RIMS Kôkyûroku Bessatsu
B58 (2016), 19–32

Continued fractions: natural extensions and invariant measures

By

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Abstract

We give a heuristic technique to find a model for the natural extension of a piecewise homographic, or more generally projective, map on a domain of $\mathbb{R}$ or $\mathbb{R}^d$. In case of success, this gives explicit formula for an invariant density.

§ 1. Introduction

Continued fractions have been studied for a very long time, in particular as a mean to find best rational approximations to real numbers. Most of this work was considered as pure number theory and algebra, although it turned out to be very useful in the study of gears (See [Roc] for a clear explanation of their use by Huyghens around 1680 to build a mechanical model of the solar system). But Ford in 1917, followed by Artin in 1924, [For, Art] showed that the continued fraction map was related to the geodesic flow on the modular surface; this relation gives new meaning to a large number of questions, as for example the well-known Gauss measure, which is invariant by the Gauss map.

In this short paper, we explain a heuristic method to find geometric models for generalized continued fraction and their natural extension, and to deduce explicit formulas for the invariant density. We rely on the notations given in the paper by Tom Schmidt.

In Section 2, we first treat the case of the classical continued fraction, to explain the principles of the method. We then give the general technique for continued fractions given by piecewise homographies in dimension 1, and show a few examples of application; we state some open problems for that case. in Section 3, we briefly outline how this method can be extended in higher dimension, and give a few recent examples.

I would like to thank the anonymous referee for many corrections and suggestions which greatly improved the paper.

Received February 9, 2016. Revised March 28, 2016. Accepted 1 May 2016.
2010 Mathematics Subject Classification(s): 37A45, 37C40, 37E05, 11K50, 30B70
Key Words: *Continued fractions, Gauss measure, invariant measure, natural extension, geodesic flow, iterated function systems*
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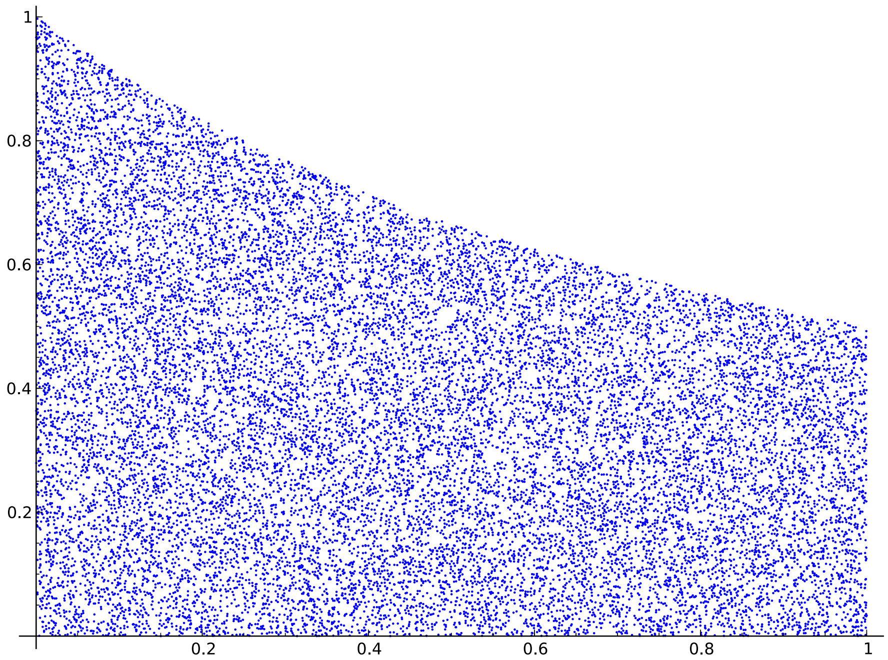

Figure 1. 20 000 points of an orbit of $\widetilde{T}$

§ 2. Continued fractions in one dimension

§ 2.1. The example of the classical continued fraction

We consider the Gauss map defined by $T(x) = \left\{\frac{1}{x}\right\}$ on the interval $[0,1]$, where $\{x\} = x - [x]$ is the fractional part of x. As is well-known, this map is related to Euclid's Algorithm : subtracting an interval of length x from an interval of length 1, and renormalizing the resulting intervals by division by x. If we interpret this operation as an induction, it is natural to define a dual stacking operation, and a short computation (see [Ar1] for details) gives a heuristic model for a natural extension as $\tilde{T}(x,y) = \big(T(x), x - x^2 y\big)$.

A simple experiment (computing the orbit under this map of a random point in $(0,1) \times \mathbb{R}$, see Fig.1) hints that there is an attractive invariant set for $\tilde{T}$ given by $K = \{(x,y) | 0 \le x \le 1, 0 \le y \le \frac{1}{1+x}\}$. Direct inspection shows that the map $\tilde{T}$ is bijective on K (except for a set of measure 0). Since the Jacobian of this map is defined almost everywhere, with value 1, the map $\tilde{T}$ preserves Lebesgue measure; projection on the first coordinate gives the density of the Gauss measure as the height $\frac{1}{1+x}$ of the set K.

