Submitted to Mechanics Research Communications

An order statistics approach to multiobjective structural optimization considering robustness and confidence of responses

Makoto OHSAKI [1*], Makoto YAMAKAWA [2], Wenliang FAN [3], Zhengliang LI [3]

[1*] Department of Architecture and Architectural Engineering, Kyoto University
Kyoto-Daigaku Katsura, Nishikyo, Kyoto 615-8540, Japan
[2] Department of Architecture, Tokyo University of Science, Tokyo 125-8585, Japan
[3] School of Civil Engineering, Chongqing University, Chongqing 400045, China
*Corresponding author: ohsaki@archi.kyoto-u.ac.jp
Tel.:+81-75-383-2923; fax: +81-75-383-2923
Received

Abstract

A design method is proposed for reducing the worst value of representative response of structure under random variation of structural parameters. To reduce the computational cost for finding the worst value, the requirement of 'worst' is relaxed to the value corresponding to the specified quantile of the distribution function. The order statistics is used for defining the level of robustness of the approximate worst response value for specified confidence. Obviously, a larger estimate of response is required for ensuring larger robustness with the same confidence. Therefore, a trade-off relation exists between the order of response generated by random parameter values and the robustness in estimation of the worst response. A multiobjective optimization problem is formulated to minimize the representative responses with various order; thus, the solutions with various levels of robustness are simultaneously obtained. A numerical example of a 20-story shear frame is presented for minimizing the maximum interstory drift against seismic motions under constraint on the total amount of damping coefficient, where uncertainty is considered in the story stiffness and the floor mass as well as the damping coefficient that is also the design variable. It is shown that the distribution of additional damping coefficients depends on the level of robustness in estimation of the worst response value.

Keywords: order statistics, robustness, multiobjective optimization, seismic response, shear frame

1. Introduction

In the process of designing structures, it is desirable to decide the shape and stiffness considering uncertainty in design loads, material parameters, etc. [1, 2]. Therefore, in the field of structural optimization, extensive research has been carried out for design methods such as robust optimization [3–6], which maximizes a robustness function, and worst-case design [7, 8]. However, there exist many definitions of robustness, and if the worst response is considered, computational cost for obtaining the exact worst (extreme) value is very large even when simple interval variables are used for representing uncertainty [7, 9, 10].

Reliability-based design is a standard approach to incorporate various types of uncertainty in the process of structural design [11−14]. Although approximation methods such as dimension reduction method [15] are available, computational cost for finding the failure point is still large even for small-scale structures, if limit state functions corresponding to multiple failure modes are considered. Thus, in the approach based on reliability index with the failure probability, the stochastic variables are often implicitly assumed to be normally distributed or evaluated through various approximations [16, 17]. Recently, stochastic structural mechanics including stochastic perturbation method has made significant progress, and stochastic finite element method has become a powerful tool [18]. However, design optimization with such sophisticated approach has not been fully explored.

Order statistics is developed for evaluating the statistical properties of the *k*th best/worst value in a set of specified number of random variables [19–21]. The unique feature of tolerance intervals and confidence intervals of quantiles of order statistics is that it is independent of the type of distribution of random parameter, and ensures that the upper bound of 100γth $(0 \le \gamma \le 1)$ quantile value can be obtained with smaller computational cost than the upper bound of the worst value $(\gamma = 1)$ with the specified confidence. Such approach is known as distribution-free tolerance intervals. Therefore, it can be effectively utilized for obtaining approximate upper bounds of the worst responses of structures subjected to various types of external loads including seismic loads.

The upper bound of 100γth quantile value of response for specified confidence can be quantitatively calculated using the theory of order statistics, and the upper bound obviously increases if the parameter γ is increased. Therefore, if the structure is designed such that the approximate worst representative response is less than the specified value with large value of γ, then the structure has a large probability of its representative response less than the specified value. Hence, γ may be regarded as a parameter of robustness as shown in Fig.1.

