Multiplayer Games as Extension of Misère Games(Summary)

Koki Suetsugu*

Combinatorial game theory is a study of mathematical constructions of perfect information games in which there are no chance, nor hidden informations. In this thesis, we discuss multiplayer games. Most of the notions and results presented in this thesis are discussed in [15] by the author.

Among the early results of combinatorial game theory is a winning strategy for *NIM* by Bouton[4] in 1902. NIM is a two-player game with some heaps of stones, and the current player chooses one of the heaps and takes out some stones. The winner of NIM is the player who takes out the last stone. We denote the number of stones in the heaps like $(3, 2)$ to express NIM positions.

Mathematically, a game is represented as a well-founded directed acyclic graph. Nodes are called *positions* and a direct successor of a position is called an *option* at the position, and a position without an option is called an *end position*. We call such a graph a *game tree*. A play of a game starts by selecting a current position and a current player. Each player selects one option of the current position and update the current position to it alternatively. A game ends when it arrives at an end position. We say a game is in *normal play* if we define that the winner of the game is the player who moves last and the loser is the player who comes to be the current player of an end position (like NIM). This kind of game in which both players have the same set of options (like NIM) is called an *impartial game* and a game which is not impartial (like go, shogi or chess) is called a *partisan game*. In this thesis, we study only impartial games.

For the readers who are interested in this subject, we introduce some textbooks on which impartial and partisan games are discussed, [1], [2] and [3].

We say a player has a *winning strategy* if she can make herself the winner regardless of her opponent's move. In an impartial game, we say that a game position is an *N-position* or a *P-position* if the next player (i.e. the current player) or the previous player (i.e. the other player) has a winning strategy, respectively. The following is one of the most important facts for a two-player impartial game.

Theorem 1. *A game position of an impartial two-player game is an N-position or a P-position.*

*Graduate School of Human and Environmental Studies, Kyoto University, Japan

We can analyze whether a NIM position is an N-position or a P-position in a simple way, by calculating modulo-2 sum without carry which is denoted by $\oplus$ (NIM sum).

Theorem 2 (Bouton[4]). *A NIM position $(n_1, n_2, \ldots, n_k)$ is a P-position if and only if $n_1^{<2>} \oplus n_2^{<2>} \oplus \cdots \oplus n_k^{<2>} = 0$. Here, $n^{<2>}$ is the binary notation of n.*

In contrast to normal play, a game is called in *misère play* if the last player to move is the loser. We can also analyze misère NIM game in the following way.

Theorem 3 (Bouton[4]). *In misère NIM game, a position $(n_1, n_2, \ldots, n_k)$ is a P-position if and only if*

$$\begin{cases} n_1^{<2>} \oplus n_2^{<2>} \oplus \ldots \oplus n_k^{<2>} = 0 & (\exists j.\ n_j > 1) \\ n_1^{<2>} \oplus n_2^{<2>} \oplus \ldots \oplus n_k^{<2>} = 1 & (\forall j.\ n_j \leq 1). \end{cases}$$

We study multiplayer games in this thesis. Multiplayer combinatorial games are difficult to analyze, because of the possibility of coalitions in them. For example, consider the NIM position (1,2) in three-player NIM. If the current player moves to (1,0) or (0,2), then the second player wins. However if the current player moves to (1,1) then the second player moves to (1,0) and the third player wins. So, the current player has no winning strategy but she can choose whether the second player becomes the winner or the third player does.

With this observation, people usually study multiplayer games by adding some assumptions to determine the game result. Li[13] defined a rank system. Krawec[5, 6] introduced alliance matrix and Liu et al.[7, 8, 9, 10, 11, 12] studied some multiplayer games with Krawec's definitions.

Following the earlier results, we introduce the notion of a preference-based play to multiplayer games in this thesis. We assume that each player has her own preference order, which is a total ordering of all the players. We study the situation where each player knows the content of each other's preference, and it is common knowledge that every player knows the preferences of the other players, that is, each player knows that other players know the preferences of other players, and so on. We call such a play a preference-based play. In this thesis, we study the case that each player behaves optimally so that her most preferred player will move last, and if she cannot, then she behaves so that her second preferred player will move last, ..., and so on. We call the preference order which is equal to the play order but starting with the i-th player an i-misère play. This notion contains Li's rank-based play as the special case $i = 0$. One of the main contributions of this thesis is the characterization of losing positions of multiplayer i-misère games. From this characterization, we obtained a characterization of losing positions of multiplayer NIM in i-misère play. This result contains Theorem 2, 3 and the Li's result as special cases.

