H_2 Analysis of LTI Systems via Conversion to Externally Positive Systems

Yoshio Ebihara

***Abstract*—Motivated by recent advances in the study of linear time-invariant (LTI) positive systems, we explore analysis techniques of general, not necessarily positive, LTI systems using positive system theory. Even though a positive system is characterized by its peculiar property that its impulse response is nonnegative, we often deal with nonnegative impulse responses even in general LTI system analysis. A typical example is the computation of the H_2 norm where we focus on squared impulse responses. To deal with such products of impulse responses in a systematic fashion, in this paper, we first establish a construction technique of an LTI system whose impulse response is given by the product of impulse responses of two different LTI systems. Then, as the main result, we reduce the H_2 norm computation problem of a general LTI system into the L_∞-induced norm computation problem (or L_1 problem in short) of a positive system, by which we can derive various formulas for the H_2 norm computation.**

***Index Terms*—system conversion, positive system, H_2 norm, L_∞-induced norm.**

I. Introduction

Analysis and synthesis of linear time-invariant (LTI) positive systems have attracted growing attention recently. An LTI system is said to be internally positive if its state and output are nonnegative for any nonnegative initial state and nonnegative input [9], [13]. Since internally positive systems frequently appear in the fields of engineering, economics, chemistry, pharmacy, etc., and since convex optimization works particularly well for the analysis and synthesis of internally positive systems, intensive research efforts have been made along this direction, see, e.g., [18], [19], [26], [11], [14], [23], [25], [2], [16], [4], [8]. As remarkable results, it is known that the H_∞ norm, the L_1-induced norm, and the L_∞-induced norm of an LTI SISO internally positive system coincide with its steady state gain and hence can be computed very efficiently. On the other hand, as a milder class of positive systems, the class of externally positive systems is also well known. An LTI system is said to be externally positive if its output is nonnegative for any nonnegative input under zero initial state [13], [9]. This property can be restated equivalently that its impulse response is nonnegative. Due to this nonnegativity property, again the L_∞-induced norm of an LTI SISO externally positive system can be computed exactly. We emphasize that the exact computation of L_∞-induced norm of a general, not necessarily positive system is very hard since we need to integrate the absolute value of its impulse response (i.e., we need to compute the L_1 norm of the impulse response and this is the reason why the L_∞-induced norm computation problem is sometimes called the L_1 problem). When we deal with positive systems, it is not necessary to take the absolute value since the impulse response is inherently nonnegative, and due to this nonnegativity property we can obtain an analytic formula for the L_∞-induced norm.

Even though an externally positive system is characterized by the peculiar property that its impulse response is nonnegative, we often deal with nonnegative impulse responses even in general, not necessarily positive, LTI system analysis. A typical example is the computation of the H_2 norm where we focus on squared impulse responses. This observation motivates us to compute the H_2 norm of general LTI systems using positive system theory. To this end, we need to deal with such products of impulse responses in a systematic fashion. Therefore, in this paper, we first establish a construction technique of an LTI system whose impulse response is given by the product of impulse responses of two (different) LTI systems. This enables us to construct an externally positive system whose impulse response is the square of that of the original system. Then, as the main result, we reduce the H_2 norm computation problem of a general LTI system into the L_∞-induced norm computation problem (i.e., L_1 problem) of an externally positive system, by which we derive a closed-form formula for the H_2 norm computation. As is expected, this result is not necessarily new and can be viewed as an alternative representation of the well-known Gramian-based H_2 norm characterization [27]. However, this treatment enables us to derive various formulas for the H_2 norm computation and they are novel to the best of the author's knowledge. In particular, we derive a novel linear matrix inequality (LMI) for the H_2 norm computation by newly deriving an LMI for the L_∞-induced norm computation of externally positive systems, where we fully rely on duality-based arguments [17], [20], [15], [6]. Our results clarify that positive system theory works well for the computation of the H_2 norm of general LTI systems, and this sheds new light on the H_2 analysis of LTI systems.

We use the following notation. We denote by $\mathbb{R}$ and $\mathbb{R}_+$ the set of real and nonnegative real numbers, respectively. The set of Hurwitz stable matrices of size n is denoted by $\mathbb{H}^n$. The set of symmetric, positive semidefinite, and positive definite matrices of size n are denoted by $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$, respectively. For $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the set of the eigenvalues of A and $\mathrm{He}\{A\} := A + A^T$. For a vector $v \in \mathbb{R}^n$, we denote by $\|v\|_\infty$ its ∞-norm, i.e., $\|v\|_\infty = \max_i |v_i|$. For a vector function $v: \mathbb{R}_+ \to \mathbb{R}^n$, we denote by $\|v\|_\infty$ its L_∞-norm, i.e., $\|v\|_\infty = \sup_{0\le t<\infty} \|v(t)\|_\infty$. For $A_1 \in \mathbb{R}^{n_1\times m_1}$ and $A_2 \in \mathbb{R}^{n_2\times m_2}$, we denote by $A_1 \otimes A_2$ their Kronecker product.

