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A new approach to the characterization of closed forms in the nongradient method

By

Yukio Kametani* and Makiko Sasada**

Abstract

We announce our recent result on the characterization of closed forms on configuration spaces associated to interacting particle systems. In the context of the study of hydrodynamic limit, closed forms on an infinite configuration space in a L^2 space are well studied and their characterization theorem plays an essential role if our model is non-gradient. In this article, we report that closed forms in the set of local functions can be characterized by a similar way as L^2 functions but its proof is very simple and completely different from that for L^2 functions. With this new observation, we also have an alternative proof of the original characterization theorem in the L^2 space, which does not require the sharp estimate of the spectral gap, for the class of lattice gases that are reversible under the Bernoulli measures. Moreover, we extend these characterization theorems for the models in a crystal lattice from $\mathbf{Z}^d$.

§ 1. Introduction

This article is a research announcement on our study about a new aspect of one-forms on configuration spaces. The goal of the study is to find a simpler and intuitive proof of the hydrodynamic limit for non-gradient models.

To prove the hydrodynamic limit for non-gradient models, applying the gradient replacement, introduced by Varadhan and Quastel in [10] and [6], is a standard and unique strategy so far. Its essential part is the so-called characterization of closed forms (cf. [1], [3]). This part requires a very complicated argument with a sharp spectral gap estimate. Even though the statement of the characterization theorem of closed forms

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*Keio University, 3-14-1, Hiyoshi, Kohoku-ku, Yokohama-shi, Kanagawa, 223-8522, Japan.
e-mail: `kametani@math.keio.ac.jp`
**The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 606-8502, Japan.
e-mail: `sasada@ms.u-tokyo.ac.jp`

is almost same for a wide class of models, we need to change the details of the proof depending on the specific model and it is not straightforward. Also, to show the sharp spectral gap estimate for each model is usually a tough work.

In this article, we aim to understand the common structure of the characterization of closed forms among different models. In particular, we reveal why the dimension and the explicit expression of the set of harmonic forms (namely closed but not exact forms) do not depend on the details of the model, which we can guess from the previous works on the non-gradient models ([1],[2],[3],[7],[10]). For this purpose, we introducing a CW complex (or de Rham complex) associated to the configuration space and reconsider the characterization of closed forms from algebraic and geometric points of view. We do not give the definition of the complex in this article, but we report new observations and results obtained by the study of this complex. They are the followings:

(i) The typical characterization theorem of closed forms required in the context of hydrodynamic limit is about the closed forms in $L^2(\nu)$ where ν is a probability measure on a configuration space. The theorem claims that any closed form (precisely any germ of closed form, see below) is decomposed as a sum of an exact form and a harmonic form. Moreover, the space of harmonic forms is explicitly given. In this article, we study the closed forms which are local functions, and prove the similar decomposition of them by the exact forms and the harmonic forms. The space of harmonic forms are common for L^2 functions and local functions. The proof is very simple and able to apply very general models directly.

(ii) The statement and the proof of the characterization theorem for local functions do not relate to the probability measure ν nor the spectral gap estimate, so it turns out to be purely an algebraic problem.

(iii) From the characterization theorem for local functions, we know the dimension and the explicit expression of the set of harmonic forms. In fact, the dimension is exactly the first cohomology group of an abelian group acting the configuration space.

(iv) Using the idea of the proof of the characterization theorem for local functions, for the case where the model has a good duality, we give an alternative proof of the characterization theorem for $L^2(\nu)$ functions where we do not use the spectral gap estimate. The example of the model having a nice duality is the lattice gas reversible under Bernoulli measures studied in [1].

(v) With our new observations, we can generalize these results of the characterization theorem of closed forms for the interacting particle systems in a crystal lattice instead of $\mathbf{Z}^d$. As mentioned in [9], the hydrodynamic limit for a non-gradient system in a crystal lattice is an important open problem. Our result gives a way to attack the problem.

The rest of this article is organized as follows. In Section 2, we give the precise statement of the above observations in the context of lattice gas reversible under Bernoulli measures. In Section 3, we give other examples where we can apply our results on the characterization theorems for local functions. For the models with a discrete state space, we can directly apply the method in Section 2. For the models with a continuous state space, we need to change the framework slightly but the main ideas work in the same way. In Section 4, we give the conclusion and some remarks.