In this thesis, we assume that there are m players $P_0, P_1, \ldots, P_{m-1}$ and they play in this order. For simplicity, any arithmetic in the subscript (e.g. P_{i+k}) is done modulo m.

As we noted above, we introduce the notion of a preference-based play to multiplayer games. We write the preference order of a player as $P_{i(0)} > P_{i(1)} > \cdots > P_{i(m-1)}$ if she wants player $P_{i(0)}$ to be the last player to move, and if that is impossible, she wants player $P_{i(1)}$ to play last, ..., and it is the worst result that player $P_{i(m-1)}$ becomes the last player to move.

Definition 1 (Preference matrix)**.** *Following Krawec[5], we introduce an $m \times m$ matrix notation to express the preference orders of all the players.*

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m-1} \\ \vdots & \vdots & & \vdots \\ A_{m-1,0} & A_{m-1,1} & \cdots & A_{m-1,m-1} \end{bmatrix}$$

Here, $A_{j,k}$ is the index of the k-th preferred player of P_j relative to j with the most preferred player called the 0-th preferred player. That is, the preference order of P_j is $P_{j+A_{j,0}} > P_{j+A_{j,1}} > \cdots > P_{j+A_{j,m-1}}$.

Our notion of preference-based play is similar to the notion of alliance by Krawec[5]. The difference is that the objective of alliance-based play is to let the specified player have no moves, where the objective of our play is to let the specified player play last. The two notions are convertible and we can express our results with the notion of alliance-based play. However, then the constructions in this thesis becomes complicated and we cannot state our result(Theorem 6) as a generalization of Li's result. The following notion of a last moving player is a reformulation of Krawec's notion of game value and Theorem 4 is his result expressed with the position of the last moving player.

Definition 2. *For a game position G, we write $opt(G)$ as the set of all options at G. That is, for every $G' \in opt(G)$, one can move from G to G'.*

Definition 3. *We define the function l to be*

$$l(G,t) = \begin{cases} m-1 & if\ G\ is\ an\ end\ position \\ A_{t,q} & otherwise \end{cases}$$

for $q = min\{j \in \mathbb{N} \mid l(G', t+1) + 1 = A_{t,j}\ with\ G' \in opt(G)\}$.

Theorem 4 (Krawec)**.** *If every player plays optimally, then $P_{t+l(G,t)}$ moves last in the game that starts with position G and player P_t. If the starting position G is an end position, then we consider P_{t-1} to have moved last.*

By using this theorem, it is guaranteed that for multiplayer games in preference-based play, the result of each game is uniquely determined as using Theorem 1 for two-player games.

Definition 4. *We say a multiplayer game is* preference-impartial *if the game is impartial and its preference matrix satisfies, $A_{i,k} = A_{j,k}$ for every $i, j, k < m$. In this case, we abbreviate the preference matrix as:*

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \end{bmatrix}$$

Definition 5. *In a preference-impartial game, $l(X,t) = l(X,t')$ for any t and t'. Therefore we simply describe it as a unary function $l(X)$.*

$$l(G) = \begin{cases} m-1 & if\ G\ is\ an\ end\ position \\ A_q & otherwise \end{cases}$$

for $q = min\{j \in \mathbb{N} \mid l(G') + 1 = A_j \ with\ G' \in opt(G)\}$.

For $G = (n_1, n_2, \ldots, n_k)$, we will abbreviate $l(G)$ as $l(n_1, n_2, \ldots, n_k)$ when they will be no confusion.

Definition 6. *If a game is a preference-impartial game and its preference matrix is*

$$\begin{bmatrix} i & (i+1) & \cdots & (m-1) & 0 & 1 & \cdots & (i-1) \end{bmatrix},$$

*then we say that it is in i-*misère play*.*

In i-misère play, each player wants the i-th player to be the last moving player. The reason why we call it misère is that we obtain two-player misère play when $m = 2$ and $i = 1$. In a preference impartial game, according to Theorem 4, if every player plays optimally, then $P_{l(G)}$ plays last in the game G starting with player P_0. Therefore, the person whose preference order starts (resp. ends) with $P_{l(G)}$ obtains the most pleasant (resp. unpleasant) result. We call them the winner and the loser of the game, and say that a position is a *winning* (resp. *losing*) *position* if the starting player is the winner (resp. loser). Note that an N-position is a winning position and a P-position is a losing position for two-player games. In an i-misère game, a position is a winning position if $l(G) = i$ and is a losing position if $l(G) = i - 1$.