There are other models for the natural extension; a well-known one, explained in the paper by Tom Schmidt, is defined on the square $[0,1] \times [0,1]$ by $(x,y) \mapsto (\frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \frac{1}{\lfloor \frac{1}{x} \rfloor + y})$; since the natural extension is unique up to measurable conjugacy, it must be conjugate to $\widetilde{T}$, and it is indeed easy to check that this map, which we will denote by $\overline{T}$, is conjugate to $\tilde{T}$ by the map ϕ given by $\phi(x,y) = (x, \frac{y}{1+xy})$. Figure 2 shows the first 50 000 points of the orbit of a random point. This orbit seems to densely fill the unit square, but in an uneven way: indeed, it is known that this map preserves the density $\frac{1}{(1+xy)2}$, with lower density near (1,1), as appears on the

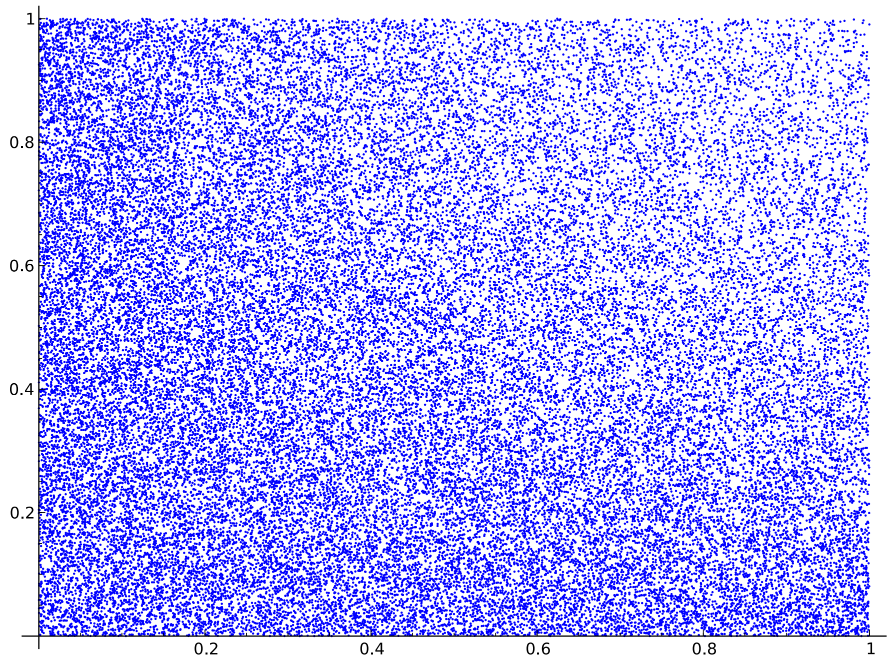

Figure 2. 50 000 points of an orbit of $\overline{T}$

figure.

As we will show below, the map $\widetilde{T}$ can be seen (up to a covering of degree 2) as a first return map of the geodesic flow on the modular surface.

The numerical experiments are very stable; even if the starting value is chosen to be exterior to K, the orbit converges exponentially fast to K, and except of course for periodic or finite orbits, all starting points seem to show the same invariant set.

The goal of this short paper is to explain how this heuristic procedure can be generalized to other maps, and why it works.

§ 2.2. Natural extension for a measure preserving map

Consider a differentiable map S on an interval (or more generally on a domain of $\mathbb{R}^n$); finding an invariant density for this map amounts to solving the Ruelle equation:

$$\phi(x) = \sum_{y;Sy=x} \frac{\phi(y)}{|J_s(y)|}$$

where J_S is the derivative, or more generally the Jacobian of S.

This solution is difficult to find in general because the point x has numerous preimages, and thus due to iteration the combinatorics become complicated. It is much easier when the map is bijective, as the equation becomes $\phi(Sy) = \frac{\phi(y)}{|J_s(y)|}$, and trivial if furthermore $J_S = 1$, that is, S preserves Lebesgue measure.

We see that in this case, when we have a mesure preserving dynamical system (X, S, μ), it is a natural idea in many contexts to look for a bijective extension, that is, a measure preserving dynamical system (X', S', μ') which is bijective and of which the initial system is a factor.

It was proved by Rohlin (see [Roh]) that there is a canonical such system, the natural extension, through which any other bijective extension factors. The construction, inspired by the passage from the one-sided shift to the two-sided shift, is very simple, and consists in considering the inverse limit, that is, the set of infinite sequences $(x_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$ such that, for all $n \in \mathbb{Z}$, $x_{n+1} = S(x_n)$; intuitively, we consider all possible backwards orbits. The shift map on this set is obviously bijective, and there is a natural invariant measure derived from μ.

This construction is canonical, but rather abstract, and does not allow to get new informations on the invariant measure. Hence, it is interesting to find geometric explicit models of this natural extension; in the case of piecewise homographic maps, this can be done in some cases by using ideas coming from consideration of the geodesic flow on hyperbolic surfaces.

§ 2.3. Heuristics for piecewise homographic continued fractions

2.3.1. Coordinates on $SL(2,\mathbb{R})$ We briefly recall here some remarks on $SL(2,\mathbb{R})$, see [ArSc1], Sec. 3 for more details.

