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Mahler's classification and a certain class of p-adic numbers

By

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Abstract

In this paper, we study a relation between digits of p-adic numbers and Mahler's classification. We show that an irrational p-adic number whose digits are automatic, primitive morphic, or Sturmian is an S-, T-, or U_1-number in the sense of Mahler's classification. Furthermore, we give an algebraic independence criterion for p-adic numbers whose digits are Sturmian.

§ 1. Introduction

Let p be a prime. We denote by $|\cdot|_p$ the absolute value of the p-adic number field $\mathbb{Q}_p$ normalized to satisfy $|p|_p = 1/p$. We denote by $\lfloor x \rfloor$ the integer part and $\lceil x \rceil$ the upper integer part of a real number x. We set $P := \{0, 1, \dots, p-1\}$.

Let $\mathcal{A}$ be a finite set. Let $\mathcal{A}^*, \mathcal{A}^+$, and $\mathcal{A}^{\mathbb{N}}$ denote the set of finite words over $\mathcal{A}$, the set of nonempty finite words over $\mathcal{A}$, and the set of infinite words over $\mathcal{A}$, respectively. We denote by $|W|$ the length of a finite word W over $\mathcal{A}$. For any integer $n \geq 0$, write $W^n = WW \dots W$ (n times repeated concatenation of the word W) and $\overline{W} = WW \dots W \dots$ (infinitely many times repeated concatenation of the word W). Note that W^0 is equal to the empty word. More generally, for any real number $w \geq 0$, write $W^w = W^{\lfloor w \rfloor} W'$, where W' is the prefix of W of length $\lceil (w - \lfloor w \rfloor)|W| \rceil$. Let $\mathbf{a} = (a_n)_{n \geq 0}$ be a sequence over the set $\mathcal{A}$. We identify $\mathbf{a}$ with the infinite word $a_0 a_1 \dots a_n \dots$. An infinite word $\mathbf{a}$ over $\mathcal{A}$ is said to be *ultimately periodic* if there exist finite words $U \in \mathcal{A}^*$ and $V \in \mathcal{A}^+$ such that $\mathbf{a} = U\overline{V}$.

We recall the definition of automatic sequences, primitive morphic sequences, and Sturmian sequences. Let $k \geq 2$ be an integer. A *k-automaton* is a sextuplet

$$A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau),$$

where Q is a finite set, $\Sigma_k = \{0, 1, \dots, k-1\}$, $\delta : Q \times \Sigma_k \to Q$ is a map, $q_0 \in Q$, Δ is a finite set, and $\tau : Q \to \Delta$ is a map. For an integer $n \geq 0$, we set $W_n := w_m w_{m-1} \dots w_0$,

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where $\sum_{i=0}^{m} w_i k^i$ is the k-ary expansion of n. For $q \in Q$, we define recursively $\delta(q, W_n)$ by $\delta(q, W_n) = \delta(\delta(q, w_n w_{n-1} \dots w_1), w_0)$. A sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is said to be k-*automatic* if there exists a k-automaton $A = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(\delta(q_0, W_n))$ for all $n \geq 0$. A sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is said to be *automatic* if there exists an integer $k \geq 2$ such that $\mathbf{a}$ is k-automatic.

Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is said to be a *morphism* if $\sigma(UV) = \sigma(U)\sigma(V)$ for all $U, V \in \mathcal{A}^*$. We define the *width* of σ by $\max_{a\in\mathcal{A}} |\sigma(a)|$. We say that σ is k-*uniform* if there exists an integer $k \geq 1$ such that $|\sigma(a)| = k$ for all $a \in \mathcal{A}$. In particular, we call a 1-uniform morphism a *coding*. A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is said to be *primitive* if there exists an integer $n \geq 1$ such that a occurs in $\sigma^n(b)$ for all $a, b \in \mathcal{A}$. A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is said to be *prolongable* on $a \in \mathcal{A}$ if $\sigma(a) = aW$ where $W \in \mathcal{A}^+$, and $\sigma^n(W)$ is not empty word for all $n \geq 1$. We say that a sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is *primitive morphic* if there exist finite sets $\mathcal{A}, \mathcal{B}$, a primitive morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ which is prolongable on some $a \in \mathcal{A}$, and a cording $\tau : \mathcal{A}^* \to \mathcal{B}^*$ such that $\mathbf{a} = \lim_{n\to\infty} \tau(\sigma^n(a))$.

Let $0 < \theta < 1$ be an irrational real number and ρ be a real number. For an integer $n \geq 1$, we put $s_{n,\theta,\rho} := \lfloor (n+1)\theta + \rho \rfloor - \lfloor n\theta + \rho \rfloor$ and $s'_{n,\theta,\rho} := \lceil (n+1)\theta + \rho \rceil - \lceil n\theta + \rho \rceil$. Note that $s_{n,\theta,\rho}, s'_{n,\theta,\rho} \in \{0, 1\}$. We also put $\mathbf{s}_{\theta,\rho} := (s_{n,\theta,\rho})_{n\geq 1}$ and $\mathbf{s}'_{\theta,\rho} := (s'_{n,\theta,\rho})_{n\geq 1}$. A sequence $\mathbf{a} = (a_n)_{n\geq 1}$ is called *Sturmian* if there exist an irrational real number $0 < \theta < 1$, a real number ρ, a finite set $\mathcal{A}$, and a coding $\tau : \{0, 1\}^* \to \mathcal{A}^*$ with $\tau(0) \neq \tau(1)$ such that $(a_n)_{n\geq 1} = (\tau(s_{n,\theta,\rho}))_{n\geq 1}$ or $(\tau(s'_{n,\theta,\rho}))_{n\geq 1}$. Then we call θ (resp. ρ) the slope (resp. the intercept) of $\mathbf{a}$.

