Variational proof of the existence of brake orbits in the planar 2-center problem

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Abstract

The restricted three-body problem is an important subject that deals with significant issues referring to scientific fields of celestial mechanics, such as analyzing asteroid movement behavior and orbit designing for space probes. The 2-center problem is its simplified model. The goal of this paper is to show the existence of brake orbits, which means orbits whose velocities are zero at some times, under some particular conditions in the 2-center problem by using variational methods.

1 Introduction & main theorem

The n-center problem is given by the following ODEs:

$$\ddot{\boldsymbol{q}} = -\sum_{k=1}^{n} \frac{m_k}{|\boldsymbol{q} - \boldsymbol{a}_k|^3}(\boldsymbol{q} - \boldsymbol{a}_k) \qquad (\boldsymbol{q} \in \mathbb{R}^d), \tag{1}$$

where $\boldsymbol{a}_k \in \mathbb{R}^d$ is a constant vector. A solution $\boldsymbol{q}(t)$ of (1) is called a brake orbit if there are real numbers T_1 and T_2 ($T_2 > T_1$) such that

$$\dot{\boldsymbol{q}}(T_1) = \dot{\boldsymbol{q}}(T_2) = \boldsymbol{0} \tag{2}$$

and $\boldsymbol{q}(t)$ is not a stationary solution. A brake orbit is a periodic orbit with period $2(T_2 - T_1)$. The fact is shown in Section 2.

The 2-center problem is a simplified model of the restricted three-body problem [7]. The 2-center problem is integrable, but its first integrals are complicated(for further details, see [1]). We can not immediately know what types of periodic solutions exist.

For various Lagrange systems, it has been researched for a long time to find periodic solutions with variational methods. In the n-center problem, it is shown that there exist periodic orbits that move around one or several primaries ([8],[10]). The brake orbits we prove to exist in this paper do not wind around particles.

Brake orbits are a special type of periodic orbits. Chen[3] proved that brake orbits exist in the planar isosceles three-body problem using collision manifold. In [5], Moeckel, Montgomery and Venturelli show the existence of brake orbits using variational methods with respect to the Jacobi-Maupertuis functional. The Lagrangian actional functional have not been used to find brake orbits.

In this paper, we will show that brake orbits exist in the planar 2-center problem by minimizing the Lagrangian action functional. We can set $m_1 = 1$ and $\boldsymbol{a}_1 = -\boldsymbol{a}_2 = (1, 0)$ without loss of generality for the planar 2-center problem as stated in Section 3. More precisely, we shall prove the following theorem:

Theorem 1.1. *If $(m, T) \in D$, then a $4T$-periodic brake orbit $\boldsymbol{q}(t)(= (q_1(t), q_2(t)))$ exists in the planar 2-center problem. The orbit is orthogonal to the x-axis at $t = 0$ and has zero velocity at $t = T$. The orbit $\boldsymbol{q}(t)$ satisfies $(q_1(t), q_2(t)) = (q_1(-t), -q_2(-t))$. Here, the set D is defined by*

$$D := \{(m, T) \mid T > \alpha(m)\,, f(m, T, c) \geq 0\,(^{\exists}c \geq 0)\}$$

where

$$\alpha(m) = \frac{\sqrt{2}\pi m^{1/4}}{(1 + \sqrt{m})^2}$$

and

$$f(m, T, c) = \frac{3}{2}\pi^{2/3}T^{1/3} + \frac{\pi^{2/3}(1 + m)^{-1/3}m}{2(1 + \pi^{2/3}(1 + m)^{-1/3}T^{-2/3})}T^{1/3}$$

$$- \left(\frac{2}{3}c^2T^{1/3} + \int_0^T \frac{1}{\sqrt{(1 - b)^2 + c^2t^{4/3}}} + \frac{m}{\sqrt{(1 + b)^2 + c^2t^{4/3}}}dt\right).$$

Figure 1 shows the domain D drawn with MATLAB.

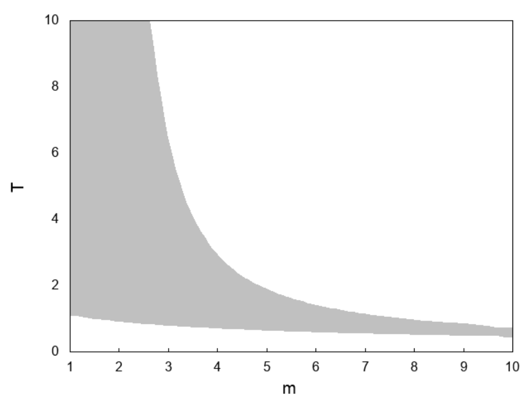

Figure 1: the domain D

Remark 1. We can expand the theorem to a larger domain than D. See appendix.

This paper is organized as follows. Section 2 contains some of well-known facts about brake orbits and variational methods. In Section 3, we introduce the variational settings in the planar 2-center problem and set the boundary condition. In Section 4, we complete the proof of Theorem 1.1 by eliminating the possibility that minimizer is a equilibrium solution or a collision path. In Appendix, we extend the theorem to a domain larger than D.

