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Linearly recurrent sequences and S-adic sequences

By

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Abstract

The main part of this article is a survey of a complete characterization of linearly recurrent sequences in terms of S-adic sequences which is due to F. Durand (2000,2003). However, we also discuss *almost minimal* sequences in connections with S-adic sequences.

§ 1. Introduction

The class of minimal subshifts arising from primitive substitutions has been extensively studied by several authors from viewpoints of ergodic theory; see for example [24, 23]. F. Durand, B. Host and C. Skau [14] found that the class is included in the class of linearly recurrent subshifts. Some properties of the linearly recurrent subshifts have been investigated, some of which are the same as substitution minimal subshifts; see for example [14, 12, 13, 6, 5]. Then, our goal is to neatly review and present F. Durand's argument [12, 13] to obtain a characterization of linearly recurrent sequences in terms of S-adic sequences, which affirms that if S is a proper and primitive sequence of finitely many morphisms then an associated S-adic sequence is linearly recurrent, and vice-versa (Theorem 4.4). We will also see that any uniformly recurrent sequence is an S-adic sequence for some (weakly) primitive sequence S of morphisms, and vice-versa (Theorem 3.4). Our manner in developing the argument is slightly different from F. Durand's. One of its reasons comes from the fact that infinite sequences which we will discuss are bilateral. However, one can find nothing new until Section 4. In Section 5, the author would like to discuss almost minimal sequences. Such sequences are exactly those sequences which generate almost minimal subshifts in the sense of [8]. Similarly to Theorem 3.4, we will see that any almost minimal sequence is an S-adic sequence for some (*weakly*) *almost primitive* sequence S of morphisms, and vice-versa (Theorem 5.5). The notion of almost primitivity was introduced by the author for substitutions [26]. The Cantor substitution is an example of almost primitive substitution, which appears in [16, 23]. One can find such substitutions also in [15].

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§ 2. Return words, morphisms and S-adic sequences

We use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Let A be a finite alphabet. An element of A is called a *letter*. A nonempty *word* w over A is a finite sequence of letters in A. We may write $w = w_1 w_2 \dots w_n$ with $w_i \in A$. The integer n is called the *length* of the word w, and denoted by $|w|$. Let A^+ denote the set of nonempty words over A. Put $A^* = A^+ \cup \{\Lambda\}$, where Λ is the empty word. Suppose that $u \in A^+$ and $v \in A^+ \cup A^{\mathbb{Z}}$. If $u = v_{[i,i+|u|)} := v_i v_{i+1} \dots v_{i+|u|-1}$, then the integer i is called an *occurrence* of u in v. Denote by $|v|_u$ the number of occurrences of u in v. For a letter $a \in A$, we put $|v|_{\neg a} = \sum_{b \in A \setminus \{a\}} |v|_b$.

A word $u \in A^*$ is called a *factor* of a word $v \in A^*$ if $v = pus$ for some words $p, s \in A^*$, which is designated by $u \prec v$. The words p and s are respectively called a *prefix* and *suffix* of v, which are designated by $p \prec_{\mathrm{p}} v$ and $s \prec_{\mathrm{s}} v$ respectively. By definition, the empty word Λ is a factor of any word. We also naturally define prefixes and suffixes of unilaterally infinite sequences over A.

Define the *language* $\mathcal{L}(x)$ of a sequence $x \in A^{\mathbb{Z}}$ by

$$\mathcal{L}(x) = \bigcup_{i \le j} \{w \in A^*; w \prec x_{[i,j]}\}.$$

Set $\mathcal{L}_n(x) = \{w \in \mathcal{L}(x); |w| = n\}$ for $n \in \mathbb{N}$ and $\mathcal{L}(x)^+ = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(x)$. The sequence x is said to be *uniformly recurrent*, or *almost periodic* [22, Section 4.1], if any word $u \in \mathcal{L}(x)$ occurs in x with bounded gap, i.e. there exists $g \in \mathbb{N}$, depending on u, such that $u \prec x_{[i,i+g)}$ for all $i \in \mathbb{Z}$. The sequence x is said to be *ultimately periodic* if there exist $m, n \in \mathbb{N}$ and a word u such that each of $x_{(-\infty,-m]}$ and $x_{[n,+\infty)}$ is written as an infinite repetition, or concatenation, of only one word u. The sequence x is said to be *periodic* if there exists a word u for which $x = u^\infty.u^\infty$, where the dot means the separation between negative and nonnegative coordinates. Observe that if a uniformly recurrent sequence is ultimately periodic then it is periodic.

Let $u, v \in A^*$ be such that $uv \ne \Lambda$. We say that $i \in \mathbb{Z}$ is an *occurrence of* $u.v$ in x if $x_{[i-|u|,i+|v|)} = uv$. In particular, an occurrence of $\Lambda.v$ is called an occurrence of v. Two occurrences i and j of $u.v$ in x are said to be *consecutive* if no elements of $\{n \in \mathbb{Z}; i < n < j\}$ are occurrences of $u.v$ in x. A word $w \in A^+$ is called a *return word to* $u.v$ in x [14, Section 5] if $w = x_{[i,j)}$ for some consecutive occurrences i and j of $u.v$ in x. Let $\mathcal{R}_{x,u.v}$ denote the set of return words to $u.v$ in x. We abbreviate $\mathcal{R}_{x,\Lambda.v}$ to $\mathcal{R}_{x,v}$. When the context is clear, the symbol $\mathcal{R}_{x,v}$ is abbreviated to $\mathcal{R}_v$. Observe that a word w is a return word to $u.v$ in x if and only if $uwv \in \mathcal{L}(x)$ and uv occurs exactly twice in uwv as a prefix and a suffix. The set $\mathcal{R}_{u.v}$ is a *circular code* [14, Definition 9], i.e. $\mathcal{R}_{u.v}$ is a code and, in addition, it holds that

$$w_1 w_2 \dots w_k = s w'_1 w'_2 \dots w'_{k'} t \ (w_i, w'_j \in \mathcal{R}_{u.v}, s \in A^+, t \in A^*, ts \in \mathcal{R}_{u.v}) \Rightarrow t = \Lambda.$$

If a circular code decompose a sequence in $A^{\mathbb{Z}}$, then the decomposition is unique.

Let A and B be finite alphabets. A map $\sigma : A \to B^+$ is called a *morphism*. If $B = A$, then σ is called a *substitution* on A.

Remark 1. Whenever we assume a morphism $\sigma : A \to B^+$, we assume that a given letter in B occurs in $\sigma(a)$ for some letter $a \in A$.

We extend the domain of σ to A^+ by defining $\sigma(w) = \sigma(w_1)\sigma(w_2)\dots\sigma(w_n)$ if $w = w_1w_2\dots w_n$ with $w_i \in A$ for every i. The domain A^+ is naturally extended to $A^{\mathbb{Z}}$. If $\tau : B \to C^+$ is another morphism, then the composition $\tau\sigma : A \to C^+$ is also defined in a natural way. The *incidence matrix* M_σ of the morphism σ is defined to be an $A \times B$ matrix whose (a,b)-entry is $|\sigma(a)|_b$.

We always assume that any sequence $S = \{\sigma_0, \sigma_1, \dots\}$ of morphisms is pairwise composable, i.e. for every $n \in \mathbb{Z}_+$, the composition $\sigma_n\sigma_{n+1}$ is well-defined as a morphism. We may write $S = \{\sigma_n : A_{n+1} \to {A_n}^+\}_{n\in\mathbb{Z}_+}$. Set $S^{(k)} = \{\sigma_n\}_{n\geq k}$ and $\sigma_{k,\ell} = \sigma_k\sigma_{k+1}\dots\sigma_\ell$ for $\ell \geq k \geq 0$. Define the *language* $\mathcal{L}(S)$ of the sequence S of morphisms by

$$\mathcal{L}(S) = \bigcup_{n=0}^{\infty} \bigcup_{a\in A_{n+1}} \{w \in {A_0}^*; w \prec \sigma_{0,n}(a)\}.$$

Set $\mathcal{L}_n(S) = \{w \in \mathcal{L}(S); |w| = n\}$ for $n \in \mathbb{N}$. A sequence $S' = \{\sigma'_k : B_{k+1} \to {B_k}^+\}_{k\in\mathbb{Z}_+}$ of morphisms is called a *telescoping* of the sequence S if there exist $0 = n_0 < n_1 < n_2 < \dots$ such that $B_k = A_{n_k}$ and $\sigma'_k = \sigma_{n_k,n_{k+1}-1}$ for all k. It follows from the standing assumption in Remark 1 that $\mathcal{L}(S') = \mathcal{L}(S)$.

The sequence S is said to be *weakly primitive* [15, p. 8] if for any $n \in \mathbb{Z}_+$ there exists an integer $m \geq n$ such that $a \prec \sigma_{n,m}(b)$ for all $a \in A_n$ and $b \in A_{m+1}$, i.e. the composition $M_{\sigma_m}M_{\sigma_{m-1}}\dots M_{\sigma_n}$ of incidence matrices is positive. The sequence S is said to be *primitive* (with constant $n_0 \in \mathbb{N}$) [12, p. 1065] if

$$n \in \mathbb{Z}_+, a \in A_{n+n_0}, b \in A_n \Rightarrow b \prec \sigma_{n,n+n_0-1}(a).$$

Clearly, the primitivity implies the weak primitivity. The primitivity is called *strong primitivity* in [2, Definition 5.1]. We also say that a morphism $\sigma : A \to B^+$ is primitive with constant 1 if the incidence matrix M_σ is positive.

