A New Classification of Strange Attractors of Chaos from Mutual Information

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Abstract. A new criterion to classify chaotic dynamics using mutual information is proposed. The mutual information (MI) is examined for four model dynamical systems, i.e., Hénon map, standard map, driven damped pendulum and Duffing equation (Ueda's Japanese attractor). A similar algorithm for calculating MI proposed by A.M.Fraser and H.L.Swinney (Phys.Rev.A33(1986)1134) is used and the temporal variation of MI is examined for a long time interval, e.g., more than 500 times of the average period T of each Poincaré cycle of the dynamical system. Two characteristic features in the teporal variation of MI are found out. In one of the two cases MI decays to zero rapidly in a few times of T less than 10T, in the other case MI also decays to some fineite value MI_∞ within 10T in the earlier stage, and MI fluctuates around MI_∞ which does not vanish even for greater than 500T in the present calculation. The numerical value of MI_∞ persists around 13% of an initial value of MI. In the present study, Hénon map, driven damped pendulum and Duffing equation belong to the lapidly decaying-out type of MI, but standard map belongs to the persistent type with finite MI_∞. Thus the dynamical feature of chaotic behavior could be classified into two types using a mutual information.

1. Introduction

In recent two decades much progress has been made in characterizing dynamical properties of low-dimensional chaos, e.g., autocorrelation function, power spectrum, Lyapunov exponent, q-phase transition of local expansion rate of nearby orbits, and temporal variation of mutual information[1] were successfully introduced.

Shaw firstly proposed that a minimum in mutual information MI should be a good criterion for the choice of time delay in phase-portrait reconstruction from time-series data.[1] And the criterion was shown to be far superior to choosing a zero of the autocorrelation function by Fraser and Swinney.[1] For their purposes they examined the time variation of MI only for a few times of average period of Poincaré cycles of a dynamical system.

Recently the present authors carried out the numerical calculation of MI of time series of chaotic dynamical systems for a quite long time interval of 500T, T is an average period of Poincaré cycles of the system. And there would be two kind of characteristic features of temporal behavior in MI. One is the rapid decay type and the other is persistent with finite value MI_{∞} of MI. In section 2, a formalism of MI given by Fraser and Swinney[1] is briefly reviewd. And four exampls of actual numerical calculation are illustrated in section 3. Finally result and some remarks are given.

2. Mutual information

Fraser and Swinney[1] gave some formalism of MI as follows. Let S denote the whole system which consists of a set of pssible messages $s_1, s_2, \cdots, s_n$, and the associated probabilities $P_s(s_1), P_s(s_2), \cdots, P_s(s_n)$. P_s maps massages to probabilities. The subscript is necessary because more than one such function will be considered at a time. The average amount of information gained from a measurement that specifies s is the entropy H of a system,

$$H(S) = -\sum_i P_s(s_i) \log P_s(s_i). \tag{1}$$

Here if the log is taken to be the base two, H is in units of bits. In the following, for simplicity, we will consider with only the discrete case. We are interested in measuring how dependent the value of $x(t+T)$ are on the values of $x(t)$. By making the assignment $[s, q] = [x(t), x(t+T)]$, we can consider a general coupled system (S, Q). Given that s has been mesured and found to be $S = \{s_1, s_2, \cdots, s_n\} = \{x(t_1), x(t_2), \cdots, x(t_n)\}$, and q with a delay time T to be $Q = \{q_1, q_2, \cdots, q_n\} = \{x(t_1+T), x(t_2+T), \cdots, x(t_n+T)\}$. Then the mutual information between S and Q is given by

$$MI(Q,S) = \sum_{i,j} P_{sq}(s_i, q_j) \log \left[P_{sq}(s_i, q_j)/P_s(s_i)P_q(q_j) \right], \tag{2}$$

where $P_{sq}(s_i, q_j)$ is the joint probability that a measurement of q will yield q_j, given that the measured value of s is s_i. Thus the mutual information measurs the general dependece of two variables.

