mathcal{N}^{\text{exch}} = \frac{\pi^h}{2}\prod_{i=1}^{4}\frac{1}{\Gamma(\Delta_i)}\,, \tag{4.2.17}$$

and $\mathcal{M}_{\Delta}(s)$ is the Mellin amplitude for a scalar field exchange, and it has the following spectral integral:

$$\mathcal{M}_{\Delta}(s) = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\frac{1}{\nu^2+(\Delta-h)^2}\frac{\Gamma\left(\frac{\Delta_{12}^{+}-h\pm i\nu}{2}\right)\Gamma\left(\frac{\Delta_{34}^{+}-h\pm i\nu}{2}\right)\Gamma\left(\frac{h\pm i\nu}{2}-s\right)}{4\Gamma(\pm i\nu)\Gamma\left(\frac{\Delta_{12}^{+}}{2}-s\right)\Gamma\left(\frac{\Delta_{34}^{+}}{2}-s\right)} \tag{4.2.18}$$

We will give the derivation of this expression in the next section, however, here we use this result and consider its conformal block expansion. Note that in the case of a scalar exchange, the Mellin amplitude depends on only s. Therefore we can perform the t-integration in the same way as

(4.2.8). Then the diagram becomes:

$$\mathcal{A}^{\text{exch}}_{\Delta}(u,v) = \mathcal{N}^{\text{exch}} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{1}{\nu^2+(\Delta-h)^2} \frac{\Gamma\left(\frac{\Delta_{12}^{+}-h\pm i\nu}{2}\right)\Gamma\left(\frac{\Delta_{34}^{+}-h\pm i\nu}{2}\right)}{4\Gamma(\pm i\nu)} \\ \times \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\Gamma\left(\frac{h\pm i\nu}{2}-s\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(s+n\pm a)\Gamma(s+n\pm b)}{\Gamma(2s+2n)\,(2s+n-1)_n}\, g^{(2d)}_{2s+2n,0}(z,\bar{z})\,. \tag{4.2.19}$$

Now we can perform the s-integration explicitly. Again, due to the asymptotic property of $g^{(2d)}_{2s+2n,0}(z,\bar{z})$, the contour is taken in the right half plane. There are two types of poles which come from the gamma functions at $s=(h\pm i\nu)/2+m$ where $(m\in\mathbb{N})$, however, these two contribution can be combined after evaluating the integral, and we obtain the following expression:

$$\mathcal{A}^{\text{exch}}_{\Delta}(u,v) \\ = \mathcal{N}^{\text{exch}} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{1}{\nu^2+(\Delta-h)^2} \frac{\Gamma\left(\frac{\Delta_{12}^{+}-h\pm i\nu}{2}\right)\Gamma\left(\frac{\Delta_{34}^{+}-h\pm i\nu}{2}\right)}{2\Gamma(\pm i\nu)} \\ \times \sum_{n,m=0}^{\infty} \frac{(-1)^{n+m}}{n!\; m!}\Gamma\left(-i\nu-m\right) \frac{\Gamma(\frac{h+i\nu}{2}+n+m\pm a)\Gamma(\frac{h+i\nu}{2}+n+m\pm b)}{\Gamma(h+i\nu+2n+2m)\,(h+i\nu+2m+n-1)_n}\, g^{(2d)}_{h+i\nu+2n+2m,0}(z,\bar{z})\,. \tag{4.2.20}$$

In this expression, we have only spectrum integral in ν, and the integration contour is closed in the lower half plane. The spectrum of conformal block expansion is determined by the integrand. Naively, we can expect that this expression has a spectrum like $\Delta+2n+2m$, however, there is an identity truncating the redundant summation, and we can see the pole structure more clearly. To see this, we focus on the summation in the third line of (4.2.20). This summation in n and m can be rewritten in n and $N\equiv n+m$.

$$\text{(The third line)} = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} S_N(\nu) \frac{\Gamma(\frac{h+i\nu}{2}+N\pm a)\Gamma(\frac{h+i\nu}{2}+N\pm b)}{\Gamma(h+i\nu+2N)} g^{(2d)}_{h+i\nu+2N,0}(z,\bar{z})\,. \tag{4.2.21}$$

Here $S_N(\nu)$ includes the m-summation:

$$S_N(\nu) \equiv \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!} \frac{\Gamma(-i\nu-N+n)}{(h+i\nu+2N-n-1)_n} \\ = \frac{-\pi}{\Gamma(i\nu+2N)\,\sin(i\pi\nu)} \sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!}(-1)^{N-n}(i\nu+1+N-n)_{N-1}\,. \tag{4.2.22}$$

In the second line we substitute h as $h = 1$, therefore this result is correct only for two-dimensional case. Then the n-summation is reduced to Kronecker's delta through the following relation:

$$\sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!}(-1)^{N-n}(x+n)_{N-1} = \frac{1}{x-1}\,\delta_{N,0}\,. \tag{4.2.23}$$

Applying the above relation, we can rewrite $S_N(\nu)$ as:

$$S_N(\nu) = \frac{-\pi}{\Gamma(i\nu+1)\,\sin(i\pi\nu)}\delta_{N,0}\,. \tag{4.2.24}$$

Substituting these results, we can simplify the integrand drastically:

$$\begin{aligned}\mathcal{A}^{\text{exch}}_{\Delta}(u,v) = \mathcal{N}^{\text{exch}} &\int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\,\frac{1}{\nu^2+(\Delta-h)^2}\,\frac{i\nu}{2\Gamma(1+i\nu)^2} \\ &\times \Gamma\left(\frac{\Delta^+_{12}-h\pm i\nu}{2}\right)\Gamma\left(\frac{\Delta^+_{34}-h\pm i\nu}{2}\right)\Gamma\left(\frac{h+i\nu}{2}\pm a\right)\Gamma\left(\frac{h+i\nu}{2}\pm b\right) g^{(2d)}_{h+i\nu,0}(z,\bar{z})\end{aligned} \tag{4.2.25}$$

Here the integration contour can be closed in the lower half ν-plane. In this expression, we can easily see the pole structure of the integrand, and there are three types of poles:

$$\text{Single trace pole:} \qquad \nu = -i(\Delta-h)\,, \tag{4.2.26}$$

$$\text{Double trace pole 1:} \qquad \nu = -i(\Delta^+_{12}-h+2n)\,, \tag{4.2.27}$$

$$\text{Double trace pole 2:} \qquad \nu = -i(\Delta^+_{34}-h+2n)\,, \qquad \text{where} \;\; n = 0,1,2... \tag{4.2.28}$$

This spectrum is consistent with that we observed before using the bulk interpretation of CPW. Performing this pole integration, we can easily obtain the conformal block expansion. For example, from the single trace pole, we obtain the following contribution:

$$\mathcal{A}^{\text{exch}}_{\Delta}(u,v)\Big|_{\text{Single tr.}} = 2\pi^h\,C_{12\Delta}\,C_{34\Delta}\,g^{(2d)}_{\Delta,0}(z,\bar{z})\,. \tag{4.2.29}$$

Now the coefficient has the factorized form, and $C_{ij\Delta}$ is defined as:

$$C_{ij\Delta} \equiv \frac{\Gamma\left(\frac{\Delta^+_{ij}-\Delta}{2}\right)\Gamma\left(\frac{\Delta^+_{ij}+\Delta-2}{2}\right)\Gamma\left(\frac{\Delta}{2}\pm a\right)}{4\Gamma(\Delta_i)\Gamma(\Delta_j)\Gamma(\Delta)}\,. \tag{4.2.30}$$

As for the double trace contributions we can also compute and obtain the OPE coefficients.

4.3 Conformal Partial Wave

In this section, we derive the Mellin representation of conformal partial wave with spin-J tensor. In the Mellin space, it is characterized by a polynomial which is so-called the Mack polynomial. In the embedding space, a CPW can be described as the following kinematic integral:[1]:

$$\Psi^{\Delta_i}_{h+i\nu,J}(P_i) \equiv \frac{1}{P_{12}^{\gamma_{12}}P_{34}^{\gamma_{34}}}\int_{\mathbb{R}^d} dP_0 \left(\prod_{i=1}^{4}\frac{1}{P_{0i}^{\gamma_{0i}}}\right)\frac{1}{J!(h-1)_J}(2P_0\cdot\mathcal{P}_{12}\cdot\mathcal{D}_{Z_0})^J(2P_0\cdot\mathcal{P}_{34}\cdot Z_0)^J\,,$$

where $\mathcal{P}_{12}$ and $\mathcal{P}_{34}$ are defined as:

$$\mathcal{P}_{12}^{AB} \equiv P_1^AP_2^B - P_2^AP_1^B\,,\quad \mathcal{P}_{34}^{AB} \equiv P_3^AP_4^B - P_4^AP_3^B\,, \tag{4.3.2}$$

and γ_{ij} are given as:

$$\begin{aligned}
\gamma_{12} &= \frac{\Delta_1+\Delta_2-h-i\nu+J}{2}\,,\ \gamma_{34} = \frac{\Delta_3+\Delta_4-h+i\nu+J}{2}\\
\gamma_{01} &= \frac{\Delta_{12}+h+i\nu+J}{2}\,,\ \gamma_{02} = \frac{-\Delta_{12}+h+i\nu+J}{2}\,,\\
\gamma_{03} &= \frac{\Delta_{34}+h-i\nu+J}{2}\,,\ \gamma_{04} = \frac{-\Delta_{34}+h-i\nu+J}{2}\,.
\end{aligned} \tag{4.3.3}$$

The contraction of symmetric traceless transverse tensors is implemented through the derivative $\mathcal{D}_{Z_0}$ and polarization vector Z_0^A, which is evaluated in terms of the Gegenbauer polynomial:

$$\begin{aligned}
&\frac{1}{J!(h-1)_J}(2P_0\cdot\mathcal{P}_{12}\cdot\mathcal{D}_{Z_0})^J(2P_0\cdot\mathcal{P}_{34}\cdot Z_0)^J\\
&\quad=\tilde{\sum_r}(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{34}\cdot P_0)^{J-2r}(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{12}\cdot P_0)^r(-4P_0\cdot\mathcal{P}_{34}\cdot\mathcal{P}_{34}\cdot P_0)^r\\
&\quad=\tilde{\sum_r}\frac{(J-2r)!}{2^{J-2r}}P_{12}^rP_{34}^r\sum_{\sum k_{ij}=J-2r}(-1)^{k_{24}+k_{13}}\prod_{(ij)}\frac{P_{ij}^{k_{ij}}}{k_{ij}!}\prod_i P_{0i}^{J-r-\sum_j k_{ji}}\,,
\end{aligned} \tag{4.3.4}$$

where the symmetric indices $k_{ij}=k_{ji}$ comes form the expansion of the factor $(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{34}\cdot P_0)^{J-2r}$, and (ij) runs over (13), (14), (23) and (24), and label the all possible four-fold non-negative

[1] Here the differential operator $\mathcal{D}_{Z_0^A}$ is defined as in (2.2.13):

$$\mathcal{D}_{Z_0^A} = \left(h-1+Z_0\cdot\frac{\partial}{\partial Z_0}\right)\frac{\partial}{\partial Z_0^A} - \frac{1}{2}Z_{0,A}\frac{\partial^2}{\partial Z_0\cdot\partial Z_0}. \tag{4.3.1}$$

integer partitions of $J-2r$ [2]. For the summation of r, we introduced a short-hand notation:

$$\tilde{\sum_r} = \sum_{r=0}^{[J/2]} (-1)^r \frac{J!(J+h-1)_{-r}}{2^{2r} r!(J-2r)!}\,, \tag{4.3.6}$$

where the square parentheses $[J/2]$ is the Gauss's symbol. Summarizing, the boundary integration over P_0 now becomes:

$$\begin{aligned}\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = & \tilde{\sum_r} \frac{(J-2r)!}{2^{J-2r}} \frac{1}{P_{12}^{\gamma_{12}-r} P_{34}^{\gamma_{34}-r}} \\ & \times \sum_{\sum k_{ij}=J-2r} (-1)^{k_{24}+k_{13}} \prod_{(ij)} \frac{P_{ij}^{k_{ij}}}{k_{ij}!} \int_{\mathbb{R}^d} dP_0 \left(\prod_{i=1}^{4} P_{0i}^{-\gamma_{0i}+J-r-\sum_j k_{ji}} \right) .\end{aligned} \tag{4.3.7}$$

We can now use the Symanzik star-formula [3] to rewrite $\mathcal{I}_{\nu,J}(P_i)$ into the desired Mellin transformation:

$$\begin{aligned}\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = & \tilde{\sum_r} \frac{(J-2r)!}{2^{J-2r}} \frac{1}{P_{12}^{\gamma_{12}-r} P_{34}^{\gamma_{34}-r}} \sum_{\sum k_{ij}=J-2r} (-1)^{k_{24}+k_{13}} \prod_{(ij)} \frac{P_{ij}^{k_{ij}}}{k_{ij}!} \\ & \times \pi^h \left(\prod_{i=1}^{4} \frac{1}{\Gamma\left(\gamma_{0i}-J+r+\sum_j k_{ij}\right)} \right) \int_{-i\infty}^{i\infty} \frac{d\bar{\delta}_{12} d\bar{\delta}_{13}}{(2\pi i)^2} \prod_{i<j} \Gamma(\bar{\delta}_{ij}) P_{ij}^{-\bar{\delta}_{ij}} .\end{aligned} \tag{4.3.8}$$

Here the parameters $\{\bar{\delta}_{ij}\}$ satisfy the following conditions:

$$\sum_{j(\neq i)} \bar{\delta}_{ij} = \gamma_{0i} - J + r + \sum_j k_{ij}\,, \tag{4.3.9}$$

and we can further unify the powers of P_{ij} using $\delta_{ij} = \delta_{ji}$, $i \neq j$ defined as:

$$\delta_{12} = \bar{\delta}_{12} + \gamma_{12} - r\,, \quad \delta_{34} = \bar{\delta}_{34} + \gamma_{34} - r\,, \quad \delta_{(ij)} = \bar{\delta}_{(ij)} - k_{(ij)}\,, \tag{4.3.10}$$

which again satisfy the constraints $\sum_{j(\neq i)} \delta_{ij} = \Delta_i$, $i = 1,2,3,4$. In terms of $\{\delta_{ij}\}$, $\Psi^{\Delta_i}_{h+i\nu,J}$ is

[2] Here we used the following identity about the Gegenbauer polynomial $C_J^{(h-1)}(x)$:

$$\begin{aligned}\frac{1}{J!(h-1)_J} (X\cdot \mathcal{D}_Z)^J (Y\cdot Z_0)^J &= \frac{J!}{2^J (h-1)_J} (X^2Y^2)^{\frac{J}{2}} C_J^{(h-1)}(x) \qquad \text{where } x = \frac{X\cdot Y}{(X^2Y^2)^{\frac{1}{2}}} \\ &= \sum_{r=0}^{[J/2]} (-1)^r \frac{J!(J+h-1)_{-r}}{2^{2r} r!(J-2r)!} (X\cdot Y)^{J-2r} (X^2Y^2)^r .\end{aligned} \tag{4.3.5}$$

[3] Note here that $\sum_i \left(-\gamma_{0i} + J - r - \sum_j k_{ji}\right) = -d$. Therefore we can use the result for the boundary integration in appendix B.

expressed as:

$$\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = \pi^h \left(\prod_{i=1}^{4}\frac{1}{\Gamma(\gamma_{0i})}\right)\tilde{\sum_r}\frac{(J-2r)!}{2^{J-2r}}\sum_{\sum k_{ij}=J-2r}(-1)^{k_{24}+k_{13}}\prod_{i=1}^{4}\Big(\gamma_{0i}-J+r+\sum_j k_{ij}\Big)_{J-r-\sum_j k_{ij}}$$
$$\times\int_{-i\infty}^{i\infty}\frac{d\delta_{12}d\delta_{13}}{(2\pi i)^2}\frac{\Gamma(\bar\delta_{12})\Gamma(\bar\delta_{34})}{\Gamma(\delta_{12})\Gamma(\delta_{34})}\prod_{(ij)}\frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!}\prod_{i<j}\Gamma(\delta_{ij})P_{ij}^{-\delta_{ij}}\,, \tag{4.3.11}$$

where the additional Pochhammer symbols arise from the shifts in (4.3.10). If we now identify the Mellin momenta s and t as:

$$t=-\delta_{23}\,,\quad 2s=\Delta_1+\Delta_2-2\delta_{12}\,, \tag{4.3.12}$$

then the relation between δ_{ij} and (s,t) are given in (4.2.5), and we obtain the following form:

$$\Psi^{\Delta_i}_{h+i\nu,J}(P_i) = \pi^h \left(\prod_{i=1}^{4}\frac{1}{\Gamma(\gamma_{0i})}\right)\int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}\frac{\Gamma\left(\frac{h\pm i\nu-J}{2}-s\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2}{2}-s\right)\Gamma\left(\frac{\Delta_3+\Delta_4}{2}-s\right)}\tilde{P}_{\nu,J}(s,t)\prod_{i<j}\Gamma(\delta_{ij})P_{ij}^{-\delta_{ij}}\,, \tag{4.3.13}$$

where $\tilde{P}_{\nu,J}(s,t)$ is the following polynomial:

$$\tilde{P}_{\nu,J}(s,t) = \sum_{r=0}^{[J/2]}(-1)^r\frac{J!(J+h-1)_{-r}}{2^{J-2r}r!}\left(\frac{h\pm i\nu-J}{2}-s\right)_r$$
$$\times\sum_{\sum k_{ij}=J-2r}(-1)^{k_{24}+k_{13}}\prod_{(ij)}\frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!}\prod_{i=1}^{4}\Big(\gamma_{0i}-J+r+\sum_j k_{ij}\Big)_{J-r-\sum_j k_{ij}}\,. \tag{4.3.14}$$

Notice that we have absorbed the shift by r in $\bar\delta_{12}$ and $\bar\delta_{34}$ in (4.3.10) by introducing additional r-dependent Pochhammer symbols in (4.3.14).

Mellin Representation of Exchange Diagram

In the previous chapter, we conclude that the exchange diagrams can be written as a spectral integration with a CPW, and from the above discussion, the Mellin representation for CPW is obtained. Now we can also get the Mellin representation for exchange diagram immediately. For example for the case of a scalar exchange, substituting the Mellin representation of CPW into (3.5.34), we can obtain (4.2.16).