The author is with the Department of Electrical Engineering, Kyoto University, Kyotodaigaku-Katsura, Nishikyo-ku, Kyoto 615-8510, Japan, he was also with CNRS; LAAS; 7, avenue du Colonel Roche, F-31077 Toulouse, France, in 2011.

For $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $A_2 \in \mathbb{R}^{n_2 \times n_2}$, we denote by $A_1 \oplus A_2$ their Kronecker sum, i.e., $A_1 \oplus A_2 := A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$.

II. Preliminaries and Motivation

Let us consider the LTI system G described by

$$\begin{array}{l} G: \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t), \\ z(t) & = & Cx(t), \end{array} \right. \\ A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m},\ C \in \mathbb{R}^{l \times n}. \end{array} \tag{1}$$

The transfer function and the impulse response of the system G are given respectively by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] = C(sI - A)^{-1}B, \tag{2}$$

$$g(t) = C\exp(At)B\ (t \geq 0). \tag{3}$$

The definition of external positivity for the system G and a related result are now reviewed.

Definition 1: [9], [13] The system G given by (1) is said to be externally positive if its output is nonnegative for any nonnegative input under zero initial state.

Proposition 1: [9], [13] The system G given by (1) is externally positive if and only if its impulse response g given by (3) is nonnegative, i.e., $g(t) \geq 0\ (\forall t \geq 0)$.

On the other hand, the definitions of the H_2 and the L_∞-induced norms of the system G are given as follows.

Definition 2: [27] Suppose the LTI system G given by (1) is asymptotically stable, i.e., $A \in \mathbb{H}^n$. Then, the H_2 norm of G is defined by

$$\begin{array}{rcl} \|G\|_2 & := & \sqrt{\int_0^\infty \mathrm{trace}(g(t)^T g(t))dt} \\ & = & \sqrt{\int_0^\infty \sum_{i=1}^{l} \sum_{j=1}^{m} g_{ij}(t)^2 dt} \end{array} \tag{4}$$

where g is the impulse response of G given by (3).

Definition 3: Suppose the LTI system G given by (1) is asymptotically stable. Then, the L_∞-induced norm of G is defined by

$$\|G\|_{\infty,\infty} := \sup_{\|w\|_\infty \leq 1} \|z\|_\infty. \tag{5}$$

On the basis of the above preliminaries, we now clarify the motivation and the goal of this paper. For the case where G is stable and SISO (i.e., $m = l = 1$), we have from (4) that

$$\|G\|_2 = \sqrt{\int_0^\infty g(t)^2 dt}. \tag{6}$$

It is also elementary to verify that

$$\|G\|_{\infty,\infty} = \int_0^\infty |g(t)|dt.$$

Namely, the L_∞-induced norm $\|G\|_{\infty,\infty}$ coincides with the L_1 norm of the impulse response g. In particular, if the system G is externally positive, the above integration can be done by skipping the operation of taking the absolute value and hence we readily obtain

$$\|G\|_{\infty,\infty} = \int_0^\infty g(t)dt = -CA^{-1}B. \tag{7}$$

The relationship between (6) and (7) clearly shows that, if we can construct an externally positive and stable LTI system G_{sq} with impulse response g^2 from a given stable LTI system G with impulse response g, we can compute the H_2 norm $\|G\|_2$ by the closed-form formula (7) using the coefficient matrices of G_{sq}. The goal of this paper is to establish such a system operation technique to convert a given LTI system into an externally positive system, by which we can derive various closed-form formulas for the H_2 norm computation of general LTI systems. As clarified later on, one of such closed-form formulas can be viewed as an alternative representation of the well-known Gramian-based H_2 norm characterization [27].

III. Conversion to Externally Positive Systems

In this section, we first establish a system operation technique by which we can construct an LTI system whose impulse response is given by the product of impulse responses of two (different) LTI systems. The next theorem provides such a system operation, where we rely on the useful properties of the Kronecker product [12].

Theorem 1: Let us consider LTI SISO systems G_1 and G_2 given respectively by

$$\begin{array}{l} G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & 0 \end{array} \right],\ A_1 \in \mathbb{R}^{n_1 \times n_1}, \\ G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & 0 \end{array} \right],\ A_2 \in \mathbb{R}^{n_2 \times n_2}. \end{array} \tag{8}$$

The impulse responses of G_1 and G_2 are given respectively by

$$g_1(t) = C_1 \exp(A_1 t)B_1\ (t \geq 0),$$
$$g_2(t) = C_2 \exp(A_2 t)B_2\ (t \geq 0).$$