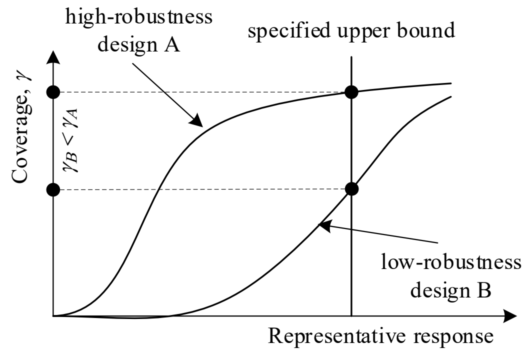


Fig. 1. Relation between parameter γ and robustness.

In the robust design process, it is desired to minimize the responses corresponding to various levels of robustness so that the appropriate robustness level can be found in view of the trade-off relation between the structural cost (volume) and robustness. Multiobjective programming problem is an optimization problem for obtaining solutions called Pareto optimal solutions for several objective functions that have trade-off relations [22]. There exist numerous researches on multiobjective programming; however, most of them have only two objective functions, and it is very difficult to evaluate the properties of Pareto optimal solutions for a problem with many objective functions [23].

In this paper, a new multiobjective programming problem is formulated for minimizing the upper-bound responses with various levels of robustness based on the distribution-free tolerance intervals of order statistics. A numerical example of a 20-story shear frame is presented for minimizing the maximum interstory drift against seismic motions under constraint on the total amount of damping coefficient, where uncertainty is considered in the story stiffness, story mass, and the damping coefficient of the frame. Pareto optimal solutions for minimizing 20 largest responses among 150 randomly generated responses are found using a multiobjective genetic algorithm.

2. Approximate worst response based on order statistics

Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_t) \in \Omega$ denote a vector of uncertain parameters representing, e.g., yield stress, Young's modulus, cross-sectional area, and external load, where t is the number of uncertain parameters, and Ω is the prescribed uncertainty set. The vector of uncertain parameters are probabilistic values distributed in the t-dimensional region Ω, and are assumed to be continuous and bounded.

The representative response of a structure such as maximum absolute value of stress under specified loading conditions is denoted by $g(\boldsymbol{\theta})$. We consider a problem of finding the worst (largest) value of $g(\boldsymbol{\theta})$ for specified region Ω of uncertain parameters. However, finding the strictly worst value demands much computational cost even for the simple case where Ω represents a set of interval regions.

To alleviate the large computational cost for finding the worst value, we can relax the requirement of 'worst' response to the 100γth $(0 \le \gamma \le 1)$ quantile response; e.g., γ may be equal to 0.9, 0.95, 0.99, etc. For this purpose, the order statistics can be effectively used.

Let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots, \boldsymbol{\theta}_n$ denote a set of n t-dimensional vectors of uncertain parameters generated using the same probability distribution on Ω. Accordingly, the representative responses denoted by $Y_1 = g(\boldsymbol{\theta}_1)$, $Y_2 = g(\boldsymbol{\theta}_2)$, …, $Y_n = g(\boldsymbol{\theta}_n)$ are random variables that have the same probability distribution, for which the cumulative distribution function is denoted by $F(g(\boldsymbol{\theta})) = \Pr\{Y \le g(\boldsymbol{\theta})\}$. These responses are renumbered in decreasing order as $Y_{1,n} \ge Y_{2,n} \ge \ldots \ge Y_{n,n}$. The kth response $Y_{k,n}$ among n responses is a probabilistic value called kth *order statistics*. Note that they are renumbered in increasing order in usual order statics. However, we align them in decreasing order to evaluate an approximate maximum response.

According to the theorem of order statistics, we can select k and n $(1 \le k \le n)$ so that there exist real values α and γ $(0 < \alpha < 1; 0 < \gamma < 1)$ to state that "The probability of $g(\boldsymbol{\theta})$ so that its $100\gamma\%$ is less than $Y_{k,n}$ is at least $100\alpha\%$," i.e.,

$$\Pr\{F(Y_{k,n}) \ge \gamma\} \ge \alpha \tag{1}$$

From Eq. (1), $(-\infty, \mathrm{Y}_{k,n})$ is known as one-sided tolerance interval that is independent of F. Hence, the interval is called distribution-free interval.