§ 2. lattice gas reversible under Bernoulli measures

§ 2.1. Characterization theorems in $\mathbf{Z}^d$

Let $S = \{0, 1\}$, $d \in \mathbf{N}$. We denote the infinite configuration space of the lattice gas by $\chi = S^{\mathbf{Z}^d}$ and the set of all bonds in $\mathbf{Z}^d$ by $(\mathbf{Z}^d)^*$. An element of the configuration space is denoted by $\eta = (\eta_x)_{x\in\mathbf{Z}^d} \in \chi$.

For $x \in \mathbf{Z}^d$, define the shift operator $\tau_x : \mathbf{Z}^d \to \mathbf{Z}^d$ as the $(-x)$-shift of $\mathbf{Z}^d$, that is, $\tau_x(i) = i-x$. For a bond $b = \{x, y\} \in (\mathbf{Z}^d)^*$, define the exchange operator $\pi_b : \mathbf{Z}^d \to \mathbf{Z}^d$ as the exchange of x and y. Then τ_x and π_b naturally induce automorphisms of $\{0, 1\}^{\mathbf{Z}^d}$. Precisely, $(\tau_x\eta)_y = \eta_{y-x}$ and for $b = \{x, y\}$

$$(\pi_b\eta)_z = \begin{cases} \eta_z & \text{if } z \neq x, y \\ \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y. \end{cases}$$

For $f : \chi \to \mathbf{R}$, let $\nabla_b f$ be the partial difference of f with respect to b defined by

$$\nabla_b f = \pi_b f - f$$

where $\pi_b f(\eta) = f(\pi_b\eta)$.

Now, we introduce closed forms and exact forms on χ with respect to these partial differences. For this purpose, we first define chains and closed chains in χ.

Definition 2.1. A finite sequence $(\eta(k))_{k=0}^n (n = 1, 2, \dots)$ in χ is called a chain, if there exist n bonds $b(1), b(2), \dots, b(n)$ in $(\mathbf{Z}^d)^*$ such that

$$\eta(k) = \pi_{b(k)}\eta(k-1) \quad k = 1, 2, \dots, n. \tag{2.1}$$

Moreover, if $\eta(0) = \eta(n)$, the sequence is called a closed chain.

Definition 2.2. A family of functions $(\Psi^b)_{b\in(\mathbf{Z}^d)^*}$ on χ is called a closed form or closed if for every closed chain $(\eta(k))_{k=0}^n (n = 1, 2, \dots)$ satisfying (2.1),

$$\sum_{k=1}^{n} \Psi^{b(k)}(\eta(k-1)) = 0.$$

Definition 2.3. A family of functions $(\Psi^b)_{b\in(\mathbf{Z}^d)^*}$ on χ is called an exact form or exact if there exists a function F such that

$$\Psi^b = \nabla_b F$$

for all $b \in (\mathbf{Z}^d)^*$.

Then, by the definition, the following lemma holds.

Lemma 2.4. *A family of functions $(\Psi^b)_{b\in(\mathbf{Z}^d)^*}$ is closed if and only if it is exact.*

We have a following useful lemma to check the closedness of a given family of functions and also to see the similarity of the definition of closedness for continuous configuration spaces.

Lemma 2.5. *A family of functions $(\Psi^b)_{b\in(\mathbf{Z}^d)^*}$ is closed if and only if*

$$\nabla_b \Psi^b = -2\Psi^b \tag{2.2}$$

for any $b \in (\mathbf{Z}^d)^*$ and

$$\nabla_b \Psi^{b'} = \nabla_{b'} \Psi^b \tag{2.3}$$

for any $b, b' \in (\mathbf{Z}^d)^*$.

Next, we introduce the notion of germs of closed forms and exact forms, which correspond one-to-one with shift-invariant forms.

Let e_i stands for the unit vector to the i-th direction and $e_i^* = \{0, e_i\} \in (\mathbf{Z}^d)^*$ for $i = 1, 2, \dots, d$.