Applying so-called Subspace Theorem, Adamczewski and Bugeaud [3] established a new transcendence criterion for p-adic numbers.

Theorem 1.1. *Let $\mathbf{a} = (a_n)_{n\geq 0}$ be a non-ultimately periodic sequence over P. Set $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. If the sequence $\mathbf{a}$ is automatic, primitive morphic, or Sturmian, then the p-adic number ξ is transcendental.*

In this paper, we study p-adic numbers which satisfy the assumption of Theorem 1.1 in more detail. For $\xi \in \mathbb{Q}_p$ and an integer $n \geq 1$, we define $w_n(\xi)$ (resp. $w_n^*(\xi)$) to be the supremum of the real number w (resp. w^*) which satisfy

$$0 < |P(\xi)|_p \leq H(P)^{-w-1} \quad (\text{resp. } 0 < |\xi - \alpha|_p \leq H(\alpha)^{-w^*-1})$$

for infinitely many integer polynomials $P(X)$ of degree at most n (resp. algebraic numbers $\alpha \in \mathbb{Q}_p$ of degree at most n). Here, $H(P)$, which is called the *height* of $P(X)$, is defined by the maximum of the usual absolute values of the coefficients of $P(X)$ and $H(\alpha)$, which is called the *height* of α, is defined by the height of the minimal polynomial of α over $\mathbb{Z}$. We set

$$w(\xi) := \limsup_{n\to\infty} \frac{w_n(\xi)}{n}, \quad w^*(\xi) := \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

A p-adic number ξ is said to be an

$$\begin{gathered} A\text{-}number \text{ if } w(\xi)=0; \\ S\text{-}number \text{ if } 0<w(\xi)<+\infty; \\ T\text{-}number \text{ if } w(\xi)=+\infty \text{ and } w_n(\xi)<+\infty \text{ for all } n; \\ U\text{-}number \text{ if } w(\xi)=+\infty \text{ and } w_n(\xi)=+\infty \text{ for some } n. \end{gathered}$$

Mahler [17] first introduced the classification. A p-adic number is algebraic if and only if it is an A-number. Almost all p-adic numbers are S-numbers in the sense of Haar measure. It is known that there exist uncountably many T-numbers. Liouville numbers are U-numbers, for example $\sum_{n=1}^{\infty} p^{n!}$. Replacing w_n and w with w_n^* and w^*, we define A^*-, S^*-, T^*-, and U^**-number* as above. It is known that the two classification of p-adic numbers coincide. Let $n \geq 1$ be an integer. For a U-number (resp. a U^*-number) $\xi \in \mathbb{Q}_p$, we say that ξ is a U_n*-number* (resp. a U_n^**-number*) if $w_n(\xi)$ is infinite and $w_m(\xi)$ are finite (resp. $w_n^*(\xi)$ is infinite and $w_m^*(\xi)$ are finite) for all $1 \leq m < n$. The detail is found in [9, Section 9.3].

We now state the main results.

Theorem 1.2. *Let $\mathbf{a}=(a_n)_{n\geq 0}$ be a non-ultimately periodic sequence over P. Set $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. If the sequence $\mathbf{a}$ is automatic, primitive morphic, or Sturmian with its slope whose continued fraction expansion has bounded partial quotients, then the p-adic number ξ is an S- or T-number. Furthermore, if the sequence $\mathbf{a}$ is Strumian with its slope whose continued fraction expansion has unbounded partial quotients, then the p-adic number ξ is a U_1-number.*

Theorem 1.2 is an extension of Theorem 1.1 and an analogue of Théorèmes 3.1, 4.2, and 5.1 in [5].

Theorem 1.3. *Let $\theta > 1$ be a real number whose continued fraction expansion has bounded partial quotients, $\theta' > 1$ be a real number whose continued fraction expansion has unbounded partial quotients, and ρ, ρ' be real numbers. Then the p-adic numbers*

$$\sum_{n=1}^{\infty} p^{\lfloor n\theta+\rho \rfloor}, \quad \sum_{n=1}^{\infty} p^{\lfloor n\theta'+\rho' \rfloor}$$

are algebraically independent.

Theorem 1.3 is an analogue of Corollaire 3.2 in [5].

This paper is organized as follows. In Section 2, we state Theorems 2.9 and 2.10, and prove the main results assuming Theorems 2.9 and 2.10. We prepare some lemmas to prove Theorems 2.9 and 2.10 in Section 3. In Section 4, we prove Theorems 2.9 and 2.10.

§ 2. Extension of the main results

Let $\mathbf{a}=(a_n)_{n\geq 0}$ be a sequence over a finite set $\mathcal{A}$. The k*-kernel* of $\mathbf{a}=(a_n)_{n\geq 0}$ is the set of all sequences $(a_{k^i m+j})_{m\geq 0}$, where $i \geq 0$ and $0 \leq j < k^i$.

Eilenberg [14] characterized k-automatic sequences.