A morphism $\sigma : A \to B^+$ is said to be *proper* [13, p. 664] if there exist $r, \ell \in B$ such that $\ell \prec_{\mathrm{p}} \sigma(b)$ and $r \prec_{\mathrm{s}} \sigma(b)$ for all $b \in A$. If each σ_n is proper, then the sequence S is said to be *proper*. A sequence $x \in A^{\mathbb{Z}}$ is called an *S-adic sequence* if for all $i, j \in \mathbb{Z}$, there exist $n \in \mathbb{Z}_+$ and $a \in A_{n+1}$ such that $x_{[i,j]} \prec \sigma_{0,n}(a)$. If S is primitive (resp. proper), then the S-adic sequence x is said to be *primitive* (resp. *proper*). This definition of an S-adic sequence is different from the original one [12, Subsection 2.5]. We do not demand the finiteness (as a set) of the sequence S.

Define an *S-adic subshift* X_S by $X_S = \{x = (x_i)_i \in A^{\mathbb{Z}}; x_{[i,j]} \in \mathcal{L}(S)\}$. Hence, an S-adic sequence is exactly a sequence belonging to the S-adic subshift X_S.

If $\mathcal{L}_n(S) \neq \emptyset$ for all $n \in \mathbb{N}$, then there necessarily exists an S-adic sequence. To see this, it is sufficient to utilize the existence of a sequence $\{w_n\}_{n\in\mathbb{Z}_+} \in \prod_{n=0}^{\infty} \mathcal{L}_{2n+1}(S)$ satisfying that 2 is an occurrence of w_n in w_{n+1} for all $n \in \mathbb{Z}_+$.

A sequence $\{x^{(n)}\}_n \in \prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$ is said to be *consistent* for the sequence S if $\sigma_n(x^{(n+1)}) = x^{(n)}$ for all $n \in \mathbb{Z}_+$. Instead, if $\sigma_n(x^{(n+1)})$ coincides with $x^{(n)}$ up to a shift by some digits, then $\{x^{(n)}\}_n$ is said to be *quasi-consistent*. The sequence S is said to be *everywhere-growing* if $\lim_{n\to\infty} |\sigma_{0,n}(a_{n+1})| = \infty$ for any sequence $\{a_n\}_{n\in\mathbb{Z}_+} \in \prod_{n=0}^{\infty} A_n$. See also [2, Definition 3.1].

Lemma 2.1. *If a sequence $S = \{\sigma_n : A_{n+1} \to {A_n}^+\}_{n\in\mathbb{Z}_+}$ of morphisms is everywhere-growing, then the sequence S has a consistent sequence in $\prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$.*

Proof. For $\ell \in \mathbb{N}$, let W_ℓ be the set of sequences $\{b_n a_n\}_n \in \prod_{n=0}^{\infty} \mathcal{L}_2(S^{(n)})$ satisfying that $a_n \prec_{\mathrm{p}} \sigma_n(a_{n+1})$ and $b_n \prec_{\mathrm{s}} \sigma_n(b_{n+1})$ for all integers n with $0 \le n < \ell$. Since $\{W_\ell\}_\ell$ is a decreasing sequence of nonempty closed sets in a compact metric space $\prod_{n=0}^{\infty} \mathcal{L}_2(\sigma^{(n)})$ endowed with the product topology of discrete topologies, there exists a sequence $\{b_n a_n\}_{n\in\mathbb{Z}_+} \in \bigcap_{\ell=1}^{\infty} W_\ell$. The assumption of the lemma allows us to define an $S^{(n)}$-adic sequence $x^{(n)}$ by $x^{(n)} = \lim_{k\to\infty} \sigma_{n,k}(b_{k+1}).\sigma_{n,k}(a_{k+1})$. Clearly, $x^{(n)} = \sigma_n(x^{(n+1)})$ for every $n \in \mathbb{Z}_+$. $\square$

Remark 2. Without assuming the everywhere-growing property, to deduce the conclusion of Lemma 2.1, it is sufficient to assume a weaker condition that if a sequence $\{a_n\}_{n\in\mathbb{Z}_+} \in \prod_{n=0}^{\infty} A_n$ satisfies at least one of the conditions:

(1) $a_n \prec_{\mathrm{p}} \sigma_n(a_{n+1})$ for all $n \in \mathbb{Z}_+$;

(2) $a_n \prec_{\mathrm{s}} \sigma_n(a_{n+1})$ for all $n \in \mathbb{Z}_+$,

then $\lim_{n\to\infty} |\sigma_{0,n}(a_{n+1})| = \infty$.

Remark 3. Any sequence $x \in A^{\mathbb{Z}}$ over a finite alphabet A is an S-adic sequence for some sequence S of finitely many morphisms. This is verified by applying an idea used in the proof of [20, Proposition 2.1], as follows. Putting $\mathcal{L}_n(x) = \{w_{n,1}, w_{n,2}, \dots, w_{n,p(n)}\}$ for $n \in \mathbb{N}$, define a sequence $y \in A^{\mathbb{Z}_+}$ by

$$y = w_{1,1}w_{1,2}\dots w_{1,p(1)}w_{2,1}w_{2,2}\dots w_{2,p(2)}w_{3,1}\dots$$

Let $\ell \notin A$ be a letter. For $i \in \mathbb{Z}_+$, define a substitution $\sigma_i : A \cup \{\ell\} \to (A \cup \{\ell\})^+$ by

$$\sigma_i(a) = \begin{cases} \ell y_i & \text{if } a = \ell; \\ a & \text{otherwise.} \end{cases}$$

Since $y_i = y_j$ implies $\sigma_i = \sigma_j$, the sequence $S = \{\sigma_i\}_{i\in\mathbb{Z}_+}$ is finite. Observe that S satisfies the condition in Remark 1. Put $m_n = \sum_{k=1}^{n} k \cdot \sharp\mathcal{L}_k(x)$. It follows that for $x_{[-n,n)} \prec \sigma_{0,m_{2n}-1}(\ell)$ for all $n \in \mathbb{N}$, so that x is an S-adic sequence.

§ 3. Uniformly recurrent sequences

Now, we go back into return words. Suppose that x is a uniformly recurrent sequence over a finite alphabet A. Let $uv \in \mathcal{L}(x)^+$. Since $\max\{|w|; w \in \mathcal{R}_{u.v}\}$ is not greater than the greatest value of distances between consecutive occurrences of $u.v$ in x, the set $\mathcal{R}_{u.v}$ of return words to $u.v$ in x is finite. Put $\mathcal{R}_{u.v} = \{w_1, w_2, \dots, w_n\}$ and $R_{u.v} = \{1, 2, \dots, n\}$. Define a morphism $\phi_{u.v} : R_{u.v} \to A^+$ by $\phi_{u.v}(i) = w_i$ for all i. Suppose that $u \prec_{\mathrm{s}} u'$, $v \prec_{\mathrm{p}} v'$ and $u'v' \in \mathcal{L}(x)^+$. Since for any $i \in R_{u'.v'}$, it holds that

$$u\phi_{u'.v'}(i)v \prec u'\phi_{u'.v'}(i)v' \in \mathcal{L}(x)^+, uv \prec_{\mathrm{p}} u\phi_{u'.v'}(i)v \text{ and } uv \prec_{\mathrm{s}} u\phi_{u'.v'}(i)v,$$

the word $\phi_{u'.v'}(i)$ is written, in a unique way, as a concatenation $\phi_{u.v}(j_1)\phi_{u.v}(j_2)\dots\phi_{u.v}(j_n)$ for some $j_1, j_2, \dots, j_n \in R_{u.v}$. This allows us to define a morphism $\sigma : R_{u'.v'} \to R_{u.v}^+$ by $\sigma(i) = j_1 j_2 \dots j_n$. We then have a commutative diagram in Figure 1. Observe that the morphism σ satisfies the condition in Remark 1.

$$\begin{tikzcd} R^+_{u.v} \arrow[d, "\phi_{u.v}"'] & R_{u'.v'} \arrow[l, "\sigma"'] \arrow[dl, "\phi_{u'.v'}"] \\ A^+ & \end{tikzcd}$$

Figure 1.

Lemma 3.1 ([11, Lemma 3.2]). *If a uniformly recurrent sequence $x \in A^{\mathbb{Z}}$ is aperiodic, i.e. non-periodic, and words $\{u_n, v_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(x)$ satisfy that*

(1) $u_n v_n \in \mathcal{L}(x)^+$, $u_n \prec_{\mathrm{s}} u_{n+1}$ *and* $v_n \prec_{\mathrm{p}} v_{n+1}$ *for all* $n \in \mathbb{N}$*;*

(2) $\lim_{n\to\infty} |u_n v_n| = \infty$,

then $m_n = \min\{|w|; w \in \mathcal{R}_{u_n.v_n}\} \uparrow \infty$ *as* $n \to \infty$.

Proof. We may assume that $u_n \prec_{\mathrm{s}} x_{(-\infty,-1]}$ and $v_n \prec_{\mathrm{p}} x_{[0,+\infty)}$ for all n. Since any word in $\mathcal{R}_{u_{n+1}.v_{n+1}}$ is a concatenation of words in $\mathcal{R}_{u_n.v_n}$, the sequence $\{m_n\}_n$ is increasing. Assume that $\{m_n\}_n$ is bounded. There exist $n_0 \in \mathbb{N}$ and $\{w_n\}_{n\geq n_0} \subset \mathcal{R}_{u_n.v_n}$ such that $|w_n|$ is constant for all integers $n \geq n_0$. There exist integers $n_0 \leq n_1 < \dots$ such that $w_{n_i} = w_{n_{i+1}}$ for all $i \in \mathbb{Z}_+$. Put $w = w_{n_0}$. We may assume that $v_{n_i} \neq \Lambda$ for all $i \in \mathbb{Z}_+$. Since $v_{n_i} \prec_{\mathrm{p}} w v_{n_i}$ for all $i \in \mathbb{Z}_+$ and $\lim_{i\to\infty} |v_{n_i}| = \infty$, $x_{[0,+\infty)}$ is an infinite repetition of w, so that x is periodic. This is a contradiction. □

Define Statement (A) including variables ♠ and ♡ for strings by `there exists a ♠ sequence` S `of morphisms which has a ♡ sequence` $\{x^{(n)}\}_n \in \prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$ `with` $x^{(0)} = x$. See also [14, Section 5] and [13, Section 4] in connection with the proof of the following lemma.