Consider the set $SL(2,\mathbb{R})$ of matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$; it admits a unique (up to a normalization) Haar measure m. The set of matrices such that $\gamma = 0$ has measure 0, and off of this set, we can take α, γ, δ as coordinates (since $\beta = \frac{\alpha\delta-1}{\gamma}$) and a simple computation shows that Haar measure is given in these coordinates by $\frac{d\alpha\, d\gamma\, d\delta}{\gamma}$.

Remark. We should be considering the group $PSL(2,\mathbb{R})$), quotienting by the center $\{Id, -Id\}$, since the action of $-Id$ is trivial in all the applications we consider. It means that we will consider matrices up to a change of sign, so we can always suppose, of a set of measure 0, that $\gamma > 0$. We will forget this technical detail in the rest of the paper, and assume $\gamma > 0$.

The subgroup of diagonal matrices $\begin{pmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{pmatrix}$ acts on the right on $SL(2,\mathbb{R})$, giving rise to the so-called geodesic flow g_t. Using this flow, we can take other coordinates, with a more obvious geometric interpretation. An interesting possibility is to consider the subset Σ of matrices with $\gamma = 1$, hence of the form write $\begin{pmatrix}x & xy-1\\ 1 & y\end{pmatrix}$. This set intersects any orbit of the geodesic flow in exactly one point (unless $\gamma = 0$), hence almost any matrix can be written:

$$\begin{pmatrix}x & xy-1\\ 1 & y\end{pmatrix} \cdot g_t = \begin{pmatrix}xe^{t/2} & (xy-1)e^{-t/2}\\ e^{t/2} & ye^{-t/2}\end{pmatrix},$$

where t is the time for the geodesic flow to meet the surface $\Sigma \subset SL(2,\mathbb{R})$. It is readily computed that, in these coordinates, the Haar measure becomes $dX\, dY\, dt$.

2.3.2. First-return map for the geodesic flow

If we consider a discrete subgroup Γ of the group $SL(2,\mathbb{R})$, and the geodesic flow on the quotient $\Gamma\backslash SL(2,\mathbb{R})$, one can compute the first return map to Σ given by the action of a matrix $A \in \Gamma$. Starting from a matrix M in Σ, with $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, we get:

$$A.\begin{pmatrix} x & xy-1 \\ 1 & y \end{pmatrix} = \begin{pmatrix} ax+b & * \\ cx+d & y(cx+d)-c \end{pmatrix};$$

we then renormalize by the geodesic flow g_t, with $e^{-t/2} = cx+d$ so that the lower left coefficient becomes 1, and find

$$A.M.g_t = \begin{pmatrix} \frac{ax+b}{cx+d} & * \\ 1 & y(cx+d)^2 - c(cx+d) \end{pmatrix}$$

.

This gives us a formula for the first return map to the section Σ, using the (x, y) coordinates, as:

$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, y(cx+d)^2 - c(cx+d)\right)$$

2.3.3. A model for the natural extension

Suppose that we have a generalized continued fraction map S, defined on an interval I, which is piecewise homographic, that is, locally defined by an homography. We can use the formula above; if S is given on interval I_k by $S(x) = \frac{ax+b}{cx+d}$, we define a map $\widetilde{S}$ on $I_k \times \mathbb{R}$ by the previous formula.

In the case of the classical continued fraction, the matrices we consider are of the type $\begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix}$, and we recover the map $(x, y) \mapsto (\frac{1}{x} - a, yx^2 - x)$ of the previous section, up to a sign (because the matrix has determinant -1).

If we can find a compact set K, of nonzero measure, which is invariant by $\widetilde{S}$, it is then easy to prove that $\widetilde{S}$ is bijective (up to a set of measure 0) and preserves Lebesgue measure; by projection, we see that S preserve the density ϕ given by $\phi(x) = m(\{y|(x, y) \in K\})$, where m is the Lebesgue measure on $\mathbb{R}$.

§ 2.4. Experimental results

We have studied this question for several examples of piecewise homographic maps; in all cases, we see that this map leaves invariant a unique compact set which projects over the interval of definition of the algorithm.

It is often easy to determine explicitly the invariant set, to prove that it has positive measure, and then to find the explicit invariant density; in the paper by Tom Schmidt in the same issue, one can see such an example for the Rosen continued fraction; see [ArSc1] for the natural extension of the α-continued fraction, and [ArSc2] for the Rosen and the Veech algorithm; in this paper, the natural extension is used to show the relation between both algorithms, by showing that they come from first-return maps on different sections of the same flow (apart from a finite degree covering) .

We show here two examples.

2.4.1. Hei-chi Chan continued fraction In 2004, Hei-chi Chan defined in [Cha] a continued fraction related to the binary expansion of x as the map $S : x \to \frac{2^{-a(x)}}{x} - 1$, where $a(x) =$