Then, the LTI SISO system G_{pr} defined by

$$G_{\mathrm{pr}}(s) = \left[\begin{array}{c|c} A_{\mathrm{pr}} & B_{\mathrm{pr}} \\ \hline C_{\mathrm{pr}} & 0 \end{array} \right] := \left[\begin{array}{c|c} A_1 \oplus A_2 & B_1 \otimes B_2 \\ \hline C_1 \otimes C_2 & 0 \end{array} \right]$$

has the impulse response of the form

$$g_{\mathrm{pr}}(t) = g_1(t)g_2(t)\ (t \geq 0). \tag{9}$$

Proof of Theorem 1: It is elementary to see that

$$\begin{array}{rcl} \exp(A_{\mathrm{pr}}t) & = & \exp\left((A_1 \oplus A_2)t\right) \\ & = & \exp\left((A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2)t\right) \\ & = & \exp((A_1 t) \otimes I_{n_2}) \exp(I_{n_1} \otimes (A_2 t)) \\ & = & (\exp(A_1 t) \otimes I_{n_2})(I_{n_1} \otimes \exp(A_2 t)) \\ & = & \exp(A_1 t) \otimes \exp(A_2 t). \end{array}$$

It follows that for $t \geq 0$ we have

$$\begin{array}{rl} g_{\mathrm{pr}}(t) & = C_{\mathrm{pr}} \exp(A_{\mathrm{pr}} t) B_{\mathrm{pr}} \\ & = (C_1 \otimes C_2)((\exp(A_1 t) \otimes \exp(A_2 t))(B_1 \otimes B_2) \\ & = (C_1 \exp(A_1 t)B_1)(C_2 \exp(A_2 t)B_2) \\ & = g_1(t)g_2(t). \end{array}$$

This completes the proof. ■

The next corollary directly follows from Theorem 1.

Corollary 1: Let us consider an LTI SISO systems G given by (1) with impulse response (3). Then, the LTI SISO system G_{sq} defined by

$$G_{\mathrm{sq}}(s) = \left[\begin{array}{c|c} A_{\mathrm{sq}} & B_{\mathrm{sq}} \\ \hline C_{\mathrm{sq}} & 0 \end{array} \right] := \left[\begin{array}{c|c} A \oplus A & B \otimes B \\ \hline C \otimes C & 0 \end{array} \right] \tag{10}$$

has the impulse response of the form

$$g_{\mathrm{sq}}(t) = g(t)^2\ (t \geq 0). \tag{11}$$

This corollary shows that we can construct an externally positive and stable LTI system G_{sq} with impulse response

g^2 from a given stable LTI system G with impulse response g. Note that $A_{\rm sq} = A \oplus A \in \mathbb{H}^{n^2}$ holds if and only if $A \in \mathbb{H}^n$ holds. This can be readily verified since $\sigma(A_{\rm sq}) = \{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in \sigma(A)\}$. See [12] for details.

IV. H_2 Norm Characterization via Reduction to L_1 Problem

A. SISO Case

For the case where the LTI system G given by (1) is SISO, the next result readily follows from (6), (11), (7), and (10).

Lemma 1: Suppose the LTI system G given by (1) is asymptotically stable and SISO, i.e., $m = l = 1$. Then, we have

$$\begin{aligned}\|G\|_2 &= \sqrt{\|G_{\rm sq}\|_{\infty,\infty}} \\ &= \sqrt{-(C \otimes C)(A \oplus A)^{-1}(B \otimes B)}\end{aligned} \tag{12}$$

where the system $G_{\rm sq}$ is given by (10).

B. MIMO Case

To deal with the case where the system G given by (1) is MIMO, let us partition $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{l\times n}$ as

$$B = [\ B_1 \ \cdots \ B_m\ ], \quad C = [\ C_1^T \ \cdots \ C_l^T\ ]^T. \tag{13}$$

Namely, B_j $(j = 1, \cdots, m)$ stands for the j-th column of B, and C_i $(i = 1, \cdots, l)$ stands for the i-th row of C. Then, it is straightforward to see from (12) that

$$\int_0^\infty g_{ij}(t)^2 dt = -(C_i \otimes C_i)(A \oplus A)^{-1}(B_j \otimes B_j). \tag{14}$$

With this in mind, let us define a SISO externally positive system $G_{\rm sq}$ as

$$\begin{aligned} G_{\rm sq}(s) &= \left[\begin{array}{c|c} A_{\rm sq} & B_{\rm sq} \\ \hline C_{\rm sq} & 0 \end{array}\right] \\ &:= \left[\begin{array}{c|c} A \oplus A & \displaystyle\sum_{j=1}^m B_j \otimes B_j \\ \hline \displaystyle\sum_{i=1}^l C_i \otimes C_i & 0 \end{array}\right]. \end{aligned} \tag{15}$$

We note that (15) reduces to (10) when $m = l = 1$, and $G_{\rm sq}$ given above is certainly an externally positive system since its impulse response $g_{\rm sq}$ is nonnegative as

$$\begin{aligned} g_{\rm sq}(t) &= \left(\sum_{i=1}^l C_i \otimes C_i\right) \exp((A \oplus A)t) \left(\sum_{j=1}^m B_j \otimes B_j\right) \\ &= \sum_{i=1}^l \sum_{j=1}^m (C_i \otimes C_i) \exp((A \oplus A)t)(B_j \otimes B_j) \\ &= \sum_{i=1}^l \sum_{j=1}^m g_{ij}(t)^2 \ \ (t \geq 0). \end{aligned} \tag{16}$$

By using (15) and (16) we can readily obtain the next theorem that is the first main result of this paper.

