RIMS Kôkyûroku Bessatsu
B58 (2016), 231–242

Dual substitutions over $\mathbb{R}_{>0}-$powered symbols

By

Jun-ichi TAMURA* and Shin-ichi YASUTOMI**

Abstract

We will consider dual substitutions of substitutions over $\mathbb{R}_{>0}-$powered symbols which are introduced in [12]. We give a class of weighted substitutions including the dual substitutions. We also introduce weighted tips which generalize the unit tip in a stepped surface (discrete plane).

§ 1. Introduction

Let $\mathscr{A}$ be an alphabet of d letters $\{1,2,\ldots,d\}$. In [12] substitutions over $\mathbb{C}-$powered symbols are considered. Its crucial point of the weighted substitution is to attach a weight to each letter. We use the terminology $[a_i : \alpha]$ for $a_i \in \mathscr{A}, \alpha \in \mathbb{C}^\times$ instead of the terminology a_i^α in [12]. Consider, for example a Fibonacci like substitution σ defined on the set of finite words over $\mathscr{A} = \{1,2\}$ which replaces $[1:1]$ with $[1:1][2:\sqrt{2}-1]$ and $[2:1]$ with $[1:1]$. Iterating σ on $[1:1]$, we have $[1:1] \to [1:1][2:\sqrt{2}-1] \to [1:1][2:\sqrt{2}-1][1:\sqrt{2}-1] \to [1:1][2:\sqrt{2}-1][1:\sqrt{2}-1][1:\sqrt{2}-1][2:(\sqrt{2}-1)^2]\ldots$. In [7, 8, 9] substitutions over $\mathbb{R}_{>0}-$powered symbols and related dynamical systems are considered.

In [12], we introduced geometric realization/representation in the unitary space $\mathbb{C}^d$ for infinite words over complex valued symbols, and we gave some new results related to Diophantine approximation of complex numbers and Rauzy sets in $\mathbb{C}^d$, which are the same as Rauzy fractals for ordinary substitutions. A dual substitution associated to an ordinary substitution which was introduced in [1] is an excellent tool in symbolic dynamical system and related areas (see [10]). In this paper we will consider dual substitutions of substitutions over $\mathbb{R}_{>0}-$powered symbols and introduce a class of substitutions including the dual substitution. We also give a generalization of the unit tip in a stepped surface (discrete plane), which is called a weighted tip. We remark that substitutions over $\mathbb{R}_{>0}-$powered symbols give algorithms by which we have algebraic integral points near certain hyperplane. For example, by iterating the above σ we have algebraic integral points in $\mathbb{Z}[\sqrt{2}] \times \mathbb{Z}[\sqrt{2}]$ close to some line(see [12]). Similarly, by iterating

Recieved October 31, 2015. Revised March 1, 2016. Accepted May 1, 2016.
2010 Mathematics Subject Classification(s): 68R15,37B10
Key Words: *substitution, dual substitution, substitution over p stepped surface, discrete plane, substitution over* $\mathbb{R}_{>0}-$ *powered symbols.*

*Institute for Mathematics and Computer Science Tsuda College, 2-1-1 Tsudamachi, Kodaira-shi,Tokyo 187-8577, Japan.
e-mail: `jtamura@tsuda.ac.jp`
**Faculty of Science, Toho university, 2-2-1 Miyama Funabashi-shi Chiba 274-8510 Japan.
e-mail: `shinichi.yasutomi@sci.toho-u.ac.jp`

the weighted dual maps of σ we have algebraic integral points in $\mathbb{Z}[\sqrt{2}]\times\mathbb{Z}[\sqrt{2}]$ close to some line. From this number theoretical point of view we serve an open problem.

§ 2. Dual substitution

In this section we consider a dual substitution for a substitution following [1]. We denote by $\mathscr{A}_R^*$ the free monoid on $\mathscr{A}\times\mathbb{R}_{>0}$, that is, $\mathscr{A}_R^* = \cup_{n=1}^{\infty}(\mathscr{A}\times\mathbb{R}_{>0})^n$. We denote an element w in $\mathscr{A}_R^*$ by $w=[w_1:\omega_1]\cdots[w_n:\omega_n]$, where $w_k\in\mathscr{A}$ and $\omega_k\in\mathbb{R}_{>0}$. For $w\in\mathscr{A}_R^*$ and $\alpha\in\mathbb{R}_{>0}$ we define $w^{\alpha}:=[w_1:\alpha\omega_1]\cdots[w_n:\alpha\omega_n]$. For $w\in\mathscr{A}_R^*$ and $i\in\mathscr{A}$, we define $|w|_i:=\sum_{w_k=i}\omega_k$. A map $f:\mathscr{A}_R^*\to\mathbb{R}^d$ is defined by

$$f(w):={}^t(|w|_1,\dots,|w|_d).$$

Then, we have $f(ww')=f(w)+f(w')$ for $w,w'\in\mathscr{A}_R^*$(see [12]).

The reader is referred to [1, 11] for a dual map of a substitution. A substitution σ over $\mathscr{A}_R^*$ is a homomorphism of $\mathscr{A}_R^*$ into $\mathscr{A}_R^*$ which satisfies that for every $u_1,u_2\in\mathscr{A}_R^*$, $\sigma(u_1u_2)=\sigma(u_1)\sigma(u_2)$ and for every $\alpha\in\mathbb{R}_{>0}$ and $u\in\mathscr{A}_R^*$, $\sigma(u^{\alpha})=\sigma(u)^{\alpha}$. Let $\sigma([i:1])=[a_1^{(i)}:\omega_1^{(i)}]\cdots[a_{l_i}^{(i)}:\omega_{l_i}^{(i)}]$ for $i=1,\dots,d$. We define $P(\sigma)_k^{(i)}(=P_k^{(i)})$ and $S(\sigma)_k^{(i)}(=S_k^{(i)})$ by $\sigma([i:1])=P(\sigma)_k^{(i)}[a_k^{(i)}:\omega_k^{(i)}]S(\sigma)_k^{(i)}$ for $k=1,2,\dots,l_i$. The incidence matrix $I_\sigma=(m_{kl})$ is $d\times d$ matrix with entries $m_{kl}=|\sigma([l:1])|_k$. We suppose that I_σ is regular and primitive.

