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Subball complexity and Sturmian colorings of regular trees

By

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Abstract

The subball complexity of colorings of regular trees is a generalized of the subword complexity or factor complexity of infinite words. The Sturmian word which exhibits the smallest subword complexity among non-eventually periodic word can be obtained in geometric way using the irrational circle rotation or the projection of a line of irrational slope. We survey the Sturmian coloring or trees and the subball complexity of colorings of a tree associated to isometries of the hyperbolic plane with a tessellation of the hyperbolic plane.

§ 1. Introduction

For an infinite sequence $\mathbf{u} = u_1 u_2 \ldots$, the subword complexity

$$p_{\mathbf{u}}(n) = \#\{u_{j+1}u_{j+2}\ldots u_{j+n} : j \geq 0\}$$

is defined as the number of different subwords of length n in $\mathbf{u}$. Hedlund and Morse showed that $p_{\mathbf{u}}(n)$ is bounded if and only if $\mathbf{u}$ is eventually periodic [6]. A sequence $\mathbf{u}$ is called Sturmian if $p_{\mathbf{u}}(n) = n + 1$. Refer [11] and [5] for the references on Sturmian words.

There is a well-known family of sequences coming from rotations of circle as follows. Consider the tiling of the real line $\mathbb{R}$ by unit length intervals $\{[n, n+1) : n \in \mathbb{Z}\}$ and a map $t \mapsto at + b$ from $\mathbb{R}$ to itself. There exists an integer j such that each interval $[n, n+1)$ is partitioned into j or $j+1$ subintervals of the form $\{[an+b, a(n+1)+b) : n \in \mathbb{Z}\} \cap \{[n, n+1) : n \in \mathbb{Z}\}$. Consider the sequence $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ with $u_n \in \mathcal{A} = \{j, j+1\}$, which is given by the number of such subintervals of $[n, n+1)$. It is well known that this two-sided sequence u_n is periodic if only if a is rational [6].

The subword complexity was generalized on the regular tree in [7]. Let T be a locally finite tree, VT its vertex set and ET the set of oriented edges of T. Let $\mathcal{A}$ be a countable set which

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will be called the alphabet. By a coloring of the tree T, we mean a vertex coloring, i.e. any map $\phi : VT \to \mathcal{A}$. Let $G = \mathrm{Aut}(T)$ be the automorphism group of T. A periodic coloring is a coloring which is Γ-invariant for some cocompact subgroup $\Gamma \subset \mathrm{Aut}(T)$. In [8], periodic colorings of regular trees induced from some tessellations of the hyperbolic plane were discussed.

We define subball complexity $b_\phi(n)$ of a coloring ϕ as the number of colored n-balls in the tree colored by ϕ. It is shown in [7] that a coloring ϕ on a regular tree is periodic if and only if its subball complexity $b_\phi(n)$ is bounded. Therefore, the smallest possible subball compelxity function for non-periodic coloring ϕ is $b_\phi(n) = n + 2$ since the 0-ball is colored by letters. Sturmian colorings are defined as colorings with minimal unbounded subball complexity, i.e. with $b_\phi(n) = n + 2$. For the tree colorings of various subball complexity including intermediate growth complexity function, refer [10].

There is a well-known family of sequences coming from rotations of circle as follows. Consider the tiling of the real line $\mathbb{R}$ by unit length intervals $\{[n, n+1) : n \in \mathbb{Z}\}$ and a map $t \mapsto at + b$ from $\mathbb{R}$ to itself. There exists an integer j such that each interval $[n, n+1)$ is partitioned into j or $j+1$ subintervals of the form $\{[an+b, a(n+1)+b) : n \in \mathbb{Z}\} \cap \{[n, n+1) : n \in \mathbb{Z}\}$. Consider the sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathcal{A} = \{j, j+1\}$, which is given by the number of such subintervals of $[n, n+1)$. It is well known that this two-sided sequence u_n is periodic if only if a is rational [6].

In a similar way we have a coloring on a regular tree. A k-regular tree is a dual graph is a hyperbolic tessellation generated by an ideal k-polygon. We associate a coloring ϕ_g of a k-regular tree ($k \geq 3$) for any isometry g of the hyperbolic plane, given a specific hyperbolic tessellation $\mathcal{D}$ generated by a discrete subgroup Γ of the group of isometries on the hyperbolic plane $\mathbb{H}^2$. Suppose that each vertex of elements of $\mathcal{D}$ lies on the boundary of the hyperbolic plane so that the dual graph of $\mathcal{D}$ is a tree.

Let Γ_r be a group generated by the reflections in the edges of a generalized ideal polygon. In [8], it was shown that an isometry $g \in \mathrm{Isom}(\mathbb{H}^2)$ is a commensurator of Γ_r if and only if its associated coloring ϕ_g is periodic. An element $g \in \mathrm{Isom}(\mathbb{H}^2)$ is called a *commensurator of* Γ if and only if $g\Gamma g^{-1} \cap \Gamma$ is a subgroup of Γ and of $g\Gamma g^{-1}$ of finite index. Commensurator subgroup plays an important role in the study of rigidity of locally symmetric spaces and more generally in geometric group theory([13], [9], [1]). This is a result analogous to the rotation case in the sense that the group of commensurators of $\mathrm{SL}_2(\mathbb{Z})$ is a group containing $\mathrm{SL}_2(\mathbb{Q})$ with finite index [14]. If $g \in \mathrm{Isom}(\mathbb{H}^2)$ is not a commensurator of Γ, then it corresponds the irrational rotation which produces Sturmian word. However, it is not clear how to obtain Sturmian tree coloring.