Theorem 2: Suppose that the LTI system G given by (1) is asymptotically stable. Then, we have

$$\begin{aligned}\|G\|_2 &= \sqrt{\|G_{\rm sq}\|_{\infty,\infty}} \\ &= \sqrt{-\left(\sum_{i=1}^l C_i \otimes C_i\right)(A \oplus A)^{-1}\left(\sum_{j=1}^m B_j \otimes B_j\right)}\end{aligned} \tag{17}$$

where the system $G_{\rm sq}$ is given by (15).

Proof of Theorem 2: The result (17) readily follows from (4), (16), (7), and (15) as

$$\begin{aligned}\|G\|_2 &= \sqrt{\int_0^\infty \sum_{i=1}^l \sum_{j=1}^m g_{ij}(t)^2 dt} \\ &= \sqrt{\int_0^\infty g_{\rm sq}(t) dt} \\ &= \sqrt{-\left(\sum_{i=1}^l C_i \otimes C_i\right)(A \oplus A)^{-1}\left(\sum_{j=1}^m B_j \otimes B_j\right)}.\end{aligned}$$

■

This theorem clearly shows that the H_2 norm of an n-dimensional LTI system can be obtained by computing the L_∞-induced norm (i.e., by solving the L_1 problem) of an n^2-dimensional SISO externally positive LTI system. In particular, the result (17) provides a closed-form formula for the computation of the H_2 norm of LTI systems.

V. Connection to Gramian-based H_2 Norm Computation

A. Controllability and Observability Gramians

In the last section we derived the closed-form formula (17) for the H_2 norm computation. It is expected that this formula is an alternative representation of the well-known Gramian-based H_2 norm characterization [27]. For the stable system G given by (1), recall that

$$\|G\|_2^2 = \mathrm{trace}(CXC^T) \tag{18}$$

holds where $X \in \mathbb{S}_+^n$ is the controllability Gramian determined as the unique solution of the Lyapunov equation

$$AX + XA^T + BB^T = 0. \tag{19}$$

From [12], we see that the controllability Gramian X is given by

$$\mathrm{vec}(X) = -(A \oplus A)^{-1}\left(\sum_{j=1}^m (B_j \otimes B_j)\right) \tag{20}$$

where $\mathrm{vec}(X) \in \mathbb{R}^{n^2}$ is the column-expansion of X. It is also true that

$$\mathrm{trace}(CXC^T) = \left(\sum_{i=1}^l (C_i \otimes C_i)\right) \mathrm{vec}(X).$$

Therefore we can conclude that (17) also follows from the standard Gramian-based approach. Here, if we apply (17) to the dual system $G_{\rm d}$ given by

$$G_{\rm d}(s) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & 0 \end{array}\right], \tag{21}$$

we can readily obtain an alternative representation of the H_2 norm as

$$\|G\|_2 = \sqrt{-\left(\sum_{j=1}^{m} B_j^T \otimes B_j^T\right)(A^T \oplus A^T)^{-1}\left(\sum_{i=1}^{l} C_i^T \otimes C_i^T\right)}. \quad (22)$$

Again, we can verify that this result follows from the standard approach based on the observability Gramian. To summarize, the closed-form formulas (17) and (22) can be viewed as alternative representations of the well-known Gramian-based H_2 norm characterizations.

B. Cross Gramian

In the preceding section we derived Theorem 2 by focusing on the squared impulse response $g(t)^2$ as shown in Corollary 1 and Lemma 1. However, in view of the treatments of MIMO systems, it is natural to consider $g(t)g(t)^T$ or $g(t)^Tg(t)$, even if these are of course identical to $g(t)^2$ in SISO cases. This treatment leads us to another formula for the H_2 norm computation.

To see this, let us focus on $g(t)^Tg(t)$. Then we readily obtain the next corollary and lemma from Theorem 1 and (7), respectively.

Corollary 2: Let us consider an LTI SISO systems G given by (1) with impulse response (3). Then, the LTI SISO system $G_{\rm ip}$ defined by

$$G_{\rm ip}(s) = \left[\begin{array}{c|c} A_{\rm ip} & B_{\rm ip} \\ \hline C_{\rm ip} & 0 \end{array}\right] := \left[\begin{array}{c|c} A^T \oplus A & C^T \otimes B \\ \hline B^T \otimes C & 0 \end{array}\right] \quad (23)$$

has the impulse response of the form

$$g_{\rm ip}(t) = g(t)^T g(t) \ \ (t \ge 0). \quad (24)$$

Lemma 2: Suppose the LTI system G given by (1) is asymptotically stable and SISO. Then, we have

$$\begin{aligned} \|G\|_2 &= \sqrt{\|G_{\rm ip}\|_{\infty,\infty}} \\ &= \sqrt{-(B^T \otimes C)(A^T \oplus A)^{-1}(C^T \otimes B)} \end{aligned} \quad (25)$$

where the system $G_{\rm ip}$ is given by (23).