Let $\mathscr{F}$ be the free $\mathbb{Z}$ module over the set $\mathbb{R}^d\times\mathscr{A}\times\mathbb{R}_{>0}$. We define the endomorphism $E_1(\sigma)$ on $\mathscr{F}$ as follows: for $(\bar{x},i,\omega)\in\mathbb{R}^d\times\mathscr{A}\times\mathbb{R}_{>0}$

$$E_1(\sigma)(\bar{x},i,\omega):=\sum_{k=1}^{l_i}(I_\sigma(\bar{x})+\omega f(P_k^{(i)}),a_k^{(i)},\omega\omega_k^{(i)}).$$

We denote by the $\mathscr{F}^*$ and $(\bar{x},i^*,\omega)$ the dual space of $\mathscr{F}$ and its element which maps $(\bar{x},i,\omega)$ to 1, and to 0 otherwise. We denote by $E_1^*(\sigma)$ the dual map of $E_1(\sigma)$. From the definition of $E_1^*(\sigma)$ we have the following lemma.

Lemma 2.1. *For substitutions σ_1,σ_2 over $\mathscr{A}_R^*$, $E_1^*(\sigma_1\sigma_2)=E_1^*(\sigma_2)\circ E_1^*(\sigma_1)$ holds.*

Lemma 2.2. *For a substitution σ over $\mathscr{A}_R^*$, we have*

$$E_1^*(\sigma)(\bar{x},i^*,\omega)=\sum_{j\in\mathscr{A}}\sum_{a_k^{(j)}=i}(I_\sigma^{-1}(\bar{x}-\frac{\omega}{\omega_k^{(j)}}f(P_k^{(j)})),j^*,\frac{\omega}{\omega_k^{(j)}}).$$

Proof. It is not difficult to see that for each j with $1\le j\le d$, $(I_\sigma(\bar{y})+\omega_2 f(P_k^{(j)}),a_k^{(j)},\omega_2\omega_k^{(j)})$ are different from each other for $k = 1, \ldots, l_j$. Hence we have

$$(\bar{x}, i^*, \omega)(E_1(\sigma)(\bar{y}, j, \omega_2)) = 1$$
$$\Updownarrow$$
$$(\bar{x}, i^*, \omega) \sum_{k=1}^{l_j} (I_\sigma(\bar{y}) + \omega_2 f(P_k^{(j)}), a_k^{(j)}, \omega_2 \omega_k^{(j)}) = 1$$
$$\Updownarrow$$
$$\bar{x} = I_\sigma(\bar{y}) + \omega_2 f(P_k^{(j)}), i = a_k^{(j)} \text{ and } \omega = \omega_2 \omega_k^{(j)},$$

which implies Lemma 2.2. □

Since I_σ is primitive, I_σ has an eigenvalue λ which is positive, simple and bigger in modulus than the other eigenvalues. ${}^tI_\sigma$ has a positive eigenvector $\bar{v}$ associated with λ. Let $\mathscr{P}$ be the plane which is orthogonal to $\bar{v}$.

Let $\{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. We say that $(\bar{x}, i^*, \omega) \in \mathscr{F}^*$ is a weighted tip, if $\bar{x} \cdot \bar{v} < 0$ and $(\bar{x} + \omega e_i) \cdot \bar{v} \geq 0$, where for $u_1, u_2 \in \mathbb{R}^d$, $u_1 \cdot u_2$ is the scalar product of two vectors.

We denote by $\mathscr{S}$ the set of all weighted tips in $\mathscr{F}^*$. We denote by $\overline{\mathscr{S}}$ the $\mathbb{Z}$ module over $\mathscr{S}$. We say an element in $\overline{\mathscr{S}}$ is geometric if its every coefficient represented by the basis $\mathscr{S}$ is 0 or 1.

Lemma 2.3. *If $(\bar{x}, i^*, \omega) \in \mathscr{S}$, then $E_1^*(\sigma)(\bar{x}, i^*, \omega) \in \overline{\mathscr{S}}$ and $E_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric.*

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathscr{S}$. By Lemma 2.2 we will check that for $j \in \mathscr{A}$, $k \in \{1, 2, \ldots, l_j\}$ with $\sigma([j:1]) = P_k^{(j)}[i : \omega_k^{(j)}]S_k^{(j)}$

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})), j^*, \frac{\omega}{\omega_k^{(j)}}) \in \mathscr{S}.$$

We see

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)}))) \cdot \bar{v} = (\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})) \cdot \frac{1}{\lambda}\bar{v} = \bar{x} \cdot \frac{1}{\lambda}\bar{v} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)}) \cdot \frac{1}{\lambda}\bar{v} < 0.$$

We have

$$\begin{aligned}(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})) + \frac{\omega}{\omega_k^{(j)}} e_j) \cdot \bar{v} &= (\bar{x} + \frac{\omega}{\omega_k^{(j)}} f(S_k^{(j)}) + \omega e_i) \cdot \frac{1}{\lambda}\bar{v} \\ &= (\bar{x} + \omega e_i) \cdot \frac{1}{\lambda}\bar{v} + \frac{\omega}{\omega_k^{(j)}} f(S_k^{(j)}) \cdot \frac{1}{\lambda}\bar{v} \geq 0.\end{aligned}$$