Lemma 2.1. *Let $k \geq 2$ be an integer. Then a sequence is k-automatic if and only if its k-kernel is finite.*

We say that the sequence $\mathbf{a} = (a_n)_{n\geq 0}$ is *k-uniform morphic* if there exist finite sets $\mathcal{A}, \mathcal{B}$, a k-uniform morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ which is prolongable on some $a \in \mathcal{A}$, and a coding $\tau : \mathcal{A}^* \to \mathcal{B}^*$ such that $\mathbf{a} = \lim_{n\to\infty} \tau(\sigma^n(a))$. Then we call $\mathcal{A}$ the *initial alphabet* associated with $\mathbf{a}$.

Cobham [12] showed another characterization of k-automatic sequences using k-uniform morphic sequences.

Lemma 2.2. *Let $k \geq 2$ be an integer. Then a sequence is k-automatic if and only if it is k-uniform morphic.*

The *complexity function* of the sequence $\mathbf{a}$ is given by

$$p(\mathbf{a}, n) := \mathrm{Card}\{a_i a_{i+1} \ldots a_{i+n-1} \mid i \geq 0\}, \quad \text{for } n \geq 1.$$

Let ρ be a real number. We say that $\mathbf{a}$ satisfies *Condition* $(*)_\rho$ if there exist sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and a sequence of nonnegative real numbers $(w_n)_{n\geq 1}$ such that

(i) the word $U_n V_n^{w_n}$ is the prefix of $\mathbf{a}$ for all $n \geq 1$,

(ii) $|U_n V_n^{w_n}|/|U_n V_n| \geq \rho$ for all $n \geq 1$,

(iii) the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing.

The *Diophantine exponent* of $\mathbf{a}$, first introduced in [2], is defined to be the supremum of a real number ρ such that $\mathbf{a}$ satisfy Condition $(*)_\rho$. We denote by $\mathrm{Dio}(\mathbf{a})$ the Diophantine exponent of $\mathbf{a}$. It is immediate that

$$1 \leq \mathrm{Dio}(\mathbf{a}) \leq +\infty.$$

We recall known results about Diophantine exponents and complexity function for automatic sequences, primitive morphic sequences, and Strumian sequences.

Adamczewski and Cassaigne [1] estimated the Diophantine exponent of k-automatic sequences.

Lemma 2.3. *Let $k \geq 2$ be an integer. Let $\mathbf{a}$ be a non-ultimately periodic and k-automatic sequence. Let m be a cardinality of the k-kernel of $\mathbf{a}$. Then we have*

$$\mathrm{Dio}(\mathbf{a}) < k^m.$$

Mossé's result [19] implies the following lemma.

Lemma 2.4. *Let $\mathbf{a}$ be a non-ultimately periodic and primitive morphic sequence. Then the Diophantine exponent of $\mathbf{a}$ is finite.*

Adamczewski and Bugeaud [5] established a relation between Strumian sequences and Diophantine exponents.

Lemma 2.5. *Let* $\mathbf{a}$ *be a Strumian sequence with slope* θ. *Then the continued fraction expansion of* θ *has bounded partial quotients if and only if the Diophantine exponent of* $\mathbf{a}$ *is finite.*

It is known that automatic sequences, primitive morphic sequences, and Sturmian sequences have low complexity.

Lemma 2.6. *Let* $k \geq 2$ *be an integer and* $\mathbf{a}$ *be a* k*-automatic sequence. Let* d *be a cardinality of the internal alphabet associated with* $\mathbf{a}$. *Then we have for all* $n \geq 1$

$$p(\mathbf{a}, n) \leq kd^2 n.$$

Proof. See [7, Theorem 10.3.1] or [12]. □

Lemma 2.7. *Let* $\mathbf{a}$ *be a primitive morphic sequence over a finite set of cardinality of* $b \geq 2$. *Let* v *be the width of a primitive morphism* σ *which generates the sequence* $\mathbf{a}$. *Then we have for all* $n \geq 1$

$$p(\mathbf{a}, n) \leq 2v^{4b-2} b^3 n.$$

Proof. See [7, Theorem 10.4.12]. □

Lemma 2.8. *Let* $\mathbf{a}$ *be a Sturmian sequence. Then we have for all* $n \geq 1$

$$p(\mathbf{a}, n) = n + 1.$$

Proof. See [7, Theorem 10.5.8]. □

Theorem 2.9. *Let* $\mathbf{a} = (a_n)_{n \geq 0}$ *be a non-ultimately periodic sequence over* P. *Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. *Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then the p*-adic number* ξ *is an* S-, T-, *or* U_1*-number.*

Theorem 2.9 is an analogue of Théorème 1.1 in [5]. There is a real continued fraction analogue of Theorem 2.9 in [10, Theorem 3.2].

Theorem 2.10. *Let* $\mathbf{a} = (a_n)_{n \geq 0}$ *be a non-ultimately periodic sequence over* P. *Set* $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. *Assume that there exist integers* $n_0 \geq 1$ *and* $\kappa \geq 2$ *such that for all* $n \geq n_0$,

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then the Diophantine exponent of $\mathbf{a}$ *is finite if and only if* ξ *is not a* U_1*-number. Furthermore, if the Diophantine exponent of* $\mathbf{a}$ *is finite, then we have*

$$w_1(\xi) \leq 8(\kappa + 1)^2 (2\kappa + 1) \operatorname{Dio}(\mathbf{a}) - 1. \tag{1}$$

There are various versions of Theorem 2.10: b-ary expansion for real numbers [5], continued fraction expansion for real numbers [10], formal power series over a finite field, and its continued fraction expansion [20].