We can further rewrite the H_2 norm characterization in this lemma. If we trace back the arguments around (19) and (20), it is not hard to see that $-(A^T \oplus A)^{-1}(C^T \otimes B) = {\rm vec}(F)$ holds where $F \in \mathbb{R}^{n\times n}$ is the column expansion of the unique solution for the Sylvester equation $AF+FA+BC=0$. On the other hand, we can readily obtain $(B^T \otimes C){\rm vec}(F) = CFB$. To summarize, we see that the next result holds.

Lemma 3: Suppose the LTI system G given by (1) is asymptotically stable and SISO. Then, we have

$$\|G\|_2 = \sqrt{CFB} \quad (26)$$

where F is the unique solution of the Sylvester equation

$$AF + FA + BC = 0. \quad (27)$$

Remark 1: The matrix F that satisfies the Sylvester equation (27) is called the cross-Gramian [10] and the H_2 norm characterization (26) is also known [10]. At a quick glance, it must be hard (or at least not easy) to see that CFB given by (26) and (27) is nonnegative. From the viewpoint of the present paper, however, it is easy to see that this nonnegativity is ensured by the external positivity of the system $G_{\rm ip}$ defined by (23).

The rest of this subsection is devoted to the extension of the results in Lemma 3 to MIMO cases. To achieve this, let us focus on the partition (13). Then, from Lemma 3, we can readily see that

$$\int_0^\infty g_{ij}(t)^T g_{ij}(t) dt = C_i F_{ij} B_j$$

where F_{ij} $(i = 1, \cdots, l, j = 1, \cdots, m)$ is the unique solution of the Sylvester equation $AF_{ij} + F_{ij}A + B_jC_i = 0$. We thus obtain the next theorem that provides another formula for the H_2 norm computation of LTI MIMO systems.

Theorem 3: Suppose that the LTI system G given by (1) is asymptotically stable. Then, we have

$$\|G\|_2 = \sqrt{\sum_{i=1}^{l}\sum_{j=1}^{m} C_i F_{ij} B_j} \quad (28)$$

where F_{ij} $(i = 1, \cdots, l,\ j = 1, \cdots, m)$ is the unique solution of the Sylvester equation

$$AF_{ij} + F_{ij}A + B_jC_i = 0. \quad (29)$$

Remark 2: Let us apply the result in Theorem 3 to the dual system $G_{\rm d}$ given by (21). Then, from the elementary fact that $\|G\|_2 = \|G_{\rm d}\|_2$, we have

$$\|G\|_2 = \sqrt{\sum_{j=1}^{m}\sum_{i=1}^{l} B_j^T J_{ji} C_i^T} \quad (30)$$

where J_{ji} $(i = 1, \cdots, l,\ j = 1, \cdots, m)$ is the unique solution of the Sylvester equation

$$A^T J_{ji} + J_{ji} A^T + C_i^T B_j^T = 0. \quad (31)$$

However, by comparing (29) and (31), it is clear that $J_{ji} = F_{ij}^T$ and hence (30) reduces to (28). Namely, the dual system representation leads to the same characterization as the original system representation. This is in stark contrast with the Gramian-based H_2 norm characterizations [27] where the dual system leads to different (observer-Gramian-based) characterization.

VI. New Formulas via LMI

It is well known that the system G given by (1) is asymptotically stable and satisfies $\|G\|_2 < \gamma$ if and only if there exists $X \in \mathbb{S}_{++}^n$ such that $AX + XA^T + BB^T \prec 0$ and ${\rm trace}(CXC^T) < \gamma^2$, see, e.g., [24]. This LMI characterization is useful in dealing with a wide variety of problems, including robust H_2 performance analysis of LTI systems affected by parametric uncertainties [3]. This motivates us to explore another LMI-based characterization of $\|G\|_2$ on the basis of the externally positive system representation (15). From (17), such an LMI can readily be obtained if we characterize the L_∞-induced norm of SISO externally positive systems by LMIs. We can indeed derive such LMIs as we see in the next theorem.

Theorem 4: Suppose the LTI system G given by (1) is SISO and externally positive. Then, for a given $\gamma > 0$, the next conditions are equivalent.

(i) $A \in \mathbb{H}^n$ and $\|G\|_{\infty,\infty} = G(0) < \gamma$.