Thus, we have $(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_k^{(j)}} f(P_k^{(j)})), j^*, \frac{\omega}{\omega_k^{(j)}}) \in \mathscr{S}$. Secondly, we will show that $E_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric. We suppose that for $j_1, j_2 \in \mathscr{A}$, $k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1:1]) = P_{k_1}^{(j_1)}[i : \omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2:1]) = P_{k_2}^{(j_2)}[i : \omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$(I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)})), j_1^*, \frac{\omega}{\omega_{k_1}^{(j_1)}}) = (I_\sigma^{-1}(\bar{x} - \frac{\omega}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})), j_2^*, \frac{\omega}{\omega_{k_2}^{(j_2)}}),$$

which implies that $j_1 = j_2$, $\omega_{k_1}^{(j_1)} = \omega_{k_2}^{(j_2)}$ and $f(P_{k_1}^{(j_1)}) = f(P_{k_2}^{(j_2)})$. Then, it is not difficult to see that $k_1 = k_2$. Thus, we have Lemma 2.3. □

Lemma 2.4. *If* $(\bar{x_1}, i_1^*, \omega_1), (\bar{x_2}, i_2^*, \omega_2) \in \mathscr{S}$ *and* $(\bar{x_1}, i_1^*, \omega_1) \neq (\bar{x_2}, i_2^*, \omega_2)$*, then* $E_1^*(\sigma)((\bar{x_1}, i_1^*, \omega_1) + (\bar{x_2}, i_2^*, \omega_2))$ *is geometric.*

Proof. We suppose that $(\bar{x_1}, i_1^*, \omega_1), (\bar{x_2}, i_2^*, \omega_2) \in \mathscr{S}$ and $E_1^*(\sigma)((\bar{x_1}, i_1^*, \omega_1) + (\bar{x_2}, i_2^*, \omega_2))$ is not geometric. By Lemma 2.2 and Lemma 2.3 there exist $j_1, j_2 \in \mathscr{A}$ and $k_1, k_2 \in \mathbb{Z}$ such that $a_{k_1}^{(j_1)} = i_1$, $a_{k_2}^{(j_2)} = i_2$ and $(I_\sigma^{-1}(\bar{x_1} - \frac{\omega_1}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)})), j_1^*, \frac{\omega_1}{\omega_{k_1}^{(j_1)}}) = (I_\sigma^{-1}(\bar{x_2} - \frac{\omega_2}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})), j_2^*, \frac{\omega_2}{\omega_{k_2}^{(j_2)}})$. Then, we have $j_1 = j_2$, $\frac{\omega_1}{\omega_{k_1}^{(j_1)}} = \frac{\omega_2}{\omega_{k_2}^{(j_2)}}$ and $\bar{x_1} - \frac{\omega_1}{\omega_{k_1}^{(j_1)}} f(P_{k_1}^{(j_1)}) = \bar{x_2} - \frac{\omega_2}{\omega_{k_2}^{(j_2)}} f(P_{k_2}^{(j_2)})$. We assume $k_1 \leq k_2$ without loss of generality. In the case of $k_1 = k_2$ we see $(\bar{x_1}, i_1^*, \omega_1) = (\bar{x_2}, i_2^*, \omega_2)$. Therefore, $k_1 < k_2$ holds. We have

$$\bar{x_2} = \bar{x_1} + \omega_1 e_{i_1} + \frac{\omega_1}{\omega_{k_1}^{(j_1)}} \left(f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}] \cdots [a_{k_2-1}^{(j_1)} : \omega_{k_2-1}^{(j_1)}]) - f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}]) \right),$$

which implies $\bar{x_2} \cdot \bar{v} \geq 0$. This is a contradiction.

□

We define $\mathscr{G}$ as the set of all geometric elements in $\overline{\mathscr{S}}$.

By Lemma 2.3 and Lemma 2.4 we have

Proposition 2.5. $E_1^*(\sigma)\overline{\mathscr{S}} \subset \overline{\mathscr{S}}$ *and* $E_1^*(\sigma)\mathscr{G} \subset \mathscr{G}$.

§ 3. Certain Extension

Now, we define a class of endomorphisms $T_\alpha^*(\sigma)$ on $\mathscr{F}^*$ for $\alpha \in \mathbb{R}$ by

$$T_\alpha^*(\sigma)(\bar{x}, i^*, \omega) := \sum_{j \in \mathscr{A}} \sum_{a_k^{(j)} = i} (I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})), j^*, \omega(\omega_k^{(j)})^{\alpha-1}).$$

We note that $E_1^*(\sigma) = T_0^*(\sigma)$. We call the above α the weight of $T_\alpha^*(\sigma)$.

Lemma 3.1. *If* $(\bar{x}, i^*, \omega) \in \mathscr{S}$*, then* $T_\alpha^*(\sigma)(\bar{x}, i^*, \omega) \in \overline{\mathscr{S}}$.

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathscr{S}$. We will check that for $j \in \mathscr{A}$, k with $\sigma([j:1]) = P_k^{(j)}[i : \omega_k^{(j)}]S_k^{(j)}$

$$(I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})), j^*, \omega(\omega_k^{(j)})^{\alpha-1}) \in \mathscr{S}. \tag{3.1}$$

We see

$$(I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)}))) \cdot \bar{v} = ((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})) \cdot \frac{1}{\lambda} \bar{v}$$

$$= (\omega_k^{(j)})^{\alpha-1} (\omega_k^{(j)} \bar{x} \cdot \frac{1}{\lambda} \bar{v} - \omega f(P_k^{(j)}) \cdot \frac{1}{\lambda} \bar{v}). \tag{3.2}$$