4.4 Series Expansion of Conformal Block

In (4.3.13), we have derived the Mellin representation of CPW, and from this expression, performing the Mellin integrations, we can obtain the expansion form of the conformal block for d-dimensions and general spin J. The Mellin integral For CPW has the following form:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = \mathcal{F}(x_i)\,\pi^h \int_{-i\infty}^{i\infty} \frac{dsdt}{(2\pi i)^2} u^s\, v^t \sum_{r,k} \left(\prod_i \frac{(1-\gamma_{0i})_{J-r-\sum_j k_{ij}}}{\Gamma(\gamma_{0i})}\right) \Gamma\left(\frac{h \pm i\nu - J + 2r}{2} - s\right)$$
$$\times \Gamma(-t+k_{23})\Gamma(-a-b+k_{14}-t)\Gamma(a+s+t+k_{24})\Gamma(b+s+t+k_{13})\,. \qquad (4.4.1)$$

Here we have used the following a short-hand notation for the summation:

$$\sum_{r,k} \equiv \sum_{r=0}^{[J/2]} (-1)^r \frac{J!(J+h-1)_{-r}}{2^J\, r!} \sum_{\sum k_{ij}=J-2r} (-1)^{k_{24}+k_{13}} \prod_{(ij)} \frac{1}{k_{ij}!}\,. \qquad (4.4.2)$$

In the above expression, the integrand depends on only cross ratios u and v, this is consistent with the fact that CPW transforms in the same way as four-point functions under the conformal transformation as discussed in 2.6 . After performing t-integration picking up poles at $t = k_{23} + n_t$ and $t = -a-b+k_{14}+n_t$ ($n_t = 0, 1, 2, ...$), the infinite summation can be written as a hypergeometric series. Then the Mellin integral (4.3.13) becomes:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = \mathcal{F}(x_i)\pi^h \sum_{r,k} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\ u^s\ \Gamma\left(\frac{h \pm i\nu - J + 2r}{2} - s\right)\left(\prod_i \frac{(1-\gamma_{0i})_{J-r-\sum_j k_{ij}}}{\Gamma(\gamma_{0i})}\right)$$
$$\times\ v^{\frac{-a-b+k_{14}+k_{23}}{2}} \left[v^{\frac{\omega_{23}}{2}}\ {}_2\mathbf{F}_1 \begin{pmatrix} \kappa_2(s)\ ,\ \kappa_3(s) & \\ 1+\omega_{23} & ; v \end{pmatrix} + v^{\frac{\omega_{14}}{2}}\ {}_2\mathbf{F}_1 \begin{pmatrix} \kappa_1(s)\ ,\ \kappa_4(s) & \\ 1+\omega_{14} & ; v \end{pmatrix} \right] \qquad (4.4.3)$$

where we have introduced some useful notations:

$$\begin{aligned} \kappa_1(s) &= -a+s+k_{13}+k_{14}\,, \quad \kappa_2(s) = a+s+k_{23}+k_{24}\,, \\ \kappa_3(s) &= b+s+k_{13}+k_{23}\,, \qquad \kappa_4(s) = -b+s+k_{14}+k_{24}\,, \\ \omega_{23} &= -\omega_{14} = a+b+k_{23}-k_{14}\,, \end{aligned} \qquad (4.4.4)$$

and the function ${}_2\mathbf{F}_1$ is defined as a hypergeometric function including some factors:

$${}_2\mathbf{F}_1 \begin{pmatrix} a\ ,\ b & \\ c & ; x \end{pmatrix} = \Gamma(a)\Gamma(b)\Gamma(1-c)\ {}_2F_1 \begin{pmatrix} a\ ,\ b & \\ c & ; x \end{pmatrix}\,. \qquad (4.4.5)$$

We can perform the s-integral also picking up the relevant poles in the right half plane; $s = \frac{h\pm i\nu-J+2r}{2} + n_s$ $(n_s = 0,1,2,...)$. Again, after the integration, it produces an infinite summation, however, these two infinite summations which come from s and t integral can be combined as an Appell function. Then the final expression is given by:

$$\Psi^{\Delta_i}_{h+i\nu,J}(x_i) = \mathcal{F}(x_i)\,\pi^h \sum_{r,k}\sum_{\sigma_s=\pm}\left(\prod_i \frac{(1-\gamma_{0i})_{J-r-\sum_j k_{ij}}}{\Gamma(\gamma_{0i})}\right) u^{\tau_{\sigma_s\nu}} v^{\frac{-a-b+k_{14}+k_{23}}{2}} \tag{4.4.6}$$
$$\times\left[v^{\frac{\omega_{23}}{2}}\mathbf{F}_4\left(\begin{matrix}\kappa_2(\tau_{\sigma_s\nu}),\ \kappa_3(\tau_{\sigma_s\nu})\\ 1+\omega_{23},\ 1+i\sigma_s\nu\end{matrix};u,\,v\right) + v^{\frac{\omega_{14}}{2}}\mathbf{F}_4\left(\begin{matrix}\kappa_1(\tau_{\sigma_s\nu}),\ \kappa_4(\tau_{\sigma_s\nu})\\ 1+\omega_{14},\ 1+i\sigma_s\nu\end{matrix};u,\,v\right)\right]$$

where $\tau_{\sigma_s\nu} = \frac{h\pm i\nu-J+2r}{2}$ and the summation $\sum_{\sigma_s=\pm}$ is taken for $\pm i\nu$ which correspond to the contributions coming form two types of pole at $s = \frac{h\pm i\nu-J+2r}{2} + n_s$. The function $\mathbf{F}_4$ is defined as the Appell function with some factors:

$$\mathbf{F}_4\left(\begin{matrix}a_1,\ a_2\\ b_1,\ b_2\end{matrix};x,\,y\right) = \Gamma(a_1)\Gamma(a_2)\Gamma(1-b_1)\Gamma(1-b_2)F_4\left(\begin{matrix}a_1,\ a_2\\ b_1,\ b_2\end{matrix};x,\,y\right), \tag{4.4.7}$$

and F_4 is the Appell function defined by

$$F_4\left(\begin{matrix}a_1,\ a_2\\ b_1,\ b_2\end{matrix};x,\,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}}{m!n!(b_1)_m(b_2)_n}x^m\,y^n\,. \tag{4.4.8}$$

Here comparing this expression with the relation (2.6.5) and from the asymptotic behavior at $u\to 0$, we can identify the fist term in the summation $\sum_{\sigma_s=\pm}$, namely terms when $\sigma_s = +$, corresponds to the direct conformal block $G_{h+i\nu,J}$, and the second term associated with $\sigma_s = -$ corresponds to the shadow conformal block $g_{h-i\nu,J}$. After multiplying the inverse of the coefficient $K^{\Delta_3,\Delta_4}_{h-i\nu,J}$, we obtain the expression for the conformal block in general dimension and spin J.

$$g^{\Delta_i}_{\Delta,J}(x_i) = \mathcal{F}(x_i)\frac{\pi^h}{K^{\Delta_3,\Delta_4}_{d-\Delta,J}}\sum_{r,k}\left(\prod_i \frac{(1-\gamma_{0i})_{J-r-\sum_j k_{ij}}}{\Gamma(\gamma_{0i})}\right) u^{\frac{\Delta-J+2r}{2}} v^{\frac{-a-b+k_{14}+k_{23}}{2}} \tag{4.4.9}$$
$$\times\left[v^{\frac{\omega_{23}}{2}}\mathbf{F}_4\left(\begin{matrix}\kappa_2(\frac{\Delta-J+2r}{2}),\ \kappa_3(\frac{\Delta-J+2r}{2})\\ 1+\omega_{23},\ 1+\Delta-h\end{matrix};u,\,v\right) + v^{\frac{\omega_{14}}{2}}\mathbf{F}_4\left(\begin{matrix}\kappa_1(\frac{\Delta-J+2r}{2}),\ \kappa_4(\frac{\Delta-J+2r}{2})\\ 1+\omega_{14},\ 1+\Delta-h\end{matrix};u,\,v\right)\right].$$

We can check that in two or four dimensions, this expression has the same series expansion as the conformal blocks determined by the conformal Casimir equation discussed in 2.5.

4.5 Mellin Representation with Spinning Fields

In section 4.3, we have derived the Mellin representation of CPW, and using this expression we could immediately acquire the Mellin representation for exchange diagrams. In this section, we will discuss the spinning CPW defined by gluing two three-point functions:

$$\mathcal{I}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(P_i,Z_i) \equiv \frac{1}{J!\,(h-1)_J}\int_\partial dP_0 \begin{bmatrix} \Delta_1 & \Delta_2 & h+i\nu \\ l_1 & l_2 & J \\ n_{20} & n_{01} & n_{12} \end{bmatrix} \cdot \begin{bmatrix} \Delta_3 & \Delta_4 & h-i\nu \\ l_3 & l_4 & J \\ n_{40} & n_{03} & n_{34} \end{bmatrix}, \tag{4.5.1}$$

where we have used the box notation introduced in (2.3.30) and the spinning CPW is parametrized by integers $\mathbf{n}_{L,R}$ defined as $\mathbf{n}_L = (n_{20}, n_{01}, n_{12})$ and $\mathbf{n}_R = (n_{40}, n_{03}, n_{34})$. This integration can also be evaluated and expressed into Mellin representation by using the generalization of Symanzik star-formula given in appendix B, and we relegate most of the calculation details and some definitions of notations to the next section 4.5.1. Let us first state the final result:

$$\begin{aligned}\mathcal{I}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(P_i,Z_i) &= \pi^h \left(\prod_{i=1}^{4}\frac{m_i!}{\Gamma(\tilde{\gamma}_{0i})}\right)\sum_{\{a_{ij},b_{ij}\}}\int_{-i\infty}^{i\infty}\frac{d\delta_{12}d\delta_{13}}{(2\pi i)^2} \\ &\times \tilde{P}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(\delta_{ij},a_{ij},b_{ij})\frac{\Gamma(\hat{\delta}_{12})\Gamma(\hat{\delta}_{34})}{\Gamma(\delta_{12})\Gamma(\delta_{34})}\prod_{i\neq j}\frac{\tilde{V}_{ij}^{a_{ij}}}{a_{ij}!}\prod_{i<j}\frac{\tilde{H}_{ij}^{b_{ij}}}{b_{ij}!}\Gamma(\delta_{ij})P_{ij}^{-\delta_{ij}}\,,\end{aligned} \tag{4.5.2}$$

and explain various quantities and definitions in turns. Here the Mellin variables δ_{ij} satisfy the modified constraints:

$$\sum_{j(\neq i)}\delta_{ij} = \Delta_i - l_i \equiv \tilde{\tau}_i\,, \tag{4.5.3}$$

where we have introduced the twist parameters $\tilde{\tau}_i$ and we can solve the constraint (4.5.3) as in (4.2.5) with Δ_i replaced by $\tilde{\tau}_i$. This also leads us to the natural identifications of Mellin momenta for the spinning Mellin amplitude as:

$$s = \tilde{\tau}_3 - \tilde{\tau}_4 - 2\delta_{13}\,, \quad t = \tilde{\tau}_1 + \tilde{\tau}_2 - 2\delta_{12}\,. \tag{4.5.4}$$

The arguments $\hat{\delta}_{12}$ and $\hat{\delta}_{34}$ in the explicit Γ-functions in (4.5.2) are:

$$\hat{\delta}_{12} = \delta_{12} - \frac{\tilde{\tau}_1+\tilde{\tau}_2-\tilde{\tau}_0^+}{2}\,, \qquad \hat{\delta}_{34} = \delta_{34} - \frac{\tilde{\tau}_3+\tilde{\tau}_4-\tilde{\tau}_0^-}{2}\,, \tag{4.5.5}$$

where $\tilde{\tau}_0^\pm \equiv h \pm i\nu - J$. In terms of the Mellin momenta t, $\hat{\delta}_{12}$ and $\hat{\delta}_{34}$ can be expressed as:

$$\hat{\delta}_{12} = \frac{h+i\nu-J-t}{2}\,, \qquad \hat{\delta}_{34} = \frac{h-i\nu-J-t}{2}\,. \tag{4.5.6}$$

In (4.5.2), we have introduced the alternative basis for independent tensor structures $\{\tilde{V}_{ij}\}$ and $\{\tilde{H}_{ij}\}$:

$$\tilde{V}_{ij} \equiv \frac{-2\bar{v}_i \cdot P_j}{P_{ij}}\,, \quad \tilde{H}_{ij} \equiv \frac{-2\bar{v}_i \cdot \bar{v}_j}{P_{ij}}, \quad i,j = 1,2,3,4. \tag{4.5.7}$$

Here $\{\bar{v}_i^A\}$ are composite vectors which are constructed from antisymmetric tensor C_i^{AB}, and they are defined as:

$$\bar{v}_1^A \equiv \frac{(-2P_2 \cdot C_1)^A}{P_{12}}\,,\ \bar{v}_2^A \equiv \frac{(2P_1 \cdot C_2)^A}{P_{12}}\,,\ \bar{v}_3^A \equiv \frac{(-2P_4 \cdot C_3)^A}{P_{34}}\,,\ \bar{v}_4^A \equiv \frac{(2P_3 \cdot C_4)^A}{P_{34}}\,. \tag{4.5.8}$$

The list of the non-vanish products among P_i and $\bar{v}_i$ is given in (4.5.39) and (4.5.40), which consist of combinations of $\{V_{i,jk}\}$ and $\{H_{ij}\}$ given in (2.3.21). In particular $\tilde{V}_{ij}$ is linear in Z_i and $\tilde{H}_{ij}$ is quadratic in Z_i and Z_j, and we still preserve the transverse condition. The two sets of non-negative integers $\{a_{ij}\}$ and $\{b_{ij}\}$ labeling the powers of $\tilde{V}_{ij}$ and $\tilde{H}_{ij}$ need to satisfy the constraints:

$$\sum_{j(\neq i)} (b_{ij} + a_{ij}) = l_i\,, \tag{4.5.9}$$

this can be understood that we can only have total l_i of Z_i polarization vectors in the final expression. The $\{a_{ij}\}$ and $\{b_{ij}\}$ in the summation of (4.5.2) take values when they satisfy the constraint (4.5.9). Notice that b_{ij} are symmetric in the two indices, but a_{ij} are not, and they can be regarded as the discrete Mellin variables which incorporate the discrete spin degrees of freedom encoded in the invariant tensor structures $\{\tilde{H}_{ij}, \tilde{V}_{ij}\}$.

Now to count the number of independent parameters, both continuous and discrete, there are two types of constraints: (4.5.3) for δ_{ij} and (4.5.9) for a_{ij} and b_{ij} . There are six continuous variables δ_{ij} initially, the constraint (4.5.3) eliminate them to only two independent ones, the same number as the independent cross ratios. Similarly, due to the four constraints (4.5.9), starting with twelve a_{ij} and six b_{ij}, we are left with fourteen independent discrete variables. This number precisely corresponds to the number of independent elements of tensor structures, explicitly we have eight $\tilde{V}_{ij}$'s and six $\tilde{H}_{ij}$'s as in from (4.5.39) to (4.5.41).

It is also interesting to pause here and consider the possible interpretations of these remaining variables in the flat space limit. For the continuous Mellin variable δ_{ij}, we can again regard them as bilinear of scattering momenta $\{p_i\}$ in the flat space through the identification: $\delta_{ij} = p_i \cdot p_j$ and $p_i^2 = -\tilde{\tau}_i$. For the flat space limit of independent discrete $\{a_{ij}, b_{ij}\}$, we can count the number of independent possible tensor structures arising from the four-point spinning scattering amplitudes. Consider the fields with spins in the flat space following [64], we can also construct the corresponding elements of tensor structures as products of the scattering momenta $\{p_i\}$ and polarization vectors $\{\epsilon_i\}$, $i=1,2,3,4$ for each spinning field. There are six $\epsilon_i\cdot\epsilon_j$ and eight $\epsilon_i\cdot p_j$ because inner products like $\epsilon_i\cdot\epsilon_i=0$ or $p_i\cdot\epsilon_i=0$. The number of the independent products are the same as the discrete Mellin variables $\{a_{ij}\}$ and $\{b_{ij}\}$, which indicate they may play the similar role of enumerating the independent tensor structures even in the flat space.

Finally in (4.5.2), we have introduced the following polynomial:

$$
\begin{aligned}
&\tilde{P}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(\delta_{ij},a_{ij},b_{ij}) \equiv \tilde{\sum_{r,\beta,k}} \frac{1}{(-2)^{\sum_i m_i+n_{12}+n_{34}}} \prod_i (\tilde{\gamma}_{0i}-\kappa_i)_{\kappa_i}(m_i+1)_{\bar{\kappa}_{\bar{i}}} \\
&\times(\hat{\delta}_{12})_{d_{12}}(\hat{\delta}_{34})_{d_{34}} \frac{b_{12}!}{(b_{12}-n_{12}-\bar{k}^{\{1\bar{1}\bar{2}2\}})!}\frac{b_{34}!}{(b_{34}-n_{34}-\bar{k}^{\{3\bar{3}\bar{4}4\}})!} \\
&\times\prod_{(ij)}\frac{(\delta_{ij})_{d_{ij}+\sum_{\{A\}}k^{\{A\}}_{ij}}}{\prod_{\{A\}}k^{\{A\}}_{ij}!}\prod_{(\bar{i}j)}\frac{(a_{ij}-\sum_{\{A\}}k^{\{A\}}_{\bar{i}j}+1)_{\sum_{\{A\}}k^{\{A\}}_{\bar{i}j}}}{\prod_{\{A\}}k^{\{A\}}_{\bar{i}j}!}\prod_{(\bar{i}\bar{j})}\frac{(b_{ij}-\sum_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}+1)_{\sum_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}}}{\prod_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}!},
\end{aligned}
\tag{4.5.10}
$$

where the definition of the summation $\tilde{\sum}_{r,\beta,k}$ is given in (4.5.33), $\{\tilde{\gamma}_{0i}\}$ are given in (4.5.18), and κ_i and $\bar{\kappa}_{\bar{i}}$ are defined in (4.5.32) as summations of integers $\beta^{\{\dots\}}$ and $k^{\{\dots\}}_{\alpha\beta}$ which come from the polynomial expansions. d_{ij} are non-negative integers defined in (4.5.44) and (4.5.46). Note that when all external spins $l_i=0$, this polynomial becomes $\tilde{P}_{\nu,J}$ in (4.3.14). In this sense, we shall call $\tilde{P}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(\delta_{ij},a_{ij},b_{ij})$ generalized Mack polynomial, which is purely kinematical and can be regarded as the natural polynomial basis for expressing spinning conformal partial waves in Mellin space.

Combining various pieces, now we can define the Mellin representation for the contribution of interaction vertices labeled by $(\mathbf{k_L},\mathbf{k_R})$ to the four-point spinning Witten diagram. Substituting (4.5.2) into (3.7.26), we obtain the following[4]:

$$
\begin{aligned}
&W^{4\text{pt}}_{\Delta,J}(\mathbf{k}_L,\mathbf{k}_R)\ni \\
&\sum_{\{\mathbf{n}_L,\mathbf{n}_R\}}\sum_{\{a_{ij},b_{ij}\}}\int_{-i\infty}^{i\infty}\frac{ds\,dt}{(4\pi i)^2}\int_{-\infty}^{\infty}d\nu\; b^{(\mathbf{n}_L,\mathbf{n}_R)}_J(\nu)\mathcal{M}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(s,t;a_{ij},b_{ij})\prod_{i\neq j}\frac{\tilde{V}^{a_{ij}}_{ij}}{a_{ij}!}\prod_{i<j}\frac{\tilde{H}^{b_{ij}}_{ij}}{b_{ij}!}\prod_{i<j}\frac{\Gamma(\delta_{ij})}{P^{\delta_{ij}}_{ij}},
\end{aligned}
\tag{4.5.11}
$$

where we define the spectrum function $b^{(\mathbf{n_L},\mathbf{n_R})}_J(\nu)$ and the spinning Mellin amplitude $\mathcal{M}^{(\mathbf{n}_L,\mathbf{n_R})}_{\nu,J}(s,t;a_{ij},b_{ij})$ as:

$$
b^{(\mathbf{n}_L,\mathbf{n}_R)}_J(\nu)\equiv\frac{\pi^h g_{\Xi_1\Xi_2\Xi_0}g_{\Xi_3\Xi_4\tilde{\Xi}_0}}{2}\left(\prod_{i=1}^4\frac{m_i!}{\Gamma(\tau_i)}\mathcal{C}_{\Delta_i,l_i}\right)\frac{1}{(h-\Delta)^2+\nu^2},
\tag{4.5.12}
$$

[4] Through the split representation of the bulk-to-bulk propagator, the four-point spin-J tensor exchange diagram gives the summation over $l=0$ to J. Here we focus only on the highest contribution for $l=J$.

$$\mathcal{M}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(s,t;a_{ij},b_{ij}) \equiv \frac{2\nu^2}{\pi}\left(\prod_{i=1}^{4}\frac{\Gamma(\tau_i)}{\Gamma(\tilde{\gamma_{0i}})\mathcal{C}_{\Delta_i,l_i}}\right) \times \mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)\mathbf{b}(\mathbf{k}_R,\mathbf{n}_R)\frac{\Gamma(\hat{\delta}_{12})\Gamma(\hat{\delta}_{34})}{\Gamma(\delta_{12})\Gamma(\delta_{34})}\tilde{P}^{(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(\delta_{ij},a_{ij},b_{ij})\,, \tag{4.5.13}$$

where $g_{\Xi_1\Xi_2\Xi_0}$ and $g_{\Xi_3\Xi_4\tilde{\Xi}_0}$ are the bulk coupling constants. We can view the Mellin representation of the spinning four-point function (4.5.11) as combining the continuous integral and discrete transformations, where we have rewritten the factorials $a_{ij}!$ and $b_{ij}!$ into Γ-functions to make them on parallel footing with the continuous Mellin variables δ_{ij}. These variables need to satisfy the constraints (4.5.3) and (4.5.9), the last three products of Gamma functions in (4.5.11) form universal transformation kernel for given external twists $\{\tilde{\tau}_i\}$ and spins $\{l_i\}$. All the information about specific choice of interaction vertices encoded in the dynamical factors $\mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)\mathbf{b}(\mathbf{k}_R,\mathbf{n}_R)$, we can for example use the explicit expressions for these factors given in [58] and [59] to compute the complete spinning Mellin amplitude (4.5.13).