Proof of Theorem 1.2 assuming Theorems 2.9 and 2.10. Since the sequence $\mathbf{a}$ is automatic, primitive morphic, or Strumian, ξ is an S-, T-, or U_1-number by Lemmas 2.6, 2.7, 2.8 and Theorem 2.9. It follows from Lemmas 2.3, 2.4, 2.5 and Theorem 2.10 that ξ is a U_1-number if $\mathbf{a}$ is Strumian with its slope whose continued fraction expansion has unbounded partial quotients, and ξ is an S- or T-number otherwise. $\square$

Let θ and ρ be real numbers. For an integer $n \geq 1$, we put

$$t_n := \begin{cases} 1 & \text{if } n = \lfloor k\theta + \rho \rfloor \text{ for some integer } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$t'_n := \begin{cases} 1 & \text{if } n = \lceil k\theta + \rho \rceil \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

We also put $\mathbf{t}_{\theta,\rho} := (t_n)_{n\geq 1}$ and $\mathbf{t}'_{\theta,\rho} := (t'_n)_{n\geq 1}$. The lemma below is well-known result.

Lemma 2.11. *Let $\theta > 1$ be an irrational real number and ρ be a real number. Then we have $\mathbf{t}_{\theta,\rho} = \mathbf{s}'_{1/\theta,-(\rho+1)/\theta}$ and $\mathbf{t}'_{\theta,\rho} = \mathbf{s}_{1/\theta,-(\rho+1)/\theta}$.*

Lemma 2.12 (Mahler [17])**.** *Let ξ, η be p-adic numbers. If ξ and η are algebraically dependent, then ξ and η are in the same class.*

Proof of Theorem 1.3 assuming Theorems 2.9 and 2.10. Set $\xi := \sum_{n=1}^{\infty} p^{\lfloor n\theta+\rho\rfloor}$ and $\eta := \sum_{n=1}^{\infty} p^{\lfloor n\theta'+\rho'\rfloor}$. By the definition, the digits of ξ and η are $\mathbf{t}_{\theta,\rho}$ and $\mathbf{t}_{\theta',\rho'}$, respectively. It follows from Lemma 2.11 that $\mathbf{t}_{\theta,\rho}$ (resp. $\mathbf{t}_{\theta',\rho'}$) is Strumian with its slope whose continued fraction expansion has bounded (resp. unbounded) partial quotients. Therefore, ξ is an S- or T-number and η is a U_1-number by Theorem 1.2. Hence, we see that ξ and η are algebraically independent by Lemma 2.12. $\square$

§ 3. Preliminaries

We recall several facts about the exponents w_n and w_n^*.

Theorem 3.1. *Let $n \geq 1$ be an integer and ξ be in $\mathbb{Q}_p$. Then we have*

$$w_n^*(\xi) \leq w_n(\xi) \leq w_n^*(\xi) + n - 1.$$

Proof. See [18]. $\square$

Theorem 3.2. *Let $n \geq 1$ be an integer and $\xi \in \mathbb{Q}_p$ be not algebraic of degree at most n. Then we have*

$$w_n(\xi) \geq n, \quad w_n^*(\xi) \geq \frac{n+1}{2}.$$

Furthermore, if $n = 2$, then $w_2^*(\xi) \geq 2$.

Proof. See [17, 18]. □

We recall Liouville inequality, that is, a non trivial lower bound of differences of two algebraic numbers.

Lemma 3.3. *Let $\alpha, \beta \in \mathbb{Q}_p$ be distinct algebraic numbers of degree m, n, respectively. Then we have*

$$|\alpha - \beta|_p \geq \frac{(m+1)^{-n}(n+1)^{-m}}{H(\alpha)^n H(\beta)^m}.$$

Proof. See [21, Lemma 2.5]. □

Applying Lemma 3.3, we give an estimate for the value of w_1.

Lemma 3.4. *Let ξ be in $\mathbb{Q}_p$ and $c_0, c_1, c_2, \theta, \rho, \delta$ be positive numbers. Let $(\beta_j)_{j\geq 1}$ be a sequence of positive integers with $\beta_j < \beta_{j+1} \leq c_0 \beta_j^\theta$ for all $j \geq 1$. Assume that there exists a sequence of distinct terms $(\alpha_j)_{j\geq 1}$ with $\alpha_j \in \mathbb{Q}$ such that for all $j \geq 1$*

$$\frac{c_1}{\beta_j^{1+\rho}} \leq |\xi - \alpha_j|_p \leq \frac{c_2}{\beta_j^{1+\delta}},$$
$$H(\alpha_j) \leq c_3 \beta_j.$$

Then we have

$$\delta \leq w_1(\xi) \leq (1+\rho)\frac{\theta}{\delta} - 1.$$

Remark. There are several versions of Lemma 3.4 as in [4, 5, 6, 8, 10, 11, 13, 16, 20, 22].