2 Preliminaries

2.1 Brake orbits

Consider ordinary differential equations:

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) \quad (\boldsymbol{x} \in \mathbb{R}^n). \tag{3}$$

Definition 2.1 (Reversible)**.** Let R be an involuntary liniear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e. $R^2 = E_n$. If (3) satisfies

$$FR + RF = 0,$$

then (3) is said to be reversible with respect to R.

With a simple calculation, we get the following proposition:

Proposition 1. *In reversible systems, if $\boldsymbol{x}(t)$ is a solution of* (3)*, then so is $R\boldsymbol{x}(-t)$.*

We define

$$\mathrm{Fix}(R) = \{\boldsymbol{x}(s) \in \mathbb{R} \mid R\boldsymbol{x}(s) = \boldsymbol{x}(s)\}$$

For a solution $\boldsymbol{x}(t)$ and a real value $s \in \mathbb{R}$, $\boldsymbol{x}(s) \in \mathrm{Fix}(R)$ is satisfied if and only if $\boldsymbol{x}(s+t) = R\boldsymbol{x}(s-t)$. See [6] for more detailed explanation for reversible systems.

Consider the following Lagrangian:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}|\dot{\boldsymbol{q}}|^2 + V(\boldsymbol{q}) \qquad (\boldsymbol{q}, \dot{\boldsymbol{q}} \in \mathbb{R}^n). \tag{4}$$

The differential equations of the Lagrangian system

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{p} \\ DV(\boldsymbol{q}) \end{pmatrix} \tag{5}$$

are reversible with respect to

$$R = \begin{pmatrix} E_n & \boldsymbol{0} \\ \boldsymbol{0} & -E_n \end{pmatrix}.$$

In this case, the fixed space is $\mathrm{Fix}(R) = \{(\boldsymbol{q}, \boldsymbol{0}) \mid \boldsymbol{q} \in \mathbb{R}^n\}$.

Proposition 2. *Brake orbits of Lagrangian system* (4) *with $\dot{\boldsymbol{q}}(T_1) = \dot{\boldsymbol{q}}(T_2) = 0$ are $2(T_2 - T_1)$-periodic orbits.*

The n-center problem is a Lagrangian system with form (4),

Corollary 1. *In the n-center problem, if a solution $\boldsymbol{q}$ satisfies* (2)*, then it is a $2(T_2 - T_1)$-periodic orbit.*

2.2 Existence of the minimizer

Let $\mathcal{C}_{A,B,T}$ be the set of C^2 curves in an open set $\mathcal{D} \subset \mathbb{R}^n$ connecting from A to B :

$$\{\boldsymbol{q} \in C^2([0,T], \mathcal{D}) \mid \boldsymbol{q}(0) \in A,\ \boldsymbol{q}(T) \in B\}$$

where $A, B \subset \mathcal{D}$ are affine spaces. The action functional for (4) is defined by:

$$\mathcal{A}(\boldsymbol{q}) = \int_0^T L(\boldsymbol{q}, \dot{\boldsymbol{q}}) dt.$$

The following is well-known.

Proposition 3. *Let L be a Lagrangian of the form* (4) *and $\mathcal{A}$ be the action functional. If $\boldsymbol{q} \in \mathcal{C}_{A,B,T}$ is a critical value of $\mathcal{A}$, then $\boldsymbol{q}(t)$ satisfies the Euler-Lagrange equation in $(0, T)$. Moreover, in the case* (4)*, $\dot{\boldsymbol{q}}(0)$ is orthogonal to A and $\dot{\boldsymbol{q}}(T)$ to B. If $A = \mathcal{D}$ ($B = \mathcal{D}$ resp.) , $\dot{\boldsymbol{q}}(0) = 0$ ($\dot{\boldsymbol{q}}(T) = 0$ resp.)*

We take

$$H^1(I, \mathcal{D}) = \left\{ \boldsymbol{q} \colon I \to \mathcal{D} \mid \boldsymbol{q} \in L^2(I, \mathcal{D}), \frac{d\boldsymbol{q}}{dt} \in L^2(I, \mathcal{D}) \right\}$$

where $I = [0, T]$. The norm is defined by

$$\|\boldsymbol{q}\|_{H^1} := \sqrt{\int_0^T |\boldsymbol{q}(t)|^2 + |\dot{\boldsymbol{q}}(t)|^2 dt.}$$

Definition 2.2 (coercive)**.** Let $\Omega \subset H^1(I, \mathcal{D})$. We call the functional $\mathcal{A}|_\Omega$ coercive if $\mathcal{A}(\boldsymbol{q}) \to \infty$ as $\|\boldsymbol{q}\|_{H^1} \to \infty$ ($\boldsymbol{q} \in \Omega$).

In general, action functionals for potential systems are weakly lower semi-continuous ([4]).