Lemma 3.2. *If x is a uniformly recurrent, bilaterally infinite sequence over a finite alphabet, then Statement* (A) *with* ♠ = *"everywhere-growing, proper and primitive" and* ♡ = *"consistent". In particular, x is an S-adic sequence.*

Proof. Put $A = \mathcal{L}_1(x)$.

Consider the case where x is periodic. Let $v \in \bigcup_{n\geq 2} \mathcal{L}_n(x)$ be such that $x = v^\infty.v^\infty$. Put $A_n = A$ for $n \in \mathbb{Z}_+$. Define $\sigma_n : A_{n+1} \to A_n^+$ by $\sigma_n(a) = v$ for all $a \in A_{n+1}$. Clearly, x is an S-adic sequence, where $S = \{\sigma_n\}_{n\in\mathbb{Z}_+}$, which is everywhere-growing, proper and primitive with constant 1.

Consider the case where x is aperiodic. Fix $g_0 \in \mathbb{N}$ so that $a \prec x_{[i,i+g_0)}$ for all $a \in A$ and $i \in \mathbb{Z}$. Lemma 3.1 allows us to find $n_1 \in \mathbb{N}$ such that

$$\min\{|w|; w \in \mathcal{R}_{x_{[-n_1,-1]}.x_{[0,n_1)}}\} \geq g_0.$$

Put $u_1 = x_{[-n_1,-1]}$, $v_1 = x_{[0,n_1)}$, $\mathcal{R}_1 = \mathcal{R}_{u_1.v_1}$ and $R_1 = \{1, 2, \dots, \sharp\mathcal{R}_1\}$. Remark that $\sharp R_1 = \sharp\mathcal{R}_1 \geq 2$. Let $\sigma_0 : R_1 \to A^+$ be the morphism $\phi_{u_1.v_1}$.

It follows from the choice of g_0 that $a \prec \sigma_0(i)$ for all $a \in A$ and $i \in R_1$, so that σ_0 is primitive with constant 1. Since for all $w \in \mathcal{R}_1$, we have that $x_0 \prec_{\mathrm{p}} v_1 \prec_{\mathrm{p}} wv_1$ and $x_{-1} \prec_{\mathrm{s}} u_1 \prec_{\mathrm{s}} u_1w$, the first and last letter of w are respectively x_0 and x_{-1}. This implies that σ_0 is proper.

Fix $g_1 \in \mathbb{N}$ so that $w \prec x_{[i,i+g_1)}$ for all $w \in \mathcal{R}_1$ and $i \in \mathbb{Z}$. Put

$$m_1 = \max\{|w|; w \in \mathcal{R}_1\}.$$

Lemma 3.1 allows us to find an integer $n_2 > n_1 \vee m_1$ such that

$$\min\{|w|; w \in \mathcal{R}_{x_{[-n_2,-1]}.x_{[0,n_2)}}\} > g_1.$$

Put $u_2 = x_{[-n_2,-1]}$, $v_2 = x_{[0,n_2)}$, $\mathcal{R}_2 = \mathcal{R}_{u_2.v_2}$ and $R_2 = \{1, 2, \ldots, \sharp\mathcal{R}_2\}$. Remark that $\sharp R_2 = \sharp\mathcal{R}_2 \geq 2$. Let $\sigma_1 : R_2 \to {R_1}^+$ be the morphism making the diagram in Figure 1 commutes under $u = x_{[-n_1,-1]}$, $v = x_{[0,n_1)}$, $u' = x_{[-n_2,-1]}$, $v' = x_{[0,n_2)}$ and $\sigma = \sigma_1$. It follows from the choice of g_1 that $w' \prec w$ for all $w' \in \mathcal{R}_1$ and $w \in \mathcal{R}_2$, so that σ_1 is primitive with constant 1. Let $w \in \mathcal{R}_2$ be arbitrary. Since $v_2 \prec_{\mathrm{p}} wv_2$ and $|v_2| > m_1$, a word in $\mathcal{R}_1$ which is a prefix of w is also a prefix of v_2. This implies that the first letter of $\sigma_1(i)$ does not depend on i but is constant. This method also shows a similar fact about the last letter of $\sigma_1(i)$. Hence, σ_1 is proper.

Continuing the inductive procedure, we can find integers $1 \leq n_1 < n_2 < \ldots$ so that the sequence $S = \{\sigma_k : R_{k+1} \to R_k^+\}_{k\in\mathbb{Z}_+}$ of morphisms is proper and primitive with constant 1, where $R_0 = A$, $R_k = \{1, 2, \ldots, \sharp\mathcal{R}_{u_k.v_k}\}$, $u_k = x_{[-n_k,-1]}$ and $v_k = x_{[0,n_k)}$, and, in addition, the diagram in Figure 1 commutes under $u = x_{[-n_k,-1]}$, $v = x_{[0,n_k)}$, $u' = x_{[-n_{k+1},-1]}$, $v' = x_{[0,n_{k+1})}$ and $\sigma = \sigma_k$. Let $\ell_k, r_k \in R_k$ be such that $\ell_k \prec_{\mathrm{p}} \sigma_k(i)$ and $r_k \prec_{\mathrm{s}} \sigma_k(i)$ for all $i \in R_{k+1}$. Then, it follows that for all $i \in R_{k+1}$,

(1) $\sigma_{0,k-1}(\ell_k) = \phi_{u_k.v_k}(\ell_k) \prec_{\mathrm{p}} \phi_{u_k.v_k}\sigma_k(i) = \phi_{u_{k+1}.v_{k+1}}(i)$;

(2) $\sigma_{0,k-1}(r_k) = \phi_{u_k.v_k}(r_k) \prec_{\mathrm{s}} \phi_{u_k.v_k}\sigma_k(i) = \phi_{u_{k+1}.v_{k+1}}(i)$,

so that for all $k \in \mathbb{Z}_+$, $\sigma_{0,k-1}(\ell_k) \prec_{\mathrm{p}} x_{[0,q_k)}$ and $\sigma_{0,k-1}(r_k) \prec_{\mathrm{s}} x_{[-p_k,-1]}$, where $x_{[-p_k,-1]}, x_{[0,q_k)} \in \mathcal{R}_k$. This shows that $x = \lim_{k\to\infty} \sigma_{0,k-1}(\ell_k).\sigma_{0,k-1}(r_k)$, so that x is an S-adic sequence. Since it holds that $\sigma_{m,k-1}(\ell_k) \prec_{\mathrm{p}} \sigma_{m,k}(\ell_{k+1})$ and $\sigma_{m,k-1}(r_k) \prec_{\mathrm{s}} \sigma_{m,k}(r_{k+1})$ for all integers $k > m$, there exists an $S^{(m)}$-adic sequence

$$x^{(m)} = \lim_{k\to\infty} \sigma_{m,k-1}(r_k).\sigma_{m,k-1}(\ell_k),$$

which satisfies $x^{(m)} = \sigma_m(x^{(m+1)})$ for all $m \in \mathbb{Z}_+$, where $x^{(0)} = x$. □

Lemma 3.3 ([12, Lemma 7]). *If a weakly primitive sequence $S = \{\sigma_n : A_{n+1} \to A_n^+\}_{n\in\mathbb{Z}_+}$ of morphisms has a consistent sequence $\{x^{(n)}\}_n \in \prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$, then the S-adic sequence $x^{(0)}$ is uniformly recurrent.*

Proof. Let $u \in \mathcal{L}(S)^+$. There exist $r_0 \in \mathbb{Z}_+$ and $a \in A_{r_0+1}$ such that $u \prec \sigma_{0,r_0}(a)$. Fix an integer $r_1 > r_0$ so that $b \prec \sigma_{r_0+1,r_1}(c)$ for all $b \in A_{r_0+1}$ and $c \in A_{r_1+1}$. Since for all $c \in A_{r_1+1}$,

$$u \prec \sigma_{0,r_0}(a) \prec \sigma_{0,r_0}\left(\sigma_{r_0+1,r_1}(c)\right) = \sigma_{0,r_1}(c),$$

and the S-adic sequence $x^{(0)}$ is an infinite concatenation of words of the form $\sigma_{0,r_1}(c)$, where $c \in A_{r_1+1}$, any consecutive occurrences of the word u in $x^{(0)}$ is not greater than $2\max\{|\sigma_{0,r_1}(b)|; b \in A_{r_1+1}\}$. This implies that $x^{(0)}$ is uniformly recurrent. $\square$

A sequence of morphisms which generates a uniformly recurrent sequence is not necessarily weakly primitive; recall Remark 3. The Chacon substitution, which is defined by $a \mapsto a$ and $b \mapsto bbab$, also gives such an example.

We now obtain the following statement, which slightly strengthens [2, Theorem 5.2] and [15, Theorem 2.12]. .

Theorem 3.4. *If x is a bilaterally infinite sequence over a finite alphabet, then the following are equivalent:*

(1) *x is uniformly recurrent;*

(2) *Statement* (A) *with* $\spadesuit =$ *"proper and primitive" and* $\heartsuit =$ *"consistent";*

(3) *Statement* (A) *with* $\spadesuit =$ *"primitive" and* $\heartsuit =$ *"consistent";*

(4) *Statement* (A) *with* $\spadesuit =$ *"proper and weakly primitive" and* $\heartsuit =$ *"consistent";*

(5) *Statement* (A) *with* $\spadesuit =$ *"weakly primitive" and* $\heartsuit =$ *"consistent".*

Proof. The implications (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1) are nothing but Lemmas 3.2 and 3.3, respectively. The implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) follow from the fact that the primitivity implies the weak primitivity. The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial. $\square$

We now go into connections between minimal subshifts and Bratteli-Vershik systems on path spaces of ordered Bratteli diagrams.