(ii) There exists $P \in \mathbb{S}_{++}^n$ such that

$$\begin{bmatrix} PA + A^TP & PB + C^T \\ B^TP + C & -2\gamma \end{bmatrix} \prec 0. \quad (32)$$

(iii) There exists $X \in \mathbb{S}_{++}^n$ such that

$$\begin{bmatrix} AX + XA^T & XC^T + B \\ CX + B^T & -2\gamma \end{bmatrix} \prec 0. \quad (33)$$

Remark 3: Most of existing studies on positive system analysis using LMIs are restricted to internally positive systems [19], [4], [25], [7]. This is because we can make good use of internal positivity to derive new LMI conditions. In stark contrast, in Theorem 4, we deal with externally positive systems and the LMI results (32) and (33) are new to the best of the author's knowledge. For the proof of Theorem 4, we follow the duality-based arguments [17], [20], [15], [6] and at the final stage of the proof we make good use of the external positivity. See Appendix A for details. From Lemma 4 given there, we see that Theorem 4 is still valid even if we replace (i) with "$A \in \mathbb{H}^n$ and $\|G\|_\infty < \gamma$," i.e., the LMIs (32) and (33) also characterize the H_∞ norm of the externally positive system G.

Remark 4: The LMI (32) can be rewritten as

$$\mathrm{He}\left\{\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} A_a(\gamma)\right\} \prec 0,\ A_a(\gamma) := \begin{bmatrix} A & B \\ C & -\gamma \end{bmatrix}.$$

It follows that (i) holds if only if $A_a(\gamma)$ admits a block-diagonal Lyapunov matrix of the form $\mathrm{diag}(P,1)$. On the other hand, if G is internally positive, then (i) holds if and only if $A_a(\gamma)$ admits a purely diagonal Lyapunov matrix [4], [18]. In the next section, we show by numerical examples on an externally positive system that $A_a(\gamma)$ does not admit diagonal Lyapunov matrices even if (i) holds. Namely, there is a certain gap between internal and external positive systems.

The next corollary, which provides new LMIs that characterize the H_2 norm of the system G given by (1), readily follows from (15) and Theorems 2 and 4.

Corollary 3: Let us consider the LTI system G given by (1). Then, for a given $\gamma > 0$, the next conditions are equivalent.

(i) $A \in \mathbb{H}^n$ and $\|G\|_2 < \gamma$.

(ii) There exists $P \in \mathbb{S}_{++}^{n^2}$ such that

$$\begin{bmatrix} PA_{\mathrm{sq}} + A_{\mathrm{sq}}^T P & PB_{\mathrm{sq}} + C_{\mathrm{sq}}^T \\ B_{\mathrm{sq}}^T P + C_{\mathrm{sq}} & -2\gamma^2 \end{bmatrix} \prec 0. \tag{34}$$

(iii) There exists $X \in \mathbb{S}_{++}^{n^2}$ such that

$$\begin{bmatrix} A_{\mathrm{sq}} X + XA_{\mathrm{sq}}^T & XC_{\mathrm{sq}}^T + B_{\mathrm{sq}} \\ C_{\mathrm{sq}} X + B_{\mathrm{sq}}^T & -2\gamma^2 \end{bmatrix} \prec 0. \tag{35}$$

Here, $A_{\mathrm{sq}} \in \mathbb{R}^{n^2 \times n^2}$, $B_{\mathrm{sq}} \in \mathbb{R}^{n^2 \times 1}$, and $C_{\mathrm{sq}} \in \mathbb{R}^{1 \times n^2}$ are defined in (15).

In the new LMIs (34) and (35), the size of the Lyapunov matrix is n^2 while in the standard LMI [24] the size is n as quickly reviewed at the beginning of this section. Therefore, the LMIs (34) and (35) are computationally more demanding than the standard LMIs. However, the extra freedom of the Lyapunov matrix of size n^2 works fine, e.g., in deriving tighter upper bounds for the worst case H_2 performance analysis problems of LTI systems affected by parametric uncertainties.

Another possible application of Corollary 3 is the state-feedback H_2 control problem of positive systems. The problem is to design a state-feedback controller so that the closed-loop system remains positive and its H_2 norm is minimized. To the best of the author's knowledge, this is an open problem, and no convex (SDP) formulation is known. We expect that we can obtain numerically tractable conditions for this problem along the lines of Corollary 3. Rigorous treatments of these problems are future research topics.

VII. Numerical Examples

Let us consider the case where the coefficient matrices of the system G given by (1) are

$$A = \begin{bmatrix} -0.8 & -0.1 & -0.6 \\ 0.5 & -0.3 & -0.7 \\ -0.2 & 0.1 & 0 \end{bmatrix},\ B = \begin{bmatrix} -0.7 & 0.2 \\ -0.5 & -0.5 \\ 0.7 & -0.3 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.4 & 0 & -0.2 \\ 0.6 & -0.4 & -0.5 \end{bmatrix}.$$

Note that $\sigma(A) = \{-0.9140, -0.0930 \pm 0.2477j\}$ and hence $A \in \mathbb{H}^3$. To confirm the validity of Theorems 2, 3, and Corollary 3, we computed $\|G\|_2$ by these three methods. Then, we obtained $\|G\|_2 = 1.6673$ for every case as expected. When applying Corollary 3, we solved the SDP infimizing γ^2 subject to the LMI (34). If we solve this SDP under the additional constraint that P is diagonal, we obtained $\overline{\gamma} = 405.9977$. This result clearly shows that diagonal Lyapunov matrices cannot provide the exact result.