The values of k and n are selected based on the criterion described below. If k is decreased close to 1 for constant n, then the response close to the worst (extremal) value can be obtained.

Let $I_\gamma(a,b)$ denote the incomplete beta function defined as

$$I_\gamma(a,b) = \frac{\int_0^\gamma t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt} \tag{2}$$

Then, for specified values of k, α, and γ, the number of samples n can be determined as the smallest value satisfying the following inequality:

$$1 - I_\gamma(n-k+1, k) \ge \alpha \tag{3}$$

Accordingly, if n is to be specified as the minimum required value satisfying the inequality (3) and k is also given, the relation between the remaining two parameters α and γ are obtained from the following equation:

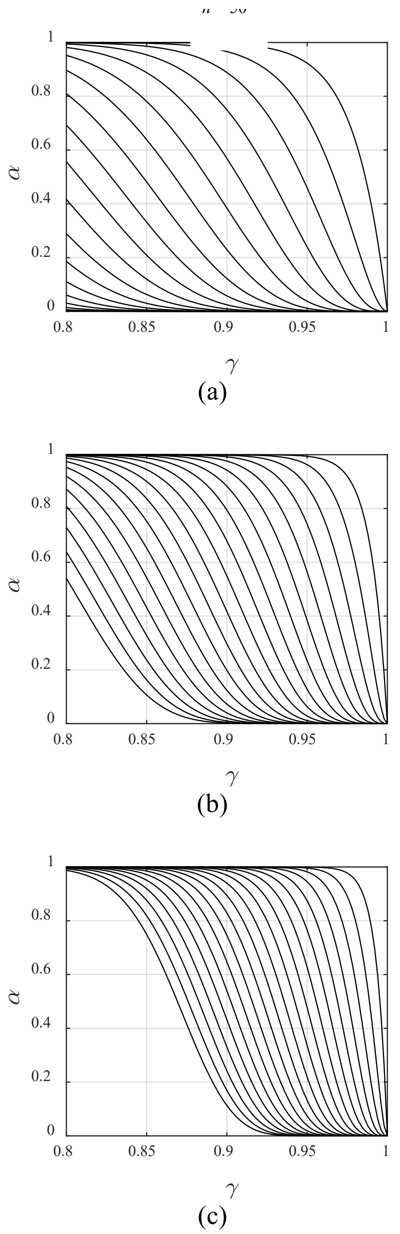


Fig. 2. Relation between α and γ for fixed value of n and various values of k; (a) $n = 50$, (b) $n = 100$, (c) $n = 150$.

$$1 - I_{\gamma}(n-k+1, k) = \alpha \tag{4}$$

Relation between α and γ is plotted in Fig. 2 for fixed value of n and various specified values of k $(=1,\ldots 20)$. The curves in top-right and bottom-left in each figure correspond to $k=1$ and 20, respectively. The following properties are observed from the figure:

1. For specified values of n and k, α is a decreasing function of γ, and has a larger value for a smaller value of k.
2. For specified values of n and α, k is a discretely decreasing function of γ.
3. For specified values of n and γ, k is a discretely decreasing function of α.
4. The curves move to top-right as n is increased.

We can also have a table for listing the values of k and γ for specified n and α. Values of γ corresponding to the various values of k for $n = 100$ and 200, respectively, are obtained by solving Eq. (4), and listed in Tables 1(a) and (b) for $\alpha = 0.9$. It is confirmed from these tables that γ is a decreasing function of k if n and α are fixed. Therefore, the order k can also be regarded as a parameter of robustness; i.e., a smaller value of k leads to a larger robustness.

3. Multiobjective optimization problem for minimizing worst response

Structural optimization problem may be simply formulated so as to minimize the structural cost represented by, e.g., the total structural volume under upper-bound constraint on the representative response and bound constraints on the design variables. If deviation of responses is not considered, the median value $Y_{(n+1)/2,n}$ (if n is an odd number) may be minimized. Alternatively, in the framework of worst-case design, an upper bound is given for the worst value $Y_{1,n}$ of the response. Our purpose here is to incorporate the responses corresponding to various levels of robustness into the structural design problem.