Definition 2.6. A family of functions $\xi = (\xi^{e_i^*})_{i=1}^d$ on χ is called a germ of closed form if the family of functions $(\xi^b)_{b\in(\mathbf{Z}^d)^*}$ is closed where

$$\xi^b = \tau_x \xi^{e_i^*} \quad \text{if} \quad b = \{x, x + e_i\}.$$

To introduce the notion of germs of exact forms, we first introduce that notion of local functions. We say a function $f : \chi \to \mathbf{R}$ is local if f depends only on a finite number of occupation variables of η.

Definition 2.7. A family of functions $\xi = (\xi^{e_i^*})_{i=1}^d$ is called a germ of exact form if there exists a local function F such that

$$\xi^{e_i^*} = \nabla_{e_i^*}\Big(\sum_{x\in\mathbf{Z}^d} \tau_x F\Big)$$

for all $i = 1, 2, \dots, d$.

Remark. Though $\big(\sum_{x\in\mathbf{Z}^d}\tau_x F\big)$ is not well defined, $\nabla_{e_i^*}\big(\sum_{x\in\mathbf{Z}}\tau_x F\big)$ is well defined since F is a local function.

Denote the set of local functions on χ by $C_\ell(\chi)$ and the set of all d-tuple of local functions by $(C_\ell(\chi))^d$, the set of all germs of closed forms by $\mathcal{C}$ and that of exact forms by $\mathcal{E}$. Notice that $\mathcal{E}\subset(C_\ell(\chi))^d$. Define $\Phi_i=(\Phi_i^{e_j^*})_{j=1}^d\in(C_\ell(\chi))^d$ by

$$\Phi_i^{e_j^*}=\nabla_{e_j^*}\big(\sum_{x\in\mathbf{Z}^d}x_i\eta_x\big)=\delta_{i,j}(\eta_{e_i}-\eta_0).$$

One of our main results is the following characterization theorem of germs of closed forms in $C_\ell(\chi)$.

Theorem 2.8. *Let $\xi=(\xi^{e_i^*})_{i=1}^d\in(C_\ell(\chi))^d$ be a germ of closed form. Then, there exist real numbers $a_1,a_2,\dots,a_d$ and a local function F such that*

$$\begin{aligned}\xi^{e_i^*}&=\sum_{j=1}^d a_j\Phi_j^{e_i^*}+\nabla_{e_i^*}\big(\sum_{x\in\mathbf{Z}^d}\tau_x F\big)\\&=\nabla_{e_i^*}\big(\sum_{x\in\mathbf{Z}^d}\sum_{j=1}^d a_j x_j\eta_x+\sum_{x\in\mathbf{Z}^d}\tau_x F\big).\end{aligned}$$

Equivalently,

$$\mathcal{C}\cap(C_\ell(\chi))^d=\mathcal{E}+\{\sum_{i=1}^d a_i\Phi_i\}.$$

Idea of the proof. Let $\chi_n=\{\eta\in\chi;\sum_{x\in\mathbf{Z}^d}\eta_x=n\}$ for $n\ge 0$ and $\chi_f=\{\eta\in\chi;\sum_{x\in\mathbf{Z}^d}\eta_x<\infty\}=\cup_{n\ge0}\chi_n$. The key of the proof is that $\xi\in(C_\ell(\chi))^d$ is completely determined by the restriction of ξ on χ_f. So, we may consider $\xi\in(C_\ell(\chi_f))^d$. Then, if $\xi\in(C_\ell(\chi_f))^d$ is closed, from Lemma 2.4, we can integrate it along a chain and construct a function $F_n:\chi_n\to\mathbf{R}$ satisfying

$$\nabla_{e_i^*}F_n=\xi^{e_i^*}$$

for $i=1,2,\dots,d$ on χ_n. By the construction, we can show that $\tau_{e_i}F_n-F_n$ is a constant for each $i=1,2,\dots,d$ on χ_n. So, we denote it by c_n^i. Using the locality of ξ, we can also show that $c_n^i=na_i$ where $a_i=c_1^i$. Define a function $\bar F:\chi_f\to\mathbf{R}$ as