§ 2. Periodic tree colorings

Let T be a k-regular tree with the vertex set VT and oriented edge set ET. We assume that ET contains $\bar{e}$, which is e with reversed orientation, if it contains e. We will denote by $[x, y]$ the edge from vertex x to vertex y. Let G be the group of all automorphisms of T, which is a locally compact topological group with compact-open topology. Consider the length metric d on T with edge length all equal to 1. An *n-ball around* x is defined by $B_n(x) = \{y \in VT \cup ET : d(x, y) \leq n\}$.

Let us fix a coloring $\phi : VT \to \mathcal{A}$. We say that two balls $B_n(x)$ and $B_n(y)$ are *equivalent* if there exists a color-preserving isomorphism from $B_n(x)$ to $B_n(y)$. We will call such an equivalence class *a colored n-ball* and denote it by $[B_n(x)]$. A colored n-ball is called *admissible* if it appears in T colored by ϕ. Let $\mathcal{B}_\phi(n)$ be the set of admissible colored n-balls. The *subball complexity* $b_\phi(n)$ *of* ϕ is defined by $b_\phi(n) = |\mathcal{B}_\phi(n)|$.

Definition 2.1. A coloring $\phi : VT \to \mathcal{A}$ is *periodic* if there exists a subgroup $\Gamma \subset G$ such that $\Gamma\backslash T$ is a finite graph and ϕ is Γ-invariant, i.e.

$$\phi(\gamma x) = \phi(x), \text{ for all } x \in VT \text{ and } \gamma \in \Gamma.$$

Let Γ be a group acting on a k-regular tree T by automorphisms. If Γ acts without torsion, then the quotient $\Gamma\backslash T$ is a k-regular graph, but in general, the quotient has a structure of a graph of groups, a graph version of orbifold quotient.

A *graph of groups* $(X, G_\bullet)$ is a graph X equipped with a group G_x for each $x \in VX \cup EX$ and an injective homomorphism $j_e : G_e \to G_{\partial_0(e)}$ from the edge group to the group of the initial vertex $\partial_0(e)$ of the edge e, for each oriented edge $e \in EX$. The quotient graph of groups associated to a group Γ acting on T is denoted by $\Gamma\backslash\backslash T$.

The *edge-indexed graph* (X, i) *of a graph of groups* $(X, G_\bullet)$ is an edge-indexed graph whose graph is the underlying graph X and for which $i(e)$ is the index of G_e in $G_{\partial_0(e)}$. If (X, i) is the edge-indexed group of $(X, G_\bullet)$, then $(X, G_\bullet)$ is called *a grouping of* (X, i). The universal cover of a graph of groups is isomorphic to the universal cover of its edge-indexed graph [2].

If a coloring ϕ is Γ-invariant, then ϕ is determined by a coloring on the edge-indexed graph of $\Gamma\backslash\backslash T$. We conclude that ϕ is periodic if and only if it is a lift of a coloring on an edge-indexed finite graph. Conversely, if ϕ is determined on an edge-indexed graph X, then there exists $\Gamma \leq G$, not necessarily a discrete subgroup, for which ϕ is Γ-invariant. In particular, if X is an edge-indexed finite graph, then ϕ is periodic. (For details on graph of groups and the edge-indexed graph of a graph of groups, see [3], [4] and [15].)

We have an analogue theorem of the classical theorem of Hedlund and Morse [6].

Theorem 2.2 ([7]). *Let $\phi : VT \to \mathcal{A}$ be a coloring. The following are equivalent.*
(i) The coloring ϕ is periodic.
(ii) The subball complexity of ϕ satisfies $b_\phi(n+1) = b_\phi(n)$ for some $n > 0$.
(iii) The subball complexity $b_\phi(n)$ is bounded.

For example, let $\Gamma = \langle a_1, \cdots, a_k : a_i^2 = 1\rangle$ and T be its Cayley graph. Then to any element g of $\mathrm{Aut}(T)$ is associated a coloring ϕ_g as follows. For each $t \in VT$, there exists a unique $\gamma_t \in \Gamma$ sending the identity to t. Let $\phi_g(t)$ be the map $\gamma_{g(t)}^{-1} \circ g \circ \gamma_t$ which sends the identity to itself, restricted to the 1-sphere of the identity. Therefore, we may consider $\phi_g : VT \to S_k$, where S_k is the symmetric group on the set of vertices of 1-sphere of identity, as a coloring with $\mathcal{A} = S_k$. The coloring ϕ_g is periodic if and only if g is an element of the commensurator of Γ [12]. More generally, if T is a locally finite tree, $G = \mathrm{Aut}(T)$ is its automorphism group, and Γ is a cocompact discrete subgroup of G, then to any automorphism is associated a coloring $\phi_g : T \to Y = \Gamma\backslash T$, which is a covering map. An automorphism g is in the commensurator group of Γ if and only if its associated coloring ϕ_g is periodic [1].