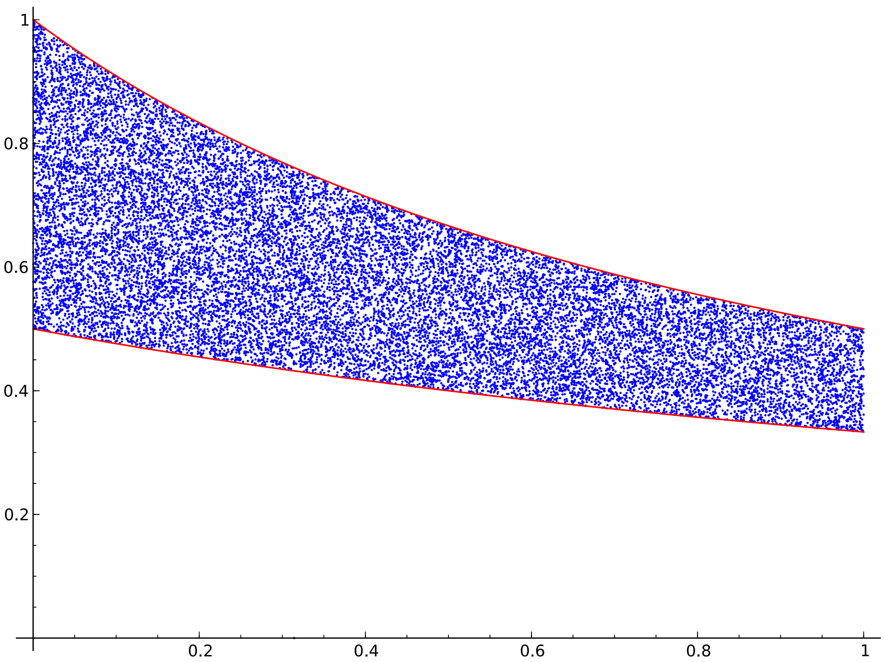

Figure 3. 20 000 points of an orbit of $\widetilde{S}$

$\{-\log_2(x)\}$. Using the formula above, and taking care of the fact that the underlying matrices have negative determinant, we consider the potential natural extension:

$$\widetilde{S}(x,y) = \left(S(x), 2^{a(x)}(x - x^2 y)\right)$$

Computing the orbit of a random point, we get Fig. 3, which appears to densely and uniformly fill the area between the curves $y = 1/(x+1)$ and $y = 1/(x+2)$; one easily checks that it is indeed a model of the natural extension, and that the map S has an invariant density given, up to a normalization, by $\frac{1}{(x+1)(x+2)}$.

In fact, this map is just an acceleration of a much simpler additive map, given by $x \mapsto 2x$ if $x < \frac{1}{2}$, and $x \mapsto \frac{1}{x} - 1$ if $x > \frac{1}{2}$. It is easy to compute the natural extension in that case also; surprisingly, it turns out to be the same as for the classical continued fraction, so that additive map has the same invariant density $\frac{1}{1+x}$ as the classical Gauss map.

2.4.2. Ralston continued fraction In 2014, David Ralston, [Ral], to solve a problem in symbolic dynamics, defined a variant of the usual continued fraction; if we denote as above by T the map $x \mapsto \{\frac{1}{x}\}$, the map g is defined by $g(x) = 1 - x$ if $x > \frac{1}{2}$, $g(x) = T^2(x)$ if the first partial quotient $\left[\frac{1}{x}\right]$ of x is even, and $g(x) = \frac{1}{1+T(x)}$ it it is odd and larger than 1 (see Fig. 4 for the graph of the map g)

As in the other cases, the explicit formula for the natural extension $\widetilde{g}$ of g is readily computed. See Fig. 5 for an orbit; we can check, as shown on the figure, that this orbit filles the region defined by $\frac{-1}{1-x} \leq y \leq \frac{1}{1+x}$, if $x < \frac{1}{2}$, and $0 \leq y \leq \frac{1}{x}$ if $x > \frac{1}{2}$. Hence there is an invariant density, given by $\frac{2}{1-x^2}$ if $x < \frac{1}{2}$, and $\frac{1}{x}$ if $x > \frac{1}{2}$. To give a complete formal proof, one has to consider the Markov partition of the natural extension $\widetilde{g}$, given explicitly by the intervals of continuity of g,

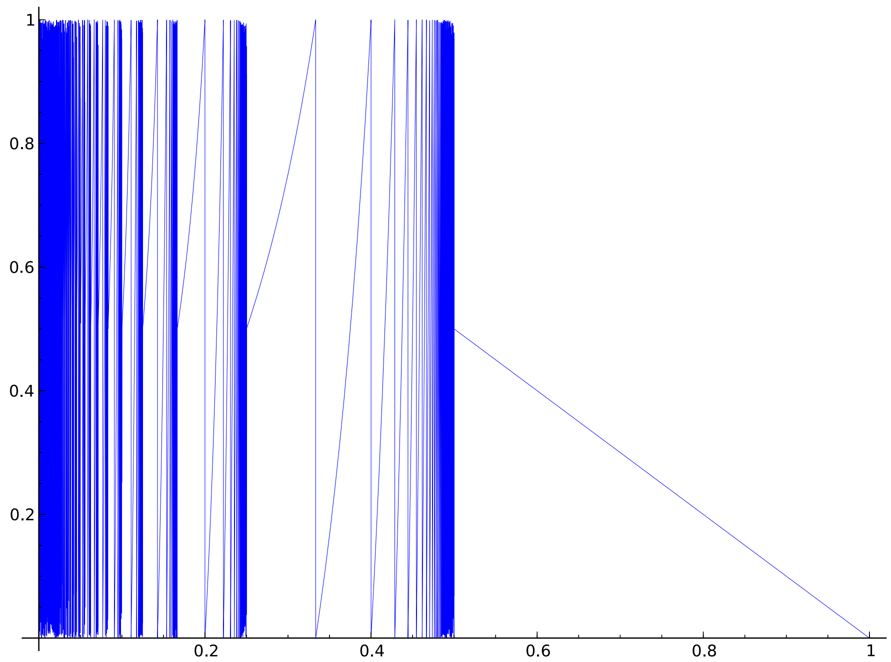

Figure 4. The graph of Ralston continued fraction g

and prove that the images of all these sets are contained in the invariant set, and disjoint up to a set of measure 0.