A pair (X, T) is called a *topological dynamical system* if T is a homeomorphism on a compact metric space X. The system (X, T) is said to be *minimal* if a T-invariant open subset of X is either $\emptyset$ or X. The condition is equivalent to saying that the orbit $\{T^n x; n \in \mathbb{Z}\}$ of any point $x \in X$ is dense; see for example [22, Theorem 1.2 in p. 136]. Given another topological dynamical system (Y, S), a homeomorphism $\phi : X \to Y$ is called a *conjugacy* between the systems if $\phi(Tx) = S(\phi x)$ for all points $x \in X$.

Let A be a finite alphabet. Endow $A^{\mathbb{Z}}$ with the product topology of the discrete topology on A. Define a homeomorphism $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $(Tx)_i = x_{i+1}$ for all $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, in other words T is the *left shift*. A (bilateral) *subshift* X over A is defined to be a T-invariant, closed subset of $A^{\mathbb{Z}}$. As usual, by restricting the action of the left shift T to X equipped with the relative topology, we consider a topological dynamical system (X, T). The topology of X is generated by the family of *cylinder subsets*. A cylinder subset $[u.v]$ of X associated with words u, v is defined by $[u.v] = \{y = (y_i)_i \in X; y_{[-|u|,|v|)} = uv\}$. The notation $[\Lambda.v]$ is abbreviated to $[v]$. The subshift X generated by a sequence $x \in A^{\mathbb{Z}}$ is defined by $X = \{y = (y_i)_i \in A^{\mathbb{Z}}; y_{[i,j]} \in \mathcal{L}(x)$ for all integers $i \leq j\}$. It is not difficult to verify that the sequence x is uniformly recurrent if and only if the associated subshift X is minimal.

A directed infinite graph (V, E) is called a *Bratteli diagram* [4] if

(1) V (resp. E) is the vertex (resp. edge) set, which is a disjoint union $\bigcup_{i=0}^{\infty} V_i$ (resp. $\bigcup_{i=1}^{\infty} E_i$) of finite subsets;

(2) V_0 is a singleton, whose element is called the *top vertex*;

(3) for every $i \in \mathbb{N}$, every edge $e \in E_i$ starts from a vertex in V_{i-1}, which is denoted by $s(e)$, and terminates at a vertex in V_i, which is denoted by $r(e)$;

(4) $s^{-1}\{v\} \neq \emptyset$ for any vertex $v \in V$;

(5) $r^{-1}\{v\} \neq \emptyset$ for any vertex $v \in V \setminus V_0$.

The maps $r, s : E \to V$ are called the *range* and *source maps*, respectively. The *incidence matrix* M_n at the level $n \in \mathbb{N}$ of the Bratteli diagram (V, E) is a $V_n \times V_{n-1}$ matrix whose (u, v)-entry is the number of edges from the vertex v to the vertex u. The Bratteli diagram is said to be *simple* if there exists a sequence $0 = n_0 < n_1 < n_2 < \ldots$ of integers such that for all $k \in \mathbb{Z}_+$, all the entries of $M_{n_{k+1}} M_{n_{k+1}-1} \ldots M_{n_k+1}$ are positive. See also [9, Corollary III.4.3]

The triple $B := (E, V, \geq)$, where $\geq$ is a partial order on the edge set E, is called an *ordered* Bratteli diagram [19] if it holds that given $e, f \in E$ are comparable with respect to $\geq$ if and only if $r(e) = r(f)$. Given a vertex $v \in V$, we indicate the order on the edges in $r^{-1}\{v\}$ by integers $0, 1, 2, \ldots, \sharp r^{-1}\{v\} - 1$ endowed with the numerical order. Let $B = (E, V, \geq)$ be an ordered Bratteli diagram. Set

$$X_B = \{x = (x_1, x_2, \ldots) \in \prod_{i=1}^{\infty} E_i; r(x_i) = s(x_{i+1}) \text{ for all } i \in \mathbb{N}\},$$

which is called an *infinite path space* of B. Put

$$\begin{aligned} X_B^{\max} &= \{x \in X_B; x_i \text{ is maximal with respect to } \geq \text{ for all } i\}; \\ X_B^{\min} &= \{x \in X_B; x_i \text{ is minimal with respect to } \geq \text{ for all } i\}. \end{aligned}$$

We say that B is *properly ordered* if (V, E) is simple and $\sharp X_B^{\max} = \sharp X_B^{\min} = 1$. Then, let $X_B^{\max} = \{x_{\max}\}$ and $X_B^{\min} = \{x_{\min}\}$. The partial order $\geq$ on E induces a partial order $\geq$ on X_B. Two infinite paths $x, x' \in X_B$ are comparable if and only if there exists $j \in \mathbb{N}$ such that $x_i = x_{i'}$ for all integers $i, i' \geq j$. If in addition $x \neq x'$ and x'_{i_0-1} is a successor of x_{i_0-1}, where $i_0 = \min\{j \in \mathbb{N}; x_i = x'_i, \forall i \geq j\}$, then x is a successor of x'. Endow X_B with the restriction of the product topology. The space X_B is a compact metric space. Assume that the ordered Bratteli diagram B is properly ordered. Define the *Vershik map* $\lambda_B : X_B \to X_B$ [25, 19, 17] by for $x \in X_B$, $\lambda_B(x)$ is the immediate successor of x if $x \neq x_{\max}$; otherwise $\lambda_B(x) = x_{\min}$. It follows that λ_B is a minimal homeomorphism. In fact, A. Vershik only formulated the map from $X \setminus X_B^{\max}$ onto $X \setminus X_B^{\min}$, having no concern with the cardinality of $X_B^{\max}$. The topological dynamical system (X_B, λ_B) is called a *Bratteli-Vershik system* [19].

With an ordered Bratteli diagram $B = (V, E, \geq)$, we associate morphisms $\sigma_n : V_{n+1} \to V_n$ for $n \in \mathbb{Z}_+$ which are defined so that $\sigma_n(a)_i = b$ if and only if there exists an edge in $r^{-1}\{a\} \cap s^{-1}\{b\}$ with order $i - 1$. See also [14, Subsection 4.1]. Let us refer to the morphism σ_n as the *morphism read on* E_n. Let $S = \{\sigma_n : A_{n+1} \to {A_n}^+\}_{n \in \mathbb{Z}_+}$ be the sequence of morphisms constructed in the proof of Lemma 3.2. Construct a properly ordered Bratteli diagram $B = (V, E, \geq)$ so that the morphism read on each E_n is exactly σ_{n+1} and $\sharp r^{-1}(v) = 1$ for all $v \in V_1$.

Proposition 3.5. *Then, the subshift X generated by the uniformly recurrent sequence x is conjugate to the Bratteli-Vershik system (X_B, λ_B).*

Proof. Put $\mathcal{P}_k = \{T^j[u_k.\phi_{u_k.v_k}(i)v_k]; 0 \leq j < |\phi_{u_k.v_k}(i)|, i \in R_k\}$, which is the so-called Kakutani-Rokhlin partition [19, 17, 14], and which is a finite partition of X into clopen sets. The family $\{\mathcal{P}_k\}_k$ generates the topology of X. The way of constructing B allows us to name the vertices in V_k the elements of a set $\{(k, i); i \in R_k\}$. The number of paths starting from the top vertex and terminating at a vertex (k, i) equals $|\phi_{u_k.v_k}(i)|$. Among the paths, let $p(k, i)$ be the path having the smallest order. Let $C(k, i)$ be the set of infinite paths in X_B which go through $p(k, i)$. Put $\mathcal{Q}_k = \{{\lambda_B}^j C(k, i); 0 \leq j < |\phi_{u_k.v_k}(i)|, i \in R_k\}$, which is a Kakutani-Rokhlin partition for (X_B, λ_B). The family $\{\mathcal{Q}_k\}_k$ generates the topology of X_B. Then, a map $\phi : X \to X_B$ defined by $\phi(T^j[u_k.\phi_{u_k.v_k}(i)v_k]) = {\lambda_B}^j C(k, i)$ is a conjugacy between the systems (X, T) and (X_B, λ_B). See also [14, Subsection 5.4]. □

§ 4. Linearly recurrent sequences

Suppose that a bilaterally infinite sequence x over a finite alphabet is uniformly recurrent. If there exists $K \in \mathbb{N}$ such that $|v| \leq K|u|$ for all $u \in \mathcal{L}(x)$ and $v \in \mathcal{R}_{x,u}$, then x is said to be *linearly recurrent* (with a constant $K \in \mathbb{N}$) [14, Definition 13]. A typical example of a linearly recurrent sequence is a fixed point of a primitive substitution. Suppose that a primitive substitution $\sigma : A \to A^+$ has a fixed point $x \in A^{\mathbb{Z}}$. Applying Perron-Frobenius Theory to the incidence matrix M_σ, which is primitive, one can see the existence of constants $C, \lambda > 0$ such that $C^{-1}\lambda^k \leq |\sigma^k(a)| \leq C\lambda^k$ for all $k \in \mathbb{Z}_+$ and $a \in A$. Let $u \in \mathcal{L}(x)^+$. There exists $k \in \mathbb{Z}_+$ for which

$$\min_{a \in A} |\sigma^k(a)| < |u| \leq \min_{a \in A} |\sigma^{k+1}(a)|.$$

Choose $w \in \mathcal{L}_2(x)$ so that $u \prec \sigma^{k+1}(w)$. Let $D \in \mathbb{N}$ be the greatest distance between two consecutive occurrences of an arbitrary word in $\mathcal{L}_2(x)$. Then, for any word $v \in \mathcal{R}_u$,

$$\begin{aligned}
|v| &\leq D \max_{a \in A} |\sigma^{k+1}(a)| \\
&\leq C^2 D \min_{a \in A} |\sigma^{k+1}(a)| \\
&\leq C^2 D \max_{a \in A} |\sigma^k(a)| \min_{a \in A} |\sigma(a)| \\
&\leq C^4 D \min_{a \in A} |\sigma^k(a)| \min_{a \in A} |\sigma(a)| \\
&\leq C^4 D \min_{a \in A} |\sigma(a)| \cdot |u|.
\end{aligned}$$

This shows that x is linearly recurrent, because $C^4 D \min_{a \in A} |\sigma(a)|$ is independent of the choice of u and v.

