SELF-EXCITED OSCILLATIONS AND THEIR BIFURCATIONS IN SYSTEMS DESCRIBED BY NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract
Experimental studies of the steady states described by some nonlinear differential-difference equations have been carried out. Various types of self-excited oscillations and their bifurcations in computer simulated systems are surveyed.

1. INTRODUCTION
This report deals with oscillatory steady states which occur in systems described by the following two differential-difference equations:
\[ \frac{d\theta(t)}{dt} + \sin \theta(t - L) = \delta, \]
\[ \frac{d^2\theta(t)}{dt^2} + \delta \frac{d\theta(t)}{dt} + (2\xi - \alpha) \cos \theta(t - L) \frac{d\theta(t - L)}{dt} + \sin \theta(t - L) = \delta. \]
These equations are mathematical models of phase-locked-loops (PLL) with time delay. Synchronized states of the PLL are represented by the equilibrium points of the equations. Firstly, the parameter regions are important from the engineering point of view, in which all initial conditions lead to quiescent steady states. The regions, i.e., pull-in ranges of the PLL, were already reported in [1]. Secondly, oscillatory steady states are of scientific interest. By using a hybrid computer, this report surveys self-excited oscillations and their bifurcations obtained by varying the system parameters.

2. EXPERIMENTAL RESULTS
In this report, steady states will mean asymptotically stable equilibrium points, periodic solutions, and so forth of (1) and (2). The trajectories of the steady states of (1) and (2) can be represented on the cylindrical phase plane \((\theta, \dot{\theta})\) as shown in Figure 1. The types of steady states in the computer simulated systems can be roughly classified as follows:
(a) Equilibrium point: There are two equilibrium points for (1) and (2).
\[ \theta = \sin^{-1} \delta, \]
\[ \theta = \pi - \sin^{-1} \delta. \]

...
The one given by (4) is always unstable, while the stability of the point given by (3) depends on the values of the system parameters.

(b) Limit cycle of the first kind: The closed orbit represented by a periodic solution
\[ \phi(t) = \phi(t), \quad \phi(t + T) = \phi(t), \]  
(5)
is called a limit cycle of the first kind provided every solution of (1) or (2) tends to this orbit as \( t \to \infty \).

(c) Limit cycle of the second kind: When \( \phi(t) \) along a limit cycle is represented by
\[ \phi(t) = \phi(t) + 2nT/T, \quad \phi(t + T) = \phi(t), \]  
(6)
the corresponding orbit is called a limit cycle of the second kind.

Examples of equilibrium points, limit cycles of the first and second kinds are illustrated in Figure 1. (c') N-kai* limit cycle of the second kind: Special cases of limit cycles of the second kind are closed orbits with phase space periods which are multiples of \( 2\pi \). In these cases the corresponding steady states take the form
\[ \phi(t) = \phi_N(t) + 2nT/T_N, \quad \phi_N(t + T_N) = \phi_N(t), \]  
(7)
for some least integer \( N \).

(d) Chaotic steady states: There sometimes appear chaotic steady states in spite of the perfectly de-

* We introduce the Japanese word "kai" here which means "times" or "rounds". It has the features of being short and gives exactly the correct meaning desired.
Figure 4-11. Steady state at the point in Figure 2 ($\delta = 0.8$ and $L = 2.0$).

Figure 3. Regions of different steady states for the system represented by (2).
(a) $\xi = 1/\sqrt{2}$ and $\beta = 0.5$.
(b) $\xi = 1/\sqrt{2}$ and $\beta = 0.1$.

Figure 4-IV. Steady states at the point IV in Figure 3b ($\delta = 0.53$ and $L = 1.5$).

Figure 4-V. Steady states at the point V in Figure 3b ($\delta = 0.1$ and $L = 1.5$).
2.3 REPRESENTATIVE EXAMPLES OF SELF-EXCITED OSCILLATIONS

In order to illustrate the regions of the \((L, \delta)\)-plane, we show the steady states at six representative sets of parameters \((L, \delta)\) indicated by I, II, ...VI in Figures 2 and 3. Figure 4-I shows a limit cycle of the first kind. The mathematical proof of the existence of the periodic solution for this case was proved by Furumochi [2]. Figure 4-II shows a limit cycle of the second kind. An example of a 2-kai limit cycle of the second kind is shown in Figure 4-III. In the cases of Figures 4-IV and V, there are two possible steady states and which one occurs depends on the initial conditions. Figure 4-VI shows a chaotic steady state with a waveform \(\dot{\delta}(t)\).

2.4 BIFURCATIONS OF THE STEADY STATES

Figure 5a shows an example of a bifurcation in which the parameter \(\delta\) is decreased with \(L = 2.0\) in (1). The locations of those parameter values are indicated by a, b, ...f in Figure 2. We can see that the limit cycle of the second kind disappears at some value of \(\delta\) between 0.58 and 0.57 and a limit cycle of the first kind then appears. This situation has some resemblance to the case of ODE systems in which limit cycles approach a saddle to saddle separatrix and then cease to exist. Increasing \(\delta\) from zero, the limit cycle of the first kind converges to the stable equilibrium point on the boundary \(P\) and the point continues to exist until \(\delta\) reaches 1.0.

Figure 5b shows another example of a bifurcation in which \(\delta\) is increased with \(L = 3.0\). The locations of the parameters are also indicated in Figure 2. This example demonstrates the great complexity of the phenomenon.

Figure 6 is another example in which \(\delta\) is increased with \(L = 2.5\) in (2). The locations of the parameters are indicated by m, n, ... s in Figure 3a. These figures show the trajectory doubling bifurcations explicitly and it seems that chaotic steady states occurs through a cascade of the bifurcations. These situations also closely resemble the case of ODE systems.

3. CONCLUSION

By using a hybrid computer, experimental studies of the steady states of nonlinear differential-difference equations (1) and (2) have been carried out. From these results, various types of self-excited oscillations and their bifurcations under variations of the system parameters have been clarified. The results reported here deserve further mathematical attention.

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REFERENCES


Figure 5a. Bifurcation for decreasing $\delta$ with $L = 2.0$ in (1).

Figure 5b. Bifurcation for increasing $\delta$ with $L = 3.0$ in (1).

Figure 6. Bifurcation for increasing $\delta$ with $L = 2.5$, $\zeta = 1/\sqrt{2}$, and $\beta = 0.5$ in (2).