§ 2.5. A fixed point theorem

One can give a good explanation for the convergence of numerical experiments. Any continued fraction algorithm is interesting only if it is expanding, so that different starting points have different orbits which diverge quickly. This means that the homography has derivative larger than 1; hence the second coordinate of the partial map $y \mapsto \tilde{T}(x, y)$, which is affine, is uniformly contracting on $\mathbb{R}$ if the continued fraction is strictly expanding. In that case, a fixed point theorem (a generalization of the classical Hutchinson theorem on iterated function systems, see [Hut]) shows that there is a unique solution for a compact set projecting onto the definition interval I of T, which is invariant by $\tilde{T}$. A formal proof will be given in a paper in preparation, [ArSc3].

This explains why, for almost every starting point in $I \times \mathbb{R}$, we get in the experiments the same orbit closure (More precisely, the closures of the tails of orbit agree, even if there could be a transient part if the starting point is not in the invariant domain, as is clear in numerical experiments).

§ 2.6. Conjugate models for the natural extension

Other models can be given for the natural extension; any system of coordinates on $SL(2, \mathbb{R})$ can give such a model. For example, here is a simple and geometric way to give such a formula. We can use the well-known fact that the unit tangent bundle of the hyperbolic plane is isomorphic to $PSL(2, \mathbb{R})$, and any orbit of the geodesic flow corresponds to a geodesic of the hyperbolic

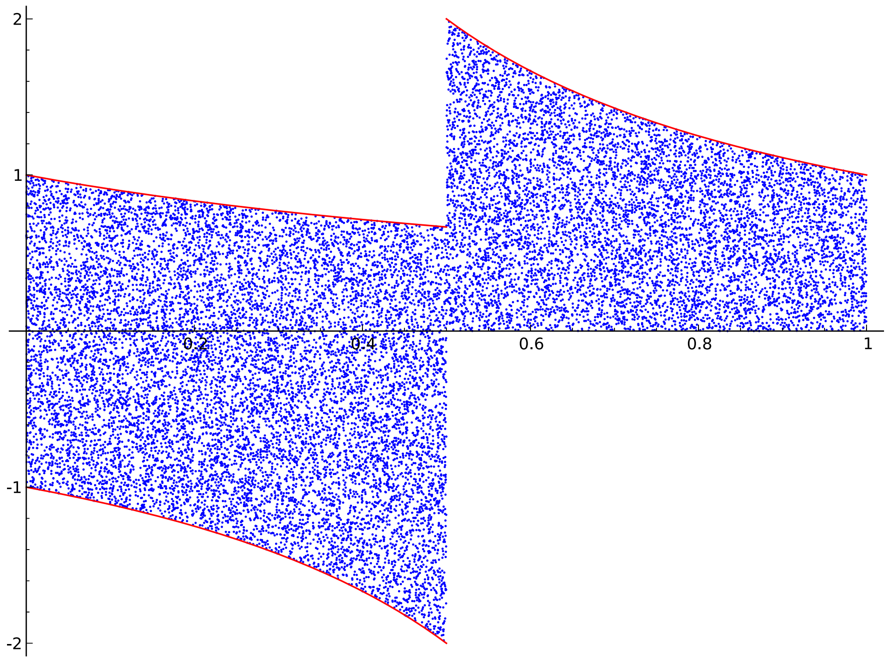

Figure 5. 20 000 points of an orbit of $\widetilde{g}$

plane. If we remove the vertical geodesics (corresponding to the set of matrices with $\gamma = 0$ seen in subsection 2.3.1), every geodesic has a distinguished point with maximal height, and horizontal tangent vector. These corresponds to matrices where the coordinates of the second row are equal, up to a sign. This allows us to give another section to the geodesic flow; the matrices are of the type:

$$\begin{pmatrix} \frac{x}{\sqrt{x-y}} & \frac{y}{\sqrt{x-y}} \\ \frac{1}{\sqrt{x-y}} & \frac{1}{\sqrt{x-y}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{x}{\sqrt{y-x}} & \frac{-y}{\sqrt{y-x}} \\ \frac{1}{\sqrt{y-x}} & \frac{-1}{\sqrt{y-x}} \end{pmatrix}$$

depending on whether the geodesic goes to the right or to the left. The parameters have a clear geometric interpretation: y is the initial point of the geodesic on the real axis, and x is the final point.

Using the geodesic flow g_t as above, we can take x, y, t as coordinates; it is easily computed that Haar measure can now be expressed as $\frac{dx\,dy\,dt}{(x-y)^2}$. The pair (x, y) gives coordinates on the hyperbolic plane: the point $\frac{x+y}{2} + i\frac{x-y}{2}$ is the top of the geodesic that joins y to $x > y$, and the measure $\frac{dx\,dy}{(x-y)^2}$ on the section appears, up to a constant, as the hyperbolic measure on the hyperbolic plane.