Lemma 4.1 ([14, Theorem 24]). *If a bilaterally infinite sequence x over a finite alphabet is linearly recurrent with a constant K, then*

(1) $u \in \mathcal{L}(x)^+, v \in \mathcal{L}_{(K+1)|u|-1}(x) \Rightarrow u \prec v$;

(2) $\sharp\mathcal{L}_n(x) \leq Kn$ *for all* $n \in \mathbb{N}$;

(3) *if x is aperiodic then x is $(K+1)$-power free, i.e.* $u^{K+1} \in \mathcal{L}(x) \Rightarrow u = \Lambda$;

(4) $|w| > \frac{1}{K}|u|$ *for all* $w \in \mathcal{R}_{x,u}$ *and* $u \in \mathcal{L}(x)^+$;

(5) $\sharp\mathcal{R}_{x,u} < K^2(K+1)$ *for all* $u \in \mathcal{L}(x)^+$.

Proof. (1) Let $u \in \mathcal{L}(x)$. The linear recurrence forces that u must have an occurrence in $x_{[i,i+K|u|)}$ for all $i \in \mathbb{Z}$, so that $u \prec v$ for all $v \in \mathcal{L}_{K|u|+|u|-1}(x)$.

(2) It follows from (1) of the lemma that all words in $\mathcal{L}_n(x)$ occur in an arbitrary word in $\mathcal{L}_{Kn+n-1}(x)$, so that $\sharp\mathcal{L}_n(x) \leq Kn$ for all $n \in \mathbb{N}$.

(3) Assume that $u^{K+1} \in \mathcal{L}(x)$ for some nonempty word u. Since $|u^{K+1}| = K|u| + |u|$, it follows from (1) that any word in $\mathcal{L}_{|u|}(x)$ is a factor of u^{K+1}, so that $\sharp\mathcal{L}_{|u|}(x) \leq |u|$. In view of [24, Proposition IV.18] due to [7], it implies that x is periodic, which is a contradiction.

(4) If there exist $u \in \mathcal{L}(x)^+$ and $w \in \mathcal{R}_{x,u}$ satisfying that $|w| \leq \frac{1}{K}|u|$, then $u \prec_{\mathrm{p}} wu$ forces $w^{K+1} \prec_{\mathrm{p}} wu$. This contradicts (3) of the lemma.

(5) Since the length of any return word to u is at most $K|u|$, it follows that $w \prec v$ for all $w \in \mathcal{R}_{x,u}$ and $v \in \mathcal{L}_{K^2|u|+K|u|-1}(x)$. Hence,

$$\sharp\mathcal{R}_{x,u} \leq \frac{K^2|u| + K|u| - 1}{\min_{w \in \mathcal{R}_{x,u}} |w|} \leq (K^2|u| + K|u| - 1) \cdot \frac{K}{|u|} < K^2(K+1).$$

□

Lemma 4.2 ([12, Lemma 8]). *If a sequence $S = \{\sigma_n : A_{n+1} \to A_n^+\}_{n \in \mathbb{Z}_+}$ of finitely many morphisms is primitive, then there exists $C > 0$ such that for all $a, b \in A_{n+1}$ and integers $n \geq m \geq 0$,*

$$\frac{|\sigma_{m,n}(a)|}{|\sigma_{m,n}(b)|} \leq C.$$

Proof. Let n_0 be a constant with which S is primitive. Since

$$\left\{ \frac{|\sigma_{m,n}(b)|}{|\sigma_{m,n}(a)|}; a, b \in A_{n+1}, m, n \in \mathbb{Z}_+, 0 \leq n - m < n_0 \right\}$$

is a finite set, it is sufficient to consider only the case where $n - m \geq n_0$. Since $\{\sigma_{k,k+n_0-1}\}_{k \in \mathbb{Z}_+}$ is a finite set, the values

$$C_1 := \max\{|\sigma_{k,k+n_0-1}(a)|; k \in \mathbb{Z}_+, a \in A_{k+n_0-1}\};$$
$$C_2 := \min\{|\sigma_{k,k+n_0-1}(a)|; k \in \mathbb{Z}_+, a \in A_{k+n_0-1}\}$$

exist and both are positive. We have that for all $a, b \in A_{n+1}$,

$$\frac{|\sigma_{m,n}(a)|}{|\sigma_{m,n}(b)|} = \frac{|\sigma_{m,n-n_0}\sigma_{n-n_0+1,n}(a)|}{|\sigma_{m,n-n_0}\sigma_{n-n_0+1,n}(b)|} \leq \frac{\|M_{m,n-n_0}(C_1\mathbf{1})\|}{\|M_{m,n-n_0}(C_2\mathbf{1})\|} = \frac{C_1}{C_2},$$

where $M_{m,n-n_0}$ is the incidence matrix of the morphism $\sigma_{m,n-n_0}$, $\mathbf{1} = {}^t(1, 1, \ldots, 1) \in \mathbb{N}^{A_{n-n_0+1}}$ and $\|\alpha\| = \sum_i |\alpha_i|$ for a vector α. □

Lemma 4.3 ([13, Lemma 3.1]). *Suppose that a primitive sequence S of finitely many morphisms has a consistent sequence $\{x^{(n)}\}_n \in \prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$ Put*

$$D_n = \max\{|u|; u \in \mathcal{R}_{x^{(n)},a.b}, ab \in \mathcal{L}_2(x^{(n)})\}.$$

If $\{D_n\}_{n\in\mathbb{N}}$ is bounded, then there exists a constant K with which all the sequences $x^{(n)}$ are linearly recurrent.

Proof. Put $S = \{\sigma_n : A_{n+1} \to A_n^+\}_{n\in\mathbb{Z}_+}$. It follows from Lemma 3.3 that $x^{(0)}$ is uniformly recurrent. Let $u \in \mathcal{L}(x^{(0)})$. We may assume that

$$|u| > \max_{n\in\mathbb{Z}_+} \min_{a\in A_{n+1}} |\sigma_n(a)|.$$

There exists $n \in \mathbb{Z}_+$ such that

$$\min_{a\in A_{n+1}} |\sigma_{0,n}(a)| < |u| \leq \min_{a\in A_{n+2}} |\sigma_{0,n+1}(a)|.$$

There exists $w \in \mathcal{L}_2(x^{(n+2)})$ such that $u \prec \sigma_{0,n+1}(w)$. Put $D = \max_{n\in\mathbb{N}} D_n$. Let C be as in Lemma 4.2. Using Lemma 4.2, we obtain that for any $v \in \mathcal{R}_{x^{(0)},u}$,

$$\begin{aligned}
|v| &\leq D \max_{a\in A_{n+2}} |\sigma_{0,n+1}(a)| \\
&\leq CD \min_{a\in A_{n+2}} |\sigma_{0,n+1}(a)| \\
&\leq CD \max_{a\in A_{n+1}} |\sigma_{0,n}(a)| \min_{a\in A_{n+2}} |\sigma_{n+1}(a)| \\
&\leq C^2D \min_{a\in A_{n+1}} |\sigma_{0,n}(a)| \min_{a\in A_{n+2}} |\sigma_{n+1}(a)| \\
&\leq C^2D \min_{a\in A_{n+2}} |\sigma_{n+1}(a)| \cdot |u| \\
&\leq C^2D \max_{a\in A_{n+1}, n\in\mathbb{Z}_+} |\sigma_n(a)| \cdot |u|.
\end{aligned}$$

This completes the proof, because $C^2D \max_{a\in A_{n+1},n\in\mathbb{Z}_+} |\sigma_n(a)|$ is independent of the choice of u and v. □

Theorem 4.4 ([13, Proposition 1.1]). *If A is a finite alphabet and $x \in A^{\mathbb{Z}}$, then x is an S-adic sequence for some proper and primitive sequence S of finitely many morphisms if and only if x is linearly recurrent.*

Proof. $\Leftarrow$) We may assume that x is aperiodic. Put $\alpha = K^2(K+1)^2$, $u_n = x_{[-\alpha^n,-1]}$, $v_n = x_{[0,\alpha^n)}$, $\mathcal{R}_n = \mathcal{R}_{x,u_n.v_n}$, $R_n = \{1, 2, \dots, \sharp\mathcal{R}_n\}$, and $\phi_n = \phi_{u_n.v_n}$. Recall Figure 1. We may assume that $\phi_n(1) \prec_{\mathrm{p}} x_{[0,+\infty)}$ for all $n \in \mathbb{Z}+$. Define a morphism $\sigma_n : R_{n+1} \to R_n^+$ for $n \in \mathbb{Z}_+$ so that $\phi_{n+1} = \phi_n\sigma_n$. Again, recall Figure 1. Set $\sigma_0 = \phi_0$. Since we know that for any $i \in R_{n+1}$, $\sigma_n(i) = j_1j_2\dots j_m \Leftrightarrow \phi_{n+1}(i) = \phi_n(j_1)\phi_n(j_2)\dots\phi_n(j_m)$, where $j_1, j_2, \dots, j_m \in R_n$, we obtain that for any $i \in R_{n+1}$,

$$|\sigma_n(i)| \leq \frac{\max_{w\in\mathcal{R}_{n+1}} |w|}{\min_{w\in\mathcal{R}_n} |w|} \leq \frac{2\alpha^{n+1}K}{\frac{2\alpha^n}{K}} = \alpha K^2.$$

This together with Lemma 4.1 (5) guarantees that $S = \{\sigma_n : R_{n+1} \to R_n^+\}_{n\in\mathbb{Z}_+}$ is finite, where $R_0 = A$. Since $\sigma_{0,n-1}(1) = \phi_n(1) \prec_{\mathrm{p}} x_{[0,+\infty)}$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} |\phi_n(1)| = \infty$ (Lemma 3.1) and x is uniformly recurrent, we conclude that x is an S-adic sequence.

