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A remark on equivalence of ensembles for surface diffusion model

By

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Abstract

The surface diffusion model is an evolutional model of random Young diagrams, which is introduced by Funaki [2] and the equivalence of ensembles for this model is also studied with some "good" conditions. In this paper we extend this result. Namely, we can prove the equivalence of ensembles for this model without "good" conditions. This result is required to give an lower bound estimate of the spectral gap estimate for this model.

§ 1. Introduction

The surface diffusion model is introduced by Funaki [2] and the equivalence of ensembles for this model is also studied with some "good" conditions. The main purpose of this paper is that we can extend this result without "good" conditions, which is required to give a lower bound estimate of the spectral gap estimate for this model.

The surface diffusion model is an evolutional model of random Young diagrams, which is given in [2]. The height function $\psi_q : [0, \infty) \to [0, \infty)$ of a two-dimensional Young diagram, which is associated with distinct partition $q = \{q_1 > q_2 > \cdots > q_K \geq 1\}$ of a positive integer M by positive integers $\{q_i\}_{i=1}^K$ (i.e., $M = \sum_i q_i$), has following expression

$$\psi_q(u) = \sum_{i=1}^{K} \mathbf{1}_{\{u < q_i\}}, \qquad u \in [0, \infty).$$

Its height difference $\eta = \{\eta_k\}_{k \in \mathbb{N}}$ is defined by

$$\eta_k = \psi_q(k-1) - \psi_q(k) \in \{0, 1\}, \qquad \text{or} \qquad \eta_k = 1 \text{ iff } k \in \{q_i\}_i.$$

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The two quantities of Young diagram K and M is represented as

$$K = \sum_k \eta_k, \quad M = \sum_k k\eta_k.$$

Here K corresponds to total height and M corresponds to total area. From the view point of the evolutional model, we treat these two quantities as conserved quantities. Furthermore we suppose that $q_1 \leq l$. We consider uniform measure on this space. The equivalence of ensembles means that this uniform measure (micro canonical ensembles) is asymptotically equivalent to some Bernoulli measure with space dependent density profile (grand canonical ensembles). In [2], the density profile (so-called Vershik curve) is determined and it is proved that the equivalence of ensembles holds true uniformly in $\rho \in (\varepsilon, 1-\varepsilon)$ and $m \in (-v/2+\varepsilon, v/2-\varepsilon)$ for every $\varepsilon > 0$, (we shall define ρ, m, v in Section 2, (2.1)). Our main result, Theorem 3.2, gives a sharp order estimate of the equivalence of ensembles uniformly in ρ, m with the same density profile. This uniform estimate yields an inequality which is used to prove a lower bound estimate for the spectral gap. To observe this theorem, we need to exclude a subset of our state space. In Section 2 we give a condition for this subset. In Section 3 we give the equivalence of ensembles and an inequality which is used to prove a lower bound estimate for the spectral gap.

§ 2. On the state space

We use notations given in [2]. For a finite set Λ in $\mathbb{Z}$ we set $\Sigma_\Lambda := \{0,1\}^\Lambda$ and its restriction

$$\Sigma_{\Lambda,K,M} := \{\eta \in \Sigma_\Lambda; \sum_{x\in\Lambda} \eta_x = K, \sum_{x\in\Lambda} x\eta_x = M\}.$$

For given $l \in \mathbb{N}$ we set $\Lambda_l := \{-l, -l+1, \ldots, l-1, l\}$. Our configuration space is $\Sigma_{\Lambda_l,K,M}$. In this setting, we have only to consider the parameters $0 \leq K \leq 2l+1$ and $|M| \leq \{K(2l+1) - K^2\}/2$. We use the following notation;

$$\rho := \frac{K}{2l+1}, \quad v := \rho(1-\rho), \quad m := \frac{M}{(2l+1)^2}. \tag{2.1}$$

Then it is easy to see that $0 \leq \rho \leq 1$ and $|m| \leq v/2$.