It is now easy to compute another possible formula for the natural extension, and a bit of work gives the trivial formula:

$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ay+b}{cy+d} \right)$$

which is just the computation of the image of the endpoints of the geodesic under the action of the matrix.

However, this formula is not convenient for our purpose, since it has no reason to be contracting on y; it is more convenient to take as coordinates on the section x and $u = -1/y$. In that case, computation gives us

$$(x, u) \mapsto \left(\frac{ax+b}{cx+d}, \frac{du-c}{a-bu}\right)$$

which is much better expressed, taking $A = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix}$, by $(x, u) \mapsto \big(A.x, A^{*-1}.u\big)$, where A^* is the transpose of A.

Some remarks are in order here. First, it is not difficult to check that this coordinate system is conjugate to the coordinate system of subsection 2.3.1 by $u = \frac{y}{1-xy}$, since u is the opposite of the ratio of the elements of the last row. Second, we recover here, in the case of the classical continued fraction, the usual extension. Third, this coordinate system has the advantage that the jacobian matrix is diagonal, and the domain has a local product structure. Last, there is an invariant density for the natural extension, related to the hyperbolic mesure, and given by the well-known formula $\frac{dx\,du}{(1+xu)^2}$.

§ 2.7. Open problems

There remain many open questions. The first one is to know whether the invariant domain has positive Lebesgue measure. Indeed, it is easy to give variants of the formula where the invariant domain has measure 0; for example, if we change the affine map on the second coordinate to a linear one, by removing the constant part, we see that the invariant compact set is just $[0, 1] \times \{0\}$, that is, a segment, with measure 0. Hence there is no *a priori* guarantee that the invariant set has positive measure.

In the case the group Γ is a co-compact, or even co-finite measure, discrete subgroup of $SL(2, \mathbb{R})$, what we get is probably a type of Bowen-Series map, and one could expect to prove that it gives a section for the geodesic flow, with positive area.

In all the examples we have shown, the set has a nice structure (simply connected, with boundaries easy to guess). This has no reason to be always the case, and indeed, for some examples, like the α-continued fraction, also known as Japanese continued fraction, for some values of α, the set has a very complicated structure, with a Cantor-like boundary. It would be interesting to find conditions which give a nice structure; for example, conditions which imply that the "slices" of the natural extension, that is, the sets $K_x = \{y|(x, y) \in K\}$ for a fixed x are intervals, or at least finite unions of disjoint intervals.

It is possible to get a (non-compact) invariant set in the case of a weakly contracting map (like the additive continued fraction) with an indifferent fixed point. In that case, it must be possible to extend the theorem by using an acceleration of the algorithm. For example, consider the additive Farey map, given by $x \mapsto \frac{x}{1-x}$ if $x < \frac{1}{2}$ and $x \mapsto 2 - \frac{1}{x}$ if $x > \frac{1}{2}$, which has two indifferent fixed points at the extremities of the interval (see Fig. 6). One can compute the natural extension; the orbit of a point is shown in Fig. 7. There is an invariant closed set, limited by $y = \frac{1}{x}$ and $y = \frac{1}{x-1}$, which is not compact, and an invariant density given by $\frac{1}{x-x^2}$, which has infinite measure. This is easy to prove, hence the technique works in at least some

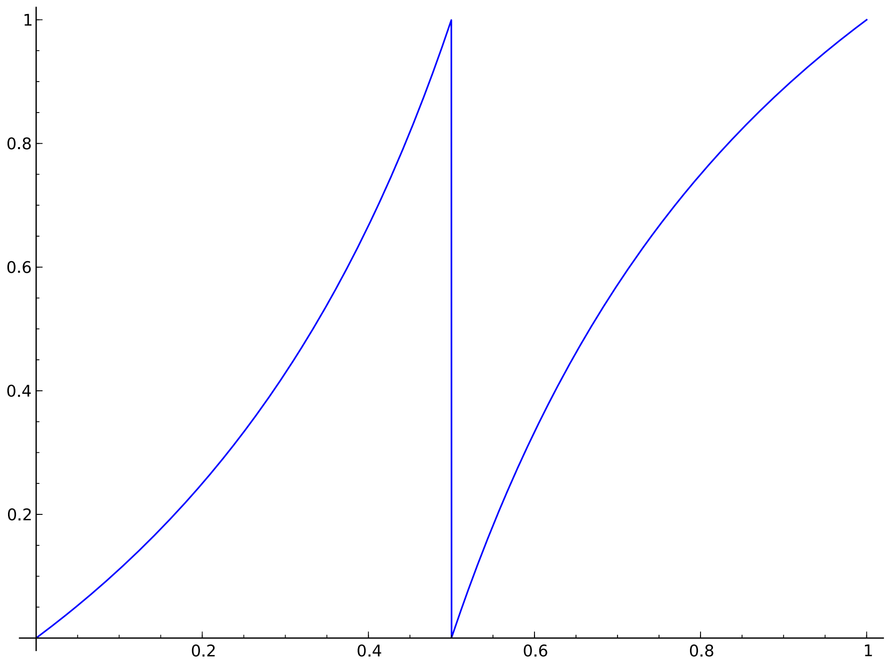

Figure 6. The graph of the Farey continued fraction g

non-contracting cases.