On the other hand,

$$\begin{aligned}&(I_\sigma^{-1}((\omega_k^{(j)})^\alpha \bar{x} - \omega(\omega_k^{(j)})^{\alpha-1} f(P_k^{(j)})) + \omega(\omega_k^{(j)})^{\alpha-1} e_j) \cdot \bar{v}\\ &= ((\omega_k^{(j)})^\alpha \bar{x} + \omega(\omega_k^{(j)})^{\alpha-1} f(S_k^{(j)}) + \omega(\omega_k^{(j)})^\alpha e_i) \cdot \frac{1}{\lambda}\bar{v}\\ &= (\omega_k^{(j)})^\alpha (\bar{x} + \omega e_i) \cdot \frac{1}{\lambda}\bar{v} + \omega(\omega_k^{(j)})^{\alpha-1} f(S_k^{(j)}) \cdot \frac{1}{\lambda}\bar{v}. \qquad (3.3)\end{aligned}$$

Since $\bar{x}\cdot\bar{v} < 0$ and $(\bar{x}+\omega e_i)\cdot\bar{v} \geq 0$, we have the inequalities (3.2) < 0 and (3.3) ≥ 0. Hence, we have $T_\alpha^*(\sigma)(\bar{x}, i^*, \omega) \in \overline{\mathscr{S}}$. □

We have following Lemma 3.2 as well as Lemma 2.4.

Lemma 3.2. *If $(\bar{x_1}, i_1^*, \omega_1), (\bar{x_2}, i_2^*, \omega_2) \in \mathscr{S}$ and $(\bar{x_1}, i_1^*, \omega_1) \neq (\bar{x_2}, i_2^*, \omega_2)$, then $T_\alpha^*(\sigma)((\bar{x_1}, i_1^*, \omega_1) + (\bar{x_2}, i_2^*, \omega_2))$ is geometric for $\alpha \neq 1$.*

Proof. Let $\alpha \neq 1$. First, we will show that $T_\alpha^*(\sigma)(\bar{x}, i_1^*, \omega_1)$ is geometric. We suppose that for $j_1, j_2 \in \mathscr{A}$, $k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1 : 1]) = P_{k_1}^{(j_1)}[i_1 : \omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2 : 1]) = P_{k_2}^{(j_2)}[i_1 : \omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$\begin{aligned}&(I_\sigma^{-1}((\omega_{k_1}^{(j_1)})^\alpha \bar{x} - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)})), j_1^*, \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1})\\ &= (I_\sigma^{-1}((\omega_{k_2}^{(j_2)})^\alpha \bar{x} - \omega_1(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})), j_2^*, \omega_1(\omega_{k_2}^{(j_2)})^{\alpha-1}),\end{aligned}$$

which implies that $j_1 = j_2$ and $k_1 = k_2$. Therefore, $T_\alpha^*(\sigma)(\bar{x}, i_1^*, \omega_1)$ is geometric. Similarly, $T_\alpha^*(\sigma)(\bar{x}, i_2^*, \omega_2)$ is geometric.

Secondly, we suppose that $(\bar{x_1}, i_1^*, \omega_1), (\bar{x_2}, i_2^*, \omega_2) \in \mathscr{S}$ and $T_\alpha^*(\sigma)((\bar{x_1}, i_1^*, \omega_1) + (\bar{x_2}, i_2^*, \omega_2))$ is not geometric. Then, there exist $j_1, j_2 \in \mathscr{A}$ and $k_1, k_2 \in \mathbb{Z}$ such that $a_{k_1}^{(j_1)} = i_1$, $a_{k_2}^{(j_2)} = i_2$ and

$$\begin{aligned}&(I_\sigma^{-1}((\omega_{k_1}^{(j_1)})^\alpha \bar{x_1} - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)})), j_1^*, \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1})\\ &= (I_\sigma^{-1}((\omega_{k_2}^{(j_2)})^\alpha \bar{x_2} - \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})), j_2^*, \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1}).\end{aligned}$$

Then, we have $j_1 = j_2$, $\omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} = \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1}$ and $(\omega_{k_1}^{(j_1)})^\alpha \bar{x_1} - \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P_{k_1}^{(j_1)}) = (\omega_{k_2}^{(j_2)})^\alpha \bar{x_2} - \omega_2(\omega_{k_2}^{(j_2)})^{\alpha-1} f(P_{k_2}^{(j_2)})$. We assume $k_1 \leq k_2$ without loss of generality. In the case of $k_1 = k_2$ we see $(\bar{x_1}, i_1^*, \omega_1) = (\bar{x_2}, i_2^*, \omega_2)$. Therefore, $k_1 < k_2$ holds. We have

$$\begin{aligned}&(\omega_{k_2}^{(j_2)})^\alpha \bar{x_2}\\ &= (\omega_{k_1}^{(j_1)})^\alpha \bar{x_1} + \omega_1(\omega_{k_1}^{(j_1)})^\alpha e_{i_1} + \omega_1(\omega_{k_1}^{(j_1)})^{\alpha-1}\left(f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}]\cdots[a_{k_2-1}^{(j_1)} : \omega_{k_2-1}^{(j_1)}]) - f([a_{k_1}^{(j_1)} : \omega_{k_1}^{(j_1)}])\right),\end{aligned}$$

which implies $\bar{x_2}\cdot\bar{v} \geq 0$. This is a contradiction.

□

For $\alpha = 1$, $T_\alpha^*(\sigma)\mathscr{G} \subset \mathscr{G}$ does not generally hold. We give a sufficient condition.