Lemma 2.3 ([9])**.** *Assume that $\mathcal{A}$ is weakly lower semi–continuous. If $\mathcal{A}|_\Omega$ is coercive, then there exists a minimizer $\boldsymbol{q}^*$ of $\mathcal{A}$ in the weak closure $\bar{\Omega}$ of Ω.*

Lemma 2.4. *Define Ω by*

$$\Omega = \{\boldsymbol{q} \in H^1(I, \mathcal{D}) \mid \boldsymbol{q}(0) \in A, \boldsymbol{q}(T) \in B\}.$$

If A is a bounded set, then $\mathcal{A}|_\Omega$ is coercive.

Proof. Here we prove this lemma, but similar proofs have appeared in some other settings (see for example [2]).

For any $\boldsymbol{q} \in \Omega$, we take

$$\delta(\boldsymbol{q}) = \max_{s_1, s_2 \in [0,T]} |\boldsymbol{q}(s_1) - \boldsymbol{q}(s_2)|.$$

By the Cauchy-Schwarz inequality,

$$\delta(\boldsymbol{q})^2 \le \left(\int_0^T |\dot{\boldsymbol{q}}| dt \right)^2 \le T \int_0^T |\dot{\boldsymbol{q}}|^2 dt.$$

By letting $\xi = \sup_{\boldsymbol{q} \in A} |\boldsymbol{q}|$,

$$|\boldsymbol{q}(t)| \le |\boldsymbol{q}(0)| + |\boldsymbol{q}(t) - \boldsymbol{q}(0)| \le \xi + \delta(\boldsymbol{q})$$

holds. Since

$$\|\boldsymbol{q}\|_{L^2}^2 = \int_0^T |\boldsymbol{q}(t)|^2 dt \le (\xi + \delta(\boldsymbol{q}))^2 T \le (\xi + \sqrt{T} \|\dot{\boldsymbol{q}}\|_{L^2})^2 T,$$

we obtain

$$\|\boldsymbol{q}\|_{H^1}^2 = \|\boldsymbol{q}\|_{L^2}^2 + \|\dot{\boldsymbol{q}}\|_{L^2}^2 \le (\xi + \sqrt{T} \|\dot{\boldsymbol{q}}\|_{L^2})^2 T + \|\dot{\boldsymbol{q}}\|_{L^2}^2.$$

Hence we get

$$\mathcal{A}(\boldsymbol{q}) \to \infty \ (\|\boldsymbol{q}\|_{H^1} \to \infty).$$

□

3 Variational setting for the 2-center problem

We consider the planar 2-center problem i.e. take $n = 2$ and $d = 2$ in (1). We fix masses and positions of the primaries as follows:

- $m_1 = 1$, $m_2 = m \geq 1$.
- Fix the position of the primaries at $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.
- $\boldsymbol{a}_1 = \boldsymbol{a} = (1, 0)$,$\boldsymbol{a}_2 = -\boldsymbol{a}$.

We can assume the above setting without loss of generality for the 2-center problem, because for any $\boldsymbol{a}_1$, $\boldsymbol{a}_2 \in \mathbb{R}^2$, $m_1 > 0$ and $m_2 > 0$, it can be reduced the above case with appropriate transformation and scaling.

We define its action functional by

$$\mathcal{A}(\boldsymbol{q}) = \int_0^T L(\boldsymbol{q}, \dot{\boldsymbol{q}})dt \tag{6}$$

where $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}|\dot{\boldsymbol{q}}|^2 + \frac{1}{|\boldsymbol{q} - \boldsymbol{a}|} + \frac{m}{|\boldsymbol{q} + \boldsymbol{a}|}$ and $\boldsymbol{q} \in H^1(I, \mathbb{R}^2)$. The planar 2-center problem is equivalent to the variational problem:

$$\mathcal{A}'(\boldsymbol{q}) = 0. \tag{7}$$

We fix a positive number T and search for a brake orbit $\boldsymbol{q}(t) = (q_1(t), q_2(t))$ satisfying

- $q_1(0) \in (-1, 1)$ and $q_2(0) = 0$.
- $\dot{\boldsymbol{q}}(T) = \boldsymbol{0}$.
- $q_1(t) = q_1(-t), q_2(t) = -q_2(-t)$.

In order to obtain such brake orbits, we take a class of curves as follows:

$$\Omega = \{\boldsymbol{q}(t) = (q_1(t), q_2(t)) \in H^1([0, T], \mathbb{R}^2) \mid -1 < q_1(0) < 1, q_2(0) = 0\}.$$

From Lemma 2.3 and 2.4, (6) has a minimizer in the weak closure $\bar{\Omega}$ of Ω. Let $\boldsymbol{q}^*(t) = (q_1^*(t), q_2^*(t))$ be a minimizer. If $\boldsymbol{q}^*$ is neither a trivial solution nor a collision solution, it is a quarter (fundamental) part of a brake orbit from Proposition 2 and 3 (See figure 2).

In fact, the system has a reversibility with respect to:

$$R \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -p_x \\ p_y \end{pmatrix} \quad \left(R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

By collorary 1 if $\boldsymbol{q}(t) = (q_1(t), q_2(t))$ is a solution, then so is $\boldsymbol{q}(t) = (q_1(-t), -q_2(-t))$. Thus, we get the entire trajectory of a $4T$-periodic brake orbit like figure 3.