Proof. Let α be a rational number with sufficiently large height. We define the integer $j_0 \geq 1$ by $\beta_{j_0} \leq c_0(4c_2c_3H(\alpha))^{\theta/\delta} < \beta_{j_0+1}$. Firstly, we consider the case $\alpha = \alpha_{j_0}$. By the assumption, we obtain

$$|\xi - \alpha|_p \geq c_1 \beta_{j_0}^{-1-\rho} \geq c_0^{-1-\rho} c_1 (4c_2c_3)^{-(1+\rho)\theta/\delta} H(\alpha)^{-(1+\rho)\theta/\delta}.$$

Next, we consider the other case. Then, by the assumption, we have

$$H(\alpha) < (4c_2c_3)^{-1}(c_0^{-1}\beta_{j_0+1})^{\delta/\theta} \leq (4c_2c_3)^{-1}\beta_{j_0}^{\delta}.$$

Therefore, we obtain

$$|\alpha - \alpha_{j_0}|_p \geq (4H(\alpha)H(\alpha_{j_0}))^{-1} > c_2 \beta_{j_0}^{-1-\delta}$$

by Lemma 3.3. Hence, it follows that

$$\begin{aligned} |\xi - \alpha|_p = |\alpha - \alpha_{j_0}|_p &\geq (4H(\alpha)H(\alpha_{j_0}))^{-1} \\ &\geq 4^{-1-\theta/\delta} c_0^{-1} c_2^{-\theta/\delta} c_3^{-1-\theta/\delta} H(\alpha)^{-1-\theta/\delta}. \end{aligned}$$

By Theorem 3.1, we have $w_1(\xi) = w_1^*(\xi)$. Thus, we obtain

$$\delta \leq w_1(\xi) \leq \max\left((1+\rho)\frac{\theta}{\delta} - 1, \frac{\theta}{\delta}\right) = (1+\rho)\frac{\theta}{\delta} - 1.$$

□

We denote by $M_{\mathbb{Q}}$ the set of all prime numbers and ∞. We denote by $|\cdot|_\infty$ the usual absolute value in $\mathbb{Q}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ and $v \in M_{\mathbb{Q}}$, we define the *norm* and the *height* of $\mathbf{x}$ by $|\mathbf{x}|_v = \max_{1 \leq i \leq n} |x_i|_v$ and $H(\mathbf{x}) = \prod_{v \in M_{\mathbb{Q}}} |\mathbf{x}|_v$.

The proof of Theorem 2.9 mainly depends on the following theorem which is so-called Quantitative Subspace Theorem and consequence of Corollary 3.2 in [15].

Theorem 3.5. *Let $\alpha \in \mathbb{Q}_p$ be an algebraic number of degree d and $0 < \varepsilon < 1$. Define linear forms*

$$L_{1\infty}(X,Y,Z) = X, \quad L_{2\infty}(X,Y,Z) = Y, \quad L_{3\infty}(X,Y,Z) = Z,$$
$$L_{1p}(X,Y,Z) = X, \quad L_{2p}(X,Y,Z) = Y, \quad L_{3p}(X,Y,Z) = \alpha X - \alpha Y - Z.$$

Then all integer solutions $\mathbf{x} = (x_1, x_2, x_3)$ of

$$\prod_{v \in \{p,\infty\}} \prod_{i=1}^{3} |L_{iv}(\mathbf{x})|_v \leq |\mathbf{x}|_\infty^{-\varepsilon}$$

with

$$H(\mathbf{x}) \geq \max\left(\left(\sqrt{d+1}H(\alpha)\right)^{1/12d}, 27^{1/\varepsilon}\right)$$

lie in the union of at most

$$2^{16}3^{39}5^{10}\varepsilon^{-9}\log(3\varepsilon^{-1}d)\log(\varepsilon^{-1}\log 3d)$$

proper linear subspaces of $\mathbb{Q}^3$.

Consider a vector hyperplane of $\mathbb{Q}^n$

$$\mathcal{H} = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid y_1 x_1 + \cdots + y_n x_n = 0\},$$

where $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}, \gcd(y_1, \ldots, y_n) = 1$. The *height* of $\mathcal{H}$, denoted by $H(\mathcal{H})$, is defined to be $|\mathbf{y}|_\infty$.

The lemma below is easily seen.

Lemma 3.6. *Let m, n be integers with $1 \leq m < n$ and $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{Z}^n$ be linearly independent vectors such that $|\mathbf{x}_1|_\infty \leq \ldots \leq |\mathbf{x}_m|_\infty$. Then there exists a vector hyperplane $\mathcal{H}$ of $\mathbb{Q}^n$ such that $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathcal{H}$ and*

$$H(\mathcal{H}) \leq m!|\mathbf{x}_m|_\infty^m.$$

Lemma 3.7. *Let $U \in P^*, V \in P^+$, and r, s be lengths of the words U, V, respectively. Put $(a_n)_{n\geq 0} := U\overline{V}$ and $\alpha := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. Then we have $H(\alpha) \leq p^{r+s}$.*

Proof. A straightforward computation shows that

$$\begin{aligned}\alpha &= \sum_{n=0}^{r-1} a_n p^n + \left(\sum_{m=0}^{s-1} a_{m+r} p^{m+r}\right)\left(\sum_{k=0}^{\infty} p^{ks}\right) \\ &= \frac{(p^s-1)\sum_{n=0}^{r-1} a_n p^n - \sum_{m=0}^{s-1} a_{m+r} p^{m+r}}{p^s - 1}.\end{aligned}$$

Therefore, we have

$$H(\alpha) \leq \max\left(p^s - 1, (p^s-1)\sum_{n=0}^{r-1} a_n p^n, \sum_{m=0}^{s-1} a_{m+r} p^{m+r}\right) \leq p^{r+s}.$$

□

In order to prove Theorem 2.10, we show the following lemma.