Proposition 3.3. *We suppose that for every $j \in \{1,2,\dots,d\}$ and $1 \le k_1, k_2 \le l_j$, if $a_{k_1}^{(j)} = a_{k_2}^{(j)}$ and $k_1 \le k_2$, $\omega_{k_1}^{(j)} \le \omega_{k_2}^{(j)}$ holds. Then, $T_1^*(\sigma)\mathscr{G} \subset \mathscr{G}$ holds.*

Proof. We suppose that $(\bar{x}, i^*, \omega) \in \mathscr{S}$. We will show that $T_1^*(\sigma)(\bar{x}, i^*, \omega)$ is geometric. We suppose that for $j_1, j_2 \in \mathscr{A}$, $k_1, k_2 \in \mathbb{Z}$ with $\sigma([j_1:1]) = P_{k_1}^{(j_1)}[i:\omega_{k_1}^{(j_1)}]S_{k_1}^{(j_1)}$ and $\sigma([j_2:1]) = P_{k_2}^{(j_2)}[i:\omega_{k_2}^{(j_2)}]S_{k_2}^{(j_2)}$

$$(I_\sigma^{-1}(\omega_{k_1}^{(j_1)}\bar{x} - \omega f(P_{k_1}^{(j_1)})), j_1^*, \omega) = (I_\sigma^{-1}(\omega_{k_2}^{(j_2)}\bar{x} - \omega f(P_{k_2}^{(j_2)})), j_2^*, \omega),$$

which implies that $j_1 = j_2$ and

$$(\omega_{k_2}^{(j_2)} - \omega_{k_1}^{(j_1)})\bar{x} = \omega(f(P_{k_2}^{(j_2)}) - f(P_{k_1}^{(j_1)})). \tag{3.4}$$

We suppose that $k_1 \ne k_2$. Then, we suppose that $k_1 < k_2$ without loss of generality. Then, $(\omega_{k_2}^{(j_2)} - \omega_{k_1}^{(j_1)})\bar{x} \cdot \bar{v} \le 0$ and $\omega_1(f(P_{k_2}^{(j_2)}) - f(P_{k_1}^{(j_1)})) \cdot \bar{v} > 0$ hold, which contradicts (3.4). Therefore, we have $k_1 = k_2$. We can prove that for $(\bar{x_1}, i_1^*, \omega_1), (\bar{x_2}, i_2^*, \omega_2) \in \mathscr{S}$ $T_1^*(\sigma)((\bar{x_1}, i_1^*, \omega_1) + (\bar{x_2}, i_2^*, \omega_2))$ is geometric as well as the proof of Lemma 3.2. □

$T_\alpha^*(\sigma)$ has the following property as well as $E_1^*(\sigma)$.

Lemma 3.4. *For substitutions σ_1, σ_2 over $\mathscr{A}_R^*$, $T_\alpha^*(\sigma_1\sigma_2) = T_\alpha^*(\sigma_2) \circ T_\alpha^*(\sigma_1)$ holds.*

Proof. Let $\sigma_1([i:1]) = [a_1^{(i)} : \omega_1^{(i)}] \cdots [a_{l_i}^{(i)} : \omega_{l_i}^{(i)}]$ for $i = 1, \dots, d$ and $\sigma_2([i:1]) = [b_1^{(i)} : \psi_1^{(i)}] \cdots [b_{l_i'}^{(i)} : \psi_{l_i'}^{(i)}]$ for $i = 1, \dots, d$.

Let $(\bar{x}, i^*, \omega) \in \mathscr{F}^*$. We see

$$\begin{aligned}
&T_\alpha^*(\sigma_2)(T_\alpha^*(\sigma_1)(\bar{x}, i^*, \omega)) \\
&= T_\alpha^*(\sigma_2)\Big(\sum_{\sigma_1([j_1:1]) = P(\sigma_1)_{k_1}^{(j_1)}[i:\omega_{k_1}^{(j_1)}]S(\sigma_1)_{k_1}^{(j_1)}} (I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^\alpha \bar{x} - \omega(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P(\sigma_1)_{k_1}^{(j_1)})), j_1^*, \omega(\omega_{k_1}^{(j_1)})^{\alpha-1})\Big) \\
&= \sum_{\sigma_1([j_1:1]) = P(\sigma_1)_{k_1}^{(j_1)}[i:\omega_{k_1}^{(j_1)}]S(\sigma_1)_{k_1}^{(j_1)}} \; \sum_{\sigma_2([j_2:1]) = P(\sigma_2)_{k_2}^{(j_2)}[j_1:\psi_{k_2}^{(j_2)}]S(\sigma_2)_{k_2}^{(j_2)}} ((\psi_{k_2}^{(j_2)})^\alpha (I_{\sigma_2}^{-1})(I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^\alpha \bar{x} \\
&- \omega(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P(\sigma_1)_{k_1}^{(j_1)}))) - (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1} I_{\sigma_2}^{-1} f(P(\sigma_2)_{k_2}^{(j_2)})), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}) \\
&= \sum_{j_1, k_1 : a_{(k_1)}^{(j_1)} = i} \; \sum_{j_2, k_2 : b_{(k_2)}^{(j_2)} = j_1} ((\psi_{k_2}^{(j_2)})^\alpha (I_{\sigma_2}^{-1})(I_{\sigma_1}^{-1}((\omega_{k_1}^{(j_1)})^\alpha \bar{x}
\end{aligned}$$

(3.5)

$$- \omega(\omega_{k_1}^{(j_1)})^{\alpha-1} f(P(\sigma_1)_{k_1}^{(j_1)}))) - (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1} I_{\sigma_2}^{-1} f(P(\sigma_2)_{k_2}^{(j_2)})), j_2^*, (\psi_{k_2}^{(j_2)})^{\alpha-1}\omega(\omega_{k_1}^{(j_1)})^{\alpha-1}).$$