Here we would like to argue that using the identification of Mellin momenta (4.5.5) and (4.5.6), the spinning Mellin amplitude (4.5.13) itself again does not contain any singularities corresponding to the double trace operators as in the scalar case. As discussed earlier, the dynamical pre-factors of the three point spinning Witten diagram $\mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)$ and $\mathbf{b}(\mathbf{k}_R,\mathbf{n}_R)$ include poles in the ν-plane corresponding to the double trace operators. Upon collision with the poles in $\Gamma(\hat{\delta}_{12})\Gamma(\hat{\delta}_{34}) = \Gamma\left(\frac{h\pm i\nu-J-t}{2}\right)$, they yield the poles in t-plane corresponding to the double trace operators. However these t-plane poles are canceled by the zeroes from $\Gamma(\delta_{12})\Gamma(\delta_{34}) = \Gamma\left(\frac{\tilde{\tau}_1+\tilde{\tau}_2-t}{2}\right)\Gamma\left(\frac{\tilde{\tau}_3+\tilde{\tau}_4-t}{2}\right)$ in the denominator. More explicitly, for example using the results in [59], the the relevant singularities are contained in the dynamical pre-factors $\mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)$ through Γ-function of the form:

$$\Gamma\left(\frac{\tilde{\tau}_1+\tilde{\tau}_2-(h+i\nu-J)}{2}+N\right)\,, \tag{4.5.14}$$

where N is some non-negative constant. Comparing with the corresponding Γ-function for the derivation of scalar Mellin amplitude $\Gamma\left(\frac{\Delta_1+\Delta_2-(h+i\nu-J)}{2}\right)$, the additional integer shift here is caused by the additional derivatives and index contractions in the interaction vertices. The poles in the ν-plane are at $h+i\nu = \tilde{\tau}_1+\tilde{\tau}_2+J+2N+2m$, and m is also a non-negative integer. When colliding with the poles in $\Gamma(\hat{\delta}_{12})$, they yield the poles in the t plane at $t = \tilde{\tau}_1+\tilde{\tau}_2+2N+2m$. These poles in the t plane are canceled with zeros which come from the Γ-function in the denominator in (4.5.13) $\Gamma(\delta_{12})$ at $t = \tilde{\tau}_1+\tilde{\tau}_2+2m'$, where $m' = 0,1,2,3,\ldots$ includes all non-negative integers. On the other hand, as similar as the scalar case, when we consider geodesic diagrams, the coefficients do not contain such Γ-functions including the double trace poles as in (4.5.14). According to this fact, spinning geodesic diagrams contain only the single trace exchange, and this is consistent with the fact that a geodesic diagram is proportional to a conformal partial wave, which is associated with the exchange of single type of primary operators. For details about spinning geodesic Witten diagrams, please see [1, 21–23].

Finally as the scalar case in (4.1.3), the spinning Mellin amplitude (4.5.13) again has Laurent expansion in t-channel, which arises from the same mechanism of the collision of the poles in ν integration $\nu = \pm i(\Delta - h)$ and poles in the t integration of the gamma functions $\Gamma(\hat{\delta}_{12})$ and $\Gamma(\hat{\delta}_{34})$. The spinning Mellin amplitude has the following expansion:

$$\int_{-\infty}^{\infty} d\nu\, b_J^{(\mathbf{n}_L,\mathbf{n}_R)}(\nu)\mathcal{M}_{\nu,J}^{(\mathbf{n}_L,\mathbf{n}_R)}(s,t;a_{ij},b_{ij}) = \sum_{m=0}^{\infty} \frac{\mathbf{Q}_{J,m}^{(\mathbf{n_L},\mathbf{n_R})}(s)}{t-(\Delta-J+2m)} + \text{regular}\,, \tag{4.5.15}$$

$$\begin{aligned}\mathbf{Q}_{J,m}^{(\mathbf{n_L},\mathbf{n_R})}(s) = {} & \pi^h g_{\Xi_1\Xi_2\Xi_0} g_{\Xi_3\Xi_4\tilde{\Xi}_0} \left(\prod_{i=1}^{4} \frac{m_i!}{\Gamma(\gamma_{0i})}\right) \frac{\Gamma(h-\Delta-m)}{\Gamma\left(\frac{\tilde{\tau}_1+\tilde{\tau}_2-\Delta+J}{2}-m\right)\Gamma\left(\frac{\tilde{\tau}_3+\tilde{\tau}_4-\Delta+J}{2}-m\right)} \\ & \times \Big[\mathbf{b}(\mathbf{k}_L,\mathbf{n}_L)\mathbf{b}(\mathbf{k}_R,\mathbf{n}_R)\tilde{P}_{\nu,J}^{(\mathbf{n}_L,\mathbf{n}_R)}(\delta_{ij},a_{ij},b_{ij})\Big]_{\substack{\nu=i(h-\Delta)\\ t=\Delta-J+2m}}\,.\end{aligned} \tag{4.5.16}$$

As expected, these t-channel poles again correspond to the exchange of symmetric traceless primary operator with twist $\tilde{\tau} = \Delta - J$ and its infinite descendants.

4.5.1 Conformal Integral for spinning cases

Here we will see the details of the calculation of general conformal integral in (4.5.1). Using the definition of the box tensor basis and the vectors $\{\bar{v}_i\}$, we can rewrite the kinematical integral as:

$$\mathcal{I}_{\nu,J}^{(\mathbf{n}_L,\mathbf{n}_R)}(P_i,Z_i) = \frac{H_{12}^{n_{12}}H_{34}^{n_{34}}}{P_{12}^{\tilde{\gamma}_{12}}P_{34}^{\tilde{\gamma}_{34}}} \int dP_0 \left(\prod_{i=1}^{4} \frac{(\bar{v}_i\cdot P_0)^{m_i}}{P_{0i}^{\tilde{\gamma}_{0i}}}\right) \mathcal{V}(\mathbf{m}_L,\mathbf{m}_R)\,. \tag{4.5.17}$$

Now $\tilde{\gamma}_{ij}$ are shifted dimensions as below:

$$\begin{aligned}\tilde{\gamma}_{12} &= \frac{\tau_1+\tau_2-\tau_0^+}{2} + m_0^+ - m_1 - m_2\,, \quad \tilde{\gamma}_{34} = \frac{\tau_3+\tau_4-\tau_0^-}{2} + m_0^- - m_3 - m_4\,, \\ \tilde{\gamma}_{01} &= \frac{\tau_1+\tau_0^+-\tau_2}{2} + m_2\,, \quad \tilde{\gamma}_{02} = \frac{\tau_2+\tau_0^+-\tau_1}{2} + m_1\,, \\ \tilde{\gamma}_{03} &= \frac{\tau_3+\tau_0^--\tau_4}{2} + m_4\,, \quad \tilde{\gamma}_{04} = \frac{\tau_4+\tau_0^--\tau_3}{2} + m_3\,,\end{aligned} \tag{4.5.18}$$

where $\tau_i = \Delta_i + l_i$, $\tau_0^{\pm} = h \pm i\nu + J$. The vectors $\{\bar{v}_i\}$ are combinations of a polarization vector and coordinates which are defined in (4.5.8). $\mathcal{V}(\mathbf{m}_L,\mathbf{m}_R)$ is introduced as the contraction part:

$$\mathcal{V}(\mathbf{m}_L,\mathbf{m}_R) \equiv \frac{1}{J!\,(h-1)_J}(2P_0\cdot\mathcal{P}_{12}\cdot\mathcal{D}_{Z_0})^{m_0^+}(2P_0\cdot C_1\cdot\mathcal{D}_Z)^{n_{01}}(2P_0\cdot C_2\cdot\mathcal{D}_Z)^{n_{20}}$$

$$\times(2P_0\cdot\mathcal{P}_{34}\cdot Z_0)^{m_0^-}(2P_0\cdot C_3\cdot Z_0)^{n_{03}}(2P_0\cdot C_4\cdot Z_0)^{n_{40}}\,, \qquad (4.5.19)$$

where $\mathbf{m}_L=(m_0^+,n_{01},n_{02})$ and $\mathbf{m}_R=(m_0^-,n_{03},n_{04})$. To calculate the contraction between $\mathcal{D}_{Z_0}$ and Z_0, we introduce the two following rank-two anti-symmetric tensors:

$$\mathcal{W}_{12}^{AB}(\mathbf{t}_L)\equiv(t_0^+\mathcal{P}_{12}+t_1C_1+t_2C_2)^{AB},\quad \mathcal{W}_{34}^{AB}(\mathbf{t}_R)\equiv(t_0^-\mathcal{P}_{34}+t_3C_3+t_4C_4)^{AB}\,, \qquad (4.5.20)$$

where two sets of triplets of real parameters $\mathbf{t}_L=(t_0^+,t_1,t_2)$ and $\mathbf{t}_R=(t_0^-,t_3,t_4)$. Using $\mathcal{W}_{12}$ and $\mathcal{W}_{34}$, we can rewrite $\mathcal{V}(\mathbf{m}_L,\mathbf{m}_R)$ as:

$$\mathcal{V}(\mathbf{m}_L,\mathbf{m}_R)=\frac{1}{J!\,(h-1)_J}\partial_{\mathbf{t}_L,\mathbf{t}_R}(2P_0\cdot\mathcal{W}_{12}\cdot\mathcal{D}_Z)^J(2P_0\cdot\mathcal{W}_{34}\cdot Z_0)^J\,. \qquad (4.5.21)$$

Here the t-differential operator $\partial_{\mathbf{t}_L,\mathbf{t}_R}$ is defined as:

$$\partial_{\mathbf{t}_L,\mathbf{t}_R}(...)\equiv\frac{1}{(J!)^2}\partial_{t_0^+}^{m_0^+}\partial_{t_1}^{n_{01}}\partial_{t_2}^{n_{02}}\partial_{t_0^-}^{m_0^-}\partial_{t_3}^{n_{03}}\partial_{t_4}^{n_{04}}\,(...)|_{\mathbf{t}_L=\mathbf{0},\mathbf{t}_R=\mathbf{0}}\,. \qquad (4.5.22)$$

Now we can easily evaluate the contraction between $\mathcal{D}_{Z_0}$ and Z_0 and it gives Gegenbauer polynomial:

$$\begin{aligned}&\mathcal{V}(\mathbf{m}_L,\mathbf{m}_R)\\&=\partial_{\mathbf{t}_L,\mathbf{t}_R}\tilde{\sum_r}(-4P_0\cdot\mathcal{W}_{12}\cdot\mathcal{W}_{34}\cdot P_0)^{J-2r}(-4P_0\cdot\mathcal{W}_{12}\cdot\mathcal{W}_{12}\cdot P_0)^r(-4P_0\cdot\mathcal{W}_{34}\cdot\mathcal{W}_{34}\cdot P_0)^r\\&=\tilde{\sum_r}\sum_{\beta\{A,L,R\}}\frac{m_0^+!n_{01}!n_{02}!m_0^-!n_{03}!n_{04}!}{(J!)^2}\frac{(J-2r)!}{\prod_{\{A\}}\beta^{\{A\}}!}\mathcal{V}^{(1234)}\frac{r!}{\prod_{\{L\}}\beta^{\{L\}}!}\mathcal{V}^{(1212)}\frac{r!}{\prod_{\{R\}}\beta^{\{R\}}!}\mathcal{V}^{(3434)}\,,\end{aligned} \qquad (4.5.23)$$

The summation $\tilde{\sum}_r$ is the same as defined in (4.3.6). In the second line, we expanded the factors $(-4P_0\cdot\mathcal{W}_{12}\cdot\mathcal{W}_{34}\cdot P_0)$, $(-4P_0\cdot\mathcal{W}_{12}\cdot\mathcal{W}_{12}\cdot P_0)$ and $(-4P_0\cdot\mathcal{W}_{34}\cdot\mathcal{W}_{34}\cdot P_0)$[5], and applied the differential operator $\partial_{\mathbf{t}_L,\mathbf{t}_R}$. Here the factors $\{\mathcal{V}^{(\cdots)}\}$ are given as:

$$\begin{aligned}\mathcal{V}^{(1234)}&=(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{34}\cdot P_0)^{\beta^{\{1234\}}}(-4P_0\cdot C_1\cdot\mathcal{P}_{34}\cdot P_0)^{\beta^{\{1\bar{1}34\}}}(-4P_0\cdot C_2\cdot\mathcal{P}_{34}\cdot P_0)^{\beta^{\{\bar{2}234\}}}\\&\times(-4P_0\cdot\mathcal{P}_{12}\cdot C_3\cdot P_0)^{\beta^{\{123\bar{3}\}}}(-4P_0\cdot\mathcal{P}_{12}\cdot C_4\cdot P_0)^{\beta^{\{12\bar{4}4\}}}(-4P_0\cdot C_1\cdot C_3\cdot P_0)^{\beta^{\{1\bar{1}3\bar{3}\}}}\\&\times(-4P_0\cdot C_2\cdot C_3\cdot P_0)^{\beta^{\{\bar{2}23\bar{3}\}}}(-4P_0\cdot C_1\cdot C_4\cdot P_0)^{\beta^{\{1\bar{1}\bar{4}4\}}}(-4P_0\cdot C_2\cdot C_4\cdot P_0)^{\beta^{\{\bar{2}2\bar{4}4\}}}\,,\\\mathcal{V}^{(1212)}&=(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{12}\cdot P_0)^{\beta^{\{1212\}}}(-8P_0\cdot\mathcal{P}_{12}\cdot C_1\cdot P_0)^{\beta^{\{121\bar{1}\}}}\\&\times(-8P_0\cdot\mathcal{P}_{12}\cdot C_2\cdot P_0)^{\beta^{\{12\bar{2}2\}}}(-8P_0\cdot C_1\cdot C_2\cdot P_0)^{\beta^{\{1\bar{1}\bar{2}2\}}}\,,\end{aligned}$$

[5]We will label the powers of the various terms obtained from the polynomial expansion by the indices $\beta^{\{\cdots\}}$. see (4.5.24) and (4.5.25).

$$
\begin{aligned}
\mathcal{V}^{(3434)} &= (-4P_0\cdot\mathcal{P}_{34}\cdot\mathcal{P}_{34}\cdot P_0)^{\beta^{\{3434\}}}(-8P_0\cdot\mathcal{P}_{34}\cdot C_3\cdot P_0)^{\beta^{\{343\bar{3}\}}}\\
&\times(-8P_0\cdot\mathcal{P}_{34}\cdot C_4\cdot P_0)^{\beta^{\{34\bar{4}4\}}}(-8P_0\cdot C_3\cdot C_4\cdot P_0)^{\beta^{\{3\bar{3}\bar{4}4\}}}\,.
\end{aligned}
\tag{4.5.24}
$$

In (4.5.23), we have used the abridged notations:

$$
\begin{aligned}
\{A\} &= \Big\{\{1234\},\ \{1\bar{1}34\},\ \{\bar{2}234\},\ \{123\bar{3}\},\ \{12\bar{4}4\},\ \{1\bar{1}3\bar{3}\},\ \{\bar{2}234\},\ \{1\bar{1}\bar{4}4\},\ \{\bar{2}2\bar{4}4\}\Big\},\\
\{L\} &= \Big\{\{1212\},\ \{121\bar{1}\},\ \{12\bar{2}2\},\ \{1\bar{1}\bar{2}2\}\Big\},\\
\{R\} &= \Big\{\{3434\},\ \{343\bar{3}\},\ \{34\bar{4}4\},\ \{3\bar{3}\bar{4}4\}\Big\}.
\end{aligned}
\tag{4.5.25}
$$

to denote the sets of indices arising from the expansions of $\mathcal{V}^{(1234)}$, $\mathcal{V}^{(1212)}$ and $\mathcal{V}^{(3434)}$ respectively. Notice that we have used combined indices ij to associate with the anti-symmetric tensor $\mathcal{P}_{ij}$, however the ordered combined indices $i\bar{i}$ and $\bar{i}i$ indicate that we can obtain C_i from $\mathcal{P}_{ij}$ or from $\mathcal{P}_{ji}$ by replacing P_j with $\bar{v}_i$. Note that seventeen possible non-negative integers $\beta^{\{\dots\}}$ need to satisfy the following nine constraints and the summations of $\beta^{\{\dots\}}$ in the last line of (4.5.23) are taken within the values where the constraints are satisfied[6]:

$$
\begin{aligned}
&\sum_{\{..12..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..12..\}\in\{L\}}\beta^{\{L\}}=m_0^+\,, \qquad \sum_{\{..1\bar{1}..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..1\bar{1}..\}\in\{L\}}\beta^{\{L\}}=n_{01}\,,\\
&\sum_{\{..\bar{2}2..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..\bar{2}2..\}\in\{L\}}\beta^{\{L\}}=n_{02}\,, \qquad \sum_{\{..34..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..34..\}\in\{R\}}\beta^{\{R\}}=m_0^-\,,\\
&\sum_{\{..3\bar{3}..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..3\bar{3}..\}\in\{R\}}\beta^{\{R\}}=n_{03}\,, \qquad \sum_{\{..\bar{4}4..\}\in\{A\}}\beta^{\{A\}}+\sum_{\{..\bar{4}4..\}\in\{R\}}\beta^{\{R\}}=n_{04}\,,\\
&\sum_{\{A\}}\beta^{\{A\}}=J-2r\,,\quad \sum_{\{L\}}\beta^{\{L\}}=r\,,\quad \sum_{\{R\}}\beta^{\{R\}}=r\,.
\end{aligned}
\tag{4.5.27}
$$

The first six constraints come from the differentiations with respect to parameters $\mathbf{t}_L$ and $\mathbf{t}_R$, and the last three constraints come from the expansion of polynomials. Next we have to expand each factor in (4.5.24) . For example, one of the factors, $(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{34}\cdot P_0)$, is expanded as:

$$
\begin{aligned}
&(-4P_0\cdot\mathcal{P}_{12}\cdot\mathcal{P}_{34}\cdot P_0)^{\beta^{\{1234\}}}\\
&=\Big(\frac{1}{2}\left(P_{14}P_{02}P_{03}-P_{13}P_{02}P_{04}-P_{24}P_{01}P_{03}+P_{23}P_{01}P_{04}\right)\Big)^{\beta^{\{1234\}}}
\end{aligned}
$$

[6] For example, the first constraint is written explicitly as follows:

$$
\beta^{\{1234\}}+\beta^{\{123\bar{3}\}}+\beta^{\{12\bar{4}4\}}+2\beta^{\{1212\}}+\beta^{\{121\bar{1}\}}+\beta^{\{12\bar{2}2\}}=m_0^+\,. \tag{4.5.26}
$$

Note here that there is factor 2 in front of $\beta^{\{1212\}}$ because it contains two set of 12 , and $\beta^{\{1\bar{1}\bar{2}2\}}$ is not included in the second summation because it does not contain $\mathcal{P}_{12}$.

$$= \frac{\beta^{\{1234\}}!}{2^{\beta^{\{1234\}}}} \sum_{\sum k_{ij}^{\{1234\}}=\beta^{\{1234\}}} (-1)^{k_{(1)(3)}^{\{1234\}}+k_{(2)(4)}^{\{1234\}}} \prod_{(ij)} \frac{P_{ij}^{k_{ij}^{\{1234\}}}}{k_{ij}^{\{1234\}}!} \prod_i P_{0i}^{\beta^{\{1234\}}-\sum_j k_{ij}^{\{1234\}}} \ . \quad (4.5.28)$$

Here we introduce new four integers $k_{ij}^{\{1234\}}$ labeling the partition of $\beta^{\{1234\}}$. The indices of (1)(3) in the third line means the first and third indexes in the bracket are substituted, for example, $k_{(1)(3)}^{\{ijkl\}} = k_{ik}^{\{ijkl\}}$, and (ij) runs over only (13), (14), (23) and (24).