Let $\mathbf{x}$ denote the vector of design variables such as stiffnesses of members, nodal coordinates, and damping coefficients of dampers. The representative response is denoted by $g(\mathbf{x}, \boldsymbol{\theta})$, which is a function of $\mathbf{x}$ and $\boldsymbol{\theta}$, and the k th largest value among n samples $g(\mathbf{x}, \boldsymbol{\theta}_i)$ $(i = 1, \ldots, n)$ of design $\mathbf{x}$ is denoted by $Y_{k,n}(\mathbf{x})$. The following problem may be solved if the order k is specified as $k = k^*$ according to an appropriate robustness level:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} && f(\mathbf{x}) \\ &\text{subject to} && Y_{k^*,n}(\mathbf{x}) \le \overline{\mathrm{g}} \\ & && \mathbf{x} \in \chi \end{aligned} \tag{5}$$

where $f(\mathbf{x})$ is a function such as the total structural volume representing the structural cost, χ is the feasible region of $\mathbf{x}$, and $\overline{\mathrm{g}}$ is the upper bound for $g(\mathbf{x}, \boldsymbol{\theta})$.

As discussed in Sec. 2, the estimated value of the maximum response becomes more robust if k is smaller. However, larger robustness leads to larger estimate of the response, and they are in trade-off relations. Therefore, it is practically important to compare the solutions with various robustness levels to decide the appropriate value of robustness. For this purpose, we formulate a multiobjective optimization problem to minimize $\tilde{k}$ $(\le n)$ order statistics $Y_{1,n}(\mathbf{x}), \ldots, Y_{\tilde{k},n}(\mathbf{x})$ as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} && Y_{1,n}(\mathbf{x}), \cdots, Y_{\tilde{k},n}(\mathbf{x}) \\ &\text{subject to} && f(\mathbf{x}) \le \overline{f} \\ & && \mathbf{x} \in \chi \end{aligned} \tag{6}$$

where $\overline{f}$ is the upper bound of the structural cost. Since we are interested in approximate worst values, an appropriate value of about $\tilde{k}$ may be, e.g., $n/10$ corresponding to the 90% quantile of the response.

4. Numerical example

Problem (6) is formulated for a 20-story shear frame with viscous dampers as shown in Fig. 3. Note that only the first and top stories are illustrated in Fig. 3, and the intermediate stories have similar properties.

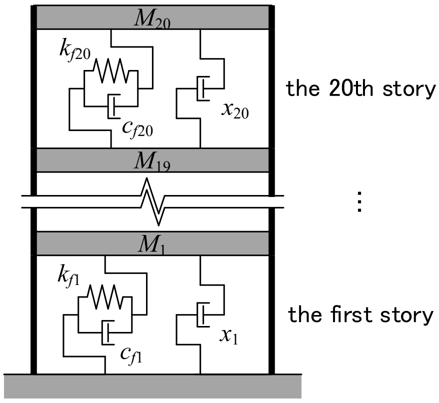


Fig. 3. A 20-story shear frame model.

Structural damping is not considered, and the damping coefficients $(c_{f1},\cdots,c_{f20})$ due to viscous dampers in the stories are taken as the design variables that are also denoted by the vector $\mathbf{x}=(x_1,\cdots,x_{20})$ in problem (6). Uncertainty is considered for story mass, stiffness, and damping coefficient that is also a design variable. The nominal value of mass at each story is 4.0×10^5 kg, and the vector of story mass is denoted as $(m_1,\cdots,m_{20})$. The nominal values of story stiffness $(K_1,\cdots,K_{20})$ are proportional to the seismic shear coefficient defined by Japanese building code, which are listed in Table 2. The fundamental natural period of the frame is 1.6 sec. for the frame with nominal values of mass and stiffness.