Lemma 3.8. *Let $\mathbf{a} = (a_n)_{n\geq 0}$ be a non-ultimately periodic sequence over P. Set $\xi := \sum_{n=0}^{\infty} a_n p^n \in \mathbb{Q}_p$. Then we have*

$$w_1(\xi) \geq \max(1, \mathrm{Dio}(\mathbf{a}) - 1). \tag{2}$$

Proof. Since ξ is irrational, we have $w_1(\xi) \geq 1$ by Theorem 3.2. Without loss of generality, we may assume that $\mathrm{Dio}(\mathbf{a}) > 1$. Take a real number δ such that $1 < \delta < \mathrm{Dio}(\mathbf{a})$. For $n \geq 1$, there exist finite words U_n, V_n and a positive rational number w_n such that $U_n V_n^{w_n}$ are the prefix of $\mathbf{a}$, the sequence $(|V_n^{w_n}|)_{n\geq 1}$ is strictly increasing, and $|U_n V_n^{w_n}| \geq \delta |U_n V_n|$. For $n \geq 1$, we set rational number

$$\alpha_n := \sum_{i=0}^{\infty} b_i^{(n)} p^i$$

where $(b_i^{(n)})_{i\geq 0}$ is the infinite word $U_n \overline{V_n}$. Since ξ and α_n have the same first $|U_n V_n^{w_n}|$-th digits, we obtain

$$|\xi - \alpha_n| \leq p^{-\delta |U_n V_n|} \leq H(\alpha_n)^{-\delta}$$

by Lemma 3.7. Hence, we have (2). □

The following lemma is a slight improvement of a part of Lemma 9.1 in [10].

Lemma 3.9. *Let $\mathbf{a} = (a_n)_{n\geq 0}$ be a sequence on a finite set $\mathcal{A}$. Assume that there exist integers $\kappa \geq 2$ and $n_0 \geq 0$ such that for all $n \geq n_0$,*

$$p(\mathbf{a}, n) \leq \kappa n.$$

Then, for each $n \geq n_0$, there exist finite words U_n, V_n over $\mathcal{A}$ and a positive rational number w_n such that the following hold:

(i) *$U_nV_n^{w_n}$ is a prefix of $\mathbf{a}$,*

(ii) $|U_n| \le 2\kappa|V_n|$,

(iii) $n/2 \le |V_n| \le \kappa n$,

(iv) *if U_n is not an empty word, then the last letter of U_n and V_n are different,*

(v) $|U_nV_n^{w_n}|/|U_nV_n| \ge 1 + 1/(4\kappa+2)$,

(vi) $|U_nV_n| \le (\kappa+1)n - 1$,

Proof. For $n \ge 1$, we denote by $A(n)$ the prefix of $\mathbf{a}$ of length n. By Pigeonhole principle, for each $n \ge n_0$, there exists a finite word W_n of length n such that the word appears to $A((\kappa+1)n)$ at least twice. Therefore, for each $n \ge n_0$, there exist finite words $B_n, D_n, E_n \in \mathcal{A}^*$ and $C_n \in \mathcal{A}^+$ such that

$$A((\kappa+1)n) = B_nW_nD_nE_n = B_nC_nW_nE_n.$$

We take these words in such way that if B_n is not empty, then the last letter of B_n is different from that of C_n.

We first consider the case of $|C_n| \ge |W_n|$. Then, there exists $F_n \in \mathcal{A}^*$ such that

$$A((\kappa+1)n) = B_nW_nF_nW_nE_n.$$

Put $U_n := B_n, V_n := W_nF_n$, and $w_n := |W_nF_nW_n|/|W_nF_n|$. Since $U_nV_n^{w_n} = B_nW_nF_nW_n$, the word $U_nV_n^{w_n}$ is a prefix of $\mathbf{a}$. It is obvious that $|U_n| \le (\kappa-1)|V_n|$ and $n \le |V_n| \le \kappa n$. By the definition, we have (iv) and (vi). Furthermore, we see that

$$\frac{|U_nV_n^{w_n}|}{|U_nV_n|} = 1 + \frac{n}{|U_nV_n|} \ge 1 + \frac{1}{\kappa}.$$

We next consider the case of $|C_n| < |W_n|$. Since the two occurrences of W_n do overlap, there exists a rational number $d_n > 1$ such that $W_n = C_n^{d_n}$. Put $U_n := B_n, V_n := C_n^{\lceil d_n/2 \rceil}$, and $w_n := (d_n+1)/\lceil d_n/2 \rceil$. Obviously, we have (i) and (iv). Since $\lceil d_n/2 \rceil \le d_n$ and $d_n|C_n| \le 2\lceil d_n/2 \rceil |C_n|$, we get $n/2 \le |V_n| \le n$. Using (iii) and $|U_n| \le \kappa n - 1$, we see (ii) and (vi). It is immediate that $w_n \ge 3/2$. Hence, we obtain

$$\begin{aligned}\frac{|U_nV_n^{w_n}|}{|U_nV_n|} &= 1 + \frac{\lceil (w_n-1)|V_n| \rceil}{|U_nV_n|} \ge 1 + \frac{w_n - 1}{|U_n|/|V_n| + 1}\\ &\ge 1 + \frac{1/2}{2\kappa+1} = 1 + \frac{1}{4\kappa+2}.\end{aligned}$$