We consider the case such that $|m|$ is close to $v/2$. Precisely, we consider the case such that $v/2 - |m| \leq C/l$ for some constant C. To simplify our notation, we assume that $\rho \geq 1/2$ and $m > 0$, i.e., $K \geq l$ and M is close to $\{K(2l+1) - K^2\}/2$. If we assume that

$$M = M(x; l, K) = \frac{K(2l+1) - K^2}{2} + (l - K) - x$$

for $l-K+1 \le x \le l$. If $M_0 = (K(2l+1)-K^2)/2$, then there is only one element $\xi \in \Sigma_{\Lambda_l,K,M_0}$ such that

$$\xi_k = \begin{cases} 0 & \text{if } -l \le k \le l-K \\ 1 & \text{otherwise.} \end{cases}$$

In our case we delete one particle at site x, $l-K+1 \le x \le l$, and add one particle at site $(l-x)$ to ξ then it is easy to see that following configuration ξ^x is created and this is an element of $\Sigma_{\Lambda_l,K,M}$;

$$(\xi^x)_k = \begin{cases} 0 & \text{if } -l \le k \le l-K-1 \text{ or } k=x \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore it is also easy to see that for any configuration $\eta \in \Sigma_{\Lambda_l,K,M}$ $\eta_k = 1$ for $k \ge x+1$. Hence in order to understand properties of $\Sigma_{\Lambda_l,K,M}$, we have only to consider $\Sigma_{\Lambda,K',M'}$ with

$$\begin{aligned} x' &= \begin{cases} x & \text{if } x+l \text{ is even} \\ x+1 & \text{otherwise,} \end{cases} \\ \Lambda &= \{k \in \mathbb{Z}; -l \le k \le x'\}, \\ K' &= K-(l-x'), \\ M' &= M - \sum_{k=x'+1}^{l} k = M - \frac{l(l+1)-x'(x'+1)}{2}. \end{aligned}$$

Namely we only see the configurations in $[-l, x']$. We set

$$l' = l'(x; l, K) = \frac{x'+l}{2}, \quad M'' = M' - (l-l')K'.$$

We also set the spatial shift τ_k by

$$\begin{aligned} &\tau_k A = A + x \quad \text{for } A \subset \mathbb{Z}, \\ &(\tau_k \eta)_l = \eta_{l-k} \quad \text{for } \eta \in \Sigma_A, \text{ and } \tau_k\eta \in \Sigma_{\tau_k A}. \end{aligned}$$

Then it is easy to see that

$$\Sigma_{\Lambda,K',M'} = \tau_{(l-l')/2}\Sigma_{\Lambda_{l'},K',M''}.$$

Namely there is a natural bijection between $\Sigma_{\Lambda_l,K,M}$ and $\Sigma_{\Lambda_{l'},K',M''}$.

Though $\rho \le 1/2$ or $m < 0$, if $|m|$ is close to $v/2$, we have similar situation. We summarize these results as following proposition:

Proposition 2.1. *Suppose that one of the following conditions holds and we set l', K', M' in each case, then we have a natural bijection between $\Sigma_{\Lambda_l,K,M}$ and $\Sigma_{\Lambda_{l'},K',M'}$.*
(1) It satisfies

$$K \geq l, \quad 1 \leq \frac{K(2l+1)-K^2}{2} - M \leq K.$$

Then we set

$$x = \begin{cases} (l-K) + \dfrac{K(2l+1)-K^2}{2} - M \\ \qquad \text{if } (l-K) + \dfrac{K(2l+1)-K^2}{2} - M + l \text{ is even}, \\ (l-K) + \dfrac{K(2l+1)-K^2}{2} - M + 1 \\ \qquad \text{otherwise}, \end{cases}$$

$$l' = \frac{l+x}{2}, \quad K' = K - (l-x), \quad M' = M - \frac{l(l+1) - x(x+1)}{2} - (l-l')K'.$$