$$\bar F=F_n-\sum_{i=1}^d a_i\sum_{x\in\mathbf{Z}^d}x_i\eta_x$$

on χ_n. Then, we have $\tau_x\bar F=\bar F$ for all $x\in\mathbf{Z}^d$ on χ_f. Since $\nabla_{e_i^*}\bar F$ is a local function, we can conclude that there exists a local function F such that $\bar F=\sum_{x\in\mathbf{Z}^d}\tau_x F$. Namely, we have real numbers $a_1, a_2, \dots, a_d$ and a local function F such that

$$\nabla_{e_i^*}(\sum_{x\in\mathbf{Z}^d}\sum_{j=1}^{d} a_j x_j \eta_x + \sum_{x\in\mathbf{Z}^d}\tau_x F) = \xi^{e_i^*}$$

for $i = 1, 2, \dots, d$ on χ_f. Since $\xi \in (C_\ell(\chi))^d$, the above equality also holds on χ.

Remark. Consider partial differential operators $\tilde{\nabla}_b$ defined for $b = \{x, x+e_i\}$ by

$$\tilde{\nabla}_b = c_{e_i^*}(\tau_x\eta)\nabla_b$$

where $c_{e_i^*} : \chi \to \mathbf{R}$ which does not depend on η_0 nor η_{e_i} for $i = 1, 2, \dots, d$. We define closed forms and exact forms associated to them. Then, if $c_{e_i^*}(\eta) > 0$ for any $\eta \in \chi$ and $i = 1, 2, \dots, d$, the characterization of germs of closed forms in $C_\ell(\chi)$ for them also follows form Theorem 2.8.

In the study of hydrodynamic limit of non-gradient models, we need the characterization of germs of closed forms in $L^2(\nu)$ where ν is a reversible or invariant measure for the model. Here, we consider the model studied in [1]. For this, fix $\rho \in [0, 1]$ and let $\nu = \nu_\rho$ be a Bernoulli measure on χ such that $\nu_\rho(\eta_x = 1) = \rho$ for all $x \in \mathbf{Z}^d$. Denote the set of all d-tuple of $L^2(\nu)$ functions by $(L^2(\nu))^d$. Notice that $(L^2(\nu))^d = \overline{(C_\ell(\chi))^d}$ where the closure is taken in $(L^2(\nu))^d$.

The next theorem is first proved in [1] with a sharp estimate of the spectral gap. In our study, we find a new proof which does not require the spectral gap estimate. The key idea is to use the duality of the partial differential operators with respect to the measure ν.

Theorem 2.9. *Let $\xi \in (L^2(\nu))^d$ be a germ of closed form. Then, there exist real numbers $a_1, a_2, \dots, a_d$ and a sequence of local functions F_n such that*

$$\begin{aligned} \xi^{e_i^*} &= \sum_{j=1}^{d} a_j \Phi_j^{e_i^*} + \lim_{n\to\infty} \nabla_{e_i^*}(\sum_{x\in\mathbf{Z}^d}\tau_x F_n) \\ &= \lim_{n\to\infty} \nabla_{e_i^*}(\sum_{x\in\mathbf{Z}^d}\sum_{j=1}^{d} a_j x_j \eta_x + \sum_{x\in\mathbf{Z}^d}\tau_x F_n) \quad \text{in } L^2(\nu). \end{aligned}$$

Equivalently,

$$\mathcal{C} \cap (L^2(\nu))^d = \overline{\mathcal{E}} + \{\sum_{i=1}^{d} a_i \Phi_i\}$$

where $\overline{\mathcal{E}}$ is the closure of $\mathcal{E}$ in $(L^2(\nu))^d$.

Idea of the new proof. We consider the set of functions $\{\prod_{x\in\Lambda}(\eta_x-\rho)\}_{\Lambda\subset\subset\mathbf{Z}^d}$ as the orthogonal basis of $L^2(\nu)$. From the expansion of the germ of closed form in $L^2(\nu)$ with respect to this basis, we can construct a closed form on χ_f. Then, from Lemma 2.4, it is integrable and we obtain a kind of "potential function" on χ_f in the same way as the proof for local functions. This potential function is decomposed by the cost of shifts and the shit-invariant function in the same way as for local functions again. But this case, this closed form on χ_f is not local, so we approximate it by local functions and we show that this approximation still holds if we reconstruct a germ of closed form on χ from them using the orthogonal basis.