VIII. Conclusion

In this paper, we showed a system operation technique by which we can construct an LTI system whose impulse response is given by the product of impulse responses of two different LTI systems. On the basis of this system operation technique, we showed that the H_2 norm of an LTI system can be obtained by computing the L_∞-induced norm of an externally-positive SISO LTI system. By this problem reduction, we derived various formulas for the H_2 norm computation of LTI systems.

Even though we have concentrated our attention on the H_2 norm computation, we expect that the system operation technique proposed in this paper is useful for the computation of the peak value of the impulse responses as well. To the best of the author's knowledge, exact computation of the peak value is hard, and only its upper bound can be computed using SDPs [22], [5]. By using the system operation, it is expected that we can obtain tighter upper bounds by solving SDPs. This topic is currently under investigation.

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Appendix A
Proof of Theorem 4

For the proof we need the next lemma.

Lemma 4: Suppose the LTI system G given by (1) is SISO, externally positive, and asymptotically stable. Then, we have $G(0) \geq |G(j\omega)|\ (\forall \omega \in \mathbb{R})$.

Proof of Lemma 4: For a given $\omega \in \mathbb{R}$, let us apply the nonnegative input $w(t) = 1 + \sin \omega t$ to the system G. Then, since G is stable, the corresponding output at the steady-state is of the form $z(t) = G(0) + |G(j\omega)| \sin(\omega t + \theta_\omega)$. Since G is externally positive, we have $z(t) \geq 0\ (t \geq 0)$ and hence $G(0) \geq |G(j\omega)|$ holds. This completes the proof. ■

We are now ready to state the proof of Theorem 4. Since the equivalence of (ii) and (iii) readily follows via an elementary congruence transformation, we prove (i) ⇔ (ii).

(i)⇐(ii): Suppose (ii) holds. Then, it is clear that $A \in \mathbb{H}^n$. Moreover, it should be noted that (32) can be rewritten equivalently as

$$\begin{bmatrix} 0 & C^T \\ C & -2\gamma \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \right\} \prec 0. \tag{36}$$

Then, multiplying by $\begin{bmatrix} -B^T A^{-T} & 1 \end{bmatrix}$ from left and by its transpose from right, we have $-CA^{-1}B - B^T A^{-T} C^T - 2\gamma < 0$ or equivalently, $G(0) < \gamma$. This completes the proof.

(i)⇒(ii): For the proof we actually prove that (i)⇒(ii)' holds where the statements of (ii)' is as follows.

(ii)' There exists $P \in \mathbb{S}^n$ such that (32) holds.

Once (i)⇒(ii)' is validated, we have $A \in \mathbb{H}^n$ and $PA + A^T P \prec 0$ and hence $P \in \mathbb{S}^n_{++}$ follows. Namely, we can conclude that (i)⇒(ii) holds.

To prove (i)⇒(ii)' by contradiction, suppose (ii)' does not hold. Then, from the LMI duality [1], [3], [21], there exists $H \in \mathbb{S}^{n+1}_+ \setminus \{0\}$ such that

$$\mathrm{trace}\left(\begin{bmatrix} PA + A^T P & PB + C^T \\ B^T P + C & -2\gamma \end{bmatrix} H \right) \geq 0 \quad \forall P \in \mathbb{S}^n. \tag{37}$$

If we partition H as

$$H =: \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix},\ H_{11} \in \mathbb{S}^n_+,\ H_{22} \in \mathbb{R}_+,$$

we can restate (37) equivalently as

$$\begin{array}{l} \mathrm{trace}\left(P\mathrm{He}\{AH_{11} + BH_{12}^T\}\right) \geq 0\ \forall P \in \mathbb{S}^n, \\ CH_{12} - \gamma H_{22} \geq 0. \end{array} \tag{38}$$

The above condition holds if and only if

$$\mathrm{He}\{AH_{11} + BH_{12}^T\} = 0,\ CH_{12} - \gamma H_{22} \geq 0. \tag{39}$$

This implies that $H_{11} \neq 0$ since otherwise we have $H_{11} = 0$, $H_{12} = 0$ from $H \in \mathbb{S}^{n+1}_+$ and hence the second inequality above does not hold due to $H_{22} > 0$. It is also true that $H_{12} \neq 0$ since otherwise we have

$$\mathrm{He}\{AH_{11}\} = 0,\ H_{11} \in \mathbb{S}^n_+ \setminus \{0\}.$$

This clearly contradicts $A \in \mathbb{H}^n$. Therefore it suffices to consider the case where $H_{11} \neq 0$, $H_{12} \neq 0$, and hence $H_{22} \neq 0$ as well.