It is also possible to find the invariant domain for matrices with determinant -1; there is a general construction by taking an orientation cover of degree 2.

Finally, and more surprising, it is also possible to work with matrices like $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, related to some algorithms of purely arithmetic nature; in that case, the underlying group Γ is very unclear, as is the interpretation of the algorithm. This is for example the case of Hei-Chi Chan algorithm.

Remark that, as we said above, there are several possible coordinate systems, which give conjugate results.

§ 3. Continued fractions in higher dimension

This method is not restricted to dimension 1, although it is in this framework that experiments are easiest (It is not easy to visualize results in dimension 4).

In higher dimensions, the basic idea is to use the formula $(x, u) \mapsto \big(A.x, A^{*-1}.u\big)$ of Sec. 2.6. For that, it is important to understand the linear structure which underlies most continued fraction algorithms. In that way, we can make some algorithms appear as first-return map of a flow to a section, and this can help to understand the properties of the algorithm, as the Gauss measure. This was already the idea in [ArNo].

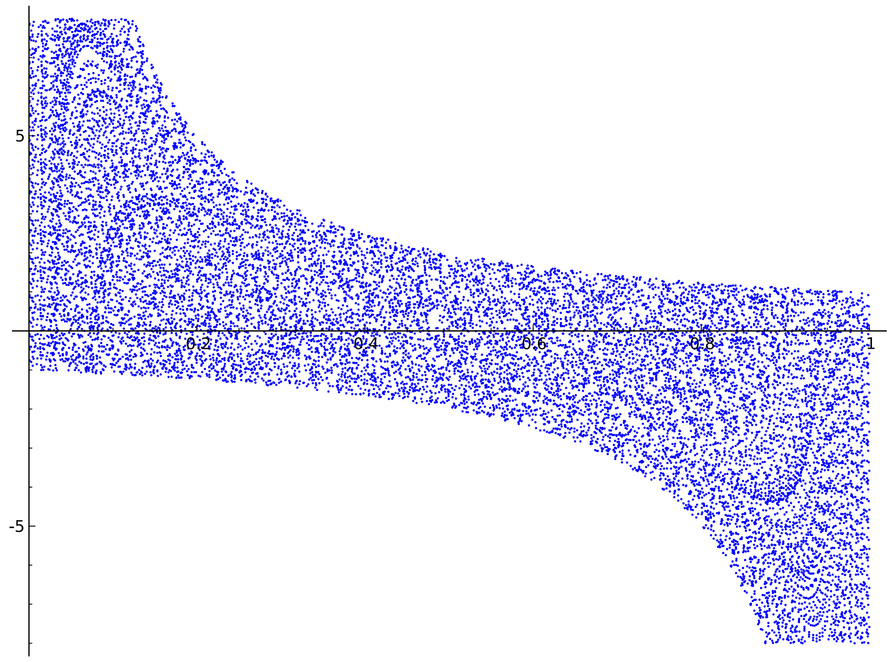

Figure 7. 50 000 points of an orbit of the natural extension of Farey continued fraction

§ 3.1. Piecewise projective continued fraction and associated linear models

Most known algorithms, in dimension $d = 1$ or higher, are given by piecewise projective maps (homographies in dimension 1); there is by definition a linear version of the algorithm, which works in dimension $d + 1$.

This is however very often hidden in the presentation of the algorithm, since the linear presentation increases the dimension by 1, so it is often avoided for supposed reasons of simplicity. For example, the usual presentation of Brun's multiplicative algorithm is as a map on the triangle $\{(x, y)|0 \leq x \leq y \leq 1\}$, defined as $(x, y) \mapsto ord(\frac{x}{y}, \{\frac{1}{y}\})$, where $ord(x, y) = (\min(x, y), \max(x, y))$. This map is given by an homography, but the underlying logic of the algorithm is not immediately clear. However, if we consider it as a projective map, it is easy to understand the corresponding linear map. If we denote by $a = [\frac{1}{y}]$ the integer part of $\frac{1}{y}$, add a last component 1 to the vector (x, y), associate to the vector $(x, y, 1)$ the reordered vector $(x, y, 1 - ay)$, and renormalize so that the largest coordinate, y, becomes 1, we recover Brun's algorithm, which appears as a projective version of a three-dimensional piecewise linear map:

$$(x, y, z) \mapsto ord\left(x, y, z - \left[\frac{z}{y}\right] y\right)$$

Moreover, this map appears naturally as an acceleration of an additive algorithm, whose linear version is simply

$$(x, y, z) \mapsto ord\,(x, y, z - y)$$

This map can be expressed very clearly in words: given a positive vector of dimension 3, subtract the second largest coordinate from the largest, one time for the additive algorithm, as many times as possible for the multiplicative algorithm, and reorder if necessary; in this presentation, it appears as a natural generalization of Euclid's algorithm. It is immediate to check that renormalisation at each step by homothety, so that $\max(x, y, z) = 1$, gives the first formulation of Brun's algorithm. It is possible to give an unsorted version of the algorithm which is even simpler; for more details, see [ArLa].