The expansions of the other factors can be obtained by replacing P_i with $\bar{v}_i$ appropriately according to the bracket index. For example, the expansion of $(-4P_0 \cdot \mathcal{P}_{12} \cdot C_3 \cdot P_0)$ is obtained by replacing P_4 with $\bar{v}_3$:

$$\begin{aligned}
&(-4P_0 \cdot \mathcal{P}_{12} \cdot C_3 \cdot P_0)^{\beta^{\{123\bar{3}\}}} \\
&= \Big(\frac{1}{2}\left((-2\bar{v}_3 \cdot P_1)P_{02}P_{03} - P_{13}P_{02}(-2\bar{v}_3 \cdot P_0) - (-2\bar{v}_3 \cdot P_2)P_{01}P_{03} + P_{23}P_{01}(-2\bar{v}_3 \cdot P_0)\right)\Big)^{\beta^{\{123\bar{3}\}}} \\
&= \frac{\beta^{\{123\bar{3}\}}!}{2^{\beta^{\{123\bar{3}\}}}} \sum_{\sum k_{\alpha\beta}^{\{123\bar{3}\}}=\beta^{\{123\bar{3}\}}} (-1)^{k_{(1)(3)}^{\{123\bar{3}\}}+k_{(2)(4)}^{\{123\bar{3}\}}} \prod_{(\bar{i}j)\in\{123\bar{3}\}} \frac{(-2\bar{v}_i \cdot P_j)^{k_{\bar{i}j}^{\{123\bar{3}\}}}}{k_{\bar{i}j}^{\{123\bar{3}\}}!} \\
&\quad \times \prod_{(ij)\in\{123\bar{3}\}} \frac{P_{ij}^{k_{ij}^{\{123\bar{3}\}}}}{k_{ij}^{\{123\bar{3}\}}!} \prod_{i\in\{123\bar{3}\}} P_{0i}^{\beta^{\{123\bar{3}\}}-\sum_\alpha k_{i\alpha}^{\{123\bar{3}\}}} \prod_{\bar{i}\in\{123\bar{3}\}} (-2\bar{v}_i \cdot P_0)^{\beta^{\{123\bar{3}\}}-\sum_\alpha k_{\bar{i}\alpha}^{\{123\bar{3}\}}} \ . \qquad (4.5.29)
\end{aligned}$$

Similarly as in the previous example, we introduced four non-negative integers $k_{\alpha\beta}^{\{123\bar{3}\}}$ to denote the four-folds partition of $\beta^{\{123\bar{3}\}}$. The indices α and β can be i and $\bar{i}$ which take values in $\{123\bar{3}\}$, such that in this case $i = 1, 2, 3$ in the third product and $\bar{i} = \bar{3}$ in the fourth product in the summation. And following these assignments, we have $k_{1\bar{3}}^{\{123\bar{3}\}}$ and $k_{2\bar{3}}^{\{123\bar{3}\}}$ from the first product and $k_{13}^{\{123\bar{3}\}}$ and $k_{23}^{\{123\bar{3}\}}$ from the second product in the summation. Comparing this case with (4.5.28), index 4 is replaced with $\bar{3}$ because this polynomial expansion is obtained by replacing P_4 with $\bar{v}_3$.

The next example is the expansion of the factor such as $(-8P_0 \cdot C_1 \cdot C_2 \cdot P_0)^{\beta^{\{1\bar{1}\bar{2}2\}}}$ in the expansion of $\mathcal{V}^{(1212)}$:

$$\begin{aligned}
(-8P_0 \cdot C_1 \cdot C_2 \cdot P_0)^{\beta^{\{1\bar{1}\bar{2}2\}}} &= (P_{12}(-2\bar{v}_1 \cdot P_0)(-2\bar{v}_2 \cdot P_0) + (-2\bar{v}_1 \cdot \bar{v}_2)P_{01}P_{02})^{\beta^{\{1\bar{1}\bar{2}2\}}} \qquad (4.5.30) \\
&= \sum_{k^{\{1\bar{1}\bar{2}2\}}+\bar{k}^{\{1\bar{1}\bar{2}2\}}=\beta^{\{1\bar{1}\bar{2}2\}}} \frac{\beta^{\{1\bar{1}\bar{2}2\}}!}{k^{\{1\bar{1}\bar{2}2\}}!\bar{k}^{\{1\bar{1}\bar{2}2\}}!} P_{12}^{k^{\{1\bar{1}\bar{2}2\}}} (-2\bar{v}_1 \cdot \bar{v}_2)^{\bar{k}^{\{1\bar{1}\bar{2}2\}}} \\
&\qquad \times P_{01}^{\bar{k}^{\{1\bar{1}\bar{2}2\}}} P_{02}^{\bar{k}^{\{1\bar{1}\bar{2}2\}}} (-2\bar{v}_1 \cdot P_0)^{k^{\{1\bar{1}\bar{2}2\}}} (-2\bar{v}_2 \cdot P_0)^{k^{\{1\bar{1}\bar{2}2\}}} .
\end{aligned}$$

For the binomial expansion of this factor, we introduced two non-negative $k^{\{1\bar{1}\bar{2}2\}}$ and $\bar{k}^{\{1\bar{1}\bar{2}2\}}$. Similarly, we introduced $k^{\{3\bar{3}\bar{4}4\}}$ and $\bar{k}^{\{3\bar{3}\bar{4}4\}}$ for the the expansion of $\beta^{\{3\bar{3}\bar{4}4\}}$. The other factors which come from $\{R\}$ or $\{L\}$ combine into a single factor. After expanding all the factors and collecting various terms, we obtain:

$$\mathcal{V}(\mathbf{m}_L, \mathbf{m}_R) = \tilde{\sum_{r,\beta,k}} \prod_{(ij)} \frac{P_{ij}^{\sum_{\{A\}} k_{ij}^{\{A\}}}}{\prod_{\{A\}} k_{ij}^{\{A\}}!} \prod_{(\bar{i}j)} \frac{(-2\bar{v}_i \cdot P_j)^{\sum_{\{A\}} k_{\bar{i}j}^{\{A\}}}}{\prod_{\{A\}} k_{\bar{i}j}^{\{A\}}!} \prod_{(\bar{i}\bar{j})} \frac{(-2\bar{v}_i \cdot \bar{v}_j)^{\sum_{\{A\}} k_{\bar{i}\bar{j}}^{\{A\}}}}{\prod_{\{A\}} k_{\bar{i}\bar{j}}^{\{A\}}!} \tag{4.5.31}$$
$$\times \prod_i P_{0i}^{\kappa_i} (-2\bar{v}_i \cdot P_0)^{\bar{\kappa}_{\bar{i}}} P_{12}^{r-\bar{k}^{\{1\bar{1}\bar{2}2\}}} P_{34}^{r-\bar{k}^{\{3\bar{3}\bar{4}4\}}} (-2\bar{v}_1 \cdot \bar{v}_2)^{\bar{k}^{\{1\bar{1}\bar{2}2\}}} (-2\bar{v}_3 \cdot \bar{v}_4)^{\bar{k}^{\{3\bar{3}\bar{4}4\}}}$$

Here κ_i and $\bar{\kappa}_{\bar{i}}$ are defined as:

$$\begin{aligned}
\kappa_i &= \sum_{\{A\}(\ni i)} (\beta^{\{A\}} - \sum_{\alpha \in \{A\}} k_{i\alpha}^{\{A\}}) + \sum_{\{B\}(\ni (ii))} \beta^{\{B\}} + \sum_{\{B\}(\ni i)} \bar{k}^{\{B\}}\,, \\
\bar{\kappa}_{\bar{i}} &= \sum_{\{A\}(\ni \bar{i})} (\beta^{\{A\}} - \sum_{\alpha \in \{A\}} k_{\bar{i}\alpha}^{\{A\}}) + \sum_{\{B\}(\ni (i\bar{i}))} \beta^{\{B\}} + \sum_{\{B\}(\ni i)} k^{\{B\}}\,.
\end{aligned} \tag{4.5.32}$$

Note that $\{B\} = \{L\} \cup \{R\}$, and α runs over i and $\bar{i}$. The summation $\tilde{\sum}$ is given by:

$$\tilde{\sum_{r,\beta,k}} = \tilde{\sum_r} \sum_{\beta^{\{A\}},\beta^{\{L\}},\beta^{\{R\}}} \frac{m_0^+! n_{01}! n_{02}! m_0^-! n_{03}! n_{04}!}{(J!)^2} \frac{(J-2r)!}{2^{J-2r}} \frac{r!}{\prod_{\{L\}} \beta^{\{L\}}!} \frac{r!}{\prod_{\{R\}} \beta^{\{R\}}!} \tag{4.5.33}$$
$$\times \sum_{\sum_{(\alpha\beta)} k_{(\alpha\beta)}^{\{A\}} = \beta^{\{A\}}} \sum_{k^{\{1\bar{1}\bar{2}2\}}, k^{\{3\bar{3}\bar{4}4\}}} (-1)^{\sum_{\{A\}} k_{(1)(3)}^{\{A\}} + k_{(2)(4)}^{\{A\}}} \frac{\beta^{\{1\bar{1}\bar{2}2\}}!}{k^{\{1\bar{1}\bar{2}2\}}! \bar{k}^{\{1\bar{1}\bar{2}2\}}!} \frac{\beta^{\{3\bar{3}\bar{4}4\}}!}{k^{\{3\bar{3}\bar{4}4\}}! \bar{k}^{\{3\bar{3}\bar{4}4\}}!}\,.$$

By substituting (4.5.31) into (4.5.17) and using the generalized Symanzik star-formula, the integration (4.5.17) can be expressed into the Mellin representation as:

$$\mathcal{I}_{\nu,J}^{(\mathbf{n}_L,\mathbf{n}_R)}(P_i, Z_i) = \pi^h \left(\prod_{i=1}^4 \frac{m_i!}{\Gamma(\tilde{\gamma}_{0i})} \right) \sum_{\{a,b\}} \int_{-i\infty}^{i\infty} \frac{d\tilde{\delta}_{12} d\tilde{\delta}_{13}}{(2\pi i)^2} \tilde{P}_{\nu,J}^{(0)(\mathbf{n_L},\mathbf{n_R})}(\tilde{\delta}_{ij}, a_{ij}, b_{ij})$$
$$\times \frac{1}{\Gamma(\tilde{\delta}_{12})\Gamma(\tilde{\delta}_{34})} \prod_{i \neq j} \frac{(-2\bar{v}_i \cdot P_j)^{a_{ij}}}{a_{ij}!} \prod_{i<j} \frac{(-2\bar{v}_i \cdot \bar{v}_j)^{b_{ij}}}{b_{ij}!} \Gamma(\tilde{\delta}_{ij}) P_{ij}^{-\tilde{\delta}_{ij}} \tag{4.5.34}$$

Now the Mellin variables $\{\tilde{\delta}_{ij}, a_{ij}, b_{ij}\}$ satisfy the conditions;

$$\sum_{j(\neq i)} (b_{ij} + a_{ij}) = l_i\,, \qquad \sum_{j(\neq i)} (\tilde{\delta}_{ij} - a_{ji}) = \Delta_i\,, \tag{4.5.35}$$

where $a_{ij} \neq a_{ji}$ and $b_{ij} = b_{ji}$ are non-negative integers. The range of summation of a_{ij} and b_{ij} restricted by the first constraint, and the second constraint of $\tilde{\delta}_{ij}$ is shifted from the scalar case

(4.2.5) by a_{ij} and b_{ij}. Here $\tilde{P}^{(0)(\mathbf{n_L},\mathbf{n_R})}_{\nu;J}$ is

$$\begin{aligned}
\tilde{P}^{(0)(\mathbf{n}_L,\mathbf{n}_R)}_{\nu,J}(\tilde{\delta}_{ij},a_{ij},b_{ij}) &\equiv \tilde{\sum_{r,\beta,k}}\frac{1}{(-2)^{\sum_i m_i+n_{12}+n_{34}}}\prod_i(\tilde{\gamma}_{0i}-\kappa_i)_{\kappa_i}(m_i+1)_{\bar{\kappa}_{\bar{i}}}\\
&\times\Gamma(\bar{\delta}_{12})\Gamma(\bar{\delta}_{34})\frac{b_{12}!}{(b_{12}-n_{12}-\bar{k}^{\{1\bar{1}\bar{2}2\}})!}\frac{b_{34}!}{(b_{34}-n_{34}-\bar{k}^{\{3\bar{3}\bar{4}4\}})!}\\
&\times\prod_{(ij)}\frac{(\tilde{\delta}_{ij})_{\sum_{\{A\}}k^{\{A\}}_{ij}}}{\prod_{\{A\}}k^{\{A\}}_{ij}!}\prod_{(\bar{i}j)}\frac{(a_{ij}-\sum_{\{A\}}k^{\{A\}}_{\bar{i}j}+1)_{\sum_{\{A\}}k^{\{A\}}_{\bar{i}j}}}{\prod_{\{A\}}k^{\{A\}}_{\bar{i}j}!}\prod_{(\bar{i}\bar{j})}\frac{(b_{ij}-\sum_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}+1)_{\sum_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}}}{\prod_{\{A\}}k^{\{A\}}_{\bar{i}\bar{j}}!}\,,
\end{aligned} \tag{4.5.36}$$

Notice that $\tilde{\delta}_{ij}$ depend on the integers a_{ij} and b_{ij} due to the condition (4.5.35). We have also defined $\bar{\delta}_{12}$ and $\bar{\delta}_{34}$ as:

$$\bar{\delta}_{12} \equiv \tilde{\delta}_{12}-\tilde{\gamma}_{12}+n_{12}+r-\bar{k}^{\{1\bar{1}\bar{2}2\}}\,, \tag{4.5.37}$$

$$\bar{\delta}_{34} \equiv \tilde{\delta}_{34}-\tilde{\gamma}_{34}+n_{34}+r-\bar{k}^{\{3\bar{3}\bar{4}4\}}\,. \tag{4.5.38}$$

In the Mellin representation, the combinations of P_i and $\bar{v}_i$ give the elements of tensor structures

$$\begin{aligned}
\bar{v}_1\cdot P_3 &= \frac{P_{23}}{P_{12}}V_{1,23}\,, \quad \bar{v}_1\cdot P_4=\frac{P_{24}}{P_{12}}V_{1,24}\,, \quad \bar{v}_2\cdot P_3=-\frac{P_{13}}{P_{12}}V_{2,13}\,, \quad \bar{v}_2\cdot P_4=-\frac{P_{14}}{P_{12}}V_{2,14}\,,\\
\bar{v}_3\cdot P_1 &= \frac{P_{14}}{P_{34}}V_{3,41}\,, \quad \bar{v}_3\cdot P_2=\frac{P_{24}}{P_{34}}V_{3,42}\,, \quad \bar{v}_4\cdot P_1=-\frac{P_{13}}{P_{34}}V_{4,31}\,, \quad \bar{v}_4\cdot P_2=-\frac{P_{23}}{P_{34}}V_{4,32}\,.
\end{aligned} \tag{4.5.39}$$

These are proportional to $V_{i,jk}$ and the products of $\bar{v}_i$ relate to H_{ij} and $V_{i,jk}$

$$\begin{aligned}
\bar{v}_1\cdot\bar{v}_2 &= \frac{H_{12}}{P_{12}}\,, \quad \bar{v}_1\cdot\bar{v}_3=\frac{-P_{12}P_{34}H_{13}+2P_{14}P_{23}V_{1,23}V_{3,41}}{P_{12}P_{34}P_{13}}\,,\\
\bar{v}_3\cdot\bar{v}_4 &= \frac{H_{34}}{P_{34}}\,, \quad \bar{v}_1\cdot\bar{v}_4=\frac{P_{12}P_{34}H_{14}-2P_{13}P_{24}V_{1,24}V_{4,31}}{P_{12}P_{34}P_{14}}\,,\\
\bar{v}_2\cdot\bar{v}_3 &= \frac{P_{12}P_{34}H_{23}-2P_{13}P_{24}V_{2,13}V_{3,42}}{P_{12}P_{34}P_{23}}\,, \quad \bar{v}_2\cdot\bar{v}_4=\frac{-P_{12}P_{34}H_{24}+2P_{14}P_{23}V_{2,14}V_{4,32}}{P_{12}P_{34}P_{24}}\,.
\end{aligned} \tag{4.5.40}$$

Now $P_i\cdot\bar{v}_j$ and $\bar{v}_i\cdot\bar{v}_j$ become the elements of tensor structure for the four point case instead of $V_{i,jk}$ and H_{ij} . Next we rescale the tensor structures as:

$$(-2\bar{v}_i\cdot\bar{v}_j)\to\tilde{H}_{ij}\equiv\left(\frac{-2\bar{v}_i\cdot\bar{v}_j}{P_{ij}}\right)\,, \qquad (-2\bar{v}_i\cdot P_j)\to\tilde{V}_{ij}\equiv\left(\frac{-2\bar{v}_i\cdot P_j}{P_{ij}}\right)\,. \tag{4.5.41}$$

Here we defined new tensor structures $\tilde{H}_{ij}$ and $\tilde{V}_{ij}$. After this rescaling, $\tilde{\delta}_{ij}$ are shifted as $\tilde{\delta}_{ij}\to\delta_{ij}$, and (4.5.34) becomes (4.5.2), and the constraint for δ_{ij} are changed as in (4.5.3). Now $\bar{\delta}_{12}$ and $\bar{\delta}_{34}$

are:

$$\begin{aligned}
\bar{\delta}_{12} &= \delta_{12} - \tilde{\gamma}_{12} + n_{12} + r - \bar{k}^{\{1\bar{1}\bar{2}2\}} + b_{12} \\
&= \delta_{12} - \frac{\tilde{\tau}_1 + \tilde{\tau}_2 - \tilde{\tau}_0^+}{2} + d_{12} \,, \qquad (4.5.42)\\
\bar{\delta}_{34} &= \delta_{34} - \tilde{\gamma}_{34} + n_{34} + r - \bar{k}^{\{3\bar{3}\bar{4}4\}} + b_{34} \\
&= \delta_{12} - \frac{\tilde{\tau}_3 + \tilde{\tau}_4 - \tilde{\tau}_0^-}{2} + d_{34} \,. \qquad (4.5.43)
\end{aligned}$$

where d_{12} and d_{34} are defined as:

$$d_{12} = r + b_{12} - n_{12} - \bar{k}^{\{1\bar{1}\bar{2}2\}}, \quad d_{34} = r + b_{34} - n_{34} - \bar{k}^{\{3\bar{3}\bar{4}4\}}. \qquad (4.5.44)$$

Note that d_{12} and d_{34} are non-negative integers because due to the factor in the Mack polynomial $\frac{1}{(b_{12}-n_{12}-\bar{k}^{\{1\bar{1}\bar{2}2\}})!} = \frac{1}{\Gamma(b_{12}-n_{12}-\bar{k}^{\{1\bar{1}\bar{2}2\}}+1)}$, if $b_{12}-n_{12}-\bar{k}^{\{1\bar{1}\bar{2}2\}} < 0$, this factor becomes zero. Therefore only b_{12} which is greater than $n_{12} + \bar{k}^{\{1\bar{1}\bar{2}2\}}$ can contribute, and for the same reason, we can regard d_{34} as a non-negative integer. Now we can decompose the gamma functions $\Gamma(\bar{\delta}_{12})$ and $\Gamma(\bar{\delta}_{34})$ as:

$$\Gamma(\bar{\delta}_{12}) = \Gamma(\hat{\delta}_{12})(\hat{\delta}_{12})_{d_{12}}, \quad \Gamma(\bar{\delta}_{34}) = \Gamma(\hat{\delta}_{34})(\hat{\delta}_{34})_{d_{34}}, \qquad (4.5.45)$$

where $\hat{\delta}_{12}$ and $\hat{\delta}_{34}$ are defined in (4.5.5). The other $\tilde{\delta}_{ij}$ are also shifted as

$$\tilde{\delta}_{ij} \to \delta_{ij} = \tilde{\delta}_{ij} - d_{ij} \,, \quad d_{ij} \equiv a_{ij} + a_{ji} + b_{ij} \,. \qquad (\, ij \in (ij)\,) \qquad (4.5.46)$$

Substituting this decomposition into (4.5.34), we obtain the Mellin representation (4.5.2).