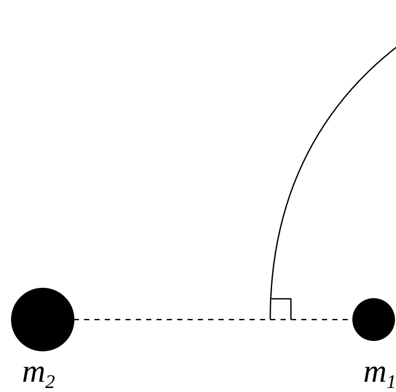

Figure 2: $\boldsymbol{q}^*(t)$ $(t \in [0, T])$

4 Proof of main theorem

4.1 Estimate of equilibrium point

Let $\boldsymbol{q}_{eq}$ denote an equilibrium point of (6), i.e.

$$\frac{1}{|\boldsymbol{q}_{\mathrm{eq}} - \boldsymbol{a}|^3}(\boldsymbol{q}_{\mathrm{eq}} - \boldsymbol{a}) + \frac{m}{|\boldsymbol{q}_{\mathrm{eq}} + \boldsymbol{a}|^3}(\boldsymbol{q}_{\mathrm{eq}} + \boldsymbol{a}) = \mathbf{0}.$$

From a simple calculation, $\boldsymbol{q}_{\mathrm{eq}}$ is determined by:

$$\boldsymbol{q}_{\mathrm{eq}} = (b, 0) \quad \left(b = \frac{\sqrt{m} - 1}{\sqrt{m} + 1}\right).$$

and the value of the action functional at $\boldsymbol{q}_{\mathrm{eq}}$ is

$$\mathcal{A}(\boldsymbol{q}_{\mathrm{eq}}) = \int_0^T \frac{1}{|\boldsymbol{q}_{\mathrm{eq}} - \boldsymbol{a}|} + \frac{m}{|\boldsymbol{q}_{\mathrm{eq}} + \boldsymbol{a}|} dt = \frac{1}{2}(1 + \sqrt{m})^2 T.$$

We will obtain a condition under which the equilibrium point is not the minimizer by estimating the second variation. The second variation $\mathcal{A}''(\boldsymbol{q})(\boldsymbol{\delta})$ is given by

$$\mathcal{A}''(\boldsymbol{q})(\boldsymbol{\delta}) = \int_0^T (\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}(t)) \nabla^2 L(\boldsymbol{q})(\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}(t))^{\mathrm{T}} dt,$$

where $\boldsymbol{q} \in H^1([0, T], \mathbb{R}^2)$ and $\boldsymbol{\delta} \in H^1([0, T], \mathbb{R}^2)$. (For details, see [11].) If there exists $\boldsymbol{\delta}$ such that $\mathcal{A}''(\boldsymbol{q})(\boldsymbol{\delta})$ is negative, then $\boldsymbol{q}$ is not the minimizer of (6). Since

$$\nabla^2 L(\boldsymbol{q}_{\mathrm{eq}}) = \begin{pmatrix} 2\gamma & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(\gamma = \frac{(1 + \sqrt{m})^4}{8\sqrt{m}}\right),$$

we obtain

$$\mathcal{A}''(\boldsymbol{q}_{\mathrm{eq}})(\boldsymbol{\delta}) = \int \dot{\delta}_1^2 + \dot{\delta}_2^2 + \gamma(2\delta_1^2 - \delta_2^2) dt. \tag{8}$$

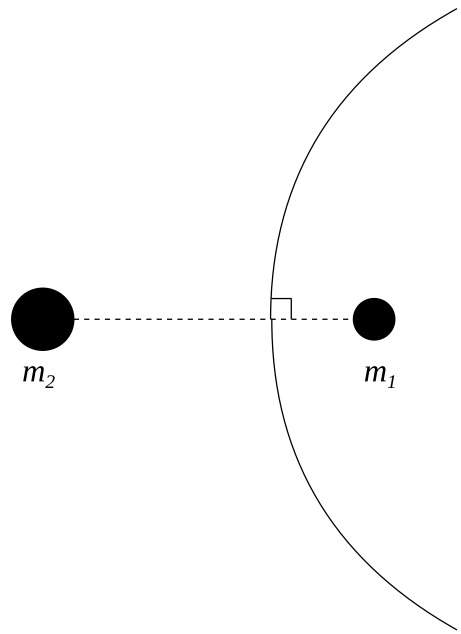

Figure 3: a whole brake orbit

We substitute

$$\boldsymbol{\delta} = (\delta_1(t), \delta_2(t)) = (0, \sin \omega t) \quad \left(\omega = \frac{\pi}{2T}\right) \tag{9}$$

into (8). Since

$$\begin{aligned}
&\int_0^T \dot{\delta}_1^2 + \dot{\delta}_2^2 + \gamma(2\delta_1^2 - \delta_2^2)dt \\
&= \omega^2 \left(\frac{T}{2} + \frac{1}{4\omega}\sin(2\omega T)\right) - \gamma\left(\frac{T}{2} - \frac{1}{4\omega}\sin(2\omega T)\right) \\
&= \frac{T}{2}(\omega^2 - \gamma),
\end{aligned}$$

the second variation of $\boldsymbol{q}_{\mathrm{eq}}$ for (9) is negative if

$$\frac{\pi}{2\sqrt{\gamma}} = \frac{\sqrt{2}\pi m^{1/4}}{(1+\sqrt{m})^2} < T.$$

From this, the following lemma is proved.