We consider geometric example of periodic coloring. Let $\mathcal{D}$ be the tessellation of the hyperbolic plane (upper-half plane) $\mathbb{H}^2$ given by the group $\Gamma_r = \langle z \mapsto -\bar{z}, z \mapsto 2 - \bar{z}\rangle$ generated by the reflections about the lines $x = 0$ and $x = 1$. Then Γ_r is isomorphic to the infinite dihedral group and its dual graph T is a 2-regular tree. The element of $\mathcal{D}$ is of the form $\{z \in \mathbb{C} : n \le \mathrm{Re}(z) < n+1\}$ for $n \in \mathbb{Z}\}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{R})$, which sends $z \in \mathbb{H}^2$ to $\frac{az+b}{cz+d}$. Then g is a commensurator of Γ_r if and only if $c = 0$ and a/d is rational.

For each vertex $x \in VT$, denote by $D_x \in \mathcal{D}$ the element of $\mathcal{D}$ dual to x. Let $\phi_g^{\#}$ be the coloring given by

$$\phi_g^{\#}(x) = \#\{gD : gD \cap D_x \neq \emptyset, D \in \mathcal{D}\}.$$

Then $\phi_g^{\#}$ is periodic if a/d is rational, $c = 0$ and Sturmian if a/d is irrational, $c = 0$ (e.g. [6], [5, Chapter 6]).

Let us generalize the above construction. Let us fix an (generalized) ideal polygon D in the hyperbolic plane $\mathbb{H}^2$, by which we mean a polygon in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ such that all vertices are on the boundary $\partial\mathbb{H}^2$. Consider the group Γ_r generated by the reflections in the edges of D, which is a discrete subgroup of finite covolume in the isometry group of $\mathbb{H}^2$. Let T be the dual graph of the tessellation $\mathcal{D}$, which is a tree since D is an ideal polygon. The tree T is the Cayley graph of the group Γ_r. For example, if $\mathcal{D}$ is the Farey tessellation, which is the tessellation by isometric copies of the ideal triangle of vertices $0, 1$, and ∞, then Γ is commensurable to $\mathrm{PSL}_2(\mathbb{Z})$ as $\Gamma \cap \mathrm{PSL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of index six. Note that $\Gamma \cap \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{PSL}_2(\mathbb{Z})/\{I, S, S^2, T, ST, S^2T\}$, where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Recall that Farey tessellation is the orbit of the geodesic triangle of vertices $0, 1, \infty$ under $\mathrm{SL}_2(\mathbb{Z})$.

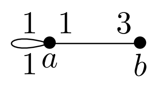
The *coloring* ϕ_g *associated to* g is the map $\phi_g : VT \to \mathcal{A}$ sending x to the isometry equivalent class of the partition element of D_x by $\mathcal{D} \vee g\mathcal{D} = \{D \cap gD' : D, D' \in \mathcal{D}\}$. Note that the coloring $\phi_g^{\#}$on a 2-regular tree is periodic if ϕ_g is periodic.

Let $\Gamma_{\mathcal{D}} \subset \mathrm{Isom}(\mathbb{H}^2)$ be the set of isometries leaving $\mathcal{D}$ invariant. Since every $D \in \mathcal{D}$ is a generalized ideal polygon with finitely many sides, Γ_r is a finite index subgroup of $\Gamma_{\mathcal{D}}$. Thus $g \in \mathrm{Isom}(\mathbb{H}^2)$ is a commensurator of Γ_r if and only if g is a commensurator of $\Gamma_{\mathcal{D}}$.

Theorem 2.3 ([8]). *Let Γ_r be a group generated by the reflections in the edges of a generalized ideal polygon. An isometry $g \in \mathrm{Isom}(\mathbb{H}^2)$ is a commensurator of Γ_r if and only if its associated coloring ϕ_g is periodic.*

Example 2.4 ([8]). Consider the Farey tessellation $\mathcal{D}$ of the hyperbolic plane, which is the tessellation with D the ideal triangle of vertices $\infty, 0, 1$. Then $\Gamma_r = \langle -z, \frac{z}{2z-1}, 2-z\rangle$ and the dual graph of Γ_r is a 3-regular tree T. For each hyperbolic element $g_1 : z \mapsto 3z$ and a parabolic element $g_2 : z \mapsto z + \frac{1}{2}$, the associated colorings ϕ_{g_1} and ϕ_{g_2} are both periodic. The periodic coloring of the edge-indexed graph of a graph of groups $\Gamma \backslash\backslash T$ for $z \mapsto 3z$ and $z \mapsto z + \frac{1}{2}$ are as follows:

$g_1 : z \mapsto 3z,$



Here, vertices a and b represent the ideal triangle partitioned as

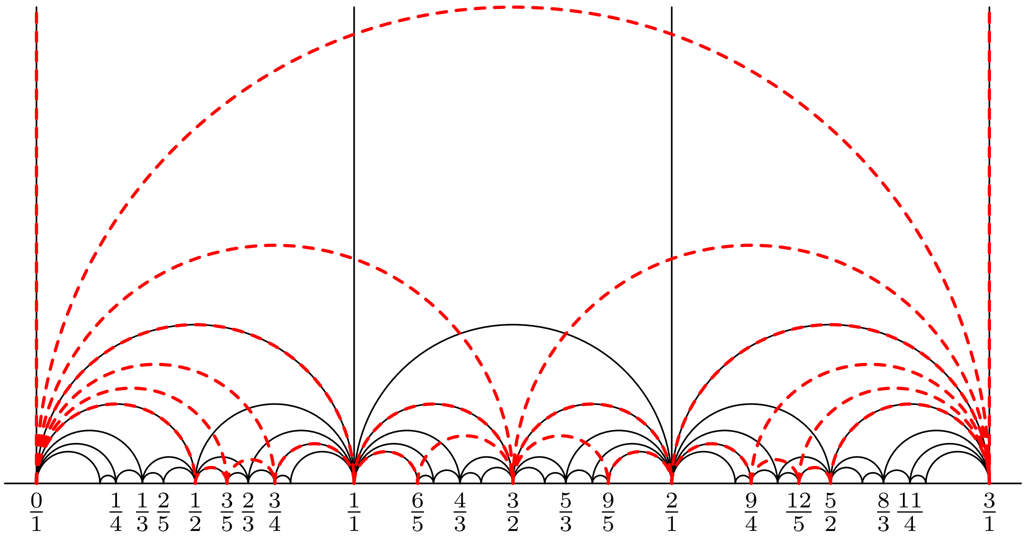
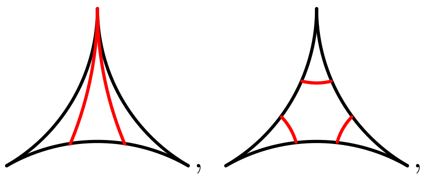


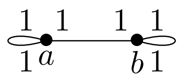
Figure 1: $z \mapsto 3z$



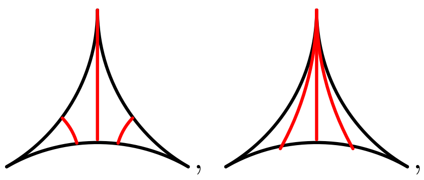
respectively. See Figure 1.

On the other hand,

$$g_2 : z \mapsto z + \frac{1}{2},$$



In this graph, vertices a and b represent the ideal triangle partitioned as



respectively. See Figure 2.

§ 3. Eventually periodic colorings

Let us fix a coloring ϕ. Let K be a finite subset of T. A coloring on a subtree U is said to have a *periodic extension to* T if there exists a periodic coloring $\bar{\phi}$ on T such that $\bar{\phi}|_U = \phi$.

Definition 3.1. A coloring $\phi : VT \to \mathcal{A}$ is called *eventually periodic* if there exists a subtree K of finite number of vertices such that $T - K = \bigcup T_i$ is a finite union of subtrees T_i such that ϕ on each T_i has a periodic extension ϕ_i to T.

For any given coloring ϕ, let Γ_ϕ be the group of automorphisms of T preserving ϕ. The vertex set of the quotient graph $X_\phi = \Gamma_\phi \backslash T$ is the set of classes of vertices in T (recall that x, y

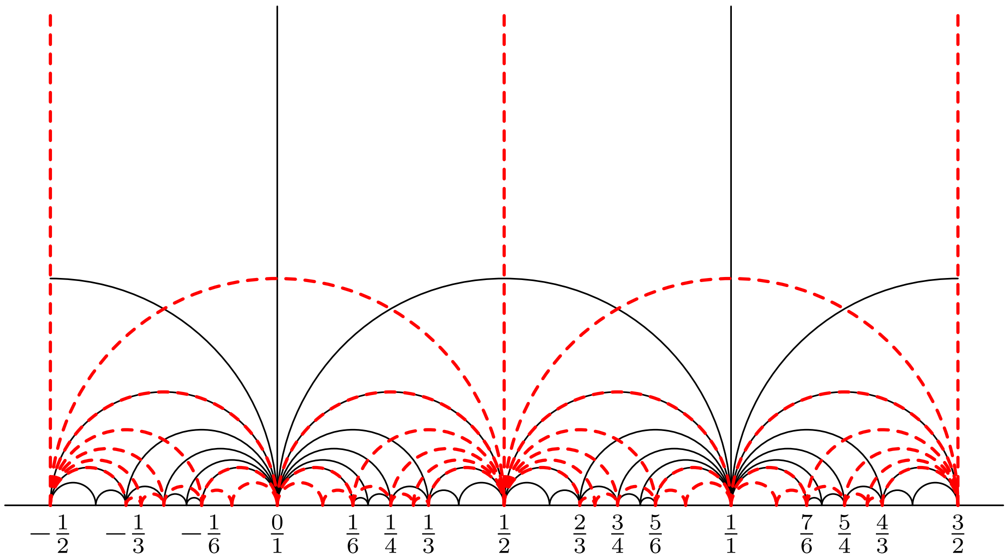


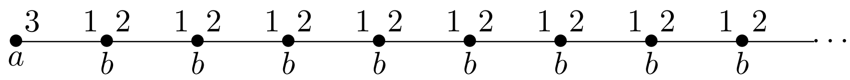
Figure 2: $z \mapsto z + \frac{1}{2}$

are in the same class if and only if $x = \gamma y$ for some $\gamma \in \Gamma_\phi$). By definition, ϕ factors through X_ϕ, i.e. there exists a coloring ϕ_{X_ϕ} on X_ϕ such that

$$\phi = \phi_{X_\phi} \circ \pi,$$

where $\pi : VT \to VX_\phi = \Gamma_\phi \backslash VT$ is the natural projection.

Example 3.2. Consider a coloring of an edge-indexed graph given as



The universal covering tree T has all vertices colored by b except one vertex which is colored by a. Admissible colored balls are the ones with vertices all colored by b except at most one vertex colored by a. These colored n-balls are determined (up to automorphisms of colored balls) by the distance $d = 0, 1, \ldots, n$ of the a vertex from the center of the ball. Therefore, $b(n) = n + 2$.