4.5.2 Simple Examples

In the previous section, we have seen the general expression of the conformal integral with spinning fields. The general expression is somewhat complicated, in this section we will consider some simple examples of Mellin representations for specific conformal integrals.

Example-1: $\mathbf{n}_L = \mathbf{n}_R = \mathbf{0}$

The first example is the conformal integral of two three point functions which contain only $V_{i,jk}$ and P_{ij}.

$$\mathcal{I}^{(\mathbf{0},\mathbf{0})}_{\nu,J}(P_i, Z_i) = \frac{1}{J!\,(h-1)_J} \int_\partial dP_0 \begin{bmatrix} \Delta_1 & \Delta_2 & h+i\nu \\ l_1 & l_2 & J \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta_3 & \Delta_4 & h-i\nu \\ l_3 & l_4 & J \\ 0 & 0 & 0 \end{bmatrix} . \qquad (4.5.47)$$

In this case, it is easier to perform the integration, and we obtain the following expression:

$$\mathcal{I}^{(\mathbf{0},\mathbf{0})}_{\nu,J}(P_i,Z_i) = \pi^h\left(\prod_{i=1}^{4}\frac{l_i!}{\Gamma(\tilde{\gamma}_{0i})}\right)\sum_{a_{ij},b_{ij}}\int_{-i\infty}^{i\infty}\frac{d\delta_{12}d\delta_{13}}{(2\pi i)^2} \\ \times\tilde{P}^{(\mathbf{0},\mathbf{0})}_{\nu.J}(\delta_{ij},a_{ij},b_{ij})\frac{\Gamma(\hat{\delta}_{12})\Gamma(\hat{\delta}_{34})}{\Gamma(\delta_{12})\Gamma(\delta_{34})}\prod_{i\neq j}\frac{\tilde{V}_{ij}^{a_{ij}}}{a_{ij}!}\prod_{i<j}\frac{\tilde{H}_{ij}^{b_{ij}}}{b_{ij}!}\Gamma(\delta_{ij})P_{ij}^{-\delta_{ij}}\,. \tag{4.5.48}$$

Now $\tilde{\gamma}_{ij}$ are the same as (4.5.18) for $m_i=l_i$, $m_0^{\pm}=J$ and $n_{ij}=0$. $\hat{\delta}_{12}$ and $\hat{\delta}_{34}$ are

$$\hat{\delta}_{12}=\delta_{12}-\tilde{\gamma}_{12}\,,\qquad \hat{\delta}_{34}=\delta_{34}-\tilde{\gamma}_{34}\,. \tag{4.5.49}$$

Here the generalized Mack polynomial $\tilde{P}^{(\mathbf{0},\mathbf{0})}_{\nu.J}(\delta_{ij},a_{ij},b_{ij})$ is given as:

$$\tilde{P}^{(\mathbf{0},\mathbf{0})}_{\nu.J}(\delta_{ij},a_{ij},b_{ij}) = \frac{1}{(-2)^{\sum_{i=1}^4 l_i}}\sum_{r=0}^{[J/2]}(-1)^r\frac{J!(J+h-1)_{-r}}{r!\,2^J}(\hat{\delta}_{12})_{r+b_{12}}(\hat{\delta}_{34})_{r+b_{34}} \\ \times\sum_{\sum k_{ij}=J-2r}(-1)^{k_{24}+k_{13}}\prod_{(ij)}\frac{(\delta_{ij})_{k_{ij}+d_{ij}}}{k_{ij}!}\prod_{i=1}^{4}\Big(\tilde{\gamma}_{0i}-J+r+\sum_j k_{ji}\Big)_{J-r-\sum_j k_{ji}}\,. \tag{4.5.50}$$

This polynomial has almost the same form as the scalar case (4.3.14), except for some additional Pochhammer symbols and the over all factors. The definition of d_{ij} is the same as in (4.5.46).

Example-2: $\mathbf{n}_L=\mathbf{n}_R=\mathbf{n}_{(J)}=(J,0,0)$

Next we consider the case with $\mathbf{n}_L=\mathbf{n}_R=\mathbf{n}_{(J)}=(J,0,0)$:

$$\mathcal{I}^{(\mathbf{n}_{(J)},\mathbf{n}_{(J)})}_{\nu,J}(P_i,Z_i)\equiv\frac{1}{J!\,(h-1)_J}\int_\partial dP_0\begin{bmatrix}\Delta_1 & \Delta_2 & h+i\nu\\ l_1 & l_2 & J\\ J & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\Delta_3 & \Delta_4 & h-i\nu\\ l_3 & l_4 & J\\ J & 0 & 0\end{bmatrix}. \tag{4.5.51}$$

Here we suppose that $l_2\geq J$ and $l_4\geq J$ to keep m_i non-negative integers. The differential operator $\mathcal{D}_{Z_0}$ and the polarization vector Z_0 are contained only in H_{02} and H_{04} respectively, and the contraction between $\mathcal{D}_{Z_0}$ and Z_0 are evaluated only through H_{02} and H_{04} instead of V_{012} and $V_{0,34}$ as in the scalar case (4.3.5). Now we have to evaluate the following combination:

$$\frac{1}{J!(h-1)_J}\left(2P_0\cdot C_2\cdot\mathcal{D}_{Z_0}\right)^J\left(2P_0\cdot C_4\cdot Z_0\right)^J\,. \tag{4.5.52}$$

This type of combination gives the Gegenbauer polynomial in general as in (4.3.5). In this case, however, because $(-4P_0\cdot C_i\cdot C_i\cdot P_0)=0$, the terms of the polynomial are reduced, and the combination is equivalent to $(-4P_0 \cdot C_2 \cdot C_4 \cdot P_0)^J$. After the similar calculation, we can obtain the following generalized Mack polynomial:

$$
\begin{aligned}
\tilde{P}^{(\mathbf{n}_{(J)},\mathbf{n}_{(J)})}_{\nu.J}(\delta_{ij}, a_{ij}, b_{ij}) &= \frac{J!}{2^J} \sum_{\sum k = J} (-1)^{k_{24}+k_{\bar{2}\bar{4}}} \frac{1}{(-2)^{\sum_i l_i - 2J}} \\
&\times (m_2+1)_{\bar{\kappa}_2} (m_4+1)_{\bar{\kappa}_4} (\tilde{\gamma}_{02} - \kappa_2)_{\kappa_2} (\tilde{\gamma}_{04} - \kappa_4)_{\kappa_4} (\hat{\delta}_{12})_{b_{12}} (\hat{\delta}_{34})_{b_{34}} \\
&\times (\delta_{13})_{d_{13}} (\delta_{14})_{d_{14}} (\delta_{23})_{d_{23}} \frac{(\delta_{24})_{d_{24}+k_{24}}}{k_{24}!} \frac{(a_{24} - k_{\bar{2}4} + 1)_{k_{\bar{2}4}}}{k_{\bar{2}4}!} \frac{(a_{42} - k_{\bar{4}2} + 1)_{k_{\bar{4}2}}}{k_{\bar{4}2}!} \frac{(b_{24} - k_{\bar{2}\bar{4}} + 1)_{k_{\bar{2}\bar{4}}}}{k_{\bar{2}\bar{4}}!} \,.
\end{aligned}
\tag{4.5.53}
$$

Note that there is no r-summation because of the reduction of the Gegenbauer polynomial, and we have only the summation in k. In this example, only $\beta^{\{\bar{2}2\bar{4}4\}}(=J)$ is non-zero, and these $k_{\alpha\beta}$ in (4.5.53) correspond to $k^{\{\bar{2}2\bar{4}4\}}_{\alpha\beta}$ in the general case. κ_i and $\bar{\kappa}_i$ are given as

$$
\begin{aligned}
\kappa_2 &= J - k_{24} - k_{2\bar{4}}\,, \quad \kappa_4 = J - k_{24} - k_{\bar{2}4}\,, \\
\bar{\kappa}_2 &= J - k_{\bar{2}4} - k_{\bar{2}\bar{4}}\,, \quad \bar{\kappa}_4 = J - k_{2\bar{4}} - k_{\bar{2}\bar{4}}\,,
\end{aligned}
\tag{4.5.54}
$$

and the others are zero. The definition of d_{ij} is the same as before.

Here we end with a couple of short comments about the possible future directions. So far, we have considered only the Mellin representation of the spinning four-point functions, it would be interesting also to consider the higher point functions involving symmetric traceless tensor fields. Although it is still difficult to obtain the explicit form, we can count the number of independent variables through the generalized Symanzik formula which is useful even for n-point functions. Counting the number of discrete Mellin variables arising in this general case, there are altogether $\frac{3n(n-1)}{2}$ of $\{a_{ij}, b_{ij}\}$[7], however among them there are also n constraints, the numbers of independent discrete Mellin variables is $\frac{n(3n-5)}{2}$. Again, this number matches with the counting of flat space independent elements of tensor structures. It would also be interesting to generalize our analysis to more general representations, while here we consider only symmetric traceless operators. There are already some works related to such a direction [51, 71, 72].

[7] Here we assumed that $n < d$, where d is the number of the Euclidean spacetime dimensions.

Chapter 5

Towards the Crossing Kernel

So far, in the previous sections, we saw how the conformal block decompositions of some diagrams are obtained, and then the conformal partial wave(CPW) and its orthogonality played a crucial role. Especially, in section 3.4, it is discussed that an exchange diagram is simply expressed in terms of CPW via the AdS harmonic function. It happened because the exchange diagram and CPW are associated with the same channel, in fact, we have considered only s-channel exchange diagram and s-channel CPW. The next natural question is what kind of decomposition can be obtained from t- or u-channel exchange diagrams, and in order to answer this question, we have to compute the inner product between CPWs in different channels, for example,

$$\left(\Psi^{(t),\Delta_i}_{h+i\nu,J}, \Psi^{(s),d-\Delta_i}_{h-i\nu',J'}\right) \tag{5.0.1}$$

where $\Psi^{(t)}$ is the t-channel CPW, the explicit definition is given later. We also denote the CPW in s-channel as $\Psi^{(s)}$ to indicate the channel explicitly. Exchange diagrams in t-channel are readily expanded in the t-channel CPW, therefore once we could compute this inner product, we can know what kind of contributions come from a t-channel diagram into s-channel conformal block decomposition. The inner product in (5.0.1) is called the crossing kernel.

The crossing kernel is important in the conformal bootstrap because through the crossing kernel, we can expand t-channel conformal block in s-channel conformal blocks. Then both sides of the crossing equation are written in terms of s-channel quantity, and it becomes easy to find solutions to this equation. In the procedure of the bootstrap approach introduced in 4, we have to consider exchange diagram in different channels and tune the coupling constant so that there is no double trace contribution in the whole summation. It is known that exchange diagram in t- or u- channel give s-channel conformal blocks for double trace operators. The crossing kernel provides the coefficients of these contributions. More explicitly, through the following equation:

$$\Psi^{(t),\Delta_i}_{h+i\nu,J}(x_i) = \sum_{J=0}^{\infty}\int_{-i\infty}^{i\infty}\frac{d\nu}{n_{\nu,J}}\left(\Psi^{(t),\Delta_i}_{h+i\nu,J},\Psi^{(s),d-\Delta_i}_{h-i\nu',J'}\right)\Psi^{(s),\Delta_i}_{h+i\nu',J'}\,, \tag{5.0.2}$$

we can know what kind contributions come from the t-channel quantities. Once we could compute the inner product, the pole structure in the ν-plane tells us the spectrum of the dimensions and the residues become the expansion coefficients. After the ν-integral, through the relation between s-channel CPW and conformal blocks, we could expand t-channel CPW in terms of s-channel conformal blocks. In this way, it seems that the crossing kernel helps to solve the crossing equation and it is a key quantity in the bootstrap problem.

There are some previous works trying to calculate this inner product [73, 74] , however, it have not yet been understood the pole structure of this inner product. In one dimension, because there is only one cross ratio, the calculation is reduced to simple Mellin integrals, and this inner product can be evaluated [75]. The technique in one dimension is used to analyzed the SYK model [76] . There we can see the poles corresponding to the double trace contributions after the integrations. Recently, it is pointed out that in the calculation of inner products, once we went to the Lorentzian space by the Wick rotation of coordinates, the range of the coordinate integral can be reduced to the integral around the cuts associated with the conformal blocks [36,37]. This is formulated as the so-called Lorentzian inversion formula. By using this technique and the analytic form of conformal blocks for two or four dimensions, the crossing kernel is calculated in [38] . There again we can see the double trace poles in the result of the calculation. However, in the general d-dimension, the pole structure of crossing kernel is not known.

In the following sections, we will see that the bulk interpretation of the crossing kernel in section 5.1, and in section 5.2, we will see that it can be represented as Mellin integrals.

5.1 Bulk interpretation

So far, we have considered s-channel CPWs which are defined as a combination three-point functions associated with points (x_1, x_2, x_0) and (x_3, x_4, x_0) , however, of cause we can defined t- or u-channel CPW gluing three-point functions in different ways. For example the t-channel CPW is defined as:

$$\Psi^{(t),\Delta_i}_{h+i\nu,J}(x_i) = \int_{\mathbb{R}^d} d^dx_0\;\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_4}(x_4)\mathcal{O}_{h+i\nu,J}(x_0)^{\mu_1\cdots\mu_J}\rangle_1\langle\tilde{\mathcal{O}}_{h-i\nu,J}(x_0)^{\mu_1\cdots\mu_J}\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_1\,. \tag{5.1.1}$$

The Mellin representation and bulk representation for t-channel CPW are obtained by exchanging $2 \leftrightarrow 4$ or $1 \leftrightarrow 3$.

We can write the bulk diagram corresponding to the crossing kernel 5.0.1 in the same way as in discussed in section 3.5. Using the bulk expression of the conformal partial wave (3.5.2) and the definition of the AdS harmonic function (3.3.1), we can obtain the crossing kernel as a bubble diagram in bulk:

$$
\begin{aligned}
\left(\Psi^{(t),\Delta_i}_{h+i\nu,J}, \Psi^{(s),d-\Delta_i}_{h-i\nu',J'}\right) = \mathcal{N}^{(t),(s)} \left[\frac{1}{J!(h-\frac{1}{2})_J}\right]^2 \left[\frac{1}{J'!(h-\frac{1}{2})'_J}\right]^2 \int \frac{dX^{12}dX^{34}dX^{14}dX^{23}}{\mathrm{vol}\,(SO(1,d+1))} \\
\times (K^{14}\cdot\nabla_{14})^J\,\Omega_{\alpha_1}(X^{12},X^{14})\,(K^{12}\cdot\nabla_{12})^{J'}\,\Omega_{\alpha_2}(X^{12},X^{23}) \\
\times (K^{23}\cdot\nabla_{23})^J\,\Omega_{\alpha_3}(X^{23},X^{34})\,(K^{34}\cdot\nabla_{34})^{J'}\,\Omega_{\alpha_4}(X^{14},X^{34}) \\
\times \Omega_{\nu,J}(X^{13},X^{24};W^{13},W^{24})\,\Omega_{\nu',J'}(X^{12},X^{34};W^{12},W^{34})\,, \qquad (5.1.2)
\end{aligned}
$$

where the coefficient $\mathcal{N}^{(t),(s)}$ is defined as follows:

$$
\mathcal{N}^{(t),(s)} = \frac{1}{\mathcal{B}^{\Delta_1,\Delta_4}_{h+i\nu,J}\mathcal{B}^{\Delta_2,\Delta_3}_{h-i\nu,J}\mathcal{B}^{d-\Delta_1,d-\Delta_2}_{h-i\nu',J'}\mathcal{B}^{d-\Delta_3,d-\Delta_4}_{h+i\nu',J'}} \left(\prod_{i=1}^{4}\frac{\pi}{\alpha_i^2}\right)\frac{\pi^2}{\nu^2\ \nu'^2}\,. \qquad (5.1.3)
$$

Here we have introduced α_i again in the same way as before: $\Delta_i = h + i\alpha_i$. The picture of this

$$
\left(\Psi^{(t),\Delta_i}_{h+i\nu,J}, \Psi^{(s),d-\Delta_i}_{h-i\nu',J'}\right) =
$$

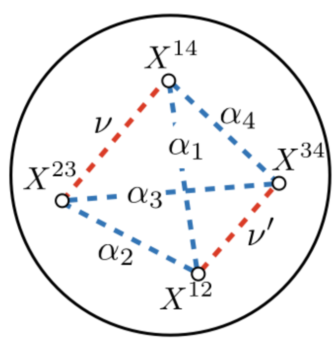


Figure 5.1: Crossing kernel as a bulk bubble diagram.

bubble diagram is given in Fig. 5.1. Because in the t-channel the combination for the three-point functions are different from the s-channel case, in this case, the inner product is represented as a tetrahedron diagram. In order to compute this diagram, we have to perform the bulk integration, however, each bulk integral has the following form:

$$
\int dY \Omega_{\nu_1}(X_1,Y)(K_Y\cdot\nabla_Y)^J\Omega_{\nu_2}(X_2,Y)\Omega_{\nu_3,J}(X_3,Y;W_3,W_Y) \qquad (5.1.4)
$$

This integral differs from the function Ξ which we have calculated in section 3.5 before. Now the scalar harmonic functions do not share the endpoints, and this integral depends on three points $X_{1,2,3}$. Therefore we cannot expand this integral in terms of the AdS harmonic function which is a basis for functions depending on two bulk points. It seems difficult to perform this integral exactly, and in the next section, we will see that this quantity can be evaluated as Mellin integrals.