(2) It satisfies

$$K \geq l, \quad -K \leq -\frac{K(2l+1)-K^2}{2} - M \leq -1.$$

Then we set

$$x = \begin{cases} (l-K) + \dfrac{K(2l+1)-K^2}{2} + M \\ \qquad \text{if } (l-K) + \dfrac{K(2l+1)-K^2}{2} + M + l \text{ is even}, \\ (l-K) + \dfrac{K(2l+1)-K^2}{2} + M + 1 \\ \qquad \text{otherwise}, \end{cases}$$

$$l' = \frac{l+x}{2}, \quad K' = K - (l-x), \quad M' = M + \frac{l(l+1) - x(x+1)}{2} + (l-l')K'.$$

(3) It satisfies

$$K \leq l, \quad 1 \leq \frac{K(2l+1)-K^2}{2} - M \leq l - K.$$

Then we set

$$x = \begin{cases} (l-K+1) - \dfrac{K(2l+1)-K^2}{2} + M \\ \qquad \text{if } (l-K+1) - \dfrac{K(2l+1)-K^2}{2} + M + l \text{ is even}, \\ (l-K+1) - \dfrac{K(2l+1)-K^2}{2} + M + 1 \\ \qquad \text{otherwise}, \end{cases}$$

$$l' = \frac{l+x}{2}, \quad K' = K, \quad M' = M - (l-l')K'.$$

(4) It satisfies

$$K \le l, \quad -(l-K) \le -\frac{K(2l+1)-K^2}{2} - M \le 1.$$

Then we set

$$x = \begin{cases} -(l-K+1) + \dfrac{K(2l+1)-K^2}{2} + M \\ \quad \text{if } -(l-K+1) + \dfrac{K(2l+1)-K^2}{2} + M + l \text{ is even}, \\ -(l-K+1) + \dfrac{K(2l+1)-K^2}{2} + M + 1 \\ \quad \text{otherwise}, \end{cases}$$

$$l' = \frac{l+x}{2}, \quad K' = K, \quad M' = M + (l-l')K'.$$

§ 3. On the equivalence of ensembles

In this section we give an extension of the result of the equivalence of ensembles given in [2]. Precisely, though $|m|$ is close to $v/2$, we have similar result.

In [2, p.591 (2.4), (2.5)], $\beta(x)$ is defined by

$$\beta(x) = \beta(x; a, b) = \frac{e^{bx}a}{e^{bx}a + (1-a)}, \quad x \in [-1, 1]$$

with two parameters $a \in (0, 1)$ and $b \in \mathbb{R}$ such that it satisfies

$$\int_{-1}^{1} \beta(x)dx = \frac{2K}{2l+1}, \quad \int_{-1}^{1} x\beta(x)dx = \frac{4M}{(2l+1)^2}.$$

If $b \neq 0$, it is easy to see that we can rewrite $\beta(x)$ as

$$\beta(x) = \frac{e^{b(x-c)}}{e^{b(x-c)} + 1}, \quad c = \frac{1}{b}\log\frac{1-a}{a}. \tag{3.1}$$

We set constants v_k, E_n, F_n, U_n, V_n and W_n by

$$v_k = v_k^n := \beta(\frac{k}{n})(1-\beta(\frac{k}{n})), \quad E_n := \sum_{k=-n}^{n} \beta(\frac{k}{n}), \quad F_n := \sum_{k=-n}^{n} k\beta(\frac{k}{n}),$$

$$U_n := \sum_{k=-n}^{n} v_k, \quad V_n := \sum_{k=-n}^{n} k^2 v_k, \quad W_n := \sqrt{\frac{V_n}{U_n}}.$$

Thanks to Proposition 2.1, we do not need to consider the case that one of the conditions