To see that S is primitive, let $i \in R_n$. It follows from Lemma 4.1 (1) that $u_n\phi_n(i)v_n \prec u$ for all $u \in \mathcal{L}_{(K+1)(|\phi_n(i)|+2\alpha^n)-1}(x)$, so that $u_n\phi_n(i)v_n \prec u_n\phi_{n+1}(j)v_n$ for all $j \in R_{n+1}$, because

$$\begin{aligned}(K+1)(|\phi_n(i)| + 2\alpha^n) - 1 &\leq (K+1)(2\alpha^n K + 2\alpha^n) - 1\\ &= 2\alpha^n(K+1)^2 - 1\\ &< 2\alpha^n\{1 + K(K+1)^2\}\\ &< |u_n\phi_{n+1}(j)v_n|.\end{aligned}$$

Since $n \in \mathbb{Z}_+$ is arbitrary, S is primitive with constant 1.

The morphism σ_0 is proper, because $x_0 \prec_{\mathrm{p}} \sigma_0(i)$ and $x_{-1} \prec_{\mathrm{s}} \sigma_0(i)$ for all $i \in R_0$. Let $i \in R_{n+1}$ for $n \in \mathbb{Z}_+$. Since $|v_{n+1}| = K^2(K+1)^2\alpha^n > (2K+1)\alpha^n \geq |\phi_n(\sigma_n(i)_1)v_n|$, we have $\phi_n(\sigma_n(i)_1)v_n \prec_{\mathrm{p}} v_{n+1}$. Hence, given $i, j \in R_{n+1}$, one of the words $\phi_n(\sigma_n(i)_1)$ and $\phi_n(\sigma_n(j)_1)$ is a prefix of the other. The words must coincide, i.e. $\phi_n(\sigma_n(i)_1) = \phi_n(\sigma_n(j)_1)$, because they are both return words to $u_n.v_n$. Since ϕ_n is injective on R_n, we conclude that $\sigma_n(i)_1 = \sigma_n(j)_1$ for all $i, j \in R_{n+1}$. The same argument shows that for arbitrary $i, j \in R_{n+1}$, the last letters of $\sigma_n(i)$ and $\sigma_n(j)$ coincide, so that σ_n is proper.

$\Rightarrow$) Since S is primitive, Lemma 2.1 allows us to have a consistent sequence $\{x^{(n)}\}_{n\in\mathbb{Z}_+} \in \prod_{n\in\mathbb{Z}_+} X_{S^{(n)}}$. We may assume that $x^{(0)} = x$, because they are both uniformly recurrent sequences belonging to X_S; see Lemma 3.3. By replacing S with its telescoping if necessary, we may assume that S is primitive with constant 1. Notice that the telescoping preserves the fact that S is proper and finite.

Let $n \in \mathbb{Z}_+$ be arbitrary. Let ℓ and r denote respectively the first and last letter of $\sigma_n(a)$ with $a \in A_{n+1}$. Since S is proper, the letters r and ℓ are independent of the choice of the letter a. In view of Lemma 4.3, it is enough to show that the sequence $\{D_n\}_n$ is bounded. If $u \in \mathcal{L}_2(x^{(n)})$, then $u = r\ell$ or $u \prec \sigma_n(a)$ for some $a \in A_{n+1}$. If $u = r\ell$, then u occurs infinitely often in $x^{(n)}$ with a gap bounded by $\max_{b\in A_{n+1}} |\sigma_n(b)|$. Assume the other case. The letter a occurs in $x^{(n+1)}$ with a gap bounded by $2\max_{b\in A_{n+2}} |\sigma_{n+1}(b)|$. Hence, the word u occurs in $x^{(n)}$ with a gap bounded by $C_n := 2\max_{b\in A_{n+2}} |\sigma_{n+1}(b)| \max_{b\in A_{n+1}} |\sigma_n(b)|$. It follows therefore that $\sup_{n\in\mathbb{Z}_+} D_n \leq \max_{n\in\mathbb{Z}_+} C_n < \infty$. This completes the proof. $\square$

As is shown in [13, Section 2], there exist a non-proper and primitive sequence S of finitely many morphisms generating a uniformly recurrent S-adic sequence which is not linearly recurrent.

Put $A = \{a, b, c\}$. Define substitutions σ and τ on A by

$$\begin{array}{lll} \sigma(a) = acb, & \sigma(b) = bab, & \sigma(c) = cbc; \\ \tau(a) = abc, & \tau(b) = acb, & \tau(c) = aac. \end{array}$$

Then, a uniformly recurrent sequence

$$x := \lim_{n\to\infty} \sigma\tau\sigma^2\tau \dots \sigma^n\tau(b.a)$$

is not linearly recurrent. Put $\rho_n = \sigma\tau\sigma^2\tau \dots \sigma^n\tau$ and

$$y = \lim_{\ell\to\infty} \sigma^{n+1}\tau\sigma^{n+2}\tau \dots \sigma^{n+\ell}\tau(b.a).$$

Hence, $x = \rho_n(y)$. We can see that $ca \in \mathcal{L}(y)$, $w \in \mathcal{R}_{y,ca} \Rightarrow \rho_n(w) \in \mathcal{R}_{x,\rho_n(ca)}$, and $\dfrac{|\rho_n(w)|}{|\rho_n(ca)|} \geq \dfrac{3^{n+2}}{2}$ for all $n \in \mathbb{Z}_+$. By [13, Proposition 2.1], the sequence has a linear complexity.

Question 1. *Is it possible to find a recurrence property for infinite sequences over a finite alphabet which characterizes the class of uniformly recurrent sequences which are written as primitive S-adic sequences with S finite?*

A Bratteli diagram is said to be *uniformly bounded* [1] if the set of its incidence matrices is finite. An ordered Bratteli diagram is said to be *uniformly bounded* if the underlying Bratteli diagram is uniformly bounded. Theorem 4.4 together with Proposition 3.5 has the following consequence.

Proposition 4.5. *If X is a bilateral subshift over a finite alphabet A, then X is linearly recurrent, i.e. generated by a linearly recurrent sequence, if and only if there exists a uniformly bounded, ordered Bratteli diagram $B = (V, E, \geq)$ such that*

(1) *the sequence of morphisms read on B is finite, proper and primitive with constant* 1*;*

(2) *the Bratteli-Vershik system (X_B, λ_B) is conjugate to X.*

Proof. It is enough to consider the case where X is aperiodic.

$\Rightarrow$) Let $S = \{\sigma_n : A_{n+1} \to A_n^+\}_{n\in\mathbb{Z}_+}$ be as in the proof of $\Leftarrow$ in Theorem 4.4. Construct an ordered Bratteli diagram $B = (V, E, \geq)$ so that

(1) there exists a single edge from the top vertex to each vertex in V_1;

(2) $V_i = A_{i-1}$ for $i \in \mathbb{N}$;

(3) σ_i is a morphism read on E_{i+2} for all $i \in \mathbb{Z}_+$.

Recall that S is proper and primitive with constant 1. Hence, the ordered Bratteli diagram B is properly ordered, and also uniformly bounded, because S is finite. In view of [14, Subsection 5.4] or Proposition 3.5, the Bratteli-Vershik system (X_B, λ_B) is conjugate to the subshift X.

$\Leftarrow$) Let $S = \{\sigma_n\}_{n\in\mathbb{Z}_+}$ be the sequence of morphisms read on B. By telescoping B and by doing a *symbol splitting* [17, p. 70] if necessary, in addition to the assumption, we may assume that

(1) the number of edges from the top vertex to every vertex in V_1 is one;

(2) a level factor X_1 [10] of (X_B, λ_B) is conjugate to X;

It is sufficient to see that X_1 is linearly recurrent. For each integer $n \in \mathbb{N}$, let $a_n \in V_n$ denote the first letter of $\sigma_n(b)$ for any (or equivalently, some) letter $b \in V_{n+1}$. Then, for each $n \in \mathbb{N}$, the limit $x^{(n)} := \lim_{n\to\infty} \sigma_1\sigma_2 \dots \sigma_n(a_{n+1})$ exists in ${V_n}^{\mathbb{N}}$. Observe that every $x^{(n)}$ is uniformly recurrent (Lemma 3.3) and that $x^{(1)}$ generates X_1. It is not hard to see that $\{D_n\}_{n\in\mathbb{N}}$ is bounded, where D_n is the largest difference between two consecutive occurrences of a word of length 2 in $x^{(n)}$, we obtain the desired conclusion, i.e. $x^{(1)}$ is linearly recurrent. □

If a substitution σ is *aperiodic* and primitive, being a specific case of Proposition 4.5, then there exists an algorithm due to [14] in which we can construct a stationary, properly ordered Bratteli diagram whose Bratteli-Vershik system is conjugate to the subshift X_S for $S = \{\sigma\}_{n\in\mathbb{Z}_+}$.