5.2 Mellin representation

Here we try to compute the crossing kernel in a different way. Substituting the original definition of t- and s-channel CPW, the inner product can be expressed as integrals of four three-point functions:

$$\begin{aligned}\left(\Psi^{(t),\Delta_i}_{h+i\nu,J},\Psi^{(s),d-\Delta_i}_{h-i\nu',J'}\right) = &\int \frac{d^dP_0...d^dP_5}{\mathrm{vol}\,(SO(d+1,1))}\frac{1}{J!(h-1)_J}\frac{1}{J'!(h-1)'_J} \\ &\times\langle\mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{h-i\nu',J'}(P_5,\mathcal{D}_{Z_5})\rangle_1\langle\tilde{\mathcal{O}}_{h+i\nu',J'}(P_5,Z_5)\mathcal{O}_{\Delta_3}(P_3)\mathcal{O}_{\Delta_4}(P_4)\rangle_1 \\ &\times\langle\mathcal{O}_{\Delta_1}(P_1)\mathcal{O}_{\Delta_4}(P_4)\mathcal{O}_{h+i\nu,J}(P_0,\mathcal{D}_{Z_0})\rangle_1\langle\tilde{\mathcal{O}}_{h-i\nu,J}(P_0,Z_0)\mathcal{O}_{\Delta_2}(P_2)\mathcal{O}_{\Delta_3}(P_3)\rangle_1\,.\end{aligned} \tag{5.2.1}$$

Here there are six integrals for P_i $(i=0,...,5)$ where P_0 and P_5 integrations come from the definitions of CPWs. In the below we will try to evaluate some of these integral for the simplest case $J=J'=0$ using the Symanzik formula and see it can be represented as four Mellin integrals. In the scalar case, the integrals become:

$$\left(\Psi^{(t),\Delta_i}_{h+i\nu,0},\Psi^{(s),d-\Delta_i}_{h-i\nu',0}\right) = \int \frac{d^dP_0...d^dP_5}{\mathrm{vol}\,(SO(d+1,1))}\frac{1}{P_{12}^{\gamma_{12}}P_{34}^{\gamma_{34}}P_{14}^{\gamma_{14}}P_{23}^{\gamma_{23}}\prod_{i=1}^4 P_{0i}^{\gamma_{0i}}P_{i5}^{\gamma_{i5}}}\,, \tag{5.2.2}$$

where γ_{ij} are given by

$$\begin{aligned}\gamma_{01} &= -a^{(t)}+\frac{h+i\nu}{2}\,,\quad \gamma_{02}=-b^{(t)}+\frac{h-i\nu}{2}\,,\quad \gamma_{03}=b^{(t)}+\frac{h-i\nu}{2}\,,\quad \gamma_{04}=a^{(t)}+\frac{h+i\nu}{2}\,, \\ \gamma_{15} &= a^{(s)}+\frac{h-i\nu'}{2}\,,\quad \gamma_{25}=-a^{(s)}+\frac{h-i\nu'}{2}\,,\quad \gamma_{35}=-b^{(s)}+\frac{h+i\nu'}{2}\,,\quad \gamma_{45}=b^{(s)}+\frac{h+i\nu'}{2}\,, \\ \gamma_{12} &= \frac{2d-\Delta^+_{12}-h+i\nu'}{2}\,,\quad \gamma_{34}=\frac{2d-\Delta^+_{34}-h-i\nu'}{2}\,,\gamma_{14}=\frac{\Delta^+_{14}-h-i\nu}{2}\,,\quad \gamma_{23}=\frac{\Delta^+_{23}-h+i\nu}{2}\,.\end{aligned} \tag{5.2.3}$$

Here $a^{(t)}=\Delta_{41}/2$ and $b^{(t)}=\Delta_{32}/2$, $a^{(s)}$ and $b^{(s)}$ are the same as a and b defined before. We can easily perform the P_1-integration and through the symanzik formula, it introduces two Mellin integrations:

$$\int dP_1\,\frac{1}{P_{12}^{\gamma_{12}}P_{15}^{\gamma_{15}}P_{14}^{\gamma_{14}}P_{01}^{\gamma_{01}}} = \mathcal{N}_1\int_{-i\infty}^{i\infty}\frac{d\delta_{02}\delta_{04}}{(2\pi i)^2}\prod_{i<j\in\{0,2,4,5\}}\Gamma(\delta_{ij})P_{ij}^{-\delta_{ij}} \tag{5.2.4}$$

Here the index i and j run over $\{0,2,4,5\}$ and δ_{ij} satisfies the conditions: $\sum_{j(\neq i)}\delta_{ij} = \gamma i1$. In the above integral, we have chosen δ_{02} and δ_{04} are the integration variables and the other δ_{ij} are written in terms of δ_{02} and δ_{04} through the conditions. The coefficient $\mathcal{N}_1$ is given by:

$$\mathcal{N}_1 = \pi^h \prod_{i\in\{0,2,4,5\}} \frac{1}{\Gamma(\gamma_{i1})} \tag{5.2.5}$$

Similarly, the P_3-integral is also evaluated:

$$\int dP_3\, \frac{1}{P_{23}^{\gamma_{23}}P_{35}^{\gamma_{35}}P_{34}^{\gamma_{34}}P_{03}^{\gamma_{03}}} = \mathcal{N}_3 \int_{-i\infty}^{i\infty} \frac{d\tilde{\delta}_{02}\tilde{\delta}_{04}}{(2\pi i)^2} \prod_{i<j\in\{0,2,4,5\}} \Gamma(\tilde{\delta}_{ij})P_{ij}^{-\tilde{\delta}_{ij}}\,, \tag{5.2.6}$$

where $\tilde{\delta}_{ij}$ satisfy $\sum_{j(\neq i)}\tilde{\delta}_{ij} = \gamma_{i3}$ and the coefficient $\mathcal{N}_3$ is give by:

$$\mathcal{N}_3 = \pi^h \prod_{i\in\{0,2,4,5\}} \frac{1}{\Gamma(\gamma_{i3})} \tag{5.2.7}$$

Now the crossing kernels for scalars is given by the following integrals:

$$\begin{aligned}\left(\Psi^{(t),\Delta_i}_{h+i\nu,0}, \Psi^{(s),d-\Delta_i}_{h-i\nu',0}\right) = \mathcal{N}_1\mathcal{N}_3 \int_{-i\infty}^{i\infty} \frac{d\delta_{02}\delta_{04}d\tilde{\delta}_{02}\tilde{\delta}_{04}}{(2\pi i)^4} \prod_{i<j\in\{0,2,4,5\}} \Gamma(\delta_{ij})\Gamma(\tilde{\delta}_{ij}) \\ \times \int \frac{dP_0dP_2dP_4dP_5}{\mathrm{vol}\,(SO(d+1,1))} \frac{1}{P_{02}^{\gamma_{02}}P_{04}^{\gamma_{04}}P_{25}^{\gamma_{25}}P_{45}^{\gamma_{45}}} \prod_{i<j\in\{0,2,4,5\}} \frac{1}{P_{ij}^{\delta_{ij}}P_{ij}^{\tilde{\delta}_{ij}}}\end{aligned} \tag{5.2.8}$$

Here we can evaluate one more coordinate integral, for example the P_2-integral can be evaluated as a usual three-point integral:

$$\begin{aligned}&\int dP_2 \frac{1}{P_{02}^{\gamma_{02}+\delta_{02}+\tilde{\delta}_{02}}P_{25}^{\gamma_{25}+\delta_{25}+\tilde{\delta}_{25}}P_{24}^{\delta_{24}+\tilde{\delta}_{24}}} \\ &\quad = \pi^h \frac{\Gamma(\hat{\delta}_{05})\Gamma(\hat{\delta}_{45})\Gamma(\hat{\delta}_{04})}{\Gamma(\gamma_{02}+\delta_{02}+\tilde{\delta}_{02})\Gamma(\gamma_{25}+\delta_{25}+\tilde{\delta}_{25})\Gamma(\delta_{24}+\tilde{\delta}_{24})} \frac{1}{P_{05}^{\hat{\delta}_{05}}P_{45}^{\hat{\delta}_{45}}P_{04}^{\hat{\delta}_{04}}}\,,\end{aligned} \tag{5.2.9}$$

where $\hat{\delta}_{ij}$ are defined as below:

$$\begin{aligned}\hat{\delta}_{05} &= \frac{\Delta_{21}+\Delta_{43}}{2} + \delta_{02} + \delta_{04} + \tilde{\delta}_{02} + \tilde{\delta}_{04}\,, \quad \hat{\delta}_{45} = \frac{\Delta_{32}+h+i\nu}{2} - \delta_{02} - \tilde{\delta}_{02}\,, \\ \hat{\delta}_{04} &= \frac{\Delta_{14}+h-i\nu}{2} - \delta_{04} - \tilde{\delta}_{04}\,.\end{aligned} \tag{5.2.10}$$

After the P_2-integration, the remaining integrals are just kinematical ones and can be evaluated by using the conformal symmetry as in section 2.7:

$$\int \frac{dP_0 dP_4 dP_5}{\mathrm{vol}\,(SO(d+1,1))} \frac{1}{P_{05}^h P_{04}^h P_{45}^h} = \frac{1}{\mathrm{vol}\,(SO(d-1))} \,. \tag{5.2.11}$$

Then the inner product becomes four Mellin integrations as follows:

$$\begin{aligned}\left(\Psi^{(t),\Delta_i}_{h+i\nu,0}, \Psi^{(s),d-\Delta_i}_{h-i\nu',0}\right) &= \frac{\mathcal{N}_1\mathcal{N}_3}{\mathrm{vol}\,(SO(d-1))} \int_{-i\infty}^{i\infty} \frac{d\delta_{02}\delta_{04}d\tilde{\delta}_{02}\tilde{\delta}_{04}}{(2\pi i)^4} \prod_{i<j\in\{0,2,4,5\}} \Gamma(\delta_{ij})\Gamma(\tilde{\delta}_{ij}) \\ &\quad \times \frac{\Gamma(\hat{\delta}_{05})\Gamma(\hat{\delta}_{45})\Gamma(\hat{\delta}_{04})}{\Gamma(\gamma_{02}+\delta_{02}+\tilde{\delta}_{02})\Gamma(\gamma_{25}+\delta_{25}+\tilde{\delta}_{25})\Gamma(\delta_{24}+\tilde{\delta}_{24})} \,. \end{aligned} \tag{5.2.12}$$

The gamma functions in the integrand depend on δ_{02} , δ_{04} , $\tilde{\delta}_{02}$ and $\tilde{\delta}_{04}$ through the equations (5.2.10) and conditions which δ_{ij} and $\tilde{\delta}_{ij}$ satisfy. Note here a similar form of integration can be found in [74] . We can perform these Mellin integral naively picking up the relevant poles coming from gamma functions, however, then the pole integrations produce infinite summations. Now we are interested in pole structure in the ν-plane of this inner product, and if there are infinite summations, it is difficult to see the correct pole structure in general. In some special cases, the result of this type of Mellin integration can be resumed and can be written in terms of gamma functions or hypergeometric functions as in section A . In this case, there are fifteen gamma functions in the numerator and three gamma functions in the denominator, and as far as we know, there is no useful mathematical formula in order to evaluate this integration. In the case including spinning CPW, we can evaluate the inner product in a similar way, and eventually, we reach the same kind of Mellin integrations.

Chapter 6

Conclusion and Future directions

In this thesis, we have discussed some kinematic aspects of d-dimensional Euclidean conformal field theory (CFT) and diagrams in the $(d+1)$-dimensional Euclidean AdS space. Recently there are some developments on analysis using orthogonal basis in CFT and AdS diagram. In the CFT side, the conformal partial wave (CPW) forms a basis for four-point functions, and in AdS space, the AdS harmonic function forms a basis for AdS propagators. Thanks to these useful function, we could compute correlation functions or AdS diagrams systematically. As other recently developed concepts, we have also discussed geodesic diagrams and Mellin representations. Here we will give a summary for each chapter and discuss future directions.

Conformal Field Theory in d-dimension

As significant progress in d-dimensional CFT, in [7,8], the conformal blocks are characterized as eigenfunctions of the conformal Casimir equation and in even dimensions, the closed forms are discovered. The CPW which is defined as a product of two three-point functions is also eigenfunction of the Casimir equation, and it forms an orthogonal basis for four-point functions. Another nice property of CPW is that it is written as a linear combination of conformal blocks [67], according to this fact, we can systematically obtain the conformal block expansion of four-point functions. This procedure is encoded as the (Euclidean) inversion formula which is formulated in [36,37]. These techniques are related to the harmonic analysis for the Lorentz group $SO(1,d+1)$ corresponding to the conformal symmetry in d-dimensional Euclidean space [35,77]. Recently, this method is applied to an analysis of the SYK model [76] or the fishnet theory [78], and also computation of anomalous dimensions [79,80].

In chapter 2, we have given a simple example to show the use of inversion formula for the case of the generalized free theory. Another direction we discussed is the extension including external symmetric traceless tensors. In this discussion, the embedding formalism is quite useful, and the possible tensor structures are parametrized by using the simple elements $V_{i,jk}$ and H_{ij}. By using the differential operators, the spinning three-point functions are constructed from three-point functions with $(0, 0, J)$ spin. The CPWs with external spins are also defined applying the differential operator to the scalar CPW [47, 49]. The extension including general representations is recently developed in [51].

Diagrams in AdS Space

In the context of the AdS/CFT correspondence, correlation functions in large N CFT are computed as tree diagrams in AdS of the dual bulk theory. In the CFT side, there are some constraints on the conformal block decomposition, unitarity, crossing symmetry, etc. There are some works trying to solve these constraints for simple theories, and find corresponding diagrams in AdS space [17–19]. From these points of view, we are interested in what kind of conformal blocks are coming from a diagram in AdS space. To answer this question, the inversion formula is also useful. We have seen that CPW can be lifted to AdS space, and there the AdS harmonic function plays a role as an AdS analog for CPW. Through the inversion formula, we have seen the conformal block expansion of tree contact and exchange diagrams. Not only to obtain the conformal block expansion of bulk diagrams but also as a calculation tool, the bulk interpretation is useful. For example, we have seen how the orthogonality of CPW is working via the bulk interpretation, then the property of AdS harmonic function made the computation transparent.

In chapter 3, we also discuss geodesic diagrams which are proposed the bulk dual of conformal block [20] . In the spectral representation, the usual exchange diagrams contain the double trace poles not only the single trace pole, however, the four-point geodesic diagrams do not contain such redundant contributions. Therefore there is only the single trace pole, and this corresponds to a conformal block [1]. The extension involving external spins is also discussed. By parameterizing the bulk interactions properly, we can see the correspondence between the bulk interactions and the tensor structures [58, 59]. In the case of geodesic diagrams, once the interactions are parametrized properly, these form a basis for CFT tensor structures [1, 23]. So far we have considered only symmetric traceless tensors, and for extensions to other representations, there are some works [24, 25].

Mellin Representation

It is pointed out in [26, 27] that CFT correlation functions have relatively simple forms in the Mellin space, and there, they become analogous form to amplitudes in frat space. From computations of AdS diagrams, through the Symanzik formula [60], Mellin integrations naturally appear, and also it has a similar form to the QFT amplitude [28] . In fact, taking the flat space limit, the tree diagrams become tree QFT amplitudes [28, 62, 65].

In section 4, we have discussed the Mellin representation of come simple tree diagram and from these expressions, we can also derive the conformal block expansion. The Mellin representation of CPW is also argued, then in the integrand, the so-called Mack polynomial appeared. Roughly speaking, this polynomial is the Mellin transformation of the Gegenbauer polynomial, and plays an essential role in the bootstrap in the Mellin space recently developed in [29, 30, 34]. The spinning extension is also discussed. Then the CPW contains polarization vectors, not only coordinates, and through the generalized Symanzik formula, the new Mellin variables for polarization vectors are naturally introduced [2].

Towards the Crossing Kernel

In section 5, we discussed the crossing kernel which is the inner product of s- and t-channel CPW. When we try to compute the s-channel conformal block decomposition of the t-channel exchange diagram, we have to evaluate this quantity. The bootstrap approach, the crossing kernel would be critical since through the crossing kernel, we can convert t-channel conformal block into s-channel one. This fact may make it easy to solve the crossing equation which gives the bootstrap conditions [10].

In another bootstrap approach in the Mellin space discussed above, the building blocks are exchange diagrams in AdS space, and a four-point function expanded as a summation of exchange diagrams in s-, t- and u-channels. The summation is taken over all possible intermediate exchanged states. These exchange diagrams contain double trace contributions in the conformal block expansion, and then the non-trivial constraints on the expansion coefficients are demanded so that there are no such redundant double trace contributions after the summation. To solve this problem, the crossing kernel seems quite useful, because it enables us to expand diagrams in different channels in terms of the same basis. However, the actual computation is very complicated as we have seen in section 5. We may obtain the expansion form from the Mellin representation, however, it contains infinite summations and it makes it difficult to see the correct pole structure. In one dimensional case, this computation works well, and the explicit form of the crossing kernel is given in [75]. Recently, via the Lorentzian inversion formula, in two and four dimensions, the crossing kernel is computed [38]. However, in general, d-dimension, it is still an open problem.

Future Works

The computation of crossing kernel is one of the most interesting directions. From the Mellin representation, we have obtained the expansion form of CPW, and this expression may help us compute the crossing kernel. Once we could succeed in computing it, the poles in the ν-plane tell us what kind of contribution comes from t-channel to s-channel. Then we may extract some non-trivial conditions for OPE coefficients from the bootstrap constraints. The extension to other representation might be interesting. In the CFT side, the generalization of the differential operators to create general representations are available [51]. These differential operators can be applied to AdS diagrams also, therefore we could discuss the AdS diagrams with general representation in a parallel way. In three dimension, there are only symmetric traceless tensor fields in the possible representation. Therefore our discussion becomes exact. It is also interesting to consider the bootstrap problem involving spinning fields in three dimensions.

Another direction is to consider loop diagrams or more complicated diagrams. In the AdS/CFT correspondence, the loop diagrams appear in the next leading order in the large N expansion. For four-point diagrams, the loop diagrams are also expanded by CPWs and through the inversion formula, we can obtain the conformal block expansion. The most simple loop diagram is computed in [28] .

In this thesis, we have considered Euclidean d-dimensional CFT and Euclidean $(d+1)$-dimensional AdS space. Almost all of the things we have discussed so far can be converted into the Lorentzian space probably. However there are some non-trivial things, for example, in the Lorentzian space, there is no geodesic connecting time-like separated boundary points. The analog of geodesic diagrams in the Lorentzian space is already discussed in [81] . It would be interesting to consider some concepts in the Lorentzian space again, CPW, inversion formula, etc.

In the end, it is desirable to develop the technique of bootstrap in d-dimensional CFT and understanding of the AdS/CFT correspondence, from kinematical points of view, we discussed in this thesis.

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Appendix A

Formulae for Hypergeometric Functions

In this appendix, some useful formulae for generalized hypergeometric functions are summarized. First of all, the generalized hypergeometric function is defined as the following series expansion.

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} x^n \,, \tag{A.0.1}
$$

where $(a)_n$ is the Pochhammer symbol defend as $(a)_n = \Gamma(a+n)/\Gamma(a)$. The simplest example is the exponential function when ${}_0F_0(-;-;x) = e^x$. By the definition, it is shown that a function ${}_{p+1}F_{q+1}$ can be created by a function of ${}_pF_q$ through the following integral:

$$
{}_{p+1}F_{q+1}\left[\begin{matrix} a_1, \ldots, a_p, c \\ b_1, \ldots, b_q, d \end{matrix} ; x\right] = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)} \int_0^1 dt\; t^{c-1}(1-t)^{d-c-1} {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; tx\right] \tag{A.0.2}
$$

This integral is called the Euler integral for generalized hypergeometric functions.

Another important expression is obtained by the Mellin transformation. The Mellin representation of hypergeometric function is given by the inverse Mellin transformation of the following gamma functions:

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; x\right] = \frac{\prod_{m=1}^{q}\Gamma(b_m)}{\prod_{m=1}^{p}\Gamma(a_m)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \, \frac{\Gamma(-s)\prod_{m=1}^{p}\Gamma(a_m+s)}{\prod_{m=1}^{q}\Gamma(b_m+s)} (-x)^s \tag{A.0.3}
$$

The integration contour is taken as in Fig.A.1 . We can show that this integral reproduce the original series expansion picking up poles which come form the gamma function $\Gamma(s)$. The integration contour is taken to above the irrelevant poles.

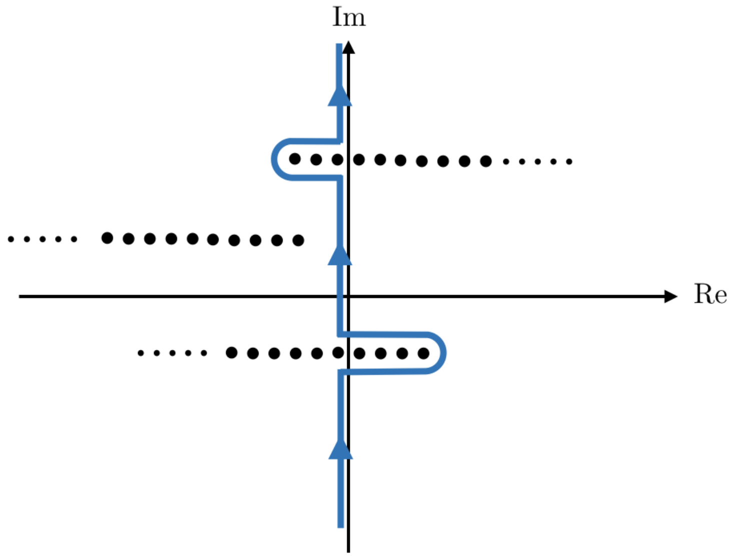

Figure A.1: The integration contour for the Mellin integral. The black points are poles coming from gamma functions. The contour is deformed so that it does not cross the series of poles.

