FORCED NEGATIVE-RESISTANCE OSCILLATOR

Chihiro Hayashi and Yoshisuke Ueda
Department of Electrical Engineering, Kyoto University
Kyoto, Japan

1. Introduction

When a periodic force is applied to a self-oscillatory system, the frequency of the self-excited oscillation, i.e., the natural frequency of the system, falls in synchronism with the driving frequency, provided these two frequencies are not far different. This phenomenon of frequency entrainment may also occur when the ratio of the two frequencies is in the neighborhood of an integer (different from unity) or a fraction. Thus, if the amplitude and frequency of the external force are appropriately chosen, the natural frequency of the system is entrained by a frequency which is an integral multiple or submultiple of the driving frequency.[1] If the ratio of the two frequencies is not in the neighborhood of an integer or a fraction, one may expect the occurrence of an almost periodic oscillation.[2]

In this paper, we consider a self-oscillatory system with nonlinear restoring force governed by

$$\frac{d^2v}{dt^2} - \mu(1-\gamma v^2)\frac{dv}{dt} + v^3 = B\cos\nu t \quad (1)$$

where μ is a small positive constant and γ is positive also. Special attention is directed toward the transition between entrained oscillations and almost periodic oscillations in the case where the amplitude B and the frequency ν of the external force are varied beyond the boundary of entrainment.

2. Fundamental Equations for Almost Periodic Oscillations

When the driving frequency ν is in the neighborhood of the natural frequency of the system, we assume the solution of Eq. (1) of the form

$$v(t) = b_1(t)\sin\nu t + b_2(t)\cos\nu t \quad (2)$$

If the amplitude B and the frequency ν of the external force are given in the region of entrainment, an entrained periodic oscillation occurs; in this case the coefficients $b_1(t)$ and $b_2(t)$ in Eq. (2) are constants. On the other hand, if the external force is prescribed outside the region of entrainment, an almost periodic oscillation results. In this case the coefficients $b_1(t)$ and $b_2(t)$ in Eq. (2) would no longer be constants, but vary slowly with time t. Bearing this in mind we shall derive the relations which will determine the coefficients $b_1(t)$ and $b_2(t)$ in Eq. (2). This relations may readily be found by the method of harmonic balance; namely, upon substitution of Eq. (2) into (1), equating the coefficients of the terms containing $\cos\nu t$ and $\sin\nu t$ separately to zero gives

$$\begin{aligned} \frac{dx_1}{d\tau} &= (1-r_1^2)\,x_1 - \sigma_1 y_1 + \frac{B}{\mu\nu a_0} \\ \frac{dy_1}{d\tau} &= \sigma_1 x_1 + (1-r_1^2)\,y_1 \end{aligned} \quad (3)$$

where $x_1 = b_1/a_0 \quad y_1 = b_2/a_0 \quad r_1^2 = x_1^2 + y_1^2$

$$a_0 = \sqrt{\frac{4}{\gamma}} \qquad \omega_0 = \sqrt{\frac{3}{\gamma}} \qquad \tau = \frac{\mu}{2}t \quad (4)$$

$$\sigma_1 = \frac{\omega_0^2 r_1^2 - \nu^2}{\mu\nu}$$

Constants a_0 and ω_0 are respectively the amplitude and frequency of the self-excited oscillation. Equations (3) play an important role in our following investigation, since they are the fundamental equations for the study of the almost periodic oscillation as well as the entrained periodic oscillation.

Oscillations in the transient state are represented by integral curves on the x_1y_1 phase plane. Hence, a periodic oscillation corresponds to a singular point, and an almost periodic oscillation to a limit cycle.