Lemma 4.1. *If* $T > \dfrac{\sqrt{2}\pi m^{1/4}}{(1+\sqrt{m})^2}$, $\boldsymbol{q}_{\mathrm{eq}}$ *is not a minimizer of* $\mathcal{A}(\boldsymbol{q})$.

4.2 Estimate of collision

Lemma 4.2. *The set* Ω_{col} *is given by*

$$\Omega_{\mathrm{col}} = \{\boldsymbol{q} \in \Omega \mid \boldsymbol{q} \text{ has collisions.}\},$$

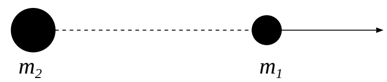

Figure 4: minimizer with collosions

then $\mathcal{A}|_{\Omega_{\mathrm{col}}}$ *is minimized by an orbit that moves along x-axis (see Figure 4).*

Proof. Assume that $\boldsymbol{q}_{\mathrm{col}}$ collides with m_1 and

$$\boldsymbol{q}_{\mathrm{col}}(t) = r(t)(\cos\theta(t), \sin\theta(t)) + (1, 0).$$

The value of action functional at $\boldsymbol{q}_{\mathrm{col}}$ is

$$\begin{aligned}\mathcal{A}(\boldsymbol{q}_{\mathrm{col}}) &= \int_0^T \frac{1}{2}|\dot{\boldsymbol{q}}_{\mathrm{col}}|^2 + \frac{1}{|\boldsymbol{q}_{\mathrm{col}} - \boldsymbol{a}|} + \frac{m}{|\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{a}|} dt \\ &= \int_0^T \frac{1}{2}(\dot{r}^2 + (r\dot{\theta})^2) + \frac{1}{|r|} + \frac{m}{\sqrt{r^2 + 4 + 2r\cos\theta}} dt \\ &\geq \int_0^T \frac{1}{2}\dot{r}^2 + \frac{1}{|r|} + \frac{m}{\sqrt{r^2 + 4 + 2r}} dt.\end{aligned}$$

This inequality becomes an equality if and only if $\theta(t)$ is identically zero. We can obtain the similar estimate in the case that $\boldsymbol{q}_{\mathrm{col}}$ collides with m_2, and it is no less than the former one since $m \geq 1$. It follows that the collision path moves on the x-axis like Figure 4. □

We will call the solution of Lemma 4.2 a collision-ejection solution of the 2-center problem and represent it by

$$\boldsymbol{q}_{\mathrm{col}}(t) = (q_{\mathrm{col}}(t), 0).$$

By Lemma 4.2, we consider only a collision-ejection solution to get a lower bound estimate for the value of the action functional for any collision path.

Lemma 4.3 ([4])**.** *Let* $\mu > 0$*,* $\rho > 0$ *be constants. For* $r \in H^1([0,T], \mathbb{R})$*, define*

$$\mathcal{B}(r) = \int_0^T \frac{\mu}{2}\dot{r}^2 + \frac{\rho}{|r|} dt. \tag{10}$$

If there exists $t_0 \in [0,T]$ *satisfying* $r(t_0) = 0$*, then the inequality,*

$$\mathcal{B}(r) \geq B(\mu, \rho, T) := \frac{3}{2}\pi^{2/3}\rho^{2/3}\mu^{1/3}T^{1/3},$$

holds and $\mathcal{B}(r) = B(\mu, \rho, T)$ *if and only if* $r(t)$ *is a collision-ejection solution of the Kepler problem. Moreover, if* $r(t)$ *is a collision-ejection solution with* $r(0) = 0$*,*

$$r(T) = 2\pi^{-2/3}\mu^{-1/3}\rho^{1/3}T^{2/3}.$$

In (10), we take $\mu = 1$ and $\rho = m + 1$. Let $\tilde{\boldsymbol{q}}(t) = (\tilde{q}(t), 0)$ where $\tilde{q}(t) - 1$ is a minimizer of (10). From Lemma 4.3, we indicate

$$\tilde{q}(T) = 2\pi^{-2/3}T^{2/3}(m+1)^{1/3} + 1. \tag{11}$$

Lemma 4.4.

$$q_{\text{col}}(T) < \tilde{q}(T).$$

Proof. Suppose

$$q_{\text{col}}(T) \geq \tilde{q}(T) \tag{12}$$

and

$$F_{\text{col}}(q) = -\frac{1}{q-1} - \frac{m}{q+1}, \; \tilde{F}(q) = -\frac{m+1}{q-1}.$$

