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Scaling limits for Glauber-Kawasaki processes

By

Kenkichi TSUNODA *

Abstract

In this paper we will give a survey on scaling limits of the empirical measures for the Glauber-Kawasaki processes. The microscopic system is described by the superposition of the Kawasaki dynamics with a spin-flip dynamics and the hydrodynamic equation is given by a reaction-diffusion equation. We especially review results on hydrodynamic limits, nonequilibrium fluctuations and large deviation principles for the Glauber-Kawasaki process.

§ 1. Introduction

The purpose of this article is to introduce the reader to mathematical results on the Glauber-Kawasaki process. De Masi et al. [5] introduced the Glauber-Kawasaki process to study the reaction-diffusion equation at the microscopic level. They showed that the macroscopic density profile of the Glauber-Kawasaki process evolves according to the autonomous reaction-diffusion equation

$$\partial_t \rho \; = \; (1/2)\Delta\rho - V'(\rho) \, , \tag{1.1}$$

where Δ is the Laplacian and V is a potential. This result is called "Hydrodynamic limit" and seems to be a starting point of the several results concerning with the Glauber-Kawasaki process. De Masi et al. [5, 6] also studied the nonequilibrium fluctuations from the solution of the reaction-diffusion equation (1.1), the phase separation and its spatial pattern.

In this paper we give a survey on scaling limits for the Glauber-Kawasaki processes. We particularly review results on hydrodynamic limits, nonequilibrium fluctuations and

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*Institute of Mathematics for Industry, Kyushu university, 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan.
e-mail: `k-tsunoda@imi.kyushu-u.ac.jp`

large deviation principles for the Glauber-Kawasaki process. This article is organized as follows: In Section 2, we introduce the Glauber-Kawasaki process which is a model of magnetizations. We also review a few results, hydrodynamic limits and nonequilibrium fluctuations, showed in [5]. In Section 3, we see the large deviations results related to the hydrodynamic limit.

§ 2. Glauber-Kawasaki process

We introduce in this section the Glauber-Kawasaki process, which is a model of magnetizations. We also review the results of the hydrodynamic limit and the nonequilibrium fluctuations.

§ 2.1. Model

Let d be a positive integer. Denote by $\mathcal{X} = \{-1,1\}^{\mathbb{Z}^d}$ the state space of our process and by $\sigma = \{\sigma(x), x \in \mathbb{Z}^d\}$ the configurations in $\mathcal{X}$. For each $x \in \mathbb{Z}^d$, $\sigma(x)$ stands for the spin sitting at site x for the configuration σ. For each $x, y \in \mathbb{Z}^d$ with $x \neq y$, we also denote by $\sigma^{x,y}$, respectively by σ^x, the configuration obtained from σ by exchanging the spins at x and y, respectively by flipping the occupation variable at site x:

$$\sigma^{x,y}(z) \ = \ \begin{cases} \sigma(y) & \text{if } z = x \ , \\ \sigma(x) & \text{if } z = y \ , \\ \sigma(z) & \text{otherwise} \ , \end{cases} \qquad \sigma^x(z) \ = \ \begin{cases} \sigma(z) & \text{if } z \neq x \ , \\ 1-\sigma(z) & \text{if } z = x \ . \end{cases}$$

We consider the superposition of the Kawasaki dynamics with a spin-flip dynamics. More precisely, the stochastic dynamics is given by the continuous-time Markov process on $\mathcal{X}$ whose generator acts on local functions $f : \mathcal{X} \to \mathbb{R}$ as

$$\mathcal{L}_\varepsilon f \ = \ \varepsilon^{-2}\mathcal{L}_E f + \mathcal{L}_G f \ ,$$

where $\mathcal{L}_E$ is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_E f)(\sigma) \ = \ (1/2) \sum_{|x-y|=1} [f(\sigma^{x,y}) - f(\sigma)] \ ,$$

and $\mathcal{L}_G$ is the generator of a spin flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\sigma) \ = \ \sum_{x\in\mathbb{Z}^d} c(x,\sigma)[f(\sigma^x) - f(\sigma)] \ .$$