□

§ 4. Proof of Theorems 2.9 and 2.10

Proof of Theorem 2.9. By Theorem 1B in [3], ξ is transcendental, that is, ξ is not an A-number. Therefore, it is sufficient to prove that if ξ is not a U_1-number, then ξ is not a

U-number. For $n \geq n_0$, we take finite words U_n, V_n over P and positive rational numbers w_n satisfying Lemma 3.9 (i)-(vi). We define a positive integer sequence $(n_k)_{k\geq 0}$ by $n_{k+1} = 4(\kappa+1)n_k$ for $k \geq 0$. We set $r_k := |U_{n_k}|, s_k := |V_{n_k}|$, and $t_k := |U_{n_k}V_{n_k}|$ for $k \geq 0$. Then a straightforward computation shows that $2t_k \leq t_{k+1} \leq ct_k, r_k \leq 2\kappa s_k$ for $k \geq 0$, and $(s_k)_{k\geq 0}$ is strictly increasing, where $c = 8(\kappa+1)^2$. For $k \geq 0$, there exists an integer p_k such that

$$\frac{p_k}{p^{s_k}-1} = \sum_{i=0}^{\infty} b_i^{(k)} p^i$$

where $(b_i^{(k)})_{i\geq 0}$ is the infinite word $U_{n_k}\overline{V_{n_k}}$. Since ξ and $p_k/(p^{s_k}-1)$ have the same first $|U_{n_k}V_{n_k}^{w_{n_k}}|$-th digits, we obtain

$$\left|\xi - \frac{p_k}{p^{s_k}-1}\right|_p \leq p^{-wt_k},$$

where $w = 1 + 1/(4\kappa+2)$. Since the sequence $(s_k)_{k\geq 1}$ is strictly increasing, we may assume that $t_0 \geq 3$.

Let $\alpha \in \mathbb{Q}_p$ be an algebraic number of degree $d \geq 2$ with $H(\alpha) \geq \max(d+1, p^{s_0}, 27^{4\kappa+2})$. We define an integer $j \geq 1$ by $p^{s_{j-1}} \leq H(\alpha) < p^{s_j}$ and a real number χ by $|\xi - \alpha|_p = H(\alpha)^{-\chi}$. Without loss of generality, we may assume that $\chi > 0$. Put $M := \max\{m \in \mathbb{Z} \mid p^{wc^{m-1}t_j} < H(\alpha)^{\chi}\}$. In what follows, we estimate an upper bound of M. Therefore, we may assume that $M \geq 1$. Then we obtain $p^{wt_{j+h}} \leq p^{wc^{M-1}t_j}$ for all $0 \leq h \leq M-1$. Therefore, we have

$$\begin{aligned}
|p^{s_{j+h}}\alpha - \alpha - p_{j+h}|_p &= \left|\alpha - \frac{p_{j+h}}{p^{s_{j+h}}-1}\right|_p \\
&\leq \max\left(\left|\xi - \frac{p_{j+h}}{p^{s_{j+h}}-1}\right|_p, |\xi-\alpha|_p\right) \leq p^{-wt_{j+h}}
\end{aligned}$$

for $0 \leq h \leq M-1$. We define linear forms by

$$\begin{gathered}
L_{1\infty}(X,Y,Z) = X, \quad L_{2\infty}(X,Y,Z) = Y, \quad L_{3\infty}(X,Y,Z) = Z, \\
L_{1p}(X,Y,Z) = X, \quad L_{2p}(X,Y,Z) = Y, \quad L_{3p}(X,Y,Z) = \alpha X - \alpha Y - Z,
\end{gathered}$$

and put $\mathbf{x}_h := (p^{s_{j+h}}, 1, p_{j+h})$ for $0 \leq h \leq M-1$. By the proof of Lemma 3.7, we obtain

$$\prod_{v\in\{p,\infty\}} \prod_{i=1}^{3} |L_{iv}(\mathbf{x}_h)|_v \leq |\mathbf{x}_h|_\infty^{-1/(4\kappa+2)}$$

for all $0 \leq h \leq M-1$. We also have

$$H(\mathbf{x}_h) = |\mathbf{x}_h|_\infty \geq p^{s_{j+h}} \geq H(\alpha) \geq \max\left(\left(\sqrt{d+1}H(\alpha)\right)^{1/12d}, 27^{4\kappa+2}\right)$$

for all $0 \leq h \leq M-1$. Hence, by Theorem 3.5, for all $0 \leq h \leq M-1$, we obtain $\mathbf{x}_h$ in the union of N proper linear subspaces of $\mathbb{Q}^3$, where $N = \lfloor 2^{25}3^{39}5^{10}(2\kappa+1)^9 \log(6(2\kappa+1)d) \log(2(2\kappa+1)\log 3d) \rfloor$.