3. Four examples of chaotic attractor

In this section the mutual information of the following four chaotic attractors is examined.

(1) Hénon map[2] is given by

$$x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n. \tag{3}$$

Figure 1 shows an example of iteration of the map with $a = 1.4, b = 0.3$ and an initial condition $(x_0, y_0) = (0, 0)$.

(2) Standard map[3] is given by

$$x_{n+1} = x_n + y_n, \quad y_{n+1} = y_n - K \sin x_n, \quad (mod\ 2\pi). \tag{4}$$

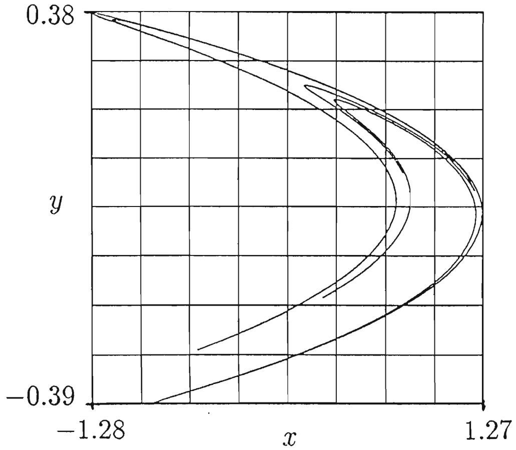

Fig.1. Hénon map Eq.(3) with a = 1.4, b = 0.3 and an initial condition $(x_0, y_0) = (0, 0)$. An equi-spaced glid with 8 × 8 squares is drawn and superposed on the map.

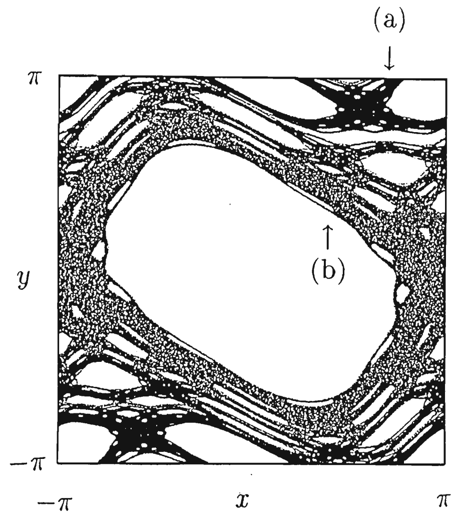

Fig.2. Standard map Eq.(4) with K = 1, and two initial conditions (x_0, y_0) = (a) $(0, 0.8\pi)$, and (b) $(0, 0.7\pi)$.

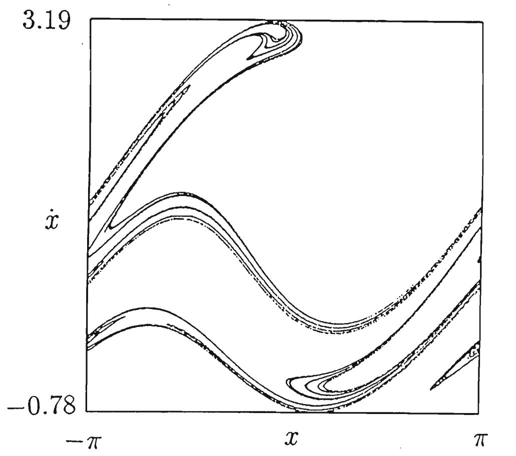

Fig.3. Poincaré section of driven damped pendulum Eq.(5) with γ = 0.22, ω = 1, A = 2.7, and an initial condition $(x, \dot{x})$ = $(-1.5, 1.3)$.