Seismic responses are evaluated for the design displacement response spectrum $S_\mathrm{d}(T,h)$, which is defined by the natural period T and the damping factor h of the frame as well as and the design acceleration response spectrum $S_\mathrm{a}(T,h)$ defined as

$$S_\mathrm{d}(T,h)=\left(\frac{2\pi}{T}\right)^2 S_\mathrm{a}(T,h) \tag{5}$$

The following definition of $S_\mathrm{a}(T,h)$ (m/s^2) in Japanese building code with damping correction coefficient $H(h)$ is used:

$$S_\mathrm{a}(T,h)=H(h)S_\mathrm{a0},\quad H(h)=\frac{1.5}{1+10h}, \tag{6a}$$

$$S_\mathrm{a0}=\begin{cases}(0.96+9T) & T<0.16\\ 2.4 & 0.16\le T<0.864\\ 2.074/T & 0.864\le T\end{cases} \tag{6b}$$

which is plotted in Fig. 4 for $h=0.02$, 0.05, and 0.10.

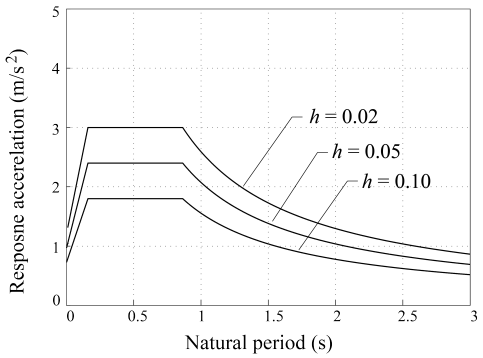


Fig. 4. Deign acceleration response spectra for damping factors 0.02, 0.05, and 0.10.

The extended CQC (complete quadratic combination) method [24, 25] is used for response evaluation of the structure with non-proportional damping matrix due to existence of viscous dampers in the stories. The maximum value among all stories of the maximum interstory drift is chosen as the representative response $g(\mathbf{x})$. All 20 eigenmodes are used for evaluation by the CQC method.

Since uncertainty is considered for the vectors of mass, stiffness, and damping coefficient, which have 20 components, respectively, the vector $\boldsymbol{\theta}$ of uncertain parameters have 60 components in total. Uniform distribution in the range of $\pm10\%$ to the mean value in 20 stories is considered in each parameter. Hence, $\boldsymbol{\theta}$ is defined as

$$\boldsymbol{\theta}=(\theta_1,\cdots,\theta_{60})\in\Omega \tag{7}$$

$$\Omega=\{(\theta_1,\cdots,\theta_{60})\,|\,-0.1\le\theta_i\le0.1;\ i=1,\cdots,60\} \tag{8}$$

The parameter is multiplied to each of the mean values of story mass, stiffness, and damping coefficient, and added to the nominal value as

$$m_i^*=m_i+\theta_i\hat{m},\quad K_i^*=K_i+\theta_{20+i}\hat{K},$$
$$c_i^*=c_i+\theta_{40+i}\hat{c}\qquad (i=1,\cdots,20) \tag{9}$$

where

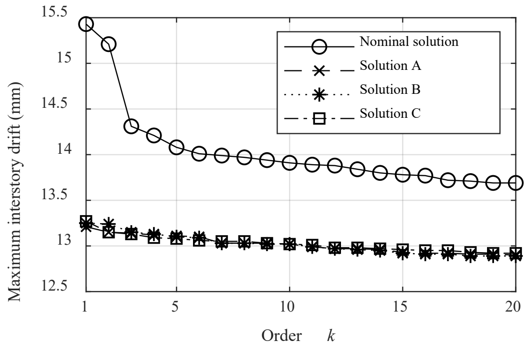
$$\hat{m}=\frac{1}{20}\sum_{i=1}^{20}m_i,\quad \hat{K}=\frac{1}{20}\sum_{i=1}^{20}K_i,\quad \hat{c}=\frac{1}{20}\sum_{i=1}^{20}c_i \tag{10}$$

Eigenvalue analysis is carried out for each solution with each set of 60 random parameter values to obtain the natural periods and eigenmodes. Then the representative response is computed using the CQC method.