In the last formula, we choose the function $c(x,\sigma)$ to be invariant under translations and local. That is,

$$c(0,\sigma) \ = \ \sum_{A\subset\Lambda_0} K_A \sigma_A \ , \qquad \sigma_A = \prod_{y\in A} \sigma(y) \ ,$$

where Λ_0 is some bounded domain with $0 \in \Lambda_0 \subset \mathbb{Z}^d$ and K_A's have to be chosen so that $c(0,\sigma) \geq 0$, $\forall \sigma \in \mathcal{X}$. By translation invariance

$$c(x,\sigma) \;=\; c(0,\tau_{-x}\sigma) \;=\; \sum_{A\subset\Lambda_0} K_A \sigma_{A+x} \;,$$

where τ_{-x} is the shift by $-x$, $(\tau_{-x}\sigma)(y) \;=\; \sigma(x+y)$. Note that the positive number $\varepsilon > 0$ appeared in the definition of $\mathcal{L}_\varepsilon$ is a scaling parameter and we here put the time-change factor ε^{-2}, which corresponds to the diffusive scaling.

Let ν_m, $-1 \leq m \leq 1$, be the Bernoulli product measure with the mean $\nu_m(\sigma(0)) = m$. Define the continuous function $F : [-1,1] \to \mathbb{R}$ by

$$F(m) \;=\; \int -2\sigma(0)\, c(0,\sigma)\, d\nu_m \;.$$

Let $D([0,\infty),\mathcal{X})$ be the space of all right-continuous trajectories from $[0,\infty)$ to $\mathcal{X}$ with left-limits, endowed with the Skorokhod topology. Let $\{\sigma_t^\varepsilon : t \geq 0\}$ be the continuous-time Markov process on $\mathcal{X}$ whose generator is given by $\mathcal{L}_\varepsilon$. For each probability measure μ on $\mathcal{X}$, let $\mathbb{P}_\mu$ be the probability measure on $D([0,\infty),\mathcal{X})$ induced by the process $\sigma_\cdot^\varepsilon$ starting from μ. Denote by $\mathbb{E}_\mu[\cdot]$ the expectation with respect to $\mathbb{P}_\mu$.

We now assume a few conditions on the initial distributions.

Definition 2.1 (Hypothesis on the Initial Measure)**.** Let $m_0 : \mathbb{R} \to [-1,1]$ be a C^3-function with uniformly bounded derivatives. Let μ^ε, $\varepsilon \in (0,1)$, be a family of measures such that

1. $\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{Z}^d} \varepsilon^{-d} |\mu^\varepsilon(\sigma(x)) - m_0(\varepsilon x)| \;=\; 0$.

2. For every $n \geq 2$ there are positive decreasing functions $\phi_n : \mathbb{N} \to [0,\infty)$ such that for all $A, B \subset \mathbb{Z}^d$, $A \cap B = \emptyset$,

$$|\mu^\varepsilon(\sigma_A\sigma_B) - \mu^\varepsilon(\sigma_A)\mu^\varepsilon(\sigma_B)| \leq \phi_n(r_{A,B}) \;,$$

where $r_{A,B} = dist(A,B)$, $n = |A| + |B|$, and

$$\sum_{x\in\mathbb{Z}^d} \phi_n(|x|) \;<\; \infty \;.$$

3. For every $k \in \mathbb{N}$, $x_1, \cdots, x_k \in \mathbb{Z}^d$ with $x_i \neq x_j$, $\forall i \neq j$,

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{Z}^d} |\mu^\varepsilon(\prod_{i=1}^k \sigma(x_i + x)) - m_0(\varepsilon x)^k| \;=\; 0 \;.$$

§ 2.2. Hydrodynamic limit

The next result is the law of large numbers for the macroscopic magnetization density and is usually called “Hydrodynamic limit”. See [5] for the proof.

Theorem 2.2. *Let μ^ε, $\varepsilon \in (0,1]$, be a family of initial measures on $\mathcal{X}$ satisfying the conditions of Definition 2.1. For $\phi \in \mathcal{S}(\mathbb{R}^d)$, define the magnetization field*

$$X_t^\varepsilon(\phi) \;=\; \varepsilon^d \sum_{x\in\mathbb{Z}^d} \phi(\varepsilon x)\sigma_t^\varepsilon(x)\ .$$

as a process on $D([0,\infty),\mathcal{S}'(\mathbb{R}^d))$ and let $\mathcal{P}^\varepsilon$ denote the law of X^ε induced from $\mathbb{P}_{\mu^\varepsilon}$. Then the sequence of measures $\{\mathcal{P}^\varepsilon, \varepsilon > 0\}$ weakly converges to the measure $\mathcal{P}$ where $\mathcal{P}$ is the measure having support on a single trajectory given by $\int dr\phi(r)m(r,t)$ and $m(r,t)$ is given by the unique solution of the Cauchy problem

$$\begin{cases} \partial_t m \;=\; (1/2)\Delta m + F(m)\ , \\ m(0,\cdot) \;=\; m_0(\cdot)\ . \end{cases} \tag{2.1}$$

Remark. It is known that Theorem 2.2 can be also proved by the entropy method introduced by Guo et al. in [10]. In this case, assumptions on the initial measures can be weakened.