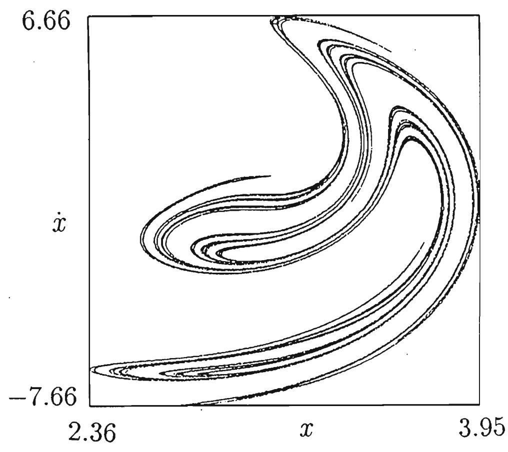

Fig.4. Poincaré section of Duffing equation (Ueda's Japanese attractor) Eq.(6) with K = 0.1, ω = 1, B = 12 and an initial condition $(x, \dot{x}) = (3.5, 0)$.

Figure 2 shows an example of iteration of the map with $K = 1$ and two initial conditions $(x_0, y_0) = (0, 0.8\pi)$ and $(0, 0.7\pi)$.
(3) The equation of motion of the driven damped pendulum[4] is given by

$$\ddot{x} + \gamma\dot{x} + \sin x = A\cos\omega t, \tag{5}$$

whose Poincaré section is shown in Fig.3 with $\gamma = 0.22, \omega = 1, A = 2.7$ and an initial condition $(x, \dot{x}) = (-1.5, 1.3)$.
(4) The Duffing equation is given by

$$\ddot{x} + k\dot{x} + x^3 = B\cos\omega t, \tag{6}$$

whose Poincaré section (which is called Ueda's Japanese attractor)[5] is shown in Fig.4 with $K = 0.1, \omega = 1, B = 12$ and an initial condition $(x, \dot{x}) = (3.5, 0)$.

At first, as illustrated in Fig.1, a phase space in which a strange attractor of Hénon map is included, is divided into an equi-spaced glid with 8×8 squares. This simple algorithm of division is slightly different from Fraser and Swinney[1], but after a few calculation steps from the initial condition the temporal behavior of MI should essentially be the same as them. Then the same algorithm as the above was also employed to the other three strange attractors shown in Fig.'s 2-4. Secondly a file of $262,144 (= 2^{18})$ initial points was setup and their orbits were traced. The temporal variation of the probability in each square was estimated and then its MI was examined at every iteration in the discret maps, i.e., Hénon map and standard map. Also the same number of trajectories (2^{18}) as sbove were traced and its MI was examined on every Poincaré section along each Poincaré cycle of the differential equations, i.e., driven damped pendulum and Duffing equation.

The temporal behavior of MI of Hénon map (Fig.1,Eq.(3)), driven damped pendulum (Fig.3,Eq.(5)) and Duffing equation (Fig.4,Eq.(6)) is shown in Fig.5, where each MI decays out to zero rapidly in approximately 15 iteration times or Poincaré cycles. As shown in Fig.6, in standard map, however, MI decays rapidly but remains around at some finite value MI_∞, which does not vanish and remains rather constant even for more than 500 times of iteration (or Poincaré cycles).

4. Concluding remarks

We have preliminarily examined the mutual information (MI) of the four examples of strange attractor and found out a new feature to characterize the chaotic dynamics. Due to the temporal dependence of the mutual information, it is proposed that there should be two types of chaotic dynamics to be classified. In one case MI fastly decays out to zero and the other MI also decays but not vanish and persists around a finite value. What is the reason why the persistency of MI happens only in the standard map? Are there any other types of the temporal dependence of MI except for the two types found out in the present investigation? These are the future problems to be considered. The present program has been just started, and we only have some preliminary results. We are now going to examine many other examples of chaotic attractor, which will be reported in the near future.