(1)-(4) in Proposition 2.1 holds. By simple computation, if $b \gg 1,\ c \geq 0$ then we have

$$\begin{aligned}
&\int_{-1}^{1} \beta(x)dx = 1 - c + o(\frac{1}{b}),\\
&\frac{1-c^2}{2} - \frac{C_1}{b^2} \leq \int_{-1}^{1} x\beta(x)dx \leq \frac{1-c^2}{2} - \frac{C_2}{b^2},\\
&C_3\frac{n}{b} \leq U_n \leq C_4\frac{n}{b},\\
&C_5\left(\frac{n}{b}\right)^3 \leq V_n \leq C_6\left(\frac{n}{b}\right)^3
\end{aligned}$$

for some positive constants C_i for $1 \leq i \leq 6$. Setting $n = l$ and comparing the conditions (1)-(4) in Proposition 2.1 and these estimates, we have $|b| \leq C\sqrt{n}$ for some constant C. Hence we have $C_1\sqrt{n} \leq U_n \leq C_2 n$ for some constants C_1, C_2. We set a sequence of independent random variables $\{Y_k^n\}_{k=-n}^n$ by

$$Y_k^n = \begin{cases} 1 & \text{with probability } \beta(\frac{k}{n}),\\ 0 & \text{with probability } 1-\beta(\frac{k}{n}). \end{cases}$$

We define a sequence of 2-dimensional random vectors $\{X_k^n\}_{k=-n}^n$ by

$${}^tX_k^n = \left(Y_k^n, \frac{k}{W_n}Y_k^n\right),$$

and 2-dimensional random vectors S_n and $\bar{S}_n$ by

$$S_n = \sum_{k=-n}^{n} X_k^n, \quad \bar{S}_n = S_n - E[S_n],$$

where $E[S_n]$ is an expectation of S_n. It is easy to see that expectation of $\bar{S}_n$ is equal to 0 and covariance matrix of $\bar{S}_n$, which is the same as that of S_n, is given by

$$\mathrm{Cov}(\bar{S}_n) = \begin{pmatrix} \sum_{k=-n}^{n} v_k & \sum_{k=-n}^{n} \frac{k}{W_n}v_k \\ \sum_{k=-n}^{n} \frac{k}{W_n}v_k & \sum_{k=-n}^{n} \frac{k^2}{W_n^2}v_k \end{pmatrix} = U_n A,$$

where

$$A = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}, \quad \lambda = \lambda_n = \frac{1}{U_n}\sum_{k=-n}^{n}\frac{k}{W_n}v_k.$$

By direct computation there is $\epsilon > 0$ such that $|\lambda| < 1-\epsilon$. Hence we can define $A^{-1/2}$.

We note that in this setting, $\bar{S}_n$ and $A^{-1/2}\bar{S}_n$ satisfy usual forth moment condition, i.e., we have $\sum_k E[\|X_k^n - E[X_k^n]\|^4] \leq CU_n$ for some constant C, where $\|\cdot\|$ is the Euclidean norm. Then the distribution of $U_n^{-1/2}A^{-1/2}\bar{S}_n$ is approximated by standard normal distribution and we also have an order estimate of error term with order $O(1/\sqrt{U_n})$. (See for example, [1, Theorem 13.3].) This observation gives an extension of the result given in [2, Propositoin 3.1, Remark 3.1].

Proposition 3.1. *We have*

$$\sup_{(K,L)} |\sqrt{U_n V_n} P(S_n = K, T_n = L) - q_0(y_1, y_2)| \leq \frac{C}{\sqrt{U_n}},$$

with some constant C where

$$q_0(u,v) = \frac{1}{2\pi\sqrt{1-\lambda^2}} \exp\{-\frac{u^2 - 2\lambda uv + v^2}{2(1-\lambda^2)}\},$$
$$y_1 = \frac{1}{\sqrt{U_n}}(K - E_n), \quad y_2 = \frac{1}{\sqrt{V_n}}(L - F_n).$$

We follow the method given in [2, proof of Theorem 2.1] and [3, proof of Corollary1.4, 1.6, 1.7], then we have following result, which is an extension of the result given in [2, Theorem 2.1].