On the other hand, we have

$$
\begin{aligned}
&T^*_\alpha(\sigma_1\sigma_2)(\bar{x}, i^*, \omega)\\
&= \sum_{\substack{\sigma_1(\sigma_2([j_2:1])) = \sigma_1(P(\sigma_2)^{(j_2)}_{k_2})\sigma_1([j_1:\psi^{(j_2)}_{k_2}])\sigma_1(S(\sigma_2)^{(j_2)}_{k_2})\\ \sigma_1([j_1:\psi^{(j_2)}_{k_2}]) = (P(\sigma_1)^{(j_1)}_{k_1})^{\psi^{(j_2)}_{k_2}}[i:\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2}](S(\sigma_1)^{(j_1)}_{k_1})^{\psi^{(j_2)}_{k_2}}}} (I^{-1}_{\sigma_1\sigma_2}((\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2})^\alpha \bar{x}\\
&- \omega(\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2})^{\alpha-1} f(\sigma_1(P(\sigma_2)^{(j_2)}_{k_2})(P(\sigma_1)^{(j_1)}_{k_1})^{\psi^{(j_2)}_{k_2}})), j_2^*, (\psi^{(j_2)}_{k_2})^{\alpha-1}\omega(\omega^{(j_1)}_{k_1})^{\alpha-1})\\
&= \sum_{j_1,k_1:a^{(j_1)}_{(k_1)}=i, \text{ and } j_2,k_2:b^{(j_2)}_{(k_2)}=j_1} (I^{-1}_{\sigma_1\sigma_2}((\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2})^\alpha \bar{x}\\
&- \omega(\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2})^{\alpha-1} f(\sigma_1(P(\sigma_2)^{(j_2)}_{k_2})(P(\sigma_1)^{(j_1)}_{k_1})^{\psi^{(j_2)}_{k_2}})), j_2^*, (\psi^{(j_2)}_{k_2})^{\alpha-1}\omega(\omega^{(j_1)}_{k_1})^{\alpha-1}) \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
&\omega(\omega^{(j_1)}_{k_1}\psi^{(j_2)}_{k_2})^{\alpha-1} f(\sigma_1(P(\sigma_2)^{(j_2)}_{k_2})(P(\sigma_1)^{(j_1)}_{k_1})^{\psi^{(j_2)}_{k_2}}\\
&= \omega(\omega^{(j_1)}_{k_1})^{\alpha-1}(\psi^{(j_2)}_{k_2})^{\alpha-1} f(\sigma_1(P(\sigma_2)^{(j_2)}_{k_2})) + \omega(\omega^{(j_1)}_{k_1})^{\alpha-1}(\psi^{(j_2)}_{k_2})^{\alpha} f(P(\sigma_1)^{(j_1)}_{k_1})\\
&= \omega(\omega^{(j_1)}_{k_1})^{\alpha-1}(\psi^{(j_2)}_{k_2})^{\alpha-1} I_{\sigma_1}(f(P(\sigma_2)^{(j_2)}_{k_2})) + \omega(\omega^{(j_1)}_{k_1})^{\alpha-1}(\psi^{(j_2)}_{k_2})^{\alpha} f(P(\sigma_1)^{(j_1)}_{k_1}). \qquad (3.7)
\end{aligned}
$$

By (3.5), (3.6)and (3.7) we have $T^*_\alpha(\sigma_2)(T^*_\alpha(\sigma_1)(\bar{x}, i^*, \omega)) = T^*_\alpha(\sigma_1\sigma_2)(\bar{x}, i^*, \omega)$. Thus we have Lemma 3.4. □

By Lemma 3.1, 3.2 and 3.4 we have following Theorem.

Theorem 3.5.

1. *For every $\alpha \in \mathbb{R}$ with $\alpha \neq 1$, $T^*_\alpha(\sigma)\overline{\mathscr{S}} \subset \overline{\mathscr{S}}$ and $T^*_\alpha(\sigma)\mathscr{G} \subset \mathscr{G}$.*
2. *For every $\alpha \in \mathbb{R}$, $T^*_\alpha(\sigma_1\sigma_2) = T^*_\alpha(\sigma_2) \circ T^*_\alpha(\sigma_1)$.*

We also define a class of endomorphisms $F^*_\alpha(\sigma)$ on $\mathscr{F}^*$ for $\alpha \in \mathbb{R}$ by

$$
F^*_\alpha(\sigma)(\bar{x}, i^*, \omega) := \sum_{j\in\mathscr{A}} \sum_{a^{(j)}_k = i} (I^{-1}_\sigma((\omega^{(j)}_k)^\alpha \bar{x} - \omega(\omega^{(j)}_k)^{\alpha-1} f(S^{(j)}_k)), j^*, \omega(\omega^{(j)}_k)^{\alpha-1}).
$$

We consider a reversed word w and a reversed substitution $\overline{\sigma}$ defined by $\overline{w} = w_l w_{l-1}\cdots w_1$ for $w = w_1 w_2 \cdots w_l$ and $\overline{\sigma}(i) = \overline{\sigma(i)}$. Then, $F^*_\alpha(\sigma) = T^*_\alpha(\overline{\sigma})$. Thus we have Theorem 3.6.

Theorem 3.6.

1. *For every $\alpha \in \mathbb{R}$ with $\alpha \neq 1$, $F^*_\alpha(\sigma)\overline{\mathscr{S}} \subset \overline{\mathscr{S}}$ and $F^*_\alpha(\sigma)\mathscr{G} \subset \mathscr{G}$.*
2. *For every $\alpha \in \mathbb{R}$, $F^*_\alpha(\sigma_1\sigma_2) = F^*_\alpha(\sigma_2) \circ F^*_\alpha(\sigma_1)$.*

We denote by U (resp.U') $\sum_{i\in\mathscr{A}}(-e_i, i^*, 1)$ (resp.$\sum_{i\in\mathscr{A}}(0, i^*, 1)$). We note that the equality $T^*_\alpha(\sigma)(U') - T^*_\alpha(\sigma)(U) = U' - U$ holds for not all $\alpha \in \mathbb{R}$. We give a sufficient condition.