Now consider an element of $\mathrm{Isom}(\mathbb{H}^2)$ which is not a commensurator of Γ_r. Then the associated coloring ϕ_g is not periodic and has infinite alphabet by Theorem 2.3. Thus, we associate coloring from g in a different way.

Example 3.3 ([8])**.** Let $\mathcal{D}$ be the tessellation of $\mathbb{H}^2$ with D a generalized ideal polygon whose edges are two geodesics $(-2, 2), (-1, 1)$ and two boundary edges $[-2, -1], [1, 2]$. The dual graph is a 2-regular tree T and we can naturally denote the element of VT dual to D_n by $n \in \mathbb{Z}$. In this case, the commensurator of Γ_r is of the form $z \mapsto az$ and $z \mapsto \frac{a}{z}$ for $a \in \mathbb{R}$. If g is considered as a map on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, then $g \in \mathrm{Comm}(\Gamma_r)$ if and only if $g(\{0, \infty\}) = \{0, \infty\}$.

Let $g \notin \mathrm{Comm}(\Gamma_r)$ and ϕ_g^0 be a coloring given by

$$\phi_g^0(m) = \begin{cases} a, & \text{if there is } D \in \mathcal{D} \text{ such that } gD \subset D_m, \\ b, & \text{otherwise.} \end{cases}$$

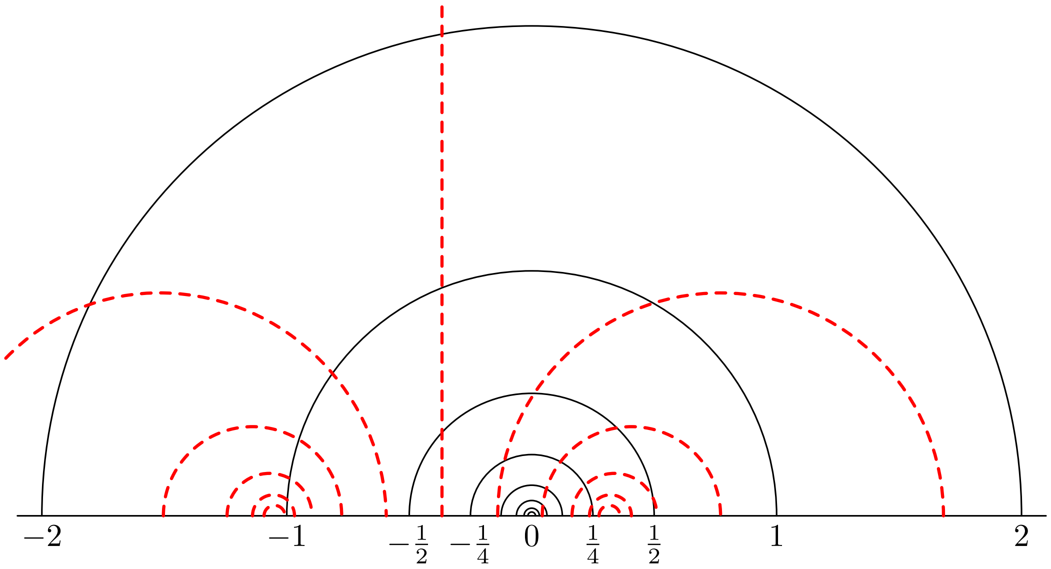
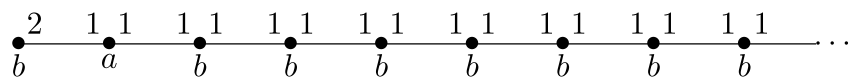


Figure 3: $z \mapsto \frac{z-\sqrt{11}}{\sqrt{10}(z+1)}$

All vertices except one or two are colored by b and remaining one or two vertices whose dual generalized ideal polygon contains $g(0)$ or $g(\infty)$ in its interior are colored by a. Hence, by omitting one or two vertices, one obtains a periodic coloring. Thus, ϕ_g^0 is an eventually periodic coloring.

In Figure 3, an example of $g : z \mapsto \frac{z-\sqrt{11}}{\sqrt{10}(z+1)}$ is presented. In this case, there are exactly two vertices colored by a, i.e. $\phi_g^0(m) = a$ for $m = -2, 0$ and $\phi_g^0(m) = b$ otherwise.

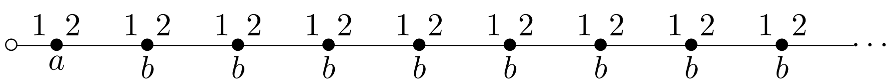


Now consider the Farey tessellation $\mathcal{D}$ and the corresponding group Γ. The dual graph of Γ is a 3-regular tree T. Let us provide two examples of colorings given by non-commensurable elements of Γ in $\mathrm{Isom}(\mathbb{H}^2)$.