Remark. Precisely, we define $(\Psi^b)_{b\in(\mathbf{Z}^d)^*}\in(L^2(\nu))^{(\mathbf{Z}^d)^*}$ is closed if they satisfy (2.2) and (2.3) in $L^2(\nu)$ sense.

Remark. Since we have Theorem 2.8, to show Theorem 2.9, it is enough to prove

$$\overline{\mathcal{C}\cap(C_\ell(\chi))^d}=\mathcal{C}\cap\overline{(C_\ell(\chi))^d}.$$

However, our new proof does not take this strategy.

Remark. Consider partial differential operators $\tilde{\nabla}_b$ defined in the above remark again. Then, if there exist $C_1,C_2>0$ such that $C_1\le c_{e_i^*}(\eta)\le C_2$ for any $\eta\in\chi$ and $i=1,2,\dots,d$, the characterization of germs of closed forms in $L^2(\nu)$ for them also follows form Theorem 2.9.

§ 2.2. Characterization theorems in a crystal lattice

In this subsection, we extend the result in the last subsection to the model in a crystal lattice.

Let X be a crystal lattice, that is an infinite locally finite connected graph $X=(V,E)$ for which a free abelian group $\Gamma\cong\mathbf{Z}^d$ acts freely, and its quotient graph $X_0=(V_0,E_0)=G\setminus X$ is a finite graph. We refer [9] for examples and properties of crystal lattices.

We denote the configuration space by $\chi=S^V=\{0,1\}^V$ and an element of it by $\eta=(\eta_x)_{x\in V}\in\chi$ as before.

By the definition, each $\sigma\in\Gamma$ defines a graph isomorphism $\tau_\sigma:X\to X$. For an edge $b=\{x,y\}\in E$ we denote by $\pi_b:V\to V$ the exchange of x and y. Then τ_σ and π_b naturally induce automorphisms of $\{0,1\}^V$.

For $f:\chi\to\mathbf{R}$, let ∇_b be the partial difference of f with respect to b as same as before. Also, the following notions are defined in a same way as before.

Definition 2.10. A finite sequence $(\eta(k))_{k=0}^n(n=1,2,\dots)$ in χ is called a chain, if there exist n edges $b(1),b(2),\dots,b(n)$ in E such that

$$\eta(k)=\pi_{b(k)}\eta(k-1)\quad k=1,2,\dots,n. \tag{2.4}$$

Moreover, if $\eta(0) = \eta(n)$, the sequence is called a closed chain.

Definition 2.11. A family of functions $(\Psi^b)_{b\in E}$ on χ is called a closed form or closed if for every finite chain $(\eta(k))_{k=0}^n (n = 1, 2, \dots)$ satisfying (2.4),

$$\sum_{k=1}^{n} \Psi^{b(k)}(\eta(k-1)) = 0.$$

Definition 2.12. A family of functions $(\Psi^b)_{b\in E}$ on χ is called an exact form or exact if there exists a function $F : \chi \to \mathbf{R}$ such that

$$\Psi^b = \nabla_b F$$

for all $b \in E$.

Then, since the crystal lattice is connected, the following lemma holds.

Lemma 2.13. *A family of functions $(\Psi^b)_{b\in E}$ is closed if and only if it is exact.*

Next, we introduce the notion of germs of closed forms and exact forms, which correspond one-to-one with τ_σ-invariant forms.

Definition 2.14. A family of functions $\xi = (\xi^e)_{e\in E_0}$ is called a germ of closed form if the family of functions (ξ^b) is closed where

$$\xi^b = \tau_\sigma \xi^e \quad \text{if} \quad b = \tau_\sigma e.$$

Definition 2.15. A family of functions $\xi = (\xi^e)_{e\in E_0}$ is called a germ of exact form if there exists a local function F such that

$$\xi^e = \nabla_e\big(\sum_{\sigma\in\Gamma} \tau_\sigma F\big)$$

for all $e \in E_0$.