Of course, we can define other positions by using the values of $l(G)$. However, it seems to be difficult to characterize the positions.

In 1978, Li[13] considered multiplayer NIM with rank system. He defined the winner of the game is the person who moves last like two-player normal play. In addition, players are assigned a rank, ranging from bottom to top in the order of $P_{k+1}, P_{k+2}, \ldots, P_{m-1}, P_0, P_1, \ldots, P_{k-1}, P_k$ when P_k is the winner and each player adopts an optimal strategy toward her own highest possible rank. That is, by definitions above, Li's theory is in 0-misère play or under following preference matrix:

$$\begin{bmatrix} 0 & 1 & \cdots & m-1 \end{bmatrix}$$

and with our terminology, the highest ranked player is the winner and the lowest ranked player is the loser.

In order to describe Li's result, we define a notion of modulo-m NIM sum.

Definition 7. *For $k \geq 2$, let SEQ_k be the set of sequences of $\{0, 1, \ldots, k-1\}$ that do not start with 0. For simplicity, we write $0 \in SEQ_k$ for the empty sequence. Note that $SEQ_k \subseteq SEQ_m$ if $k \leq m$. For a non-negative integer x, we write $x^{<k>} \in SEQ_k$ for the k-ary notation of x. That is, $x^{<k>} = (x_t, x_{t-1}, \ldots, x_1, x_0)$ if $x = \Sigma_{0 \leq s \leq t} x_s k^s$.*

Definition 8 (Generalized NIM sum). *On SEQ_m, we define the component-wise modulo-m addition operation $\oplus_m$ as follows. For $x, y \in SEQ_m$, if x and y have different length, then we prepend 0s to the shorter sequence to adjust their length and then do modulo-m addition without carry on each component and then remove the leading 0s from the result so that $x \oplus_m y$ do not start with 0. We simply write $\oplus$ for $\oplus_2$.*

By using these definitions and notations, Li's result is described in the following theorem.

Theorem 5 (Li[13]). *In 0-misère NIM, $(n_1, n_2, \ldots, n_k)$ is a losing position if and only if $n_1^{<2>} \oplus_m n_2^{<2>} \oplus_m \cdots \oplus_m n_k^{<2>} = 0$.*

We show following theorem which contains Bouton's theory and Li's theory.

Theorem 6. *In i-misère NIM, $(n_1, n_2, \ldots, n_k)$ is a losing position if and only if*

$$\begin{cases} n_1^{<2>} \oplus_m n_2^{<2>} \oplus_m \cdots \oplus_m n_k^{<2>} = 0 & (\exists j.\ n_j > 1) \\ n_1^{<2>} \oplus_m n_2^{<2>} \oplus_m \cdots \oplus_m n_k^{<2>} = i & (\forall j.\ n_j \leq 1). \end{cases}$$

This theorem also means that we can know whether a given game position G is a winning position by checking whether G has an option G' which satisfies this condition.

For the case $(m, i) = (2, 0)$ we can obtain Theorem 2, for the case $(m, i) = (2, 1)$, we can obtain Theorem 3 and for the case $(m, i) = (m, 0)$, we can obtain Theorem 5 from this theorem. That is, this theorem reveal a hidden connection between misère NIM and multiplayer NIM. We also found a characterization of the set of losing positions.

Definition 9. *For $r \geq 1$, we define $M^r(G)$ as the set of game positions which are reached in no more than r moves from G. That is, $M^1(G) = opt(G)$ and $M^r(G) = opt(M^{r-1}(G) \cup G)$ for $r > 1$. In addition, we define $M^r(G) = \phi$ for $r < 1$.*

Theorem 7. *Let S be a set of game positions of i-misère play. S is the set of losing positions if and only if*

(i) $\forall G \in S\ \forall G' \in E.\ G' \notin M^{i-1}(G)$

(ii) $\forall G, G' \in S.\ G' \notin M^{m-1}(G)$

(iii) $\forall G \notin S.\ (\exists G' \in S, G' \in M^{m-1}(G)) \vee (\exists G'' \in E, G'' \in M^{i-1}(G))$.