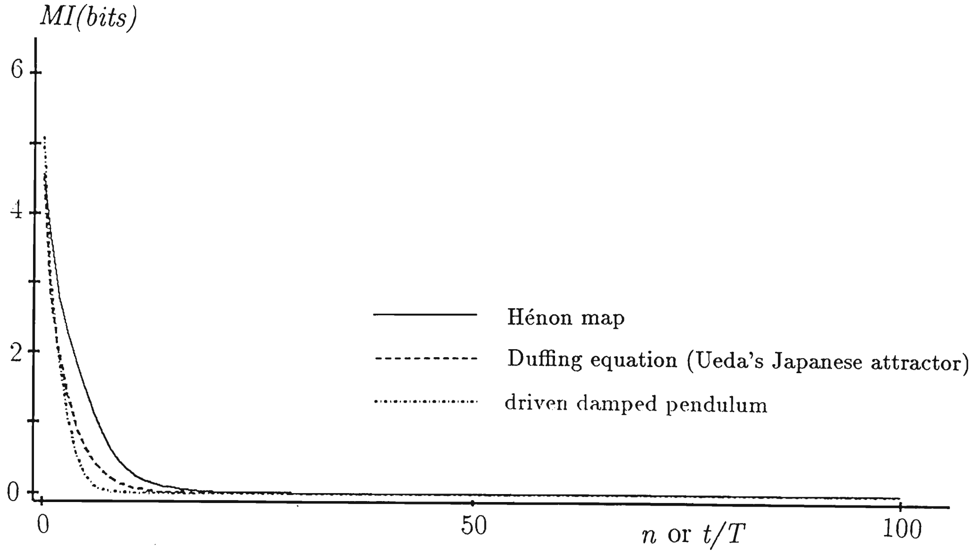

Fig.5. Rapid-decay type of mutual information MI. Hénon map (Fig.1,Eq.(3)), driven damped pendulum (Fig.3,Eq.(5)) and Duffing equation (Ueda's Japanese attractor) (Fig.4, Eq.(6)) belong to the type. The abscissa is the number of iteration n of Hénon map or the number of Poincaré cycles t/T of the differential equation.

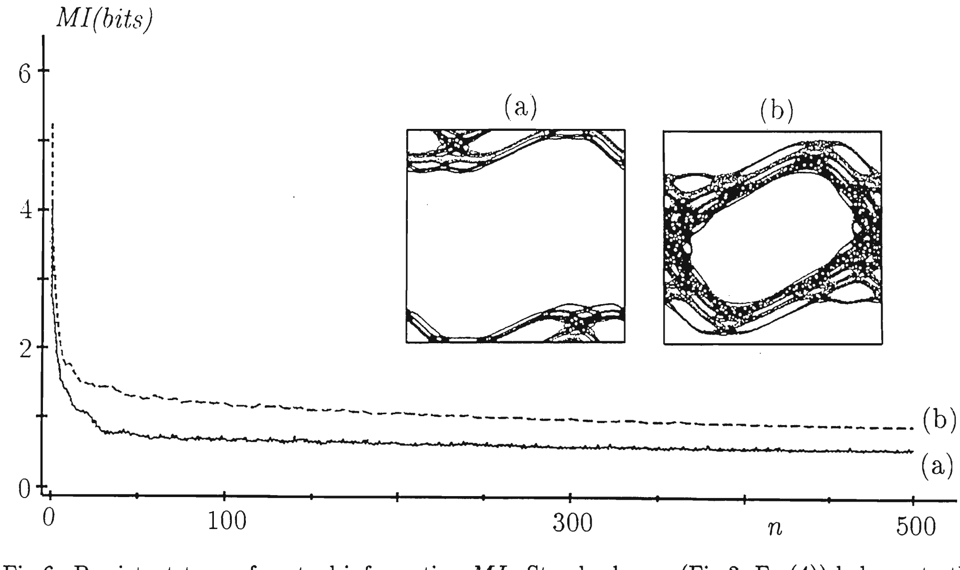

Fig.6. Persistent type of mutual information MI. Standard map (Fig.2, Eq.(4)) belongs to the type. Two initial conditions are taken, i.e., (x_0, y_0) = (a) $(0, 0.8\pi)$ and (b) $(0, 0.7\pi)$. The abscissa is the number of iteration n of the map.

References

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