With this in mind, let us consider the full-rank factorization of H as

$$H =: \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^T,\ H_1 \in \mathbb{R}^{n \times r_H},\ H_2 \in \mathbb{R}^{1 \times r_H}. \tag{40}$$

If H_1 is not of full-column rank, there exists an orthogonal matrix $V \in \mathbb{R}^{r_H \times r_H}$ such that

$$\begin{bmatrix} \widehat{H}_1 & 0 \\ \widehat{H}_{2,1} & \widehat{H}_{2,2} \end{bmatrix} := \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} V$$

where $\widehat{H}_1$ is of full-column rank. We note that $\widehat{H}_{2,1} \neq 0$ since otherwise $H_{12} = 0$. Then, if we define

$$\begin{array}{l} \widehat{H} = \begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_{2,1} \end{bmatrix} \begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_{2,1} \end{bmatrix}^T \left(= \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & \widehat{H}_{22} \end{bmatrix} \right), \\ 0 < \widehat{H}_{22} \leq H_{22}, \end{array}$$

it is very clear from (39) that

$$\mathrm{He}\{A\widehat{H}_{11} + B\widehat{H}_{12}^T\} = 0,\ C\widehat{H}_{12} - \gamma \widehat{H}_{22} \geq 0. \tag{41}$$

To summarize, we can assume w.l.o.g. that H_1 is of full-column rank in (39) and (40). From linearity, it is also true that we can assume w.l.o.g. that $H_{22} = 1$.

We now move on to the final stage for the proof. We first note that the first equation in (39) can be rewritten, equivalently, as $\mathrm{He}\left\{(AH_1 + BH_2)H_1^T\right\} = 0$. Since $H_1 \in \mathbb{R}^{n \times r_H}$

is of full column rank, this equation holds if and only if there exists a skew symmetric matrix $\Omega \in \mathbb{R}^{r_H \times r_H}$ such that $AH_1 + BH_2 = H_1\Omega$ holds [5]. Since Ω is skew symmetric, its spectral factorization is given by

$$\begin{array}{l} \Omega = U\Lambda U^*, \\ \Lambda = \mathrm{diag}(j\omega_1, \cdots, j\omega_{r_H}),\ \omega_i \in \mathbb{R}\ (i = 1, \cdots, r_H) \end{array}$$

where $U \in \mathbb{C}^{r_H \times r_H}$ is a unitary matrix. If we define

$$\begin{array}{l} \bar{H}_1 := H_1 U = [\ f_1,\ \cdots,\ f_{r_H}\ ],\ f_i \in \mathbb{C}^n, \\ \bar{H}_2 := H_2 U = [\ g_1,\ \cdots,\ g_{r_H}\ ],\ g_i \in \mathbb{C}, \end{array}$$

we can readily obtain from $AH_1 + BH_2 = H_1\Omega$ that

$$\begin{array}{l} A\bar{H}_1 + B\bar{H}_2 = \bar{H}_1\Lambda \\ \Rightarrow Af_i + Bg_i = j\omega_i f_i\ (i = 1, \cdots, r_H) \\ \Rightarrow f_i = (j\omega_i I - A)^{-1} B g_i\ (i = 1, \cdots, r_H). \end{array} \tag{42}$$

Here, it suffices to consider the case $j\omega_i \notin \sigma(A)$ $(i = 1, \cdots, r_H)$ as above since, if $j\omega_i \in \sigma(A)$, this contradicts $A \in \mathbb{H}^n$ in (i). On the other hand, from $CH_{12} - \gamma \geq 0$, we readily obtain $C[\ f_1,\ \cdots,\ f_{r_H}\ ][\ g_1,\ \cdots,\ g_{r_H}\ ]^* \geq \gamma$. From (42) and $H_{22} = 1$, it turns out that

$$\sum_{i=1}^{r_H} C(j\omega_i I - A)^{-1} B g_i g_i^* \geq \gamma, \quad \sum_{i=1}^{r_H} g_i g_i^* = 1$$

or equivalently,

$$\sum_{i=1}^{r_H} G(j\omega_i) g_i g_i^* \geq \gamma, \quad \sum_{i=1}^{r_H} g_i g_i^* = 1.$$

This clearly shows that

$$\sum_{i=1}^{r_H} |G(j\omega_i)| g_i g_i^* \geq \gamma, \quad \sum_{i=1}^{r_H} g_i g_i^* = 1.$$

Since $G(0) \geq |G(j\omega_i)|$ $(i = 1, \cdots, r_H)$ holds from Lemma 4, we obtain

$$\sum_{i=1}^{r_H} G(0) g_i g_i^* \geq \gamma, \quad \sum_{i=1}^{r_H} g_i g_i^* = 1$$

or equivalently, $G(0) \geq \gamma$. This clearly contradicts (i). We thus complete the proof.