Example 3.4 ([8])**.** Let $g : z \mapsto \alpha z$ with irrational α. Then $g \notin \mathrm{Comm}(\Gamma_r)$. Let ϕ_g^1 be a coloring given by

$$\phi_g^1(x) = \begin{cases} a, & \text{if } \exists D \in \mathcal{D} \text{ such that } D_x \cap gD \text{ contains a geodesic line,} \\ b, & \text{otherwise} \end{cases}$$

for $x \in VT$ and $D_x \in \mathcal{D}$ corresponding to x. A geodesic line is contained in $D_x \cap gD$ if only if the two ideal triangles D_x and gD have two common vertices. Since the only possible rational vertices of gD are 0 and ∞, $\phi_g^1(x) = a$ if and only if D_x corresponds to ideal triangle of vertices $(0, 1, \infty)$ or $(-1, 0, \infty)$. (See Figure 4) Therefore, ϕ_g^1 is a eventually periodic coloring and the coloring of the edge-indexed graph of a graph of groups is as follows:



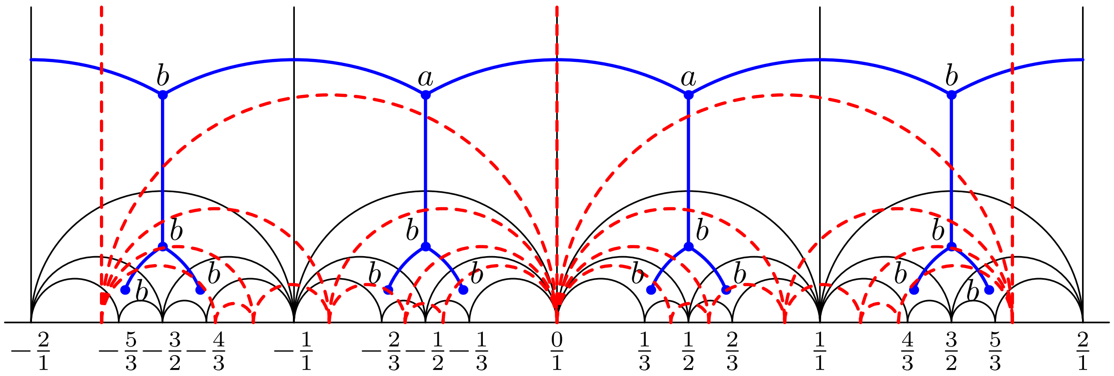


Figure 4: $z \mapsto \sqrt{3}z$

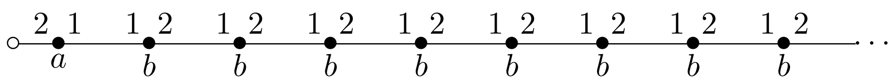
Note that we allow Γ to act with inversions (i.e. with automorphisms fixing an edge and exchanging initial and terminal vertices). The resulting graph of groups will be the usual graph of groups (for groups acting without inversions) of the first barycentric subdivision T' of T. The solid vertices in the next figure are the vertices of T, whereas the empty vertex at the left end of the ray is a vertex in $T' - T$, which was added by taking the barycentric subdivision. We omit other empty vertices if the indices of edges around it are all 1.

§ 4. Sturmian colorings

Sturmian colorings are the colorings with smallest subball complexity among all non-periodic colorings: since the coloring ϕ is on two letters, $b_\phi(0) = 2$, thus the strictly increasing condition $b_\phi(n+1) > b_\phi(n)$ implies that $b_\phi(n) \geq n+2$.

Definition 4.1. A coloring ϕ of a regular tree T is called *Sturmian* if $b_\phi(n) = n+2$.

Example 4.2. Consider a coloring of an edge-indexed graph given as



The universal covering tree T has all vertices colored by b except one geodesic $\mathbf{p}$ which is colored by a. Admissible colored balls are the ones with vertices all colored by b except vertices on $\mathbf{p}$ if there is a non-trivial intersection. These colored n-balls are determined (up to automorphisms of colored balls) by the distance $d = 0, 1, \ldots, n$ or $d > n$ of $\mathbf{p}$ from the center of the ball. Therefore, $b(n) = n+2$.

Example 4.3 ([8])**.** Let $g : z \mapsto z + \beta$ with irrational β. Then we have $g \notin \mathrm{Comm}(\Gamma_r)$. Let ϕ_g^2 be a coloring given by

$$\phi_g^2(x) = \begin{cases} a, & \text{if there is } D \in \mathcal{D} \text{ such that } D_x \cap gD \text{ is not compact,} \\ b, & \text{otherwise} \end{cases}$$

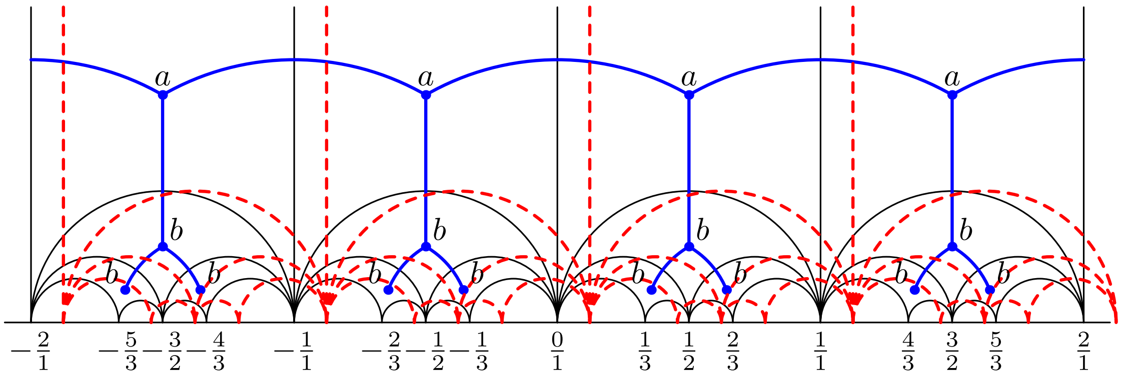
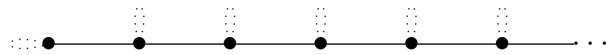