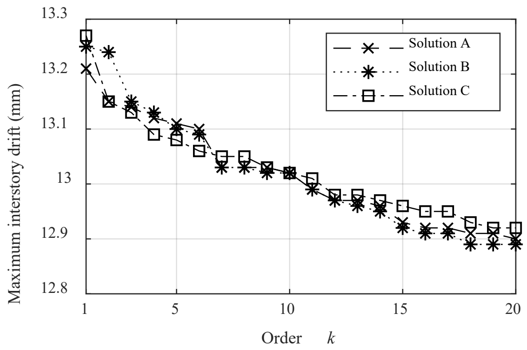
The multiobjective optimization problem is formulated as

$$\begin{aligned}&\underset{\mathbf{x}}{\text{minimize}}\quad Y_{1,n}(\mathbf{x}),\cdots,Y_{\tilde{k},n}(\mathbf{x})\\ &\text{subject to}\quad x_i\ge0,\ \sum_{i=1}^{20}x_i\le\bar{c}\end{aligned} \tag{11}$$

where $\bar{c}$ is the upper bound for the total value of damping coefficients.



(a)



(b)

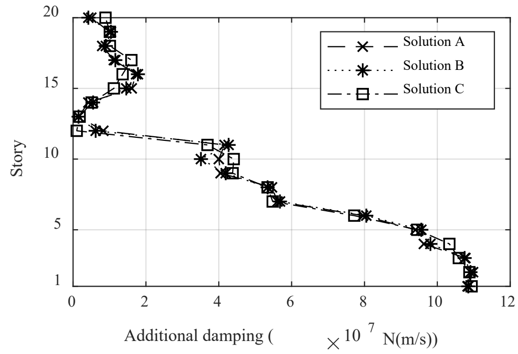
Fig. 5. Relation between maximum interstory drift and order k for nominal solution and three Pareto optimal solutions; (a) plot including nominal solution, (b) detailed plot of Pareto optimal solutions.

Pareto optimal solutions are found using NSGA-II (non-dominated sorting genetic algorithm II) [26] available in Global Optimization Toolbox of Matlab 2018a. The population size is 200, the number of generations is 117, and the elitist strategy is used; i.e., we have 200 solutions that may converge to Pareto optimal solutions.

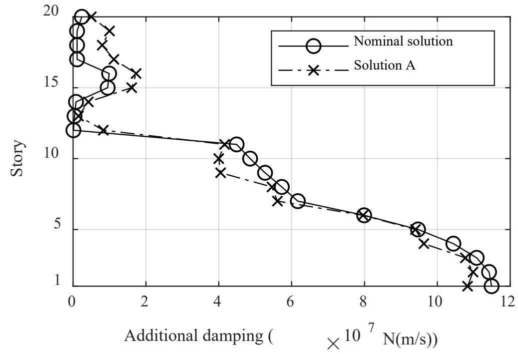
Optimization is carried out for $n=150$, $\tilde{k}=20$, and $\alpha=0.9$; therefore, as seen from Table 1(b), the robustness level γ decreases from 0.985 to 0.831 as k is increased from 1 to 20. Time-history analysis is carried out 150 times for each solution by generating 150 different sets of uncertain parameter values with uniform distribution.

As a result of optimization, the 200 solutions converged to a set of 70 different Pareto optimal solutions. Among them, the values of $Y_{1,n}(\mathbf{x}),\cdots,Y_{\tilde{k},n}(\mathbf{x})$ of three Pareto optimal solutions A, B, and C are plotted in Fig. 5(a) and (b). Note that there are nine solutions, among the Pareto optimal solutions, that minimize $Y_{k,n}(\mathbf{x})$ for different values of k in 1, …, 20.

The nominal solution is defined so that all of the uncertain parameters have the nominal values in the process of optimization. Random parameter values are assigned for the nominal solution to compute $Y_{1,n}(\mathbf{x}),\cdots,Y_{\tilde{k},n}(\mathbf{x})$ as plotted in Fig. 5(a). As seen from the figure, the nominal solution is far from optimal, and obviously the responses of all orders have large values.



(a)



(b)

Fig. 6. Distributions of additional damping coefficients; (a) Pareto optimal solutions A, B, and C, (b) solution A and nominal solution.