The unique ergodicity of a linearly recurrent subshift is shown in [12, Section 4] by means of [3, Theorem 1.2]: if a minimal subshift X is not uniquely ergodic and has an invariant probability measure μ, then $\lim_{n\to\infty} n\epsilon(n) = 0$, where $\epsilon(n) = \min\{\mu([u]); u \in \mathcal{L}_n(X)\}$. On the other hand, as is shown in [24, Theorem V.13], the unique ergodicity of the subshift generated by a fixed point of a primitive substitution is proved by means of a classical criterion [21, (5.3)] for determining whether or not a given topological dynamical system is uniquely ergodic.

Question 2. *Given a linearly recurrent sequence x, show that for any $\epsilon > 0$ and $u \in \mathcal{L}(x)$, there exist constants $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$ and integers $n \geq n_0$,*

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} 1_{[u]}(T^{k+i}x) - c \right| < \epsilon.$$

Consequently, the linearly recurrent subshift generated by the sequence x is uniquely ergodic.

§ 5. Almost minimal sequences

Definition 5.1. Let A be a finite alphabet. We say that a sequence $x \in A^{\mathbb{Z}}$ is *almost minimal* if there exists $a \in A$ such that

(1) for each $p \in \mathbb{N}$, the power a^p occurs in x with bounded gap;

(2) for every $u \in \mathcal{L}(x)^+$, there exists $n \in \mathbb{N}$ such that if $x_I = a^n$ for some closed interval I and $x_i \neq a$ for some $i \in \mathbb{Z}$ with distance 1 from I then u has an occurrence in x within distance n from the occurrence i.

Whenever we deal with almost minimal sequences, we fix the sense of the symbol a as in the definition, and set $A' = A \setminus \{a\}$ and $\mathcal{L}(x)' = \mathcal{L}(x) \setminus \{a^n\}_{n\in\mathbb{Z}_+}$. It is not difficult to see that given an almost minimal sequence x, some word in $\mathcal{L}(x)'$ occurs only finitely often in x if and only if x is ultimately periodic. An ultimately periodic sequence $\dots aaa.baaa\dots$ is a trivial example of an almost minimal sequence, which generates a subshift conjugate to a unique extension of a translation on $\mathbb{Z}$ to the one-point compactification $\mathbb{Z} \cup \{\infty\}$. Non-trivial examples are *quasi-fixed points* of *almost primitive* substitutions [26, Definition 2.5]; see also below. A topological dynamical system (X, T) is said to be *almost minimal* [8] if T has a unique fixed point and the orbit of any other point is dense in X. Given a sequence $x \in A^{\mathbb{Z}}$, x is almost minimal if and only if the associated subshift X is almost minimal [26, Lemma 3.6]. Clearly, its unique fixed point is $a^\infty.a^\infty$. We always let X' denote the complement of the fixed point in X if x is an almost minimal sequence.

Remark 4. By replacing an almost minimal sequence x with another element of X' if necessary, we may always assume that $x_{(-\infty,-1]} = a^\infty$ and $x_0 \neq a$.

Suppose that an almost minimal sequence x is not ultimately periodic. Put $A = \mathcal{L}_1(x)$, $\mathcal{R}_n = \mathcal{R}_{a^n.a^n}$, $R_n = R_{a^n.a^n}$ and $\phi_n = \phi_{a^n.a^n}$. Since any power of the letter a occurs in x with bounded gap, $\mathcal{R}_n$ is finite. Assign the numbers in R_n to the elements of $\mathcal{R}_n$ according to order of their first occurrences in $x_{[-n-1,\infty)}$, so that $\phi_n(1) = a$. As is done above, for each $n \in \mathbb{N}$, we obtain a morphism $\sigma_n : R_{n+1} \to {R_n}^+$ for which a diagram corresponding to one in Figure 1 commutes. Clearly, $\sigma_n(1) = 1$. Since any word in $\mathcal{R}_n$ is of the form $a^n w a^n$ with $w_1 \neq a$ and $w_{|w|} \neq a$, for every $i \in R_{n+1}$ there exists $\alpha \in {R_n}^+$ such that $\sigma_n(i) = 1\alpha 1$, $\alpha_1 \neq 1$ and $\alpha_{|\alpha|} \neq 1$. Hence, $S := \{\sigma_n : R_{n+1} \to {R_n}^+\}_{n \in \mathbb{Z}_+}$ is proper, where $\sigma_0 = \phi_1$ and $R_0 = A$.

Lemma 5.2. *For all $u \in \mathcal{L}(x)$, there exists $N \in \mathbb{N}$ such that if $|x_I|_{\neg a} \geq N$ for a closed interval I then $u \prec x_I$. In addition,*

$$c_n = \min\{|w|_{\neg a}; w \in \mathcal{R}'_n := \mathcal{R}_n \setminus \{a\}\} \uparrow \infty \text{ as } n \to \infty.$$

Proof. Let us verify the first assertion. For the word u, take $n \in \mathbb{N}$ as in Definition 5.1. Fix $g \in \mathbb{N}$ so that $a^n \prec x_J$ for all closed intervals J of length g. Taking $N = g \vee 2(n + |u|)$ yields the desired conclusion. To show the second half, assume that $\{c_n\}$ is bounded. For all sufficiently large n, there exist $w_n \in \mathcal{R}'_n$ such that $|w_n|_{\neg a}$ is constant. This implies the existence of a sequence in X' of the form $a^\infty.wa^\infty$ with $w \in \mathcal{L}(x)'$. However, x is not ultimately periodic. □

Corollary 5.3. *For any $u \in \mathcal{L}(x)^+$, $\mathcal{R}_u \neq \emptyset$, and $\{|v|_{\neg a}; v \in \mathcal{R}_u\}$ is bounded.*

Proof. This follows immediately from the first statement of Lemma 5.2. □

Definition 5.4. A sequence $S = \{\sigma_n : A_{n+1} \to {A_n}^+\}_{n \in \mathbb{Z}_+}$ of morphisms is said to be *almost primitive* with constant n_0 if there exists $\{a_n\}_n \in \prod_{n=0}^\infty A_n$ such that

(1) there exists $p_n \in \mathbb{N}$ such that $\sigma_n(a_{n+1}) = {a_n}^{p_n}$ for all $n \in \mathbb{Z}_+$;

(2) ${a_0}^p \in \mathcal{L}(S)$ for all $p \in \mathbb{N}$;

(3) $|\sigma_n(b)|_{\neg a_n} > 0$ for all $b \in A'_{n+1} := A_{n+1} \setminus \{a_{n+1}\}$ and $n \in \mathbb{Z}_+$;

(4) there exists $n_0 \in \mathbb{N}$ such that $b \prec \sigma_n \sigma_{n+1} \ldots \sigma_{n+n_0-1}(c)$ for all $n \in \mathbb{Z}_+$, $b \in A_n$ and $c \in A'_{n+n_0}$.

We say that S is *weakly almost primitive* if all the conditions holds in such a way that n_0 depends on n.

Now, we go back to the argument prior to Lemma 5.2. It is straightforward to see the lemma guarantees the weak almost primitivity of S. This shows that (1) $\Rightarrow$ (2) in the following theorem.

Theorem 5.5. *If a bilaterally infinite sequence x over a finite alphabet is not ultimately periodic, then the following are equivalent:*

(1) *x is almost minimal;*

(2) *Statement* (A) *with $\spadesuit =$ "proper and weakly almost primitive" and $\heartsuit =$ "quasi-consistent";*

(3) *Statement* (A) *with $\spadesuit =$ "proper and almost primitive" and $\heartsuit =$ "quasi-consistent".*

Proof. (2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (1): Replacing S with its telescoping if necessary, we may assume that S is almost primitive with constant 1. It follows that $\mathcal{L}(x) = \mathcal{L}(S)$. Given $p \in \mathbb{N}$, find $n_0 \in \mathbb{Z}_+$ such that ${a_0}^p \prec \sigma_{0,n_0}(c)$ for all $c \in A'_{n_0+1}$. This implies that ${a_0}^p \prec \sigma_{0,n_0}(w)$ for any $w \in \mathcal{L}_p(x^{(n_0+1)})$, so that ${a_0}^p$ occurs in x with bounded gap.

Given $u \in \mathcal{L}(x)^+$, choose $n \in \mathbb{Z}_+$ so that $u \prec \sigma_{0,n}(c)$ for all $c \in A'_{n+1}$. One can find $p \in \mathbb{N}$ so that if $x_{[i,i+p)} = {a_0}^p$ and $x_{i+p} \neq a_0$ then x_{i+p} is the first letter of $\sigma_{0,n}(c)$ with $c \in A_{n+1}$, skipping all the a_0. This completes the proof. $\square$

Question 3. *When the proper and almost primitive sequence S of morphisms in Theorem 5.5 is finite? Is it characterized in terms of any recurrence property for almost minimal sequences?*

For any $k \in R'_{n+1}$, the word $\phi_{n+1}(k)$ has a unique decomposition $\phi_{n+1}(k) = u_1 u_2 \dots u_m$ into words in $\mathcal{R}_n$. It follows that $u_1 = u_m = a$ and $u_2, u_{m-1} \in \mathcal{R}'_n$. If some $u_i, u_j \in \mathcal{R}'_n$ with $i < j$ are consecutive up to a's, i.e. $u_k = a$ for all k with $i < k < j$, then it is necessary that $2n + j - 1 - i < 2(n+1)$, i.e. $j - 1 - i \leq 1$, because $a^{2(n+1)}$ does not occur in $\phi_{n+1}(k)$. This shows that $|\sigma_n(k)|_1 \leq |\sigma_n(k)|_{\neg 1} + 1$.