Proposition 3.7. $T_1^*(\sigma)(U') - T_1^*(\sigma)(U) = U' - U.$

Proof. We have

$$\begin{aligned}
&T_1^*(\sigma)(U') - T_1^*(\sigma)(U) = T_1^*(\sum_{i\in\mathscr{A}}(0,i^*,1)) - T_1^*(\sum_{i\in\mathscr{A}}(-e_i,i^*,1)) \\
&= \sum_{i\in\mathscr{A}}\sum_{j\in\mathscr{A}}\sum_{a_k^{(j)}=i}(I_\sigma^{-1}(-f(P_k^{(j)})),j^*,1) - \sum_{i\in\mathscr{A}}\sum_{j\in\mathscr{A}}\sum_{a_k^{(j)}=i}(I_\sigma^{-1}(-\omega_k^{(j)}e_i - f(P_k^{(j)})),j^*,1) \\
&= \sum_{j\in\mathscr{A}}\sum_{k=1}^{l_j}(I_\sigma^{-1}(-f(P_k^{(j)})),j^*,1) - \sum_{j\in\mathscr{A}}\sum_{k=1}^{l_j}(I_\sigma^{-1}(-\omega_k^{(j)}e_{a_k^{(j)}} - f(P_k^{(j)})),j^*,1) \\
&= \sum_{j\in\mathscr{A}}\sum_{k=1}^{l_j}(I_\sigma^{-1}(-f(P_k^{(j)})),j^*,1) - \sum_{j\in\mathscr{A}}(\sum_{k=1}^{l_j-1}(I_\sigma^{-1}(-f(P_{k+1}^{(j)})),j^*,1) + (-e_j,j^*,1)) \\
&= U' - U.
\end{aligned}$$

□

§ 4. Example

Following [1], we associate the hyperface $\{\bar{x} + \omega e_i + \omega\sum_{j\neq i}\lambda_j e_j | 0 \leq \lambda_j \leq 1\}$ to $(\bar{x}, i^*, \omega)$. Let us consider the following substitution σ:

$$\begin{aligned}
\sigma[1:1] &= [1:1][1:1][2:1], \\
\sigma[2:1] &= [1:\sqrt{2}-1][3:1], \\
\sigma[3:1] &= [1:1].
\end{aligned}$$

Then, $I_\sigma = \begin{pmatrix} 2 & \sqrt{2}-1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and it is regular and primitive. The following figures are $(T^*_{weight}(\sigma))^9(U)$ for weight$= 0,1,2,3$.

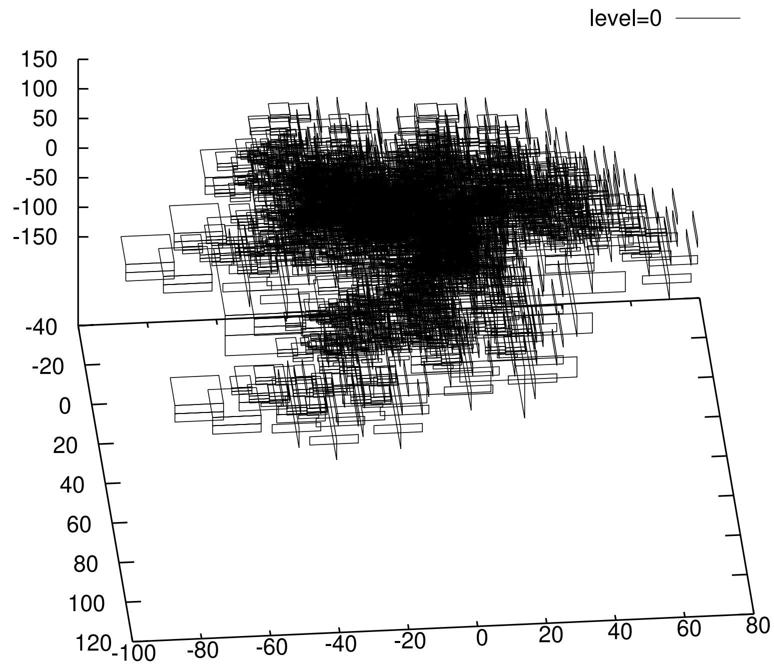

Figure 1(weight 0)

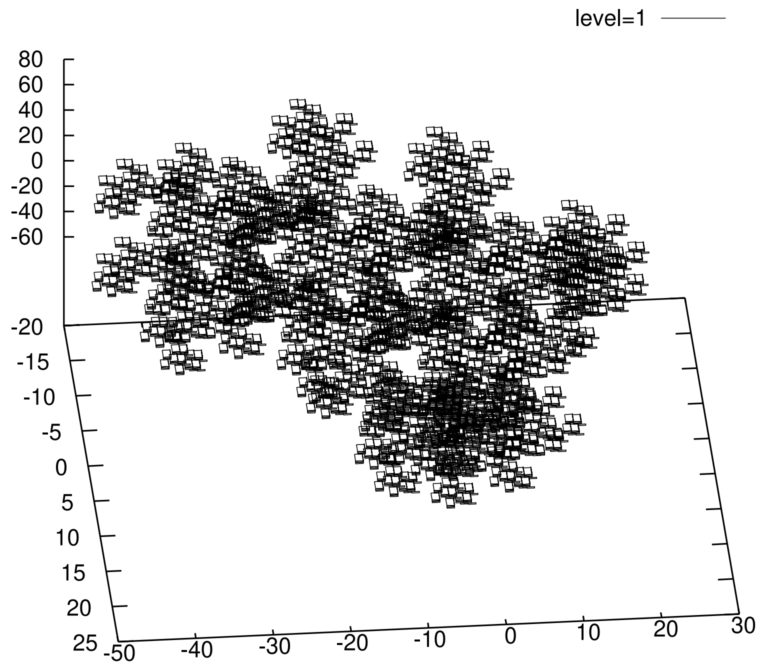

Figure 2(weight 1)