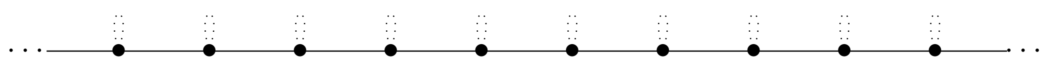
Figure 5: $z \mapsto z + \sqrt{17}$

for $x \in VT$ and $D_x \in \mathcal{D}$ corresponding to x. If $D_x \cap gD$ is not compact, then D_x and gD have at least one common vertex. Since all vertices of $g\mathcal{D}$ other than ∞ are irrational, $\phi_g^2(x) = a$ if and only if D_x has the vertex of ∞, which is the only possible common vertex of D_x with $gD \in g\mathcal{D}$. (See Figure 5.) Therefore, ϕ_g^2 is the coloring of Example 4.2.

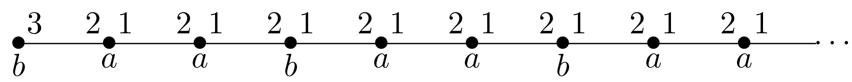
It was shown that the quotient graph of a Sturmian coloring is always linear:

Theorem 4.4 ([7])**.** *Let ϕ be a Sturmian coloring of a regular tree T. Then there exists a group Γ acting on T such that ϕ is Γ-invariant, so that ϕ is a lifting of a coloring ϕ_X on the quotient graph $X = \Gamma \backslash T$. The quotient graph $X = \Gamma \backslash T$ is one of the following two types of graphs. Here, loops are expressed by dotted lines to indicate that they may exist or not.*

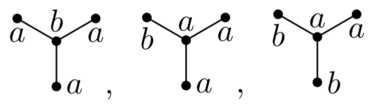


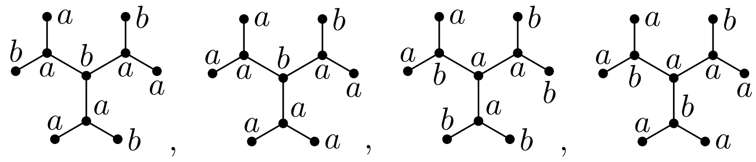


Example 4.5 ([7])**.** Consider the coloring of edge-indexed graph X:



Lift this coloring to the universal covering tree T. See Figure 6. The admissible colored balls are as follows:

$\mathcal{B}(1) = \{$  $\}$,

$\mathcal{B}(2) = \{$  $\}$.

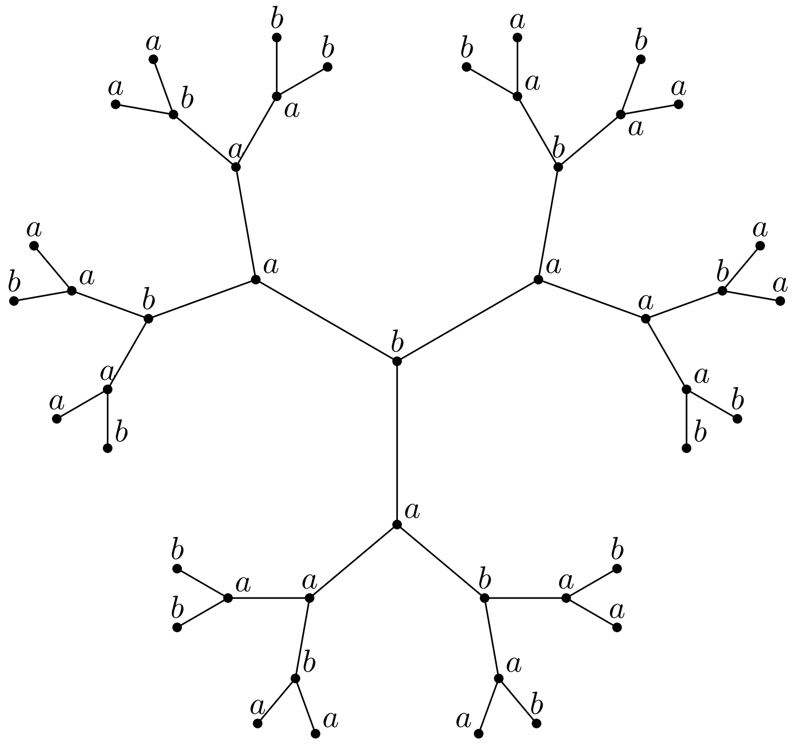
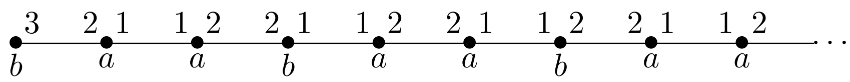
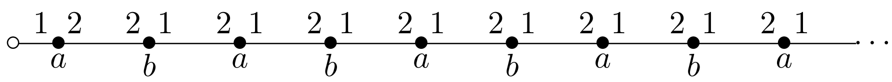


Figure 6: An example of Sturmian tree

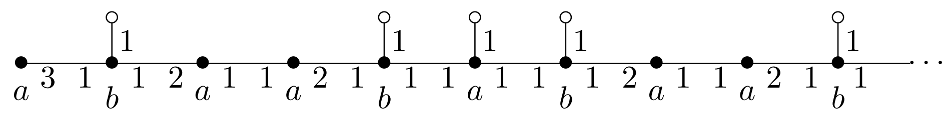
Here are more examples of Sturmian colorings which are liftings of one-sided periodic colorings on an infinite quotient ray of T.