§ 2.3. Nonequilibrium fluctuations

We next review the result on the microscopic fluctuations about the deterministic macroscopic evolution. See [5] for the proof.

Theorem 2.3. *Let μ^ε satisfy the conditions of Definition 2.1. For $\phi \in \mathcal{S}(\mathbb{R}^d)$, define the magnetization fluctuation field as*

$$Y_t^\varepsilon(\phi) \;=\; \varepsilon^{d/2} \sum_{x\in\mathbb{Z}^d} \phi(\varepsilon x)[\sigma_t^\varepsilon(x) - \mathbb{E}_{\mu^\varepsilon}(\sigma_t^\varepsilon(x))].$$

Then the process $\{Y_t^\varepsilon, t \geq 0\}$, considered as a process on $D([0,\infty),\mathcal{S}'(\mathbb{R}^d))$, weakly converges to a generalized Ornstein-Uhlenbeck process $\{Y_t, t \geq 0\}$ with distribution P. P is uniquely determined by the conditions that the processes $\{Y_t(\phi), t \geq 0\}$ are centered, that for all $G \in C_0^\infty(\mathbb{R})$

$$G(Y_t(\phi)) - \int_0^t ds\ Y_s(A_s\phi)G'(Y_s(\phi)) - \frac{1}{2}\int_0^t ds\ \|B_s\phi\|^2 G''(Y_s(\phi)) \tag{2.2}$$

is a P-martingale, and that the law of the $\{Y_0(\phi)\}$ is Gaussian with the covariance

$$P(Y_0(\phi)Y_0(\psi)) \;=\; \int_{\mathbb{R}^d} dr\ \phi(r)\psi(r)(1 - m_0(r)^2)\ .$$

In (2.2), G' and G'' denote, respectively, the first and second derivative of G and

$$\begin{aligned} A_s\phi(r) \;&=\; \frac{1}{2}\phi''(r) + \phi(r)F'(m(r,s))\ , \\ \|B_s\phi\|^2 \;&=\; \int_{\mathbb{R}^d} dr\ (\nabla\phi(r))^2(1 - m(r,s)^2) + \int_{\mathbb{R}^d} dr\ \phi(r)^2 4\nu_{m(r,s)}(c(0,\sigma))\ . \end{aligned}$$

where $m(r,t)$ is the unique solution of the Cauchy problem (2.1).

§ 3. Large deviation principles

We examine in this section the large deviation principles for empirical measure. In Section 2, we consider the Glauber-Kawasaki process as a model of magnetizations. In this section, we regard the Glauber-Kawasaki process as a model of particles. Therefore, since the settings and notation are different from one considered in Section 2, we again start from defining the processes.

§ 3.1. Dynamical large deviations

We study in this subsection the dynamical large deviation principle for the empirical measure under the setting of local equilibrium measures and deterministic initial configurations. For the former setting, the dynamical large deviation principle is shown in [11]. Recently, the dynamical large deviation principle under the latter setting is shown in [12]. We refer to [11] and [12] for more details.

We fix some notation and define the model. Let $\mathbb{T}_N$ be the one-dimensional discrete torus $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \cdots, N-1\}$. The state space of our process is given by $X_N = \{0,1\}^{\mathbb{T}_N}$. The interpretation of configurations is as follows: for a configuration η in X_N and a site x in $\mathbb{T}_N$, $\eta(x) = 1$ if there is a particle at site x and $\eta(x) = 0$ if there is no particle at site x.