Lemma 5.6. *Hence, the sequence S is finite if and only if there exists $C > 0$ such that $\sharp\mathcal{R}_n \leq C$ and $|\sigma_n(i)|_{\neg 1} \leq C$ for all $i \in R'_{n+1}$ and $n \in \mathbb{Z}_+$.*

To resolve Question 3, it might be an appropriate attempt to investigate the case where all the morphisms in the sequence S are identical. A substitution $\sigma : A \to A^+$ is said to be *almost primitive* [26, Definition 2.1] if

(1) there exists a unique letter $a \in A$ such that $\sigma(a)$ is a power of a;

(2) there exists $n \in \mathbb{N}$ such that $b \prec \sigma^n(c)$ for all $b \in A$ and $c \in A' := A \setminus \{a\}$;

(3) any power of a belongs to $\mathcal{L}(\sigma) := \mathcal{L}(S)$, where $S = \{\sigma\}_{n \in \mathbb{Z}_+}$.

The author [26] studied this class of substitutions from the viewpoints of invariant measures. Before that, a concrete almost primitive substitution, the Cantor substitution defined by $a \mapsto aaa$ and $b \mapsto bab$, was studied in [16, 23]. Another almost primitive substitution also appears in [15, Example 3.8]. The class of almost primitive substitutions was extended by [18] to a larger class. An example of such substitutions is β defined in [15, Example 3.12].

The incidence matrix M_σ has the form $\begin{bmatrix} p & O \\ * & Q \end{bmatrix}$, where Q is a primitive matrix having a dominant eigenvalue $\theta > 1$. A nonnegative, square matrix is said to be *primitive* if every entry of some power of the matrix is positive. The dominant eigenvalue means a positive eigenvalue which is greater than the absolute value of any other eigenvalue. Throughout the remainder, we use the symbols a, p, θ and A' in the above sense.

An almost primitive substitution $\sigma : A \to A^+$ gives rise to a bilaterally infinite sequence which plays a role similar to a fixed point of a primitive substitution. We may assume in view of [26, Lemma 2.4] that some $n \in \mathbb{N}$, $m \geq 2$, $b \in A'$ and $u \in A^*$ satisfy $ab \in \mathcal{L}(\sigma)$ and $\sigma^n(ab) = a^m bu$. To avoid the triviality, we deal with only the case $u \in A^+$. Taking a limit of a sequence $\{\sigma^{nk}(ab)\}_{k\in\mathbb{N}}$ of extending (or increasng) words, we obtain a bilaterally infinite sequence $x := \ldots aaaa.bu\sigma^n(u)\sigma^{2n}(u)\sigma^{3n}(u)\ldots$ over the finite alphabet A. The sequence x is a *quasi-periodic point* of x, i.e. $\sigma^n(x)$ coincides, up to a shift by some digits, with x. Since the almost primitivity guarantees that $\mathcal{L}(\sigma^n) = \mathcal{L}(\sigma)$ for all $n \in \mathbb{N}$, which does not change the associated subshift, we may assume $n = 1$ by considering σ^n instead of σ, so that x is a quasi-fixed point of σ, i.e. $\sigma(x)$ coincides with x up to a shift by some digits. Although σ is not necessarily proper, one can see that the sequence x is almost minimal. See also [26, Theorem 3.8]. Throughout the remainder, assume that x is not ultimately periodic. Put

$$M_k = \max_{b\in A} |\sigma^k(b)|, m_k = \min_{b\in A} |\sigma^k(b)|, M'_k = \max_{b\in A'} |\sigma^k(b)|_{\neg a} \text{ and } m'_k = \min_{b\in A'} |\sigma^k(b)|_{\neg a}.$$

The following lemma is a consequence of [26, Lemma 5.3].

Lemma 5.7.

(1) *If $p > \theta$, then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,*

$$c^{-1}p^k \leq m_k \leq M_k \leq cp^k \text{ and } c^{-1}\theta^k \leq m'_k \leq M'_k \leq c\theta^k.$$

(2) *If $p = \theta$, then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,*

$$c^{-1}p^k \leq m_k \leq cp^k, c^{-1}kp^k \leq M_k \leq ckp^k \text{ and } c^{-1}p^k \leq m'_k \leq M'_k \leq cp^k.$$

(3) *If $p < \theta$, then there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$,*

$$c^{-1}p^k \leq m_k \leq cp^k \text{ and } c^{-1}\theta^k \leq m'_k \leq M'_k \leq M_k \leq c\theta^k.$$

Lemma 5.8. *Suppose $p > 1$. Put $\alpha = \dfrac{\log\theta}{\log p}$.*

(1) *There exists $C \in \mathbb{N}$ such that $\max_{v\in\mathcal{R}_u} |v|_{\neg a} \leq C|u|^\alpha$ for any word $u \in \mathcal{L}(x)^+$.*

(2) *There exists* $C \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and* $v \in \mathcal{R}'_n$,

$$|v| \le \begin{cases} Cn & \text{if } p > \theta; \\ Cn\log n & \text{if } p = \theta; \\ Cn^\alpha & \text{if } p < \theta. \end{cases}$$

Proof. (1) Consider the case $p > \theta$. Let $u \in \mathcal{L}(x)^+$ be arbitrary. It follows from Corollary 5.3 that $g := \max\{|v|_{\neg a}; v \in \mathcal{R}_w, w \in \mathcal{L}_2(x)\} < \infty$. Take $k_0 \in \mathbb{N}$ so that $m_{k_0-1} \le |u| < m_{k_0}$. There exists $w \in \mathcal{L}_2(x)$ for which $u \prec \sigma^{k_0}(w)$. It follows from Lemma 5.7 that $|v|_{\neg a} \le (g+2)M'_{k_0} \le c^{1+\alpha}\theta(g+2)|u|^\alpha$ for all $v \in \mathcal{R}_u$, where c is as in Lemma 5.7. This shows the desired conclusion because $c^{1+\alpha}\theta(g+2)$ is a constant independent of the choice of u. In the other case, we can deduce the conclusion in the same way.

(2) Let $n \in \mathbb{N}$ and $v \in \mathcal{R}_n$. Fix $g \in \mathbb{N}$ so that $a \prec x_{[i,i+g)}$ for all $i \in \mathbb{Z}$. There exists $k_0 \in \mathbb{N}$ for which $p^{k_0-1} < 2n \le p^{k_0}$. Lemma 5.7 implies that $|v| \le 2gM_{k_0} \le 4cgp \cdot n$, where c is as in Lemma 5.7. In the same manner, one can verify the other assertions. $\square$

Question 4. *Is it possible to prove that if* $p > \theta$ *then there exists* $C \in \mathbb{N}$ *such that* $\min_{v\in\mathcal{R}'_n} |v|_{\neg a} \ge C^{-1}n^\alpha$ *for all* $n \in \mathbb{N}$*?*

The author believes that this property is reasonable to be expected. The Cantor substitution satisfies the property. The property implies, as shown below, that the sequence S of morphisms constructed in the proof of Theorem 5.5 (1) $\Rightarrow$ (2) is finite in the case $p > \theta$, where of course the sequence x in (1) should be now regarded as the quasi-fixed point. Suppose $p > \theta$. Let $n \in \mathbb{N}$ and $v \in \mathcal{R}_n$. Since it follows from Lemma 5.8 that $\max_{w\in\mathcal{R}_v} |w|_{\neg a} \le C|v|^\alpha \le C^{1+\alpha}n^\alpha$, any concatenation $w \in \mathcal{L}(x)$ of words in $\mathcal{R}_n$, satisfying $C^{1+\alpha}n^\alpha < |w|_{\neg a} \le C^{1+\alpha}n^\alpha + C(2n)^\alpha$, contains all the words belonging to $\mathcal{R}'_n$ as factors. If the above-mentioned expected property is the case, then

$$\sharp\mathcal{R}_n \le \sharp\mathcal{R}'_n + 1 \le \frac{|w|_{\neg a}}{\min\limits_{v\in\mathcal{R}'_n} |v|_{\neg a}} + 1 \le \frac{C^{1+\alpha}n^\alpha + C(2n)^\alpha}{C^{-1}n^\alpha} + 1 = C^{2+\alpha} + 2^\alpha C^2 + 1.$$

On the other hand, if it is the case again, then for all $i \in R'_{n+1}$,

$$|\sigma_n(i)|_{\neg 1} \le \frac{\max\limits_{v\in\mathcal{R}'_{n+1}} |v|_{\neg a}}{\min\limits_{v\in\mathcal{R}'_n} |v|_{\neg a}} \le \frac{C\{2(n+1)\}^\alpha}{C^{-1}(2n)^\alpha} = C^2\left(1+\frac{1}{n}\right)^\alpha \le 2^\alpha C^2,$$

which together with Lemma 5.6 concludes that S is finite.

Observe that, in the other case $p \le \theta$, the method as above will not show that S is finite even if the expected property is the case. However, we have an example [26] of the case $p = \theta$ in which the sequence S is finite. Let x be a quasi-fixed point of an almost primitive substitution σ defined by $a \mapsto aa$ and $b \mapsto abb$. Then, the sequence $S = \{\sigma_n : R_{n+1} \to {R_n}^+\}_{n\in\mathbb{Z}_+}$ is computed at

(1) $\sigma_0(1) = a$ and $\sigma_0(2) = abbabba$;

(2) $\sigma_n(1) = 1$ and $\sigma_n(2) = 121$ if $2^{k-1} \leq n < 2^k - 1$ for some $k \in \mathbb{N}$;

(3) $\sigma_n(1) = 1$ and $\sigma_n(2) = 12121$ if $n = 2^k - 1$ for some $k \in \mathbb{N}$.

At first sight, S is not *stationary*, which is different from the primitive case [14].

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