Assume that one of these linear subspaces of $\mathbb{Q}^3$ contains $L+1$ points of the set $\{\mathbf{x}_h \mid 0 \le h \le M-1\}$, where $L = \lceil \log_2((2\kappa+1)(4d+6+\log_p(2^{2d+1}(d+1))))\rceil$. It follows that there exist $(x,y,z) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ such that

$$xp^{s_{j+i_k}} + y + zp_{j+i_k} = 0, \quad (0 \le k \le L),$$

where $0 \le i_0 < i_1 < \ldots < i_L < M$. Since $\mathbf{x}_{i_0}$ and $\mathbf{x}_{i_1}$ are linearly independent, we chose $(x,y,z) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ such that $\max(|x|,|y|,|z|) \le 2p^{2t_{j+i_1}}$ by Lemma 3.6. Since $(s_k)_{k\ge 0}$ is strictly increasing, we have $z \neq 0$. A straightforward computation shows that

$$\begin{aligned}(1-p^{s_{j+i_k}})\alpha &= p^{s_{j+i_k}}\frac{x}{z} + \frac{y}{z} - (p^{s_{j+i_k}}\alpha - \alpha - p_{j+i_k}),\\ \alpha - \frac{y}{z} &= p^{s_{j+i_k}}\alpha + p^{s_{j+i_k}}\frac{x}{z} - (p^{s_{j+i_k}}\alpha - \alpha - p_{j+i_k})\end{aligned}$$

for all $0 \le k \le L$. Therefore, we obtain

$$\begin{aligned}|\alpha|_p = |(1-p^{s_{j+i_k}})\alpha|_p &\le \max\left(p^{-s_{j+i_k}}\left|\frac{x}{z}\right|_p, \left|\frac{y}{z}\right|_p, p^{-wt_{j+i_k}}\right)\\ &\le \max(|z|, p^{-wt_{j+i_k}}) \le 2p^{2t_{j+i_1}}.\end{aligned}$$

Hence, we have

$$\left|\alpha - \frac{y}{z}\right|_p \le \max(2p^{2t_{j+i_1}-s_{j+i_L}}, |z|p^{-s_{j+i_L}}, p^{-wt_{j+i_L}}) = 2p^{2t_{j+i_1}-s_{j+i_L}}. \tag{3}$$

It follows from Lemma 3.3 that

$$\begin{aligned}\left|\alpha - \frac{y}{z}\right|_p &\ge 2^{-d}(d+1)^{-1}H(\alpha)^{-1}H\left(\frac{y}{z}\right)^{-d}\\ &\ge 2^{-2d}(d+1)^{-1}p^{-2dt_{j+i_1}-s_j}.\end{aligned} \tag{4}$$

By the properties of $(s_k)_{k\ge 0}$ and $(t_k)_{k\ge 0}$, we have

$$s_{j+i_L} \ge \frac{t_{j+i_L}}{2\kappa+1} = \frac{1}{2\kappa+1}\frac{t_{j+i_L}}{t_{j+i_{L-1}}}\cdots\frac{t_{j+i_2}}{t_{j+i_1}}t_{j+i_1} \ge \frac{2^L t_{j+i_1}}{4\kappa+2}. \tag{5}$$

Applying (3), (4), and (5), we obtain

$$t_{j+i_1} \le \frac{(4\kappa+2)\log_p(2^{2d+1}(d+1))}{2^L-(4\kappa+2)(2d+3)} \le 2,$$

which is contradiction.

Hence, we get $M \le LN$. By the definition of M, we have

$$\begin{aligned}H(\alpha)^\chi &\le p^{wc^M t_j} \le p^{wc^{M+1}t_{j-1}}\\ &\le p^{wc^{M+1}(2\kappa+1)s_{j-1}} \le H(\alpha)^{wc^{M+1}(2\kappa+1)}.\end{aligned}$$

Therefore, we obtain

$$|\xi - \alpha|_p \ge H(\alpha)^{-wc^{LN+1}(2\kappa+1)},$$

which implies

$$w_d^*(\xi) \le \max(w_1(\xi), wc^{LN+1}(2\kappa+1)).$$

This completes the proof. $\square$

Proof of Theorem 2.10. We first assume that ξ is not a U_1-number, that is, $w_1(\xi)$ is finite. Then $\mathrm{Dio}(\mathbf{a})$ is finite by Lemma 3.8.

We next assume that $\mathrm{Dio}(\mathbf{a})$ is finite. For $n \ge n_0$, take finite words U_n, V_n and a rational number w_n satisfying Lemma 3.9 (i)-(vi). For $n \ge n_0$, we set rational numbers

$$\alpha_n := \sum_{i=0}^{\infty} b_i^{(n)} p^i$$

where $(b_i^{(n)})_{i\ge 0}$ is the infinite word $U_n\overline{V_n}$. Since ξ and α_n have the same first $|U_nV_n^{w_n}|$-th digits, we obtain

$$|\xi - \alpha_n| \le p^{-\left(1+\frac{1}{4\kappa+2}\right)|U_nV_n|}.$$

Take a real number δ which is greater than $\mathrm{Dio}(\mathbf{a})$. Note that $\delta > 1$. By the definition of the Diophantine exponent, there exists an integer $n_1 \ge n_0$ such that for all $n \ge n_1$

$$|\xi - \alpha_n| \ge p^{-\delta|U_nV_n|}.$$

We define a positive integer sequence $(n_k)_{k\ge 1}$ by $n_{k+1} = 2(\kappa+1)n_k$ for $k \ge 1$. Set $\beta_k := p^{|U_{n_k}V_{n_k}|}$ for $k \ge 1$. It follows from Lemma 3.9 (iii),(vi) that for $n \ge 1$

$$\beta_k < \beta_{k+1} \le \beta_k^{4(\kappa+1)^2}.$$

By Lemma 3.7, we have $H(\alpha_{n_k}) \le \beta_k$ for $k \ge 1$. Hence, we obtain (1) by Lemma 3.4. $\square$

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