Now

$$q_{\text{col}}(0) = \tilde{q}(0) = 1 \tag{13}$$
$$\dot{q}_{\text{col}}(T) = \dot{\tilde{q}}(T) = 0 \tag{14}$$
$$0 > F_{\text{col}}(q) > \tilde{F}(q) \tag{15}$$

holds. We take

$$t_0 := \sup\{t \in [0,T) \mid q_{\text{col}}(t) = \tilde{q}(t)\}.$$

If the inequality (12) is strict, i.e. $q_{\text{col}}(T) > \tilde{q}(T)$, $t_0 < T$ and $q_{\text{col}}(t) > \tilde{q}(t)$ holds for $t \in (t_0, T)$. In the case of the equality, i.e. $q_{\text{col}}(T) = \tilde{q}(T)$, $q_{\text{col}}(t) > \tilde{q}(t)$ is satisfied for t close to T since $\tilde{q}(T) = q_{\text{col}}(t), \dot{\tilde{q}}(T) = \dot{q}_{\text{col}}(T), \ddot{\tilde{q}}(T) < \ddot{q}_{\text{col}}(T)$. In both the cases, t_0 is less than T and $q_{\text{col}}(t) > \tilde{q}(t)$ is satisfied for $t \in (t_0, T)$.

By (12), (15), $0 > F_{\text{col}}(q_{\text{col}}(t)) > \tilde{F}(\tilde{q}(t))$ holds for $t \in (t_0, T)$. By $\ddot{q}_{\text{col}} = F_{\text{col}}, \ddot{\tilde{q}} = \tilde{F}$ and (14), for any $t \in [t_0, T)$, it holds the following inequality:

$$\dot{q}_{\text{col}}(t) = \int_T^t \ddot{q}_{\text{col}} dt = \int_T^t F_{\text{col}}(q_{\text{col}}(t)) dt < \int_T^t \tilde{F}(\tilde{q}(t)) dt = \int_T^t \ddot{\tilde{q}} dt = \dot{\tilde{q}}(t).$$

Since $q_{\text{col}}(t_0) = \tilde{q}(t_0)$, we obtain

$$0 > \int_T^{t_0} \dot{\tilde{q}}(t) - \dot{q}_{\text{col}}(t) dt = q_{\text{col}}(T) - \tilde{q}(T).$$

This contradicts (12). □

Lemma 4.5. *For any $\boldsymbol{q}_{\text{col}}$ in collision solutions,*

$$\mathcal{A}(\boldsymbol{q}_{\text{col}}) > g(m,T) := \frac{3}{2}\pi^{2/3}T^{1/3} + \frac{\pi^{2/3}(1+m)^{-1/3}m}{2(1+\pi^{2/3}(1+m)^{-1/3}T^{-2/3})}T^{1/3}$$

Proof. From Lemma 12, we have

$$\int_0^T \frac{1}{|\boldsymbol{q}_{\text{col}} + \boldsymbol{a}|} dt = \int_0^T \frac{1}{q_{\text{col}}+1} dt \geq \frac{m}{q_{\text{col}}(T)+1}\int_0^T dt = \frac{m}{q_{\text{col}}(T)+1} T$$
$$> \frac{\pi^{2/3}(1+m)^{-1/3}m}{2(1+\pi^{2/3}(1+m)^{-1/3}T^{-2/3})}T^{1/3}.$$

By (11), we get

$$\mathcal{A}(\boldsymbol{q}_{\text{col}}) = \int_0^T \frac{1}{2}|\dot{\boldsymbol{q}}_{\text{col}}|^2 + \frac{1}{|\boldsymbol{q}_{\text{col}} - \boldsymbol{a}|} dt + \int_0^T \frac{m}{|\boldsymbol{q}_{\text{col}} + \boldsymbol{a}|} dt$$
$$> \frac{3}{2}\pi^{2/3}T^{1/3} + \frac{\pi^{2/3}(1+m)^{-1/3}m}{2(1+\pi^{2/3}(1+m)^{-1/3}T^{-2/3})}T^{1/3}.$$

□

4.3 Test path vs. collision path

Lemma 4.6. *If $f(m,T,c) \geq 0$, the collision path $\boldsymbol{q}_{\mathrm{col}}$ is not a minimizer.*

Proof. We take a test path:

$$\boldsymbol{q}_c(t) = (b, ct^{\frac{2}{3}}) \quad (c \geq 0).$$

If $\mathcal{A}(\boldsymbol{q}_{\mathrm{col}}) > \mathcal{A}(\boldsymbol{q}_c)$, $\boldsymbol{q}_{\mathrm{col}}$ is not a minimizer. The value of functional with respect to the test path is

$$\mathcal{A}(\boldsymbol{q}_c) = \frac{2}{3}c^2T^{1/3} + \int_0^T \frac{1}{\sqrt{(1-b)^2 + c^2t^{4/3}}} + \frac{m}{\sqrt{(1+b)^2 + c^2t^{4/3}}}dt.$$

By Lemma 4.5, it is sufficient if $g(m,T) \geq \mathcal{A}(\boldsymbol{q}_c)$. This inequality is equivalent to $f(m,T,c) \geq 0$. □