From the Mellin transformation of hypergeometric function, we can also derive the integration formula:

$$\int_0^\infty dx\, x^{\alpha-1}{}_pF_q\left[\begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array}; -x\right] = \frac{\prod_{m=1}^q \Gamma(b_m)}{\prod_{m=1}^p \Gamma(a_m)} \frac{\Gamma(\alpha)\prod_{m=1}^p \Gamma(a_m-\alpha)}{\prod_{m=1}^q \Gamma(b_m-\alpha)} \tag{A.0.4}$$

For convenience, we will list some formulae in below. A Mellin integration with four gamma functions are computed as:

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\,. \tag{A.0.5}$$

this formula is known as the first Barnes lemma. Again the contour is taken as the same way as in Fig.A.1. We can show this formula using the following property of Gauss's hypergeometric function ${}_2F_1$:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{A.0.6}$$

As a deformation of the first lemma, the following integral including x actually gives a hypergeometric funtion:

$$\begin{aligned} &\int \frac{ds}{2\pi i}\; x^s\; \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) \\ &\qquad = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\; x^c\; {}_2F_1(a+c, b+c; a+b+c+d; 1-x) \end{aligned} \tag{A.0.7}$$

$$= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \; x^d \; {}_2F_1(a+d, b+d; a+b+c+d; 1-x)$$

In the first line, we used the the following identity:

$$\begin{aligned} {}_2F_1(a,b;c;x) = & \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} \, {}_2F_1(c-a, c-b; c-a-b+1; 1-x) \\ & + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, {}_2F_1(a,b;a+b-c+1;1-x) \, , \end{aligned} \tag{A.0.8}$$

and in the last line, we used the following identity:

$${}_2F_1(a,b;c;1-x) = x^{c-a-b} {}_2F_1(c-a, c-b; c; 1-x) \, . \tag{A.0.9}$$

As the next complicated integral, the following integral with 6 gamma functions also has a compact form:

$$\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d-s)\Gamma(-s)}{\Gamma(a+b+c+d+s)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+d)\Gamma(b+d)\Gamma(c+d)}{\Gamma(b+c+d)\Gamma(a+c+d)\Gamma(a+b+d)} \tag{A.0.10}$$

This integral is known as the second Barnes lemma. As a more general case, when the argument of the gamma function in the denominator is arbitrary, the result is written in terms of ${}_3F_2$:

$$\begin{aligned} & \int \frac{ds}{2\pi i} \quad \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c+s)\Gamma(d-s)\Gamma(-s)}{\Gamma(e+s)} \\ & \qquad = \frac{\Gamma(a+d)\Gamma(b+d)\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+d)\Gamma(e)} {}_3F_2 \left[\begin{matrix} a, & b, & e-c \\ a+b+d, & e & \end{matrix} ; 1 \right] \end{aligned} \tag{A.0.11}$$

As for ${}_3F_2$ at $x = 1$, the following relation is hold:

$${}_3F_2 \left[\begin{matrix} a, \; b, \; c \\ d, \; e, \end{matrix} ; 1 \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} {}_3F_2 \left[\begin{matrix} a, \; d-b, \; d-c \\ d, \; d+e-b-c, \end{matrix} ; 1 \right] . \tag{A.0.12}$$

A Mellin integration with eight gamma functions is computed as follows:

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \; \frac{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3+s)\Gamma(b_1-s)\Gamma(b_2-s)\Gamma(b_3-s)}{\Gamma(c_1+s)\Gamma(c_2-s)} \\ & = \frac{\Gamma(a_2-a_1)\Gamma(a_3-a_1)\Gamma(b_1+a_1)\Gamma(b_2+a_1)\Gamma(b_3+a_1)}{\Gamma(c_1-a_1)\Gamma(c_2+a_1)} {}_4F_3 \left[\begin{matrix} b_1+a_1, \; b_2+a_1, \; b_3+a_1, \; 1-c_1+a_1 \\ 1-a_2+a_1, \; 1-a_3+a_1, \; c_2+a_1 \end{matrix} ; 1 \right] \end{aligned} \tag{A.0.13}$$

$$+\frac{\Gamma(a_1-a_2)\Gamma(a_3-a_2)\Gamma(b_1+a_2)\Gamma(b_2+a_2)\Gamma(b_3+a_2)}{\Gamma(c_1-a_2)\Gamma(c_2+a_2)}\,{}_4F_3\left[\begin{matrix} b_1+a_2,\ b_2+a_2,\ b_3+a_2,\ 1-c_1+a_2 \\ 1-a_1+a_2,\ 1-a_3+a_2,\ c_2+a_2 \end{matrix};1\right]$$
$$+\frac{\Gamma(a_1-a_3)\Gamma(a_2-a_3)\Gamma(b_1+a_3)\Gamma(b_2+a_3)\Gamma(b_3+a_3)}{\Gamma(c_1-a_3)\Gamma(c_2+a_3)}\,{}_4F_3\left[\begin{matrix} b_1+a_3,\ b_2+a_3,\ b_3+a_3,\ 1-c_1+a_3 \\ 1-a_1+a_3,\ 1-a_2+a_3,\ c_2+a_3 \end{matrix};1\right]$$

Here we evaluate the contour integration around the poles at

$$s=-a_1-n\,,\quad s=-a_2-n\,,\quad s=-a_3-n\,.\qquad (\text{ where } n\in\ \mathbb{N}\ ) \tag{A.0.14}$$

Then there are gamma functions whose arguments are $\Gamma(...+n)$ and $\Gamma(...-n)$. For the gamma function including $-n$, we can apply the following formula:

$$\begin{aligned}\Gamma(x-n) &= \frac{\pi}{\sin[\pi(x-n)]}\frac{1}{\Gamma(1-x+n)} = \frac{\pi}{\sin[\pi(x-n)]\Gamma(1-x)}\frac{1}{(1-x)_n} \\ &= \frac{\Gamma(x)(-1)^n}{(1-x)_n}\,.\end{aligned} \tag{A.0.15}$$

Now we have Pochhammer symbols with the subscript n and these can be combined as a hypergeometric function.

Appendix B

The Symanzik Star Formula

In this section, we introduce the Symanzik formula. This formula play a crucial role when performing the bulk or boundary integration, and we cans that this formula naturally gives Mellin representations for $n(\geq 4)$-point integral.

B.1 Coordinate Integration

At first, we give a review of some integration formulae using the Schwinger parametrization to calculate the boundary and bulk integrals.

Boundary integration

Here we consider the following boundary integration;

$$\mathcal{I}_{\partial}(P_i) \equiv \int_{\partial} dP_0 \ \prod_{i=1}^{n} P_{i0}^{-\delta_i} \,, \tag{B.1.1}$$

where we suppose the following relation:

$$\sum_{i=1}^{n} \delta_i = d \,. \tag{B.1.2}$$

By using the Schwinger parametrization for each P_{0i}:

$$\frac{1}{P_{i0}^{\delta_i}} = \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i} \ t_i^{\delta_i} e^{-t_i P_{i0}} \,, \tag{B.1.3}$$

the integration $\mathcal{I}$ can be written as:

$$\mathcal{I}_\partial(P_i) = \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\ t_i^{\delta_i}\right) \int_\partial dP_0\ e^{2P_0\cdot Q} \tag{B.1.4}$$

where $Q \equiv \sum_{i=1}^n t_i\, P_i$. Now P_0 is parametrized as $P_0 = (1, x^2, x^\mu)$, and if we choose Q as $Q = |Q|(1,1,\mathbf{0})$ by using the Lorentz transformation, the P_0 integration can be calculated as follows;

$$\int_\partial dP_0\ e^{2P_0\cdot Q} = \int_{\mathbb{R}^d} d^d x\ e^{-|Q|(1+x^2)} = \frac{\pi^h}{|Q|^h} e^{-|Q|}\,. \tag{B.1.5}$$

Using this relation, the integration $\mathcal{I}$ becomes:

$$\mathcal{I}_\partial(P_i) = \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\ t_i^{\delta_i}\right) \frac{\pi^h}{|Q|^h} e^{-|Q|}\,. \tag{B.1.6}$$

Next we will remove the factor $|Q|^{-h}$. Firstly we insert a factor 1 into the integrand (B.1.6) :

$$1 = \int_0^\infty ds\ \delta(s - \sum_{i=1}^n t_i)\,, \tag{B.1.7}$$

and shift t_i as $t_i \to s\, t_i$.[1] Next we rescale s as $s \to |Q|s$, and then the factor $|Q|^{-h}$ is removed:[2]

$$\mathcal{I}_\partial(P_i) = \pi^h \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\ t_i^{\delta_i}\right) \int_0^\infty \frac{ds}{s}\ s^h e^{-s|Q|^2}\ \delta(1 - \sum_{i=1}^n t_i)\,. \tag{B.1.8}$$

To eliminate the s integration, we rescale t_i again as $t_i \to t_i/\sqrt{s}$, and we obtain the following form:

$$\mathcal{I}_\partial(P_i) = 2\pi^h \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\ t_i^{\delta_i}\right) e^{-\sum_{i<j} t_i t_j P_{ij}}\,. \tag{B.1.9}$$

Bulk integration

Next we consider the following bulk integration:

$$\mathcal{I}_{\text{bulk}}(P_i) \equiv \int_{\text{AdS}} dX\ \prod_{i=1}^n (-2P_i\cdot X)^{-\delta_i}\,, \tag{B.1.10}$$

As similar as the boundary case, using the Schwinger parametrization, the bulk integration becomes:

$$\mathcal{I}_{\text{bulk}}(P_i) = \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\ t_i^{\delta_i}\right) \int_{\text{AdS}} dX\ e^{2X\cdot Q}\,, \tag{B.1.11}$$

[1] Note that then Q is also shifted; $Q \to s\,Q$.
[2] Here we used the relation; $\sum_{i=1}^n \delta_i = d$.

where Q is the same as the boundary integration case: $Q \equiv \sum_{i=1}^{n} t_i\, P_i$. Q can be set as $Q = |Q|(1, 1, \mathbf{0})$ by the Lorentz transformation, and using the parametrization of X: $X = (1, x^2 + z^2, x^\mu)/z$, we can evaluate the bulk integration:

$$\begin{aligned}\int_{\mathrm{AdS}} dX\, e^{2X\cdot Q} &= \int_0^\infty \frac{dz}{z} \int_{\mathbb{R}^d} \frac{d^d x}{z^d}\, e^{-\frac{|Q|}{z}(1+x^2+z^2)} \\ &= \int_0^\infty \frac{dz}{z} \frac{\pi^h}{z^h |Q|^h} e^{-\frac{|Q|}{z}(1+z^2)} \\ &= \int_0^\infty \frac{dz}{z} \frac{\pi^h}{z^h} e^{-z-\frac{|Q|^2}{z}} \,. \end{aligned} \tag{B.1.12}$$

In the last line of (B.1.12), we scale z as $z \to |Q|^{-1} z$. Using this result and rescaling t_i as $t_i \to \sqrt{z} t_i$, we obtain the following form:

$$\mathcal{I}_{\mathrm{bulk}}(P_i) = \pi^h \Gamma\left(\frac{\sum_{i=1}^{n} \delta_i - d}{2}\right) \left(\prod_{i=1}^{n} \frac{1}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i}\, t_i^{\delta_i}\right) e^{-\sum_{i<j} t_i t_j P_{ij}} \tag{B.1.13}$$

B.2 Integration for Schwinger parameters

Here we consider the remaining integral of the Schwinger parameters in (B.1.9) and (B.1.13):

$$\tilde{\mathcal{I}}(P_i) = \left(\prod_{i=1}^{n} \int_0^\infty \frac{dt_i}{t_i}\, t_i^{\delta_i}\right) e^{-\sum_{i<j} t_i t_j P_{ij}} \,. \tag{B.2.1}$$

The exponential factors has the following Mellin transformation[3] :

$$e^{-t_i t_j P_{ij}} = \int_{-i\infty}^{i\infty} \frac{d\delta_{ij}}{2\pi i}\, \Gamma(\delta_{ij})(t_i t_j P_{ij})^{-\delta_{ij}} \,. \tag{B.2.3}$$

We use this transformation for each $[ij]$, introducing the symmetric variables:

$$\delta_{ij} (= \delta_{ji}) \qquad (\text{ for } \ (ij) \in [ij] \ ) \,, \tag{B.2.4}$$

where $[ij]$ are the possible combinations of the indexes i and j exclusive of (23) and $(1i)$ $(i \geq 2)$. For the remaining combinations, we change the variables in the following manner:

$$t_2\, t_3 = m_1 \,, \qquad t_1\, t_i = m_i \,, \quad (\text{ for } \ i = 2 \sim n \ )$$

[3] Note that this relation is easily shown due to the fact that the gamma function has poles at $-m$ $(m \in \mathbb{N})$ and the residue is

$$\mathrm{Res}_{z=-m} \Gamma(z) = \frac{(-1)^m}{m!} \,. \tag{B.2.2}$$

$$\Leftrightarrow \quad t_1 = \sqrt{\frac{m_2\, m_3}{m_1}}\,, \qquad t_i = \sqrt{\frac{m_1}{m_2\, m_3}} m_i\,. \quad (\text{ for } \ i = 2 \sim n \ )\,. \tag{B.2.5}$$

Then the measure is changed as:

$$\prod_{i=1}^{n} dt_i = \frac{1}{2m_1} \left(\frac{m_1}{m_2\, m_3} \right)^{\frac{n}{2}-1} \prod_{i=1}^{n} dm_i\,. \tag{B.2.6}$$

Then the integral $\tilde{\mathcal{I}}$ becomes:

$$\begin{aligned}\tilde{\mathcal{I}}(P_i) = & \frac{1}{2} \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{[ij]} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}} \\ & \times \left(\prod_{i=1}^{n} \int_0^\infty \frac{dm_i}{m_i} \right) \, m_1^{\delta_{23}} \left(\prod_{i=2}^{n} \, m_i^{\delta_{1i}} \right) \, e^{-m_1 P_{23} - \sum_{i=2}^{n} m_i P_{1i}}\,,\end{aligned} \tag{B.2.7}$$

where

$$[d\delta]_{\frac{n(n-3)}{2}} = \frac{\prod_{[ij]} d\delta_{ij}}{(2\pi i)^{\frac{n(n-3)}{2}}}\,, \tag{B.2.8}$$

and we define δ_{ij} where $(ij) = (23)$ or $(1i)$ in the following manner:

$$\begin{aligned}\delta_{23} &= \frac{1}{2} \left(-\delta_1 + \sum_{i=2}^{n} \delta_i - 2 \sum_{[ij]} \delta_{ij} \right)\,, \quad \delta_{12} = \frac{1}{2} \left(\delta_1 + \delta_2 - \sum_{i=3}^{n} \delta_i + 2 \sum_{[ij]} \delta_{ij} - 2 \sum_{i \geq 4} \delta_{2i} \right)\,, \\ \delta_{13} &= \frac{1}{2} \left(\delta_1 + \delta_3 - \sum_{i(\neq 1,3)}^{n} \delta_i + 2 \sum_{[ij]} \delta_{ij} - 2 \sum_{i \geq 4} \delta_{3i} \right)\,, \quad \delta_{1i} = - \sum_{j(\neq 1,i)}^{n} \delta_{ij} + \delta_i\,. \ (\text{ for } \ i \geq 4 \ )\end{aligned} \tag{B.2.9}$$

Now each m_i integral can be replaced with the Gamma function, and we obtain the Mellin integration of $\tilde{\mathcal{I}}$:

$$\tilde{\mathcal{I}}(P_i) = \frac{1}{2} \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i<j} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}}\,. \tag{B.2.10}$$

Note here from (B.2.9), it is shown that δ_{ij} satisfies the following relation:

$$\sum_{j(\neq i)} \delta_{ij} = \delta_i\,. \tag{B.2.11}$$

In this section, we demonstrated the Mellin transformation of $\tilde{\mathcal{I}}$ with a particular choice of the Mellin variables as in (B.2.4). For the other choices of $[ij]$ whose Jacobian (B.2.6) is not zero, we can drive the Mellin representation in the similar way. Combining these results, we obtain the following formulae for bulk or boundary integration[4]:

$$\int_\partial dP_0 \prod_{i=1}^n (-2P_i\cdot P_0)^{-\delta_i} = \pi^h \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)}\right) \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i<j} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}}\,, \tag{B.2.12}$$

$$\int_{\mathrm{AdS}} dX \prod_{i=1}^n (-2P_i\cdot X)^{-\delta_i} = \frac{\pi^h}{2}\Gamma\left(\frac{\sum_{i=1}^n \delta_i - d}{2}\right) \left(\prod_{i=1}^n \frac{1}{\Gamma(\delta_i)}\right) \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i<j} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}}\,.$$

B.3 Generalized Symanzik formula

Here we will generalize the Symanzik formula discussed in the previous section. A similar extension is discussed in [82].

Boundary integration

We consider the following boundary integration including vectors Y_i which satisfy $Y_i\cdot Y_i = P_i\cdot Y_i = 0$:

$$\mathcal{I}_\partial(P_i, Y_i) \equiv \int_\partial dP_0 \prod_{i=1}^n \frac{(Y_i\cdot P_0)^{\xi_i}}{(-2P_i\cdot P_0)^{\delta_i}}\,, \tag{B.3.1}$$

where ξ_i are positive integers and we assume $\sum_i(\delta_i - \xi_i) = d$. To calculate this integral, we use the Schwinger parametrization for each $(P_{i0})^{-\delta_i}$ as in (B.1.3), and for $(Y_i\cdot P_0)^{\xi_i}$ we use the following relation:

$$(Y_i\cdot X)^{\xi_i} = \xi_i! \oint \frac{d\zeta_i}{2\pi i\,\zeta_i}\, \zeta_i^{-\xi_i} e^{\zeta_i Y_i\cdot X}\,, \tag{B.3.2}$$

where the contour of integral is a small circle around the origin. Now the boundary integral becomes:

$$\mathcal{I}_\partial(P_i, Y_i) = \left(\prod_{i=1}^n \frac{\xi_i!}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i} \oint \frac{d\zeta_i}{2\pi i\,\zeta_i}\, t_i^{\delta_i}\, \zeta_i^{-\xi_i}\right) \int_\partial dP_0\; e^{2Q\cdot P_0}\,. \tag{B.3.3}$$

Here Q is given as:

$$Q \equiv \sum_{i=1}^n t_i\, P_i + \frac{1}{2}\sum_{i=1}^n \zeta_i\, Y_i\,. \tag{B.3.4}$$

[4] For the boundary integration, we assume $\sum_{i=1}^n \delta_i = d$.

The P_0 integral is evaluated similarly as in the previous section, and we obtain:

$$\mathcal{I}_\partial(P_i, Y_i) = 2\pi^h \left(\prod_{i=1}^n \frac{\xi_i!}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i} \oint \frac{d\zeta_i}{2\pi i\, \zeta_i}\; t_i^{\delta_i}\, \zeta_i^{-\xi_i} \right) e^{-|Q|^2} . \tag{B.3.5}$$

Bulk integration

Here we consider the bulk integration whose integrand is the same as (B.3.1):

$$\mathcal{I}_{\mathrm{AdS}}(P_i, Y_i) \equiv \int_{\mathrm{AdS}} dX \prod_{i=1}^n \frac{(Y_i \cdot X)^{\xi_i}}{(-2P_i \cdot X)^{\delta_i}} . \tag{B.3.6}$$

Using the Schwinger parametrization (B.1.3) and (B.3.2), X integration is evaluated as:

$$\mathcal{I}_{\mathrm{AdS}}(P_i, Y_i) = \pi^h \Gamma\left(\frac{\sum_{i=1}^n (\delta_i - \xi_i) - d}{2} \right) \left(\prod_{i=1}^n \frac{\xi_i!}{\Gamma(\delta_i)} \int_0^\infty \frac{dt_i}{t_i} \oint \frac{d\zeta_i}{2\pi i\, \zeta_i}\; t_i^{\delta_i}\, \zeta_i^{-\xi_i} \right) e^{-|Q|^2} . \tag{B.3.7}$$

Here Q is the same as (B.3.4) .