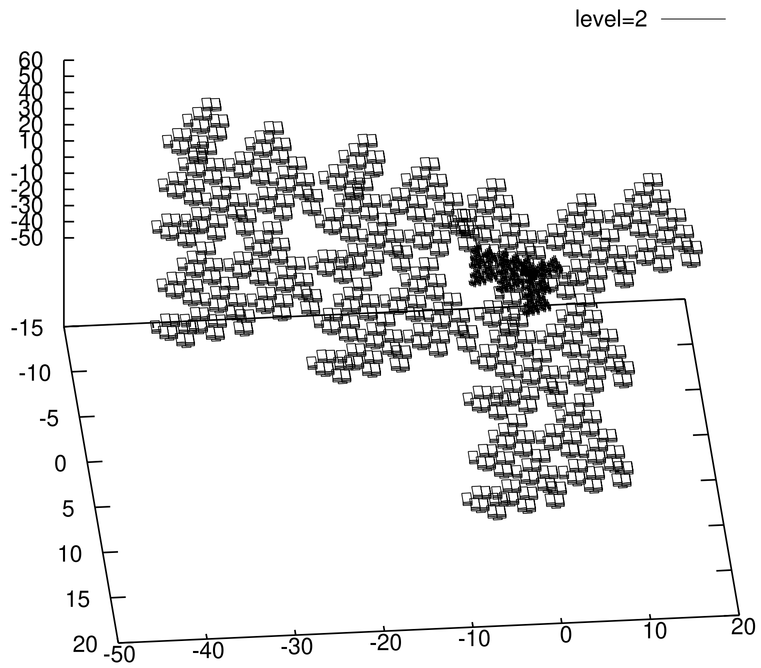

Figure 3(weight 2)

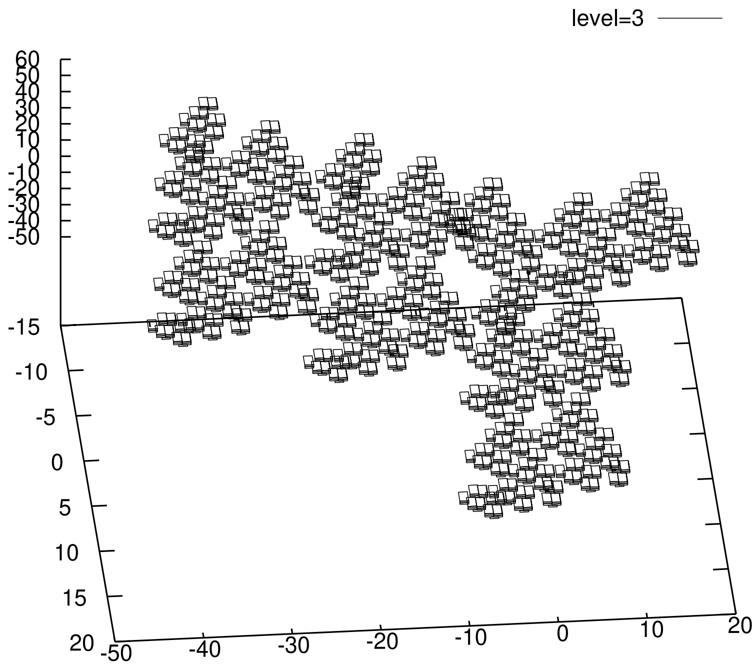

Figure 4(weight 3)

§ 5. Problem

Let $\mathbf{K}$ be a finite algebraic field over $\mathbb{Q}$ and $O_{\mathbf{K}}$ be all algebraic integers in $\mathbf{K}$. Let $E_{\mathbf{K}}$ be the unit group of $O_{\mathbf{K}}$. We suppose that $\mathbf{K}$ is generated by $\omega_k^{(i)} (i \in \mathscr{A}, 1 \le k \le l_i)$ over $\mathbb{Q}$, $\omega_k^{(i)} (i \in \mathscr{A}, 1 \le k \le l_i) \in E_{\mathbf{K}}$ holds and the determinant of I_σ is in $E_{\mathbf{K}}$. We denote by $\mathscr{G}(\mathbf{K})$ $\{(\bar{x}, i^*, \omega) \in \mathscr{G} | \bar{x} \in (O_{\mathbf{K}})^d, \omega \in O_{\mathbf{K}}\}$. By Theorem 3.5 we see $T_\alpha^*(\sigma)\mathscr{G}(\mathbf{K}) \subset \mathscr{G}(\mathbf{K})$ for every $\alpha \in \mathbb{Z}$ $(\alpha \neq 1)$. $\mathscr{G}(\mathbf{K})$ is significant in Diophantine approximation. In fact, in [6] for dual substitutions associated with a multidimensional continued fraction algorithm, the problem to determine whether $\lim_{n\to\infty} E_1^*(\sigma)^n(U)$ are equal to stepped surfaces is considered, which is used to obtain the results related to the simultaneous Diophantine approximation for certain cubic pairs in [5]. We serve a following problem:

Problem.
Does the equation $\cup_{n\in\mathbb{Z},k\in\mathbb{Z}_{\ge 0}} \Psi((T_k^*(\sigma))^n)(U)) = \Pi_1(\mathscr{G}(\mathbf{K}))$ hold , where $\Psi : \overline{\mathscr{S}} \to 2^{\mathbb{R}^d}$ is defined by $\Psi((\bar{x_1}, i_1^*, \omega_1) + \ldots (\bar{x_n}, i_n^*, \omega_n)) = \{\bar{x_1}, \ldots \bar{x_n}\}$ for $(\bar{x_1}, i_1^*, \omega_1) + \ldots (\bar{x_n}, i_n^*, \omega_n) \in \overline{\mathscr{S}}$ and $\Pi_1(\bar{x}, i^*, \omega) = \bar{x}$ for $(\bar{x}, i^*, \omega) \in \mathscr{S}$?

We remark that generating stepped surfaces (discrete planes) with substitutions are considered in [2],[3],[4].

Acknowledgement.
This work was supported by Grant-in-Aid for Scientific Research(C) Grant Number 15K00342.

The authors would like to thank the referee for their valuable comments which helped to improve the manuscript.

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