Denote the set of local functions on χ by $C_\ell(\chi)$ and the set of all $|E_0|$-tuple of local functions by $(C_\ell(\chi))^{|E_0|}$, the set of all germs of closed forms by $\mathcal{C}$ and that of exact forms by $\mathcal{E}$. Notice that $\mathcal{E} \subset (C_\ell(\chi))^{|E_0|}$. Define $\Phi_i = (\Phi_i^e)_{e\in E_0} \in (C_\ell(\chi))^{|E_0|}$ by

$$\Phi_i^e = \nabla_e\big(\sum_{v\in V_0}\sum_{\sigma\in\Gamma} \sigma_i \eta_{\tau_\sigma v}\big)$$

where $\sigma_i \in \mathbf{Z}$ is the i-th element of σ when we identify Γ with $\mathbf{Z}^d$.

Now, we have the characterization of germs of closed forms.

Theorem 2.16.

$$\mathcal{C} \cap (C_\ell(\chi))^{|E_0|} = \mathcal{E} + \{\sum_{i=1}^{d} a_i \Phi_i\}$$

The proof is almost same as that for $\mathbf{Z}^d$. With this theorem, the meaning of the dimension of the set of harmonic forms and the role of $|E_0|$ and $|V_0|$ become clear.

Next, we consider germs of closed forms in $L^2(\nu)$ as before. Let $\nu = \nu_\rho$ be a Bernoulli measure on χ such that $\nu_\rho(\eta_x = 1) = \rho$ for all $x \in V$. Denote the set of all $|E_0|$-tuple of $L^2(\nu)$ functions by $(L^2(\nu))^{|E_0|}$. Notice that $(L^2(\nu))^{|E_0|} = \overline{(C_\ell(\chi))^{|E_0|}}$.

The next theorem is a completely new result, which is essential to study the hydrodynamic limit for non-gradient models in a crystal lattice. It seems very hard to apply the standard strategy used in [1] to this model. On the other hand, our proof is easily applied to this model.

Theorem 2.17.

$$\mathcal{C} \cap (L^2(\nu))^{|E_0|} = \overline{\mathcal{E}} + \{\sum_{i=1}^{d} a_i \Phi_i\}$$

where $\overline{\mathcal{E}}$ *is the closure of* $\mathcal{E}$ *in* $(L^2(\nu))^{|E_0|}$.

§ 3. Other examples

In this section, we give examples where we can show the same result of Theorem 2.8 and 2.16. They are all new, even though it was known that the same result of Theorem 2.9 holds for some of the models. The idea of the proof for Theorem 2.8 works for all models in this section directly.

§ 3.1. Examples with a discrete state space

Here, we give examples with a discrete configuration space. For simplicity, we only give S, the state space of one site or one particle, and the partial differential operators ∇_b. Precise definition of chains, forms and the statement of theorems are given in our papers in preparation or the reference or in the reference given for each model. Notice that for general models, $\nabla_{x,y} \neq \nabla_{y,x}$.

Example 3.1 (Generalized exclusion process,[3]).
$S = \{0, 1, 2, \ldots, \kappa\}$, $\nabla_{x,y} f(\eta) = 1_{\{\eta_x \geq 1, \eta_y \leq \kappa - 1\}} \big(f(\eta^{x \to y}) - f(\eta) \big)$ where

$$(\eta^{x \to y})_z = \begin{cases} \eta_z & \text{if } z \neq x, y \\ \eta_x - 1 & \text{if } z = x \\ \eta_y + 1 & \text{if } z = y. \end{cases}$$

Example 3.2 (Zero-range process,[3]).
$S = \mathbf{N}_0 = \{0, 1, 2 \dots\}$, $\nabla_{x,y} f(\eta) = 1_{\{\eta_x \geq 1\}} \big(f(\eta^{x \to y}) - f(\eta) \big)$.

Remark. Though the zero-range process is a gradient model, it may be interesting to study the characterization theorem of germs of closed forms associated to the process.