4.4 The domain D

Here we show that the domain D is nonempty without numerical calculation. Let

$$\tilde{g}(T) := \frac{3}{2}\pi^{2/3}T^{1/3}.$$

Clearly $g(m,T) > \tilde{g}(T)$ holds, so we obtain $\tilde{g}(T) \geq \mathcal{A}(\boldsymbol{q}_{\mathrm{eq}})$, i.e. if $T < \dfrac{3\sqrt{3}\pi}{(1+\sqrt{m})^3} (= \beta(m))$, then $\boldsymbol{q}_{\mathrm{col}}$ is not a minimizer and if $T > \alpha(m)$, then $\boldsymbol{q}_{\mathrm{eq}}$ is not a minimizer. If there exists T such that $\alpha(m) < T < \beta(m)$, then

$$\emptyset \neq \{(m,T) \mid \alpha(m) < T < \beta(m)\,, f(m,T,0) \geq 0\} \subset D,$$

so D is nonempty. The inequality $\alpha(m) < \beta(m)$ is equivalent to

$$\sqrt{m}(\sqrt{m}+1)^2 - \frac{27}{2} < 0. \tag{16}$$

For $1 \leq m < 3.1164778$, (16) holds.

A

In this section, we will reconsider estimate of (6) of collisions. For all $\lambda \in (0,1)$, let

$$\mathcal{A}(q) = \mathcal{A}_1(\lambda, q-1) + \mathcal{A}_2(\lambda, q+1),$$

where

$$\mathcal{A}_1(\lambda, q) = \int_0^T \frac{1-\lambda}{2}\dot{q}^2 + \frac{1}{|q|}dt \tag{17}$$

and

$$\mathcal{A}_2(\lambda, q) = \int_0^T \frac{\lambda}{2}\dot{q}^2 + \frac{m}{|q|}dt. \tag{18}$$

By [4], we get the following estimate of (17):

$$\mathcal{A}_1(\lambda, q-1) > \frac{3}{2}\pi^{2/3}(1-\lambda)^{1/3}T^{1/3}.$$

To estimate (18), we will use a comparison of (18) and a part of the linear Kepler orbit.

We fix H and assume $-m/2 < H < 0$. Let $Q(t)$ denote a collision-ejection solution with respect to (18) satisfying $Q(t_0) = 2$, $\dot{Q}(T+t_0) = 0$ and

$$H = \frac{\lambda}{2}\dot{Q}^2 - \frac{m}{|Q|}.$$

Thus we obtain

$$\begin{aligned}\mathcal{A}_2(\lambda, q+1) &> \int_{t_0}^{T+t_0} \frac{\lambda}{2}\dot{Q}^2 + \frac{m}{|Q|}dt \\ &= \int_0^{T+t_0} \frac{\lambda}{2}\dot{Q}^2 + \frac{m}{|Q|}dt - Ht_0 - 2\int_0^{t_0} \frac{m}{|Q|}dt.\end{aligned}$$

Gordon [4] gives

$$\int_0^{T+t_0} \frac{\lambda}{2}\dot{Q}^2 + \frac{m}{|Q|}dt = \frac{3}{2}\pi^{2/3}\lambda^{1/3}m^{2/3}(T+t_0)^{1/3}.$$

Lemma A.1. *If $-H < 0$, let $T(x, H)$ denote the time from 0 to x with energy H. Then it holds the following equation:*

$$T(x,H) = m\sqrt{\frac{\lambda}{2(-H)^3}}\left\{\sin^{-1}\left(\sqrt{\frac{-xH}{m}}\right) - \sqrt{\frac{-xH}{m}\left(1+\frac{xH}{m}\right)}\right\}$$

Proof. By the definition of $T(x, H)$, we get

$$\begin{aligned}T(x,H) &= \sqrt{\frac{\lambda}{2}}\int_0^x \frac{1}{\dot{Q}}dQ = \sqrt{\frac{\lambda}{2}}\int_0^x \sqrt{\frac{Q}{HQ+m}}dQ \\ &= m\sqrt{\frac{\lambda}{2(-H)^3}}\int_0^{-\frac{H}{m}x}\sqrt{\frac{q}{1-q}}dq \\ &= m\sqrt{\frac{\lambda}{2(-H)^3}}\int_0^{\theta_0}(1-\cos 2\theta)d\theta \quad (\theta_0 = \sin^{-1}\left(\sqrt{\frac{-xH}{m}}\right)) \\ &= m\sqrt{\frac{\lambda}{2(-H)^3}}\left\{\sin^{-1}\left(\sqrt{\frac{-xH}{m}}\right) - \sqrt{\frac{-xH}{m}\left(1+\frac{xH}{m}\right)}\right\}.\end{aligned}$$

The proof is completed. ∎

The relation of T and t_0 is indicated by the above lemma:

$$\begin{aligned}T &= (T+t_0) - t_0 = T(x_{\max}, H) - T(2,H) \\ &= m\sqrt{\frac{\lambda}{2(-H)^3}}\left\{\cos^{-1}\left(\sqrt{\frac{-2H}{m}}\right) + \sqrt{\frac{-2H}{m}\left(1+\frac{2H}{m}\right)}\right\}\end{aligned}$$