We consider in the set $\mathbb{T}_N$ the superposition of the symmetric simple exclusion process (Kawasaki) with a spin-flip dynamics (Glauber). More precisely, the stochastic dynamics $\{\eta_t^N; t \geq 0\}$ is a Markov process on X_N whose generator $\mathcal{L}_N$ acts on functions $f : X_N \to \mathbb{R}$ as

$$\mathcal{L}_N f \;=\; (N^2/2)\mathcal{L}_K f + \mathcal{L}_G f \;,$$

where $\mathcal{L}_K$ is the generator of a symmetric simple exclusion process (Kawasaki dynamics),

$$(\mathcal{L}_K f)(\eta) \;=\; \sum_{x \in \mathbb{T}_N} [f(\eta^{x,x+1}) - f(\eta)] \;,$$

and where $\mathcal{L}_G$ is the generator of a spin flip dynamics (Glauber dynamics),

$$(\mathcal{L}_G f)(\eta) \;=\; \sum_{x \in \mathbb{T}_N} c(x,\eta)[f(\eta^x) - f(\eta)] \;.$$

In these formulas, $\eta^{x,x+1}$ (respectively η^x) represents the configuration obtained from η by exchanging (respectively flipping) the occupation variables $\eta(x)$, $\eta(x+1)$ (respectively $\eta(x)$):

$$\eta^x(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x \;, \\ 1-\eta(z) & \text{if } z = x \;, \end{cases} \qquad \eta^{x,y}(z) \;=\; \begin{cases} \eta(y) & \text{if } z = x \;, \\ \eta(x) & \text{if } z = y \;, \\ \eta(z) & \text{otherwise} \;. \end{cases}$$

Moreover, $c(x,\eta) = c(\eta(x-M),\cdots,\eta(x+M))$, for some $M \geq 1$ and some strictly positive cylinder function $c(\eta)$, that is, a function which depends only on a finite number of variables $\eta(y)$. Note that the exclusion dynamics has been already speeded-up by the factor N^2.

Denote by $\mathbb{T}$ the one-dimensional continuous torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$. Let $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T})$ be the space of nonnegative measures on $\mathbb{T}$, whose total mass is bounded by 1, endowed with the weak topology. For a measure π in $\mathcal{M}_+$ and a continuous function $G : \mathbb{T} \to \mathbb{R}$, denote by $\langle \pi, G\rangle$ the integral of G with respect to π:

$$\langle \pi, G\rangle \ = \ \int_{\mathbb{T}} G(u)\pi(du) \ .$$

Denote by $\mathcal{M}_{+,1}$ the closed subset of $\mathcal{M}_+$ of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_{+,1} \ = \ \{\pi \in \mathcal{M}_+(\mathbb{T}) : \pi(du) = \rho(u)du,\ 0 \leq \rho(u) \leq 1 \ a.e.\ u \in \mathbb{T}\} \ .$$

Let $L^2(\mathbb{T})$ be the space of all real square-integrable functions $G : \mathbb{T} \to \mathbb{R}$ with respect to the Lebesgue measure: $\int_{\mathbb{T}} |G(u)|^2 du < \infty$. The corresponding norm is denoted by $\|\cdot\|_2$:

$$\|G\|_2^2 \ := \ \int_{\mathbb{T}} |G(u)|^2 du \ .$$

In particular, $L^2(\mathbb{T})$ is a Hilbert space equipped with the inner product

$$\langle G, H\rangle \ = \ \int_{\mathbb{T}} G(u)H(u)du \ .$$

Fix arbitrarily $T > 0$. For a topological space X, denote by $D([0,T],X)$ be the space of all right-continuous trajectories from $[0,T]$ to X with left-limits, endowed with the Skorokhod topology. Let Q^N be the probability measure on $D([0,T],\mathcal{M}_+)$ induced by the measure-valued process π_t^N given by

$$\pi_t^N(du) \ = \ \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta_t^N(x)\delta_{x/N}(du) \ .$$

We will mention assumptions on the initial distributions of the process later.

Denote by $C^m(\mathbb{T})$, m in $\mathbb{N}_0 \cup \{\infty\}$, the set of all real functions on $\mathbb{T}$ which are m times differentiable and whose m-th derivative is continuous. For a given function G in $C^2(\mathbb{T})$, we shall denote by ∇G and ΔG the first and second derivative of G, respectively. Fix $T > 0$, and denote by $C^{m,n}([0,T]\times\mathbb{T})$, m,n in $\mathbb{N}_0\cup\{\infty\}$, the set of all real functions defined on $[0,T]\times\mathbb{T}$ which are m times differentiable in the first variable and n times in the second one, and whose derivatives are continuous.