§ 3.2. conservative Ginzburg-Landau process

In this subsection, we consider a specific model with a continuous configuration space. The model is called a conservative Ginzburg-Landau process (cf.[10]). For simplicity, we consider the model in $\mathbf{Z}^d$, but we can extend it in a crystal lattice as same as before.

Let $S = \mathbf{R}$, $\chi = \mathbf{R}^{\mathbf{Z}^d}$. We denote by $\mathcal{S}(\chi)$ the set of all functions on χ which is smooth as a function of any fixed finite number of coordinates. We denote the set of all oriented bonds by $\mathbb{E}^d$. For each oriented bond $(x, y) \in \mathbb{E}^d$ and $f \in \mathcal{S}(\chi)$, $\nabla_{x,y} f := (\partial_{\eta_x} - \partial_{\eta_y}) f$ is in $\mathcal{S}(\chi)$.

Definition 3.3. A family of functions $(\Psi^b)_{b \in \mathbb{E}^d}$ in $\mathcal{S}(\chi)$ is called a closed form or closed if

$$\Psi^{\bar{b}} = -\Psi^b$$

for any $b, \bar{b}$ for any $b \in \mathbb{E}^d$ where $\bar{b}$ is the inverse of b.

$$\nabla_b \psi^{b'} = \nabla_{b'} \psi^b$$

for any $b, b' \in \mathbb{E}^d$.

Definition 3.4. A family of functions $(\Psi^b)_{b \in \mathbb{E}^d}$ in $\mathcal{S}(\chi)$ is called an exact form or exact if there exists a function F in $\mathcal{S}(\chi)$ such that

$$\Psi^b = \nabla_b F$$

for all $b \in \mathbb{E}^d$.

Then, by the definition and by a simple topological argument, we have the following lemma again.

Lemma 3.5. *A family of functions $(\Psi^b)_{b \in \mathbb{E}^d}$ is closed if and only if it is exact.*

We introduce the notion of germs of closed forms and exact forms.

Let e_i stands for the unit vector to the i-th direction and $e_i^* = (0, e_i) \in \mathbb{E}^d$ for $i = 1, 2, \dots, d$.

Definition 3.6. A family of functions $\xi = (\xi^{e_i^*})_{i=1}^d$ in $\mathcal{S}(\chi)$ is called a germ of closed form if the family of functions $(\xi^b)_{b\in\mathbb{E}^d}$ is closed where

$$\xi^b = \tau_x \xi^{e_i^*} \quad \text{if} \quad b = (x, x+e_i)$$

and

$$\xi^b = -\tau_x \xi^{e_i^*} \quad \text{if} \quad b = (x+e_i, x).$$

Definition 3.7. A family of functions $\xi = (\xi^{e_i^*})_{i=1}^d$ is called a germ of exact form if there exists a local function F in $\mathcal{S}(\chi)$ such that

$$\xi^{e_i^*} = \nabla_{e_i^*}\Big(\sum_{x\in\mathbf{Z}^d} \tau_x F\Big)$$

for all $i = 1, 2, \ldots, d$.

Denote the set of local functions on χ in $\mathcal{S}(\chi)$ by $\mathcal{S}_\ell(\chi)$ and the set of all d-tuple of them by $(\mathcal{S}_\ell(\chi))^d$, the set of all germs of closed forms by $\mathcal{C}$ and that of exact forms by $\mathcal{E}$. Notice that $\mathcal{E} \subset (\mathcal{S}_\ell(\chi))^d$. Define $\Phi_i = (\Phi_i^{e_j^*})_{j=1}^d$ by

$$\Phi_i^{e_j^*} = \nabla_{e_j^*}\Big(\sum_{x\in\mathbf{Z}^d} x_i \eta_x\Big) = \delta_{i,j}.$$

Then, we have the following characterization theorem of germs of closed forms in $\mathcal{S}_\ell(\chi)$.

Theorem 3.8.

$$\mathcal{C} \cap (\mathcal{S}_\ell(\chi))^d = \mathcal{E} + \{\sum_{i=1}^d a_i \Phi_i\}.$$

The characterization of germs of closed forms in $L^2(\nu)$ for this model is also shown for a good class of product measure ν in [10] using the sharp estimate of the spectral gap. So far, we do not know any alternative proof for this model.