Here, E is the set of end positions.

This theorem extends the characterization of the sets of P-positions in two-player games. Actually, for the case of $m = 2$, this theorem characterize the set of P-positions in two-player normal and misère play.

Moore's game, or NIM_t, is a game in which players can choose up to t heaps and take any numbers of stones from them[14]. Therefore, NIM_1 is the original NIM.

Theorem 8 (Moore[14]). *A game position $(n_1, n_2, \ldots, n_k)$ of NIM_t is a P-position if and only if $n_1^{<2>} \oplus_{t+1} n_2^{<2>} \oplus_{t+1} \cdots \oplus_{t+1} n_k^{<2>} = 0$.*

Li[13] showed the following theorem for multiplayer NIM$_t$.

Theorem 9 (Li[13]). *In 0-misère play of m-player NIM_t, $l(n_1, n_2, \ldots, n_k) = 0$ if and only if $n_1^{<2>} \oplus_v n_2^{<2>} \oplus_v \cdots \oplus_v n_k^{<2>} = 0$ where $v = t(m-1)+1$.*

We can also extend this theorem to i-misère play.

Theorem 10. *In i-misère play of m-player NIM_t, $(n_1, n_2, \ldots, n_k)$ is a losing position if and only if*

$$\begin{cases} n_1^{<2>} \oplus_v n_2^{<2>} \oplus_v \cdots \oplus_v n_k^{<2>} = 0 & (\exists j.\ n_j > 1) \\ n_1^{<2>} \oplus_v n_2^{<2>} \oplus_v \cdots \oplus_v n_k^{<2>} = u & (\forall j.\ n_j \leq 1) \end{cases}$$

where $v = t(m-1)+1$ and

$$\begin{cases} u = 0 & (i = 0) \\ u = t(i-1)+1 & (1 \leq i \leq m-1). \end{cases}$$

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We have considered i-misère play where each player's preference order is the same as the play order. Next, we study the situation that for all player P_i, her preference order is reverse to the play order.

Definition 10. *For $0 \leq i \leq m-1$, we say that preference-based play of an m-player game is an i-*reverse play *if it is a preference-impartial game and the preference matrix is the following:*

$$\begin{bmatrix} i & (i-1) & \cdots & 1 & 0 & (m-1) & \cdots & (i+1) \end{bmatrix}$$

In i-misère play, if $l(G) = (i-1) \bmod m$, then G is a losing position. On the other hand, in i-reverse play, if $l(G) = (i-1) \bmod m$ then the secondly preferred player of the current player is going to be the last moving player of G.

In two-play normal and misère NIM. for all $j (1 \leq j \leq k)$ and for all non-negative integers $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k$, there is exactly one non-negative integer n_j such that $(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k)$ is a P-position. In i-misère play with $m > 2$, there is no such uniqueness. However, we show that there exists such a uniqueness in i-reverse play. This result suggests that i-reverse play is also a natural extension of two-player normal and misère play.

Theorem 11. *In i-reverse play, for all $j (1 \leq j \leq k)$ and for all non-negative integers $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k$, there is exactly one non-negative integer n_j such that $l(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k) = (i-1) \bmod m$.*

Finally, we study the case that each player has a different preference order for the case of three-player NIM. We have already studied the cases named 0-misère, 1-misère, and 2-misère play;

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

and the cases named 0-reverse, 1-reverse, and 2-reverse play;

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

We showed some properties of $l(G, t)$ though its characterization is an open problem for some of the cases. Now, we study the following preference orders which are not symmetric.

Semi-normal form: Each player prefers herself first. Two players secondly prefer the same player and the other player secondly prefers her next player. There are three possibilities of preference orders which are essentially the same.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Semi-reverse form: Each player prefers herself first. Two players secondly prefer the same player and the other player secondly prefers her previous player. There are three possibilities of preference orders which are essentially the same.

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Without loss of generality, we consider semi-normal form

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and semi-reverse form

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

We also obtained overall result for semi-normal NIM and semi-reverse NIM.

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