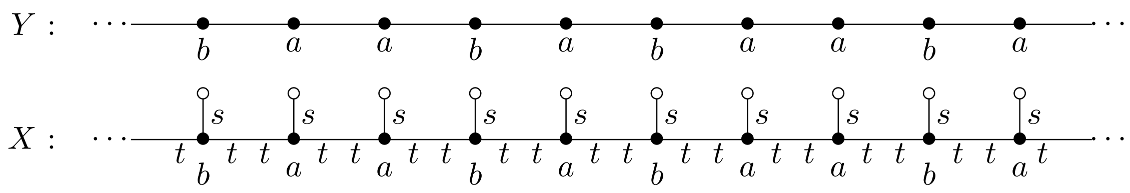
The following example indicates that the coloring on the quotient graph X can be quite arbitrary.

Example 4.6 ([7]). The following infinite edge-indexed colored graph is of Strumian coloring:



The "frequencies" of vertices of a and b in Example 4.2, 4.5 and 4.6 are rational. However in the following examples of Example 4.7 and 4.8 (ii) the "frequencies" of the vertices colored by a and b are irrational since Sturmian word has the irrational "frequencies" of a and b.

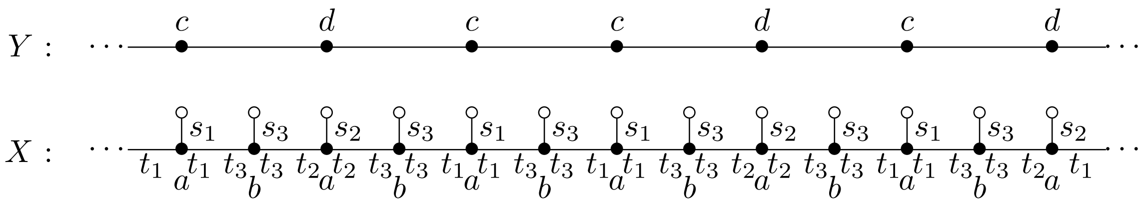
Example 4.7 (Sturmian colorings with a periodic edge configuration [7]). For a given Sturmian coloring on a 2-regular tree Y, we have the following Sturmian coloring of unbounded type on k-regular tree.



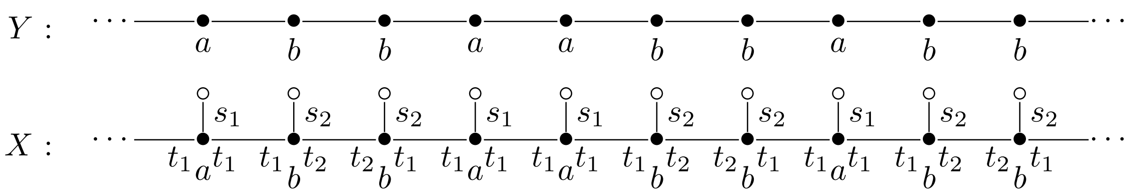
The following edge-indexed colored graph is an example of a Sturmian coloring, whose coloring on the quotient ray is periodic but whose sequence of edge indices is Sturmian.

Example 4.8 ([7]).

(i) Let ϕ_0 be a Sturmian coloring with colors $\{c, d\}$ on Y, which is associated to a bi-infinite Sturmian sequence. Then we can construct a linear graph with loops X as follows:



(ii) Let ϕ_0 be a Sturmian coloring on Y in which $a - b - a$ and $b - b - b$ are not admissible. We can construct a linear graph with loops X as follows:



Bi-infinite Sturmian words are defined as non-eventually periodic words with subword complexity $p(n) = n+1$. Note that bi-infinite words with $p(n) = n+1$ do not necessarily correspond to colorings of a 2-regular tree with $b(n) = n+2$. They could be quasi Sturmian words.

For the isometry $g(z) = \alpha z$, $\alpha \in \mathbb{C}$, and the tessellation $\mathcal{D}$ in Example 3.3, we associate a coloring ϕ given by the number of intersecting elements of $g\mathcal{D}$, $D \in \mathcal{D}$ in each D_x. The alphabet of ϕ is $\{\lfloor|\alpha|\rfloor, \lceil|\alpha|\rceil\}$, where $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$ are the floor and the ceiling functions and $|\alpha|$ is the modulus of the multiplication factor α. If $|\alpha|$ is an irrational number, then the coloring on 2-regular tree is corresponding to an infinite Sturmian word. See [5] for the detailed discussions of the Sturmian word. When $|\alpha|$ is rational, the associated coloring is periodic. However for Farey tessellation which corresponds to 3-regular tree, it is not clear how to associate an "irrational" or non commensurator isometry element g into a Sturmian coloring.

Question For $k \geq 3$, is there a natural way to associate a non-eventually periodic Sturmian coloring ϕ of the k-regular tree from an isometry and a tessellation of the hyperbolic plane?

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