Fix a measurable function $\gamma : \mathbb{T} \to [0,1]$. For each path $\pi(t,du) = \rho(t,u)du$ in $D([0,T],\mathcal{M}_{+,1})$, define the energy $\mathcal{Q} : D([0,T],\mathcal{M}_{+,1}) \to [0,\infty]$ as

$$\mathcal{Q}(\pi) \;=\; \sup_{G\in C^{0,1}([0,T]\times\mathbb{T})} \Big\{2\int_0^T dt\ \langle \rho_t, \nabla G_t\rangle - \int_0^T dt \int_{\mathbb{T}} du\ G^2(t,u)\Big\}\ .$$

It is known that the energy $\mathcal{Q}(\pi)$ is finite if and only if ρ has a generalized derivative and this generalized derivative is square integrable on $[0,T]\times\mathbb{T}$:

$$\int_0^T dt\ \int_{\mathbb{T}} du\ |\nabla\rho(t,u)|^2 < \infty\ .$$

Moreover, it is easy to see that the energy $\mathcal{Q}$ is convex and lower semicontinuous.

Let $\nu_\rho = \nu_\rho^N$, $0 \le \rho \le 1$, be the Bernoulli product measure with the density ρ. Define the continuous functions $B, D : [0,1] \to \mathbb{R}$ by

$$B(\rho) \;=\; \int [1-\eta(0)]\, c(\eta)\, d\nu_\rho\ , \quad D(\rho) \;=\; \int \eta(0)\, c(\eta)\, d\nu_\rho\ .$$

For each function G in $C^{1,2}([0,T]\times\mathbb{T})$, define the functional $\bar{J}_G : D([0,T],\mathcal{M}_{+,1}) \to \mathbb{R}$ by

$$\begin{aligned}\bar{J}_G(\pi) \;=\; & \langle \pi_T, G_T\rangle - \langle \pi_0, G_0\rangle - \int_0^T dt\ \langle \pi_t, \partial_t G_t + \frac{1}{2}\Delta G_t\rangle \\ & -\frac{1}{2}\int_0^T dt\ \langle \chi(\rho_t), (\nabla G_t)^2\rangle - \int_0^T dt\ \big\{\langle B(\rho_t), e^{G_t}-1\rangle + \langle D(\rho_t)\ , e^{-G_t}-1\rangle\big\},\end{aligned}$$

where $\chi(r) = r(1-r)$ is the mobility. Let $J_G : D([0,T],\mathcal{M}_+) \to [0,\infty]$ be the functional defined by

$$J_G(\pi) \;=\; \begin{cases} \bar{J}_G(\pi) & \text{if } \pi \in D([0,T],\mathcal{M}_{+,1})\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

We define the function $I_{dyn} : D([0,T],\mathcal{M}_+) \to [0,\infty]$ as

$$I_{dyn}(\pi) \;=\; \begin{cases} \sup J_G(\pi) & \text{if } \mathcal{Q}(\pi) < \infty\ , \\ \infty & \text{otherwise}\ , \end{cases} \tag{3.1}$$

where the supremum is taken over all functions G in $C^{1,2}([0,T]\times\mathbb{T})$.

We now clarify the assumptions on the initial distributions.

Definition 3.1 (Local equilibrium measures). Let γ be a positive and continuous function on $\mathbb{T}$. The process is said to start from local equilibrium measures if the initial distribution of the process is given by the product Bernoulli measure ν_γ with the marginals

$$\nu_\gamma(\eta; \eta(x) = 1) \;=\; \gamma(x/N)\ .$$

In this case, we define the function $I_{ini} : D([0,T],\mathcal{M}_+) \to [0,\infty]$ by

$$I_{ini}(\pi) \;=\; \sup_{\rho} \big\{ \langle \pi_0, \log \frac{\rho(1-\gamma)}{(1-\rho)\gamma} \rangle + \langle \lambda, \log \frac{1-\rho}{1-\gamma} \rangle \big\} \;,$$

where the supremum is taken over all continuous functions $\rho : \mathbb{T} \to (0,1)$ and λ stands for the Lebesgue measure on $\mathbb{T}$.