Generalized Schwinger integration

Next we consider the remaining integration over the Schwinger parameter in (B.3.5) and (B.3.7):

$$\tilde{\mathcal{I}}(P_i, Y_i) = \left(\prod_{i=1}^n \int_0^\infty \frac{dt_i}{t_i} \oint \frac{d\zeta_i}{2\pi i\, \zeta_i}\; t_i^{\delta_i}\, \zeta_i^{-\xi_i} \right) e^{-\sum_{i<j}(t_i t_j P_{ij} - \frac{1}{2}\zeta_i \zeta_j (Y_i \cdot Y_j)) + \sum_{i \neq j} t_i \zeta_j (P_i \cdot Y_j)} . \tag{B.3.8}$$

For the product $(Y_i \cdot Y_j)$ and $(P_i \cdot Y_j)$, we insert the expansion of exponential function:

$$\begin{aligned} e^{t_i \zeta_j (P_i \cdot Y_j)} &= \sum_{a_{ij}=0}^\infty \frac{1}{a_{ij}!} (t_i\, \zeta_j)^{a_{ij}} (P_i \cdot Y_j)^{a_{ij}} , \\ e^{\frac{1}{2}\zeta_i \zeta_j (Y_i \cdot Y_j)} &= \sum_{n_{ij}=0}^\infty \frac{1}{n_{ij}!} (\zeta_i\, \zeta_j)^{n_{ij}} \left(\frac{1}{2} Y_i \cdot Y_j \right)^{n_{ij}} , \end{aligned} \tag{B.3.9}$$

Here the symmetric indexes n_{ij} are symmetric: $n_{ij} = n_{ji}$, but a_{ij} is not. Then $\tilde{\mathcal{I}}$ becomes:

$$\begin{aligned} \tilde{\mathcal{I}}(P_i, Y_i) = & \sum_{a_{ij}=0}^\infty \sum_{n_{ij}=0}^\infty \left(\prod_{i=1}^n \int_0^\infty \frac{dt_i}{t_i} \oint \frac{d\zeta_i}{2\pi i\, \zeta_i}\; t_i^{\delta_i + \sum_{j(\neq i)} a_{ij}}\; \zeta_i^{-\xi_i + \sum_{j(\neq i)}(n_{ij} + a_{ji})} \right) \\ & \prod_{i \neq j} \frac{(P_i \cdot Y_j)^{a_{ij}}}{a_{ij}!} \prod_{i<j} \frac{\left(\frac{1}{2} Y_i \cdot Y_j\right)^{n_{ij}}}{n_{ij}!} e^{-\sum_{i<j} t_i t_j P_{ij}} . \end{aligned} \tag{B.3.10}$$

Now ξ_i integral is easily evaluated and this integral is zero unless $\sum_{j(\neq i)}(n_{ij} + a_{ji}) = \xi_i$. Due to this constraint, the summation over a_{ij} and n_{ij} is restricted into a finite region. For the t_i integral, we can use the Symanzik formula (B.2.10), and the integral is:

$$\tilde{\mathcal{I}}(P_i, Y_i) = \frac{1}{2} \sum_{a_{ij}, n_{ij}} \int_{-i\infty}^{i\infty} [d\delta]_{\frac{n(n-3)}{2}} \prod_{i \neq j} \frac{(P_i \cdot Y_j)^{a_{ij}}}{a_{ij}!} \prod_{i<j} \frac{\left(\frac{1}{2} Y_i \cdot Y_j\right)^{n_{ij}}}{n_{ij}!} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}} \,, \quad \text{(B.3.11)}$$

where the Mellin variables δ_{ij} , a_{ij} and n_{ij} satisfy the following relation:

$$\sum_{j(\neq i)} (\delta_{ij} - a_{ij}) = \delta_i \,, \qquad \sum_{j(\neq i)} (n_{ij} + a_{ji}) = \xi_i \,. \quad \text{(B.3.12)}$$

Note that the first condition ensures that the total power of P_i is $-\delta_i$ and the second ensure that the total power of Y_i is ξ_i .

Appendix C

Orthogonal polynomials

Here we introduce some orthogonal polynomials and formulae which appear in the main text.

C.1 Jacobi Transformation

Here we discuss the Jacobi transformation. For the more detail, please see [83]. The Jacobi transformation is an expansion associated with the Jacobi function defined as:

$$\phi_\nu^{(p,q)}(x) = {}_2F_1\left[\begin{matrix}\frac{1}{2}(p+q+1)-i\nu\ ,\ \frac{1}{2}(p+q+1)-i\nu \\ p+1\end{matrix}\ ;-x\right] \tag{C.1.1}$$

This function is related to the Jacobi polynomial:

$$P_n^{(p,q)}(x) = \frac{(p+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n\ ,\ 1+p+q+n \\ p+1\end{matrix}\ ;\frac{1}{2}(1-x)\right], \tag{C.1.2}$$

Here the factor $-n$ in the argument of ${}_2F_1$ truncates the hypergeometric series because $(-n)_m = 0$ when $m > n$, and $P_n^{(p,q)}(x)$ is a polynomial of x in degree n. The relation of these functions is given as:

$$\phi_\nu^{(p,q)}(x) = \frac{n!}{(p+1)_n} P^{(p,q)}_{\frac{1}{2}(i\nu-p-q-1)}(1-2x)\,. \tag{C.1.3}$$

According to the orthogonality and the completeness, we can defined the following functional transformation using the Jacobi function:

$$\begin{aligned}\mathcal{F}(\nu) &= \int_0^\infty dx\, w^{(p,q)}\, f(x)\, \phi_\nu^{(p,q)}(x)\,, \\ f(x) &= \int_{-\infty}^\infty \frac{d\nu}{2\pi}\frac{1}{\mathcal{N}^{(p,q)}(\nu)}\,\mathcal{F}(\nu)\,\phi_\nu^{(p,q)}(x)\,,\end{aligned} \tag{C.1.4}$$

where

$$w^{(p,q)} = x^p(1+x)^q\,, \qquad \mathcal{N}^{(p,q)}(\nu) = \frac{2\Gamma(1+p)\Gamma(\pm 2i\nu)}{\Gamma\left(\frac{1+p+q}{2} \pm 2i\nu\right)\Gamma\left(\frac{1+p-q}{2} \pm 2i\nu\right)}\,. \tag{C.1.5}$$

Note here that we have used a shot-hand notation $\Gamma(a \pm b) = \Gamma(a+b)\Gamma(a-b)$. This transformation is useful to expand a polynomial of cross ratios z and $\bar{z}$ in the k-function which is the one-dimensional conformal block because the Jacobi function $\phi_\nu^{(p,q)}(x)$ in (C.1.1) is related to the k-function through a formula of hypergeometric function[1]:

$$\phi_\nu^{(a+b,a-b)}\left(\frac{1-z}{z}\right) \equiv z^a\,\Psi_{\frac{1}{2}+i\nu}(z) = \frac{z^a}{2}\left[Q(\nu)\,k_{\frac{1}{2}+i\nu}(z) + Q(-\nu)\,k_{\frac{1}{2}-i\nu}(z)\right]\,, \tag{C.1.6}$$

where $Q(\nu)$ is defined as

$$Q(\nu) = \frac{2\Gamma(-2i\nu)\Gamma(1+a+b)}{\Gamma(\frac{1}{2}+a-i\nu)\Gamma(\frac{1}{2}+b-i\nu)}\,. \tag{C.1.7}$$

Note that the factor $\mathcal{N}^{(p,q)}(\nu)$ in (C.1.5) can be written in $Q(\nu)$

$$\mathcal{N}^{(p,q)}(\nu) = \frac{1}{2}Q(\nu)Q(-\nu)\,. \tag{C.1.8}$$

In [75, 84], they applied these technique to analyze one-dimensional CFTs.

In the below, let us consider some example of the transformation. As a simple example, firstly we will consider expanding an exponential function z^Δ in the k-function. The function in the ν-space $\mathcal{F}^{(1)}(\nu)$ is given by the following integral:

$$\mathcal{F}^{(1)}(\nu) = \int_0^1 dx\; z^{\Delta-2a-2}(1-z)^{a+b}\,\phi_\nu^{(a+b,a-b)}\left(\frac{1-z}{z}\right)\,. \tag{C.1.9}$$

Here we changed the integration variable: $x \to (1-z)/z$. By using the Mellin representation for ${}_2F_1$, the integral becomes

$$\mathcal{F}^{(1)}(\nu) = \frac{\Gamma(1+a+b)}{\Gamma\left(\frac{1}{2}+a\pm i\nu\right)}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\frac{\Gamma(-s)\Gamma\left(\frac{1}{2}+a\pm i\nu+s\right)}{\Gamma(1+a+b+s)}\int_0^1 dx\; z^{\Delta-2a-2-s}(1-z)^{a+b+s}\,. \tag{C.1.10}$$

Now the z-integral is just the beta function which is basically a combination of gamma functions, and using the Barnes first lemma (A.0.5), we can obtain a compact form:

$$\mathcal{F}^{(1)}(\nu) = \frac{\Gamma(1+a+b)\Gamma\left(\Delta-\frac{1}{2}-a\pm i\nu\right)}{\Gamma(\Delta)\Gamma(\Delta-a+b)} \tag{C.1.11}$$

[1] Here we follow the notation in [75]

Now, from the inverse transformation, we have the following expression:

$$z^{\Delta} = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{1}{\mathcal{N}^{(p,q)}(\nu)} \frac{\Gamma(1+a+b)\Gamma\left(\Delta - \frac{1}{2} \pm i\nu\right)}{\Gamma(\Delta+a)\Gamma(\Delta+b)} \, z^{-a} \phi_{\nu}^{(a+b,a-b)}\left(\frac{1-z}{z}\right) . \tag{C.1.12}$$

Here we shift Δ as $\Delta \to \Delta + a$ for later convenience. Substituting (C.1.6) and using (C.1.8), we can rewrite it as follows:

$$z^{\Delta} = \frac{\Gamma(1+a+b)}{\Gamma(\Delta+a)\Gamma(\Delta+b)} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{2}{Q(-\nu)} \Gamma\left(\Delta - \frac{1}{2} \pm i\nu\right) k_{\frac{1}{2}+i\nu}(z) . \tag{C.1.13}$$

Then the integration contour can be closed in the lower half plane and by picking up poles at $\frac{1}{2} + i\nu = \Delta + n \ (n = 0, 1, 2, ...)$, the expansion for z^{Δ} can be obtained:

$$z^{\Delta} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\Delta+a)_n(\Delta+b)_n}{(2\Delta+n-1)_n} \, k_{\Delta+n}(z) . \tag{C.1.14}$$

This formula is useful to see the conformal block expansion from a polynomial of cross ratio as in the discussions in section 2.7 and 4.2.

As a more general case, next we consider the expansion of $z^{\alpha}(1-z)^{\beta}$. Through a similar calculation as before, the function in the ν-plane $\mathcal{F}^{(2)}(\nu)$ is evaluated as follows:

$$\begin{aligned} \mathcal{F}^{(2)}(\nu) &= \int_0^1 dx \ z^{\alpha-2a-2}(1-z)^{\beta+a+b} \ \phi_{\nu}^{(a+b,a-b)}\left(\frac{1-z}{z}\right) \\ &= \frac{\Gamma(1+a+b)}{\Gamma\left(\frac{1}{2}+a \pm i\nu\right)\Gamma(\alpha+\beta-a+b)} \\ &\qquad \times \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{\Gamma\left(\frac{1}{2}+a \pm i\nu + s\right)\Gamma\left(\beta+a+b+1+s\right)\Gamma\left(\alpha-2a-1-s\right)\Gamma\left(-s\right)}{\Gamma(1+a+b+s)} \end{aligned} \tag{C.1.15}$$

Using (A.0.11) and (A.0.12) , The integration is evaluated as

$$\mathcal{F}^{(2)}(\nu) = \frac{\Gamma(1+a+b)\Gamma\left(\alpha-a-\frac{1}{2} \pm i\nu\right)}{\Gamma(\alpha)\Gamma(\alpha-a+b)} {}_3F_2\left[\begin{matrix} -\beta, \ \alpha-a-\frac{1}{2}+i\nu, \ \alpha-a-\frac{1}{2}-i\nu \\ \alpha, \ \alpha-a+b \end{matrix} ; 1\right] . \tag{C.1.16}$$

As similar as the previous case from the inverse formula, the expansion is obtained:

$$\begin{aligned} z^{\alpha}(1-z)^{\beta} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\alpha+a)_n(\alpha+b)_n}{(2\alpha+n-1)_n} {}_3F_2\left[\begin{matrix} -n \ , \ 2\alpha+n-1 \ , \ -\beta \\ \alpha+a \ , \ \alpha+b \end{matrix} ; \ 1\right] \ k_{\alpha+n}(z) \\ &= \sum_{n=0}^{\infty} \frac{i^n}{(2\alpha+n-1)_n} \ p_n(i\beta; 0, -a-b, s+a, s+b) \ k_{\alpha+n}(z) \end{aligned} \tag{C.1.17}$$

In the last line, we rewrite ${}_3F_2$ in terms of the continuous Hahn polynomial which we introduced in the next section.

C.2 Continuous Hahn Polynomial

The definition of the continuous Hahn polynomial is given as:

$$p_n(x\,;\alpha,\beta,\gamma,\delta) \equiv i^n \frac{(\alpha+\gamma)_n(\alpha+\delta)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n\ ,\ n+\alpha+\beta+\gamma+\delta-1\ ,\ \alpha+ix \\ \alpha+\gamma\ ,\ \alpha+\delta\end{matrix}\ ;\ 1\right] \tag{C.2.1}$$

Here again the factor $-n$ in the argument of ${}_3F_2$ truncates the hypergeometric series, and p_n is a polynomial of x in degree n. p_n is orthogonal for the following integral and measure:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\,\Gamma(\alpha+ix)\Gamma(\beta+ix)\Gamma(\gamma-ix)\Gamma(\delta-ix)\ p_n(x\,;\alpha,\beta,\gamma,\delta)p_m(x\,;\alpha,\beta,\gamma,\delta) \qquad \text{(C.2.2)}\\ = \frac{\Gamma(n+\alpha+\gamma)\Gamma(n+\alpha+\delta)\Gamma(n+\beta+\gamma)\Gamma(n+\beta+\delta)}{n!(2n+\alpha+\beta+\gamma+\delta-1)\Gamma(n+\alpha+\beta+\gamma+\delta-1)}\ \delta_{n,m}\,.\end{aligned}$$

This property is useful in the computation in the Mellin space. It is worthy noting that the continuous Hahn polynomial is also related to the Mack polynomial which is introduced in appendix C.4:

$$Q^{\tau+l}_{l,0}(t) = \frac{(-2i)^l\, l\,!}{(\tau+l-1)_l}p_l\left(-it\,;\frac{\tau}{2}+b,\frac{\tau}{2}+a,0,-a-b\right)\,. \tag{C.2.3}$$

C.3 Gegenbauer Polynomial

The next important polynomial is the Gegenbauer polynomial defined as

$$\begin{aligned}C^{(\alpha)}_J(x) &= \frac{\Gamma(J+2h-2)}{\Gamma(J+1)\Gamma(2h-2)}\ {}_2F_1\left[-J,\ J+2h-2;\ h-\frac{1}{2};\ \frac{1-x}{2}\right]\\ &= \sum_{r=0}^{[J/2]}(-1)^r\frac{2^{J-2r}\Gamma(\alpha+J-r)}{r!(J-2r)!\Gamma(\alpha)}x^{J-2r}\end{aligned} \tag{C.3.1}$$

In the last line, $[J/2]$ means Gauss' symbol which is the greatest integer less than or equal to $J/2$. The Gegenbauer polynomial can be regarded as a generalization of the Legendre polynomial. In fact, it is reduced to the Legendre polynomial $P_J(x)$ when $\alpha = 1/2$:

$$C^{(1/2)}_J(x) = P_J(x)\,. \tag{C.3.2}$$

This polynomial is also satisfies the orthogonality relation as follows:

$$\int_{-1}^{1} dx\ (1-x^2)^{\alpha-\frac{1}{2}} C_J^{(\alpha)}(x) C_{J'}^{(\alpha)}(x) = \frac{\pi\, 2^{1-2\alpha}\Gamma(2\alpha+J)}{J!(\alpha+J)\Gamma(\alpha)^2}\,\delta_{J,J'}\,. \tag{C.3.3}$$

The Gegenbauer polynomial appears when taking a contraction between symmetric traceless tensor structures as

$$\frac{1}{J!(h-1)_J}(X\cdot\mathcal{D}_Z)^J(Y\cdot Z)^J \ = \ \frac{J!}{2^J(h-1)_J}(X^2Y^2)^{\frac{J}{2}} C_J^{(h-1)}(x) \qquad \text{where } x = \frac{X\cdot Y}{(X^2Y^2)^{\frac{1}{2}}}\,. \tag{C.3.4}$$

Here X and Y are the $\mathbb{R}^d$ coordinates in the $(d+2)$-dimensional embedding space and $h \equiv d/2$. Finally, some useful formulae in certain limits are listed:

$$\frac{J!\,(ab)^{\frac{J}{2}}}{(h-1)_J} C_J^{(h-1)}\left(\frac{a+b}{2\sqrt{ab}}\right) \ \overset{h\to 1}{\longrightarrow}\ a^J + b^J\,, \tag{C.3.5}$$

and for $h \to 2$:

$$\frac{J!\,(ab)^{\frac{J}{2}}}{(h-1)_J} C_J^{(h-1)}\left(\frac{a+b}{2\sqrt{ab}}\right) \ \overset{h\to 2}{\longrightarrow}\ \frac{a^{J+1}-b^{J+1}}{a-b}\,. \tag{C.3.6}$$

C.4 Mack polynomial

The Mack polynomial appears as the Mellin transformation of the Gegenbauer polynomial. It is defined as:

$$\begin{aligned} \tilde{P}_{\nu,J}(s,t) \equiv &\sum_{r=0}^{[J/2]} (-1)^r \frac{J!(J+h-1)_{-r}}{2^J\, r!}\left(\frac{h\pm i\nu - J}{2} - s\right)_r \\ &\times \sum_{\sum k_{ij}=J-2r} (-1)^{k_{24}+k_{13}} \prod_{(ij)} \frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!} \prod_i \left(\gamma_{0i} - J + r + \sum_j k_{ij}\right)_{J-r-\sum_j k_{ji}}\,, \end{aligned} \tag{C.4.1}$$

The upper bound of r summation is given by the Gauss' symbol, and this summation comes from the expansion form of the Gegenbauer polynomial. As a remarkable property, at a pole value of s integration: $s = (h+i\nu-J)/2$, the Mack polynomial becomes the continuous Hahn polynomial:

$$\begin{aligned} Q_{J,0}^{h+i\nu}(t) \ &= \ \frac{2^{2J}}{(h\pm i\nu-1)_J}\tilde{P}_{\nu,J}\left(\frac{h+i\nu-J}{2}, t\right) \\ &= \ \frac{(-2i)^J\, J!}{(h+i\nu-1)_J} p_J\left(-it\,;\, \frac{h+i\nu-J}{2}+b, \frac{h+i\nu-J}{2}+a, 0, -a-b\right) \end{aligned} \tag{C.4.2}$$

This relation provides an insight that the continuous Hahn polynomial is regarded as a natural orthogonal basis in Mellin space. In general, after the Mellin integration, there are other contributions corresponding to $s = (h+i\nu-J)/2+n$, and we can define the associated polynomial $Q^{h+i\nu}_{J,n}(t)$ for general integer n. However the orthogonal relation is known only when $n = 0$. As another property, the Mack polynomial transforms in the following way under the sign reflection of a and b:

$$\tilde{P}^{\Delta_i}_{\nu,J}(s, -a-b+t) = \tilde{P}^{d-\Delta_i}_{\nu,J}(s,t) = \tilde{P}^{\Delta_i}_{\nu,J}(s,t)\Big|_{a\to -a\,, b\to -b}\,. \tag{C.4.3}$$

For more detail of the property of the Mack polynomial, please see [26, 30, 85].

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