Numerical Example

Let us consider a case in which $\mu = 0.2$, $\gamma = 8$, and $\nu = 0.9$. Figure 1 shows the amplitude characteristic of the harmonic oscillation (B vs. r_1^2). This amplitude characteristic is obtained by equating both $dx_1/d\tau$ and $dy_1/d\tau$ to zero in Eqs. (3). The dotted portions of the characteristic curve represent unstable states. Figure 2 shows the phase portraits of the system (3) for various values of B. When B is given below B_1 (see Fig. 1), a limit cycle exists which encircles an unstable spiral point. This state is shown in Fig. 2a. A coalescence of singularities occurs at $B = B_1$. Further increase in B results in the coexistence of a stable limit cycle and a stable node. When $B = B_2$, there exists a closed integral curve starting from the saddle point and comming back

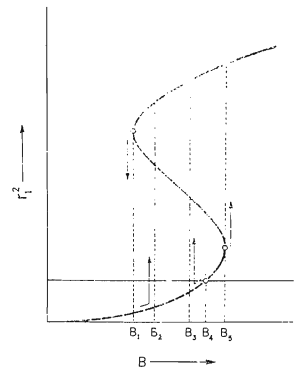

Fig. 1. Amplitude characteristic of the harmonic oscillation, showing the transition between entrained and almost periodic oscillations.

to the same point. The limit cycle disappears when B is increased beyond B_2. However, when B reaches B_3 the integral curves shows the same behavior as in Fig. 2d; see Fig. 2f. A limit cycle appears once again for $B_3<B<B_4$, and is reduced to a stable spiral at $B = B_4$. For values of B between B_4 and B_5, there exist two types of harmonic oscillations, i.e., the resonant and nonresonant oscillations. No limit cycle exists in this case. The coalescence of singularities occurs at $B = B_5$. When B is decreased, the resonant oscillation is sustained down to the value of $B = B_1$. Below B_1, only an almost periodic oscillation is obtained. Hence, it is concluded that an almost periodic oscillation occurs for $B<B_2$ and $B_3<B<B_4$, a resonant oscillation for $B_1<B$, and nonresonant oscillation for $B_4<B<B_5$. It depends on the initial condition which types of oscillation occur. The region of initial conditions leading to the different types of oscillation are bordered by the separatrices, i.e., the integral curves which tend to the saddle point with increasing τ. It should also be mentioned that, for other values of the system parameters or for different values of ν, the situation may somewhat be different from that mentioned above.

The result obtained in the present analysis is confirmed by analog-computer analysis.

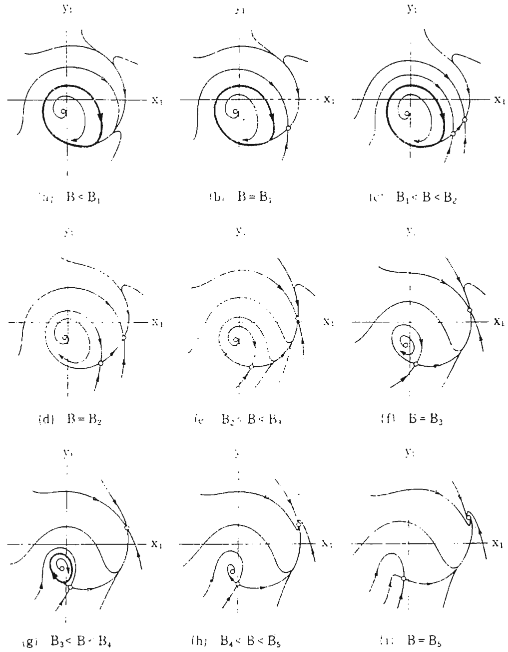

Fig. 2. Phase portraits of Eq. (3) for various values of B.

(1) C. Hayashi, H. Shibayama and Y. Nishikawa, Frequency Entrainment in a Self-Oscillatory System with External Force, Trans. IRE on Circuit Theory, CT-7: 413-422 (1960).

(2) C. Hayashi, H. Shibayama and Y. Ueda, Quasi-Periodic Oscillations in a Self-Oscillatory System with External Force, Symposium on Nonlinear Oscillations (Int. Union of Theo. and Appl. Mech.), Kiev, 1961.