Fig. 5(b) shows the detailed view of the values of $Y_{1,n}(\mathbf{x}),\cdots,Y_{\tilde{k},n}(\mathbf{x})$ of the three Pareto optimal solutions. Solution A has the smallest value for $k=1$ among all 70 Pareto optimal solutions; solution B has the smallest value for $k=16,18,19$; and solution C for $k=2,\ldots,6$. Solution C has the minimum variance of $Y_{1,n}(\mathbf{x}),\cdots,Y_{\tilde{k},n}(\mathbf{x})$ among 70 solutions. Solutions A and B have small values if k is small; however, decrease of $Y_{k,n}(\mathbf{x})$ for larger k is very small for these solutions. By contrast, for solution C, $Y_{k,n}(\mathbf{x})$ has a large value if k is small; however, it rapidly decreases as k is increased.

The values of additional damping coefficients, which are the design variables, are plotted in Fig. 6(a) for the three Pareto optimal solutions. As seen in the figure, the three Pareto optimal solutions have a similar distribution with small difference in the middle stories. Optimal additional damping coefficients of solution A and the nominal solution are compared in Fig. 6(b). As seen from the figure, the nominal solution has smaller damping ratios in the upper stories and larger damping coefficients in the lower stories than the Pareto optimal solution.

5. Conclusions

A new formulation of multiobjective optimization problem for robust design has been presented based on the distribution-free tolerance interval of order statistics. If the parameter α

for the confidence of solution and the number of solutions are fixed, the order k of the solution corresponds to the parameter γ related to the robustness of the solution. Therefore, the kth maximum value of the representative response may be regarded as an approximate worst response with the specified robustness level. Accordingly, the Pareto optimal solutions with various levels of robustness can be obtained by solving the multiobjective optimization problem of minimizing the representative responses with various order.

Effectiveness of the proposed method has been demonstrated through the example of a 20-story shear frame subjected to seismic motions. The design variables are the additional damping coefficients due to the viscous dampers in the stories, and uncertainty is considered in the mass, stiffness, and the damping coefficient that is also the design variable. It is observed in the optimization results of minimizing 20 largest response values obtained using the NSGA II that the optimal solution corresponding to the minimum response of each order in the Pareto optimal set depends on the order of response, which suggests that the optimal distribution of additional damping coefficients depends on the level of robustness to be specified by the designer.

Acknowledgements

This work was supported by NSFC-JSPS China-Japan Scientific Cooperation Project.