§ 3.3. Other examples with a continuous state space

Here, we give other examples with a continuous configuration space.

Example 3.9 (Energy conserving Ginzburg-Landau model, [4]).
$S = \mathbf{R}_+$, $\nabla_{x,y} f = (\partial_{\eta_x} - \partial_{\eta_y}) f$.

Example 3.10 (Energy conserving stochastic model,[2]).
Let $S = \mathbf{R}$, $\nabla_{x,y} f = (\eta_y \partial_{\eta_x} - \eta_x \partial_{\eta_y}) f$.

Remark. In the case of Example 3.10, since the conserved quantity is $\sum_x \eta_x^2$ instead of $\sum_x \eta_x$, the harmonic forms are given as

$$\Phi_i^{e_j^*} = \nabla_{e_j^*}\big(\sum_{x\in\mathbf{Z}^d} x_i\eta_x^2\big).$$

The role of the conserved quantity becomes also clear by our new proof.

§ 4. Conclusion and remarks on further extensions

Our study reveals universal properties of the first cohomology group of configuration spaces associated to interacting particle systems. We summarize them here in a formal way.

Consider an interacting particle system which is irreducible on a configuration space with a fixed conserved quantity. Then, the set of closed forms and that of exact forms associated to the model is identical because of the irreducibility. In other words, the first cohomology group is zero. Under such a situation, if an abelian group G acts on the configuration space and the partial differential operators compatibly, then we may consider closed (resp. exact) forms which are invariant by group actions. We call them germs of closed (resp. exact) forms. Now, we should have the decomposition of germs of closed forms into germs of exact forms and harmonic forms. The dimension of the set of germs of harmonic forms are exactly given by the dimension of the first homology of the group G. Formally, this property is well-known in the cohomology theory. Notice that our results and proofs are applicable even for finite configuration spaces such as $\chi = S^{\mathbb{T}_N^d}$ where $\mathbb{T}_N^d$ is the d-dimensional discrete torus with size N.

Finally, we give some comments on possible extensions of our result. It is not difficult to extend it to the models with two or more types of partial differential operators such as the models studied in [5] and [8]. Moreover, we can extend it to the models with two or more conserved quantities such as the model studied in [6]. For the latter case, we can show that the dimension of the germs of harmonic forms are the dimension of the first homology of the group G times the number of conserved quantities. From this, the role of the number of conserved quantities in the characterization theorem of closed forms becomes more clear.

References

[1] T. Funaki, K. Uchiyama and H. T. Yau, *Hydrodynamic Limit for Lattice Gas Reversible under Bernoulli Measures*, *Nonlinear Stochastic PDEs*, Springer-Verlag New York **77** (1996),1–44.

[2] F. Hernández, *Equilibrium Fluctuations for a Non Gradient Energy Conserving Stochastic Model*, *Commun. Math. Phys.*, **326** (2014), 687–726

[3] C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems*, 1999, Springer.

[4] C. Liverani, S. Olla and M. Sasada, *Diffusive scaling in energy Ginzburg-Landau dynamics*, arxiv1509:0611.

[5] S. Olla and M. Sasada, *Macroscopic energy diffusion for a chain of anharmonic oscillators*, *Probab. Theory and Relat. Fields*, **157** (2013), 721–775.

[6] J. Quastel, *Diffusion of color in the simple exclusion process*, *Comm. Pure Appl. Math.*, **45** (1992), 623–679.

[7] M. Sasada, *Hydrodynamic limit for two-species exclusion processes,Stoch. Proc. Appl.* **120** (2010), 494–521.

[8] M. Sasada, *Hydrodynamic limit for exclusion processes with velocity,Markov Process. Related Fields* **17** (2011), 391–428.

[9] R. Tanaka, *Hydrodynamic Limit for Weakly Asymmetric Simple Exclusion Processes in Crystal Lattices*, *Commun. Math. Phys.*, **315** (2012), 603–641.

[10] S.R.S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions II*, in Asymptotic Problems in Probability Theory, Stochastic Models and Diffusions on Fractals, Pitman Res. Notes Math. Ser., **283** (1994), 75–128.