Substituting $H = -\frac{m}{2}y$ for any $y \in (0,1)$,

$$T := \bar{T}(m.\lambda, y) = 2\sqrt{\frac{\lambda}{m}} \cdot \frac{1}{y}\left\{\frac{\cos^{-1}\left(\sqrt{y}\right)}{\sqrt{y}} + \sqrt{1-y}\right\}$$

and

$$t_0 = 2\sqrt{\frac{\lambda}{m}} \cdot \frac{1}{y}\left\{\frac{\sin^{-1}\left(\sqrt{y}\right)}{\sqrt{y}} - \sqrt{1-y}\right\}.$$

It follows that

$$T + t_0 = \frac{\pi m}{2}\sqrt{\frac{\lambda}{2(-H)^3}} = \frac{\pi}{y}\sqrt{\frac{\lambda}{my}} \tag{19}$$

and

$$-Ht_0 = \sqrt{m\lambda}\left\{\frac{\sin^{-1}\left(\sqrt{y}\right)}{\sqrt{y}} - \sqrt{1-y}\right\}.$$

Moreover, we have

$$\begin{aligned}
2\int_0^{t_0} \frac{m}{|Q|}dt &= 2m\sqrt{\frac{\lambda}{2(-H)}}\int_0^{t_0} \frac{1}{Q}\sqrt{\frac{Q}{-Q-(m/H)}}dQ \\
&= 2m\sqrt{\frac{\lambda}{2(-H)}}\int_0^{2} \sqrt{\frac{1}{q(1-q)}}dq \\
&= 2m\sqrt{\frac{\lambda}{2(-H)}}\int_0^{\theta_0} 2d\theta \quad (\theta_0 = \sin^{-1}\left(\sqrt{\frac{-2H}{m}}\right)) \\
&= 2m\sqrt{\frac{2\lambda}{-H}}\sin^{-1}\left(\sqrt{\frac{-2H}{m}}\right) = 4\sqrt{\frac{m\lambda}{y}}\sin^{-1}\left(\sqrt{y}\right).
\end{aligned}$$

Hence, since for all $\lambda \in (0,1)$ and $y \in (0,1)$,

$$\mathcal{A}_2(\lambda, q) > \sqrt{m\lambda}\left\{\frac{3}{\sqrt{y}}\cos^{-1}\left(\sqrt{y}\right) - \sqrt{1-y}\right\},$$

we get

$$\mathcal{A}(q_{\text{col}}) > \bar{g}(m, \lambda, y), \tag{20}$$

where

$$\bar{g}(m, \lambda, y) = \frac{3}{2}\pi^{2/3}(1-\lambda)^{1/3}\bar{T}(m, \lambda, y)^{1/3} + \sqrt{m\lambda}\left\{\frac{3}{\sqrt{y}}\cos^{-1}\left(\sqrt{y}\right) - \sqrt{1-y}\right\}.$$

In the same way as the proof of Lemma 4.6, if $\bar{g}(m, \lambda, y) - \mathcal{A}(\boldsymbol{q}_c) \geq 0$, then q_{col} is not a minimizer.

From the above discussion, we show:

Theorem A.2. *If $(m, T) \in D'$, then $4T$-periodic brake orbits $\boldsymbol{q}(t)(= (q_1(t), q_2(t)))$ satisfying the same condition of Theorem 1.1 exists in the planar 2-center problem. Here, the set D' is defined by*

$$D' := \left\{ (m, T) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} T > \alpha(m) \text{ and } {}^{\exists}\lambda, y \in (0,1) \text{ such that} \\ \bar{f}(m, \lambda, y, c) \geq 0 \text{ and } T = \bar{T}(m, \lambda, y). \end{array} \right\}$$

where

$$\bar{f}(m, \lambda, y, c) = \bar{g}(m, \lambda, y) - \mathcal{A}(\boldsymbol{q}_c)$$

and

$$\bar{T}(m, \lambda, y) = 2\sqrt{\frac{\lambda}{m}} \cdot \frac{1}{y} \left\{ \frac{\cos^{-1}\left(\sqrt{y}\right)}{\sqrt{y}} + \sqrt{1-y} \right\}.$$

Now, similarly as the domain D, we describe the domain D' with MATLAB.

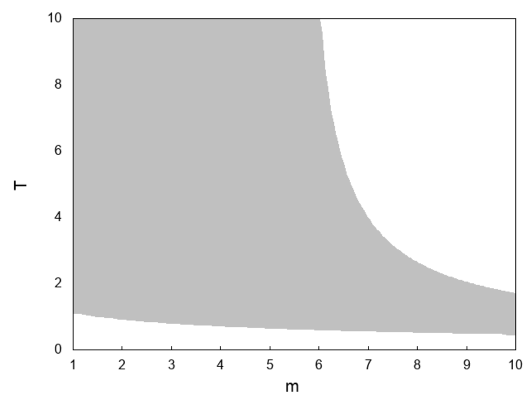

Figure 5: the domain D'

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