References

[1] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, Robust Optimization, Princeton University Press, 2009.
[2] I. Elishakoff and M. Ohsaki，Optimization and anti-optimization of structures under uncertainty，World Scientific, London, 2010.
[3] C. Zang, M. I. Friswell and J. E. Mottershead, A review of robust optimal design and its application in dynamics, Comput. Struct. 83 (2004) 315–326.
[4] N. D. Lagaros and M. Papadrakakis, Robust seismic design optimization of steel structures, Struct. Multidisc. Optim. 33 (2007) 457–469.
[5] D. Hashimoto and Y. Kanno, A semidefnite programming approach to robust truss topology optimization under uncertainty in locations of nodes. Struct Multidiscip Optim 51 (2015) 439–461.
[6] H.-G. Beyer and B. Sendhoff, Robust optimization: A comprehensive survey, Comp. Meth. Appl. Mech. Eng., Vol. 196, pp. 3190–3218, 2007.
[7] M. Ohsaki and M. Katsura, A random sampling approach to worst-case design of structures, Struct. Multidisc. Optim, 46 (2010) 27–39.
[8] A. Takezawa, S. Nii, M. Kitamura and N. Kogiso, Topology optimization for worst load conditions based on the eigenvalue analysis of an aggregated linear system, Comp. Meth. Appl. Mech. Engrg., Vol. 200, pp. 2268–2281, 2011.
[9] A. D. Dimarogonas, Interval analysis of vibrating systems, J. Sound and Vibrations, Vol. 183(4), pp. 739–749, 1995.
[10] Z. Qiu, Convex models and interval analysis method to predict the effect of uncertain-but-bounded parameters on the buckling of composite structures, Comp. Meth. Appl. Mech. Engrg.,Vol. 194, pp. 2175–2189, 2005.
[11] Y. Noh, K. K. Choi and I. Lee, Reliability-based design optimization with confidence level under input model uncertainty due to limited test data, Struct. Multidisc. Optim. 43 (2011) 443–458.
[12] A. T. Beck and W. J. D. S Gomes, A comparison of deterministic, reliability-based and risk-based structural optimization under uncertainty, Probabilistic Eng. Mech. 28 (2012) 18–29.
[13] K. K. Choi, R. V. Grandhi and R. A. Canfield, Reliability-Base Structural Design, Springer, 2007.
[14] W. L. Fan, A. H. S. Ang amd Z. L. Li, Reliability assessment of deteriorating structures using Bayesian updated probability density evolution method (PDEM), Structural Safety, Vol. 65, pp. 60–73.
[15] H. Xu and S. Rahman, A generalized dimension-reduction method for multidimensional integration in stochastic mechanics, Int. J. Numer. Meth. Eng., Vol. 61, pp. 1992–2019, 2004.
[16] A. Der Kiureghian and P.-L. Liu, Structural reliability under incomplete probability information, J. Eng. Mech. 112(1) (1986) 85−104.
[17] A. S. Nowak and K. R. Collins, Reliability of Structures, CRC Press, 2012.
[18] M. Kaminski, The stochastic perturbation method for computational mechanics, John Wiley & Sons, 2013.
[19] H. A. David and H. N. Nagaraja, Order Statistics, Wiley, 2003.
[20] M. Yamakawa and M. Ohsaki，Worst-case design of structures using stopping rules in k-adaptive random sampling approach，Proc. 10th World Congress on Struct. Multidisciplinary Opt.，ISSMO，Paper. No. 5314, Orlando，USA，2013.
[21] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja, A First Course in Order Statistics, SIAM, 2008.
[22] R. T. Marler and J. S. Arora, Survey on multi-objective optimization methods for engineering, Struct. Multidisc. Optim., Vol. 26, pp. 369–395, 2004.
[23] J. Branke, K. Deb, K. Miettinen and R. Słowiński (Eds.), Multiobjective Optimization: Interactive and Evolutionary Approaches, Lecture Note in Computer Science, No. 5252, Springer.
[24] R. N. Iyengar and C. S. Manohar, Nonstationary random critical seismic excitations, J. Eng. Mech. 113 (4) (1987) 529–541.
[25] J. N. Yang, S. Sarkani and F. X. Long, A response spectrum approach for seismic analysis of nonclassically damped structures, Engineering Structures 12 (3) (1990) 173–184.
[26] K. Deb, Multi-Objective Optimization using Evolutionary Algorithms, John Wiley & Sons, 2001.

Table 1. Population quantile for order statistics of $\alpha = 0.9$.

(a) $n = 100$

k	1	2	3	4	5	6	7	8	9	10
γ	0.977	0.962	0.948	0.934	0.922	0.909	0.897	0.885	0.873	0.862
k	11	12	13	14	15	16	17	18	19	20
γ	0.850	0.839	0.827	0.816	0.805	0.794	0.783	0.772	0.761	0.750

(b) $n = 200$

k	1	2	3	4	5	6	7	8	9	10
γ	0.989	0.981	0.974	0.967	0.960	0.954	0.948	0.942	0.936	0.930
k	11	12	13	14	15	16	17	18	19	20
γ	0.924	0.918	0.912	0.907	0.901	0.895	0.890	0.884	0.878	0.873

Table 2. Nominal values of story stiffness of shear frame model ($\times 10^9$ N/m).

i	1	2	3	4	5	6	7	8	9	10
K_i	1.244	1.234	1.220	1.202	1.180	1.154	1.123	1.088	1.048	1.004
i	11	12	13	14	15	16	17	18	19	20
K_i	0.955	0.901	0.841	0.777	0.706	0.629	0.545	0.452	0.347	0.223