Let $\nu_{\Lambda_l,K,M}$ be a uniform probability measure on $\Sigma_{\Lambda_l,K,M}$. Given $\rho \in [0,1]$, we define a Bernoulli measure ν_ρ on Σ_{Λ_l} with marginal

$$P_{\nu_\rho}(\eta_x = 1) = \rho.$$

Theorem 3.2. *Let $f_j = f_j(\eta)$, $1 \leq j \leq p$ be a function on Σ_{A_j} for $A_j \subset \Lambda_l$. Namely f_j depends only on the values of $\{\eta_x; x \in A_j\}$ for $1 \leq j \leq p$. Suppose that $\tau_{k_j} A_j \subset \Lambda_l$, and $\tau_{k_i} A_i \cap \tau_{k_j} A_j = \emptyset$ if $i \neq j$. We regard $\tau_{k_j} f_j$, $1 \leq j \leq p$ as functions on $\Sigma_{\Lambda_l,K,M}$ and on Σ_{Λ_l}. Then we have that there is a constant C which never depend on l, K, M such that*

$$|E_{\nu_{\Lambda_l,K,M}}[\prod_{j=1}^p \tau_{k_j} f_j] - \prod_{j=1}^p E_{\nu_{\beta(k_j/l)}}[f_j]| \leq \frac{C}{U_l} \prod_{j=1}^p E_{\nu_{\beta(k_j/l)}}[\{f_j - E_{\nu_{\beta(k_j/l)}}[f_j]\}^2].$$

Furthermore, if $b = 0$ or $|c| > 1$, where c is defined by (3.1), then we have that there is a constant C which never depends on l, K, M such that

$$|E_{\nu_{\Lambda_l,K,M}}[\prod_{j=1}^p \tau_{k_j} f_j] - \prod_{j=1}^p E_{\nu_{\beta(k_j/l)}}[f_j]| \leq \frac{C}{l} \prod_{j=1}^p \{\max_\eta f_j(\eta) - \min_\eta f_j(\eta)\}.$$

Proof. Since the proof is the same as that given in [3, proof of Corollary1.4, 1.6, 1.7], we omit the details. □

We suppose that b and c are given by (3.1). If $|b| \gg 1$, then U_l becomes small, for example U_l becomes of order $O(\sqrt{l})$. However if $x \in \Lambda_l \cap [cl - U_l, cl + U_l]^c$, then

$0 \leq \beta(x/l) < 1/l^2$ or $1 - 1/l^2 < \beta(x/l) \leq 1$. Precisely, suppose that $b \gg 1$. If $x < cl - U_l$ then $0 \leq \beta(x/l) < 1/l^2$ and if $x > cl + U_l$ then $1 - 1/l^2 < \beta(x/l) \leq 1$. Hence by Theorem 3.2 we can regard (with probability greater than $1 - 1/l^2$) that if $x < cl - U_l$ then $\eta_x \equiv 0$ and if $x > cl + U_l$ then $\eta_x \equiv 1$. In other words, if we consider a stochastic particle system with invariant measure $\nu_{\Lambda_l,K,M}$, then particles can move only in the interval $[cl - U_l, cl + U_l]$ and particles in $[cl - U_l, cl + U_l]^c$ are frozen (with probability greater than $1 - 1/l^2$). Here we regard that if $\eta_x = 1$ then there is a particle at site x and if $\eta_x = 0$ then site x is vacant site. This observation yields following remark:

Remark. Suppose that for $A \subset \Lambda_l \times \Lambda_l$, $A' \subset [-1,1]^2$, there is a constant C such that

$$\left| \frac{1}{l^2} \sum_{(x,y)\in A} \beta(\frac{x}{l})\beta(\frac{y}{l}) - \int_{A'} \beta(x)\beta(y)dxdy \right| \leq \frac{C}{l}.$$

Then thanks to Theorem 3.2, we have that there is a constant C' such that

$$\left| \frac{1}{l^2} E_{\nu_{\Lambda_l,K,M}} \left[\sum_{(x,y)\in A} \eta_x\eta_y \right] - \int_{A'} \beta(x)\beta(y)dxdy \right| \leq \frac{C'}{l}.$$

This bound is used to prove a lower bound estimate for the spectral gap.

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