From now on, we will consider a continued fraction algorithm as a piecewise linear map A, defined on a finite or countable collection of cones $C = \cup_i C_i \in \mathbb{R}^{d+1}$ by matrices A_i; the n-th power of this linear map associated with T^n is given locally by $A_{i_n} \dots A_2.A_1$. We are particularly interested by the projective map T naturally associated with A; this projective map can be expressed in coordinates by considering an affine hyperplane H transverse to the cones: this is the condition $z = 1$ seen above. In this setting, the projective map one usually consider is the map induced on $H \cap C$, seen as representative of a subset of the projective space, by the piecewise linear map A: given an element $v \in H \cap C$, apply the linear map A, then the unique homothety which returns $A.v$ to $H \cap C$.

We want in particular to find, if it exists, the Gauss measure for the projective map T; our method will be to find a natural extension for this map.

§ 3.2. A general model for the natural extension

3.2.1. Linear model

To find a linear model for the natural extension, we consider a matrix acting in dimension $2(d+1)$, adding $d+1$ dual parameters to the initial $d+1$ linear coordinates; we need to find a matrix which depends covariantly on A_i. A very simple idea is to consider the matrix $\widetilde{A}_i := \begin{pmatrix} A_i & 0 \\ 0 & A_i^{*-1} \end{pmatrix}$, which satisfies some interesting conditions.

First of all, it is covariant : $\widetilde{AB} = \widetilde{A}\widetilde{B}$. Second, it has determinant 1, hence it preserves Lebesgue measure. Third, it preserves the quadratic form $\sum_{i=0}^{d} x_i y_i$ for the natural coordinates (x, y) on $\mathbb{R}^{2(d+1)}$; hence, we can restrict to the subspace $\sum x_i y_i = 1$. Fourth, it commutes with the flow $\phi_t(x, y) = (e^t x, e^{-t} y)$, which is the hamiltonian flow related to the symplectic form $\sum dx_i \wedge dy_i$ and the hamiltonian $\sum x_i y_i$.

Under these conditions, since the flow and the map preserve the hamiltonian, we can define on the subspace $\Omega = \{(x, y) \in \mathbb{R}^{2d+2} | \sum x_i y_i = 1\}$ a map $\widetilde{A}$, given by the matrix above, and a linear flow ϕ_t which commutes with this map. We can consider in Ω a section $\Sigma = \{(x, y) \in \Omega | x \in H\}$ which is transverse to the flow ϕ_t.

Under these conditions, it makes sense to consider an element $(x, y) \in \Sigma$, apply the map $\widetilde{A}$, and then return to Σ using the flow ϕ_t; we obtain in this way a map $\widetilde{T} : \Sigma \to \Sigma$. There is a natural projection $\pi : \Sigma \to H \cap C$. Since the effect of $\widetilde{A}$ on the x coordinates is that of A, and the flow ϕ_t act on x by homothety, it is clear that $\widetilde{T}$ is an extension of T under the projection π.

3.2.2. The problem

The problem is of course to find an invariant domain in Σ of bounded measure on which the natural extension can be defined: the map $\widetilde{T}$ defined on Σ has no particular reason to be bijective. We would ideally like to find a compact invariant set K such that $\pi(K) = H \cap C$.

If the continued fraction is uniformly expanding, one can show that the map $\widetilde{T}$ is uniformly strictly contracting on the second coordinate, because the projective action of the matrices A_i^{*-1} is strictly contracting. In that case, the results of the previous section can be generalized, and one can find a compact invariant domain, under reasonable conditions, by a generalization of the theorem of Hutchinson [Hut]. We will prove such a statement in [ArSc3].

As in the previous section, the main problem is to prove that this invariant domain has strictly positive measure, at least in the basic case (strictly expanding on a finite number of pieces). If this can be done, since there is a natural invariant density derived from Lebesgue measure, it follows that $\widetilde{T}$ is bijective up to a set of measure 0, hence $(\widetilde{T}, K)$ is a model for the natural extension of T.

One interesting thing is that, when this works, the natural extension appears as a first-return map to a section of a flow; it would be most interesting to identify the underlying manifold, as has been done in some (but not all) cases in dimension 1. For 2-dimensional continued fraction, the natural extension should of dimension 4, and the flow on dimension 5; for example, the geodesic flow on the unit tangent bundle of a 3-dimensional manifold would have the good dimension.

Remark. A closely related result is the natural extension for the Rauzy induction on interval exchanges, which turned out to be related to the first return to some section of the Teichmller flow for surfaces. The literature on the subject, starting from the seminal papers of Rauzy, Veech and Masur 30 years ago, is huge; see [Rau, Ve1, Ve2, Mas] for the original papers, and [Yoc] for a more recent survey.

§ 3.3. Experimental results

In several cases (for example, Brun and Selmer algorithms, see [ArNo]), this method allows to find an explicit domain for the natural extension, and an explicit form for their invariant density.

Some other cases have been studied recently with Sébastien Labbé, for example the Reverse flow, see [ArLa]. However, in many cases (Jacobi-Perron, for example) the exploratory calculations seem to produce fractals sets, which are probably measure 0; see the same paper for examples of apparently fractal invariant sets for several algorithms.

When the experiments allow us to find a domain which can be explicitly described, it is in general easy to prove that it is invariant, and of positive measure. But we know at the moment no *a priori* condition that ensure that the measure of the invariant set will not be 0.

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