Definition 3.2 (Deterministic initial configurations). Let γ be a measurable function on $\mathbb{T}$. The process is said to start from deterministic initial configurations if the initial distribution of the process is given by a sequence of deterministic configurations $\{\eta^N ; N \geq 1\}$ associated to γ, in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{T}_N} G(x/N)\eta^N(x) \;=\; \int_{\mathbb{T}} G(u)\gamma(u)du$$

for every continuous functions $G : \mathbb{T} \to \mathbb{R}$. In this case, we define the function $I_{ini} : D([0,T],\mathcal{M}_+) \to [0,\infty]$ by

$$I_{ini}(\pi) \;=\; \begin{cases} 0 & \text{if } \pi(0,du) = \gamma(u)du \;, \\ \infty & \text{otherwise} \;. \end{cases} \tag{3.2}$$

For each initial distributions, we define the dynamical rate function $I_T : D([0,T],\mathcal{M}_+) \to [0,\infty]$ by

$$I_T(\pi) = I_{ini}(\pi) + I_{dyn}(\pi) \;. \tag{3.3}$$

The following large deviation principle is shown in [11, 12].

Theorem 3.3. *Fix $T > 0$. Assume that the functions B and D are concave on $[0,1]$ and one of the conditions of Definitions 3.1 and 3.2 holds with an initial profile γ. Then, the measure Q^N on $D([0,T],\mathcal{M}_+)$ satisfies a large deviation principle with the rate function I_T. That is, for each closed subset $\mathcal{C} \subset D([0,T],\mathcal{M}_+)$,*

$$\varlimsup_{N\to\infty} \frac{1}{N} \log Q^N(\mathcal{C}) \;\leq\; -\inf_{\pi\in\mathcal{C}} I_T(\pi) \;,$$

and for each open subset $\mathcal{O} \subset D([0,T],\mathcal{M}_+)$,

$$\varliminf_{N\to\infty} \frac{1}{N} \log Q^N(\mathcal{O}) \;\geq\; -\inf_{\pi\in\mathcal{O}} I_T(\pi) \;.$$

Moreover, the rate function I_T is lower semicontinuous and has compact level sets.

In the remaining subsection we briefly review the joint dynamical large deviation principle studied in [4]. We continue to consider the Glauber-Kawasaki process on a one-dimensional periodic domain although a boundary-driven system is considered in [4] .

We now consider two types of currents, namely, the conservative currents over bonds and the nonconservative currents on each sites. For each bond $(x, x+1)$, $x \in \mathbb{T}_N$, let $J_t^N(x)$ be the conservative current over the bond $(x, x+1)$, that is, the total number of particles that have jumped from x to $x+1$ minus the total number of particles that have jumped from $x+1$ to x on the time interval $[0,t]$. Define the empirical current field J_t^N as the signed measure on $\mathbb{T}$ given by

$$J_t^N(du) \;=\; \frac{1}{N} \sum_{x \in \mathbb{T}_N} J_t^N(x) \delta_{x/N}(du) \;.$$

For each site $x \in \mathbb{T}_N$, let $K_t^N(x)$ be the nonconservative current at the site x, that is, the total number of particles that have been created at x minus the total number of particles that have been annihilated at x on the time interval $[0,t]$. Define the empirical current field K_t^N as the signed measure on $\mathbb{T}$ given by

$$K_t^N(du) \;=\; \frac{1}{N} \sum_{x \in \mathbb{T}_N} K_t^N(x) \delta_{x/N}(du) \;.$$

Let $\mathcal{M}$ be the set of all singed measures on $\mathbb{T}$ endowed with the weak topology. Note that the triplet (π^N, J^N, K^N) can be regarded as a process on $D([0,T], \mathcal{M}_+ \times \mathcal{M} \times \mathcal{M})$. Let the process $\{\eta_t^N; t \geq 0\}$ start from local equilibrium measures with an initial profile γ (see Definition 3.1) and denote by $\mathbb{P}_\gamma^N$ the distribution induced from the process $\{\eta_t^N; t \geq 0\}$ on $D([0,T], X_N)$.

We now introduce the joint dynamical large deviation principle obtained in [4]. To state the result we need somewhat restricted assumptions on B and D. We assume that the functions B and D are concave on $[0,1]$. In addition to the previous assumption, we also assume that B is nondecreasing and D is nonincreasing on $[0,1]$. Under these assumptions, Bodineau and Lagouge [4] proved the following large deviation principle: For each opne subset $\mathcal{F} \subset D([0,T], \mathcal{M}_+ \times \mathcal{M} \times \mathcal{M})$

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_\gamma^N((\pi_\cdot^N, J_\cdot^N, K_\cdot^N) \in \mathcal{F}) \;\leq\; - \inf_{(\pi, J, K) \in \mathcal{F}} \mathcal{I}_T(\pi, J, K) \;,$$

and for each open subset $\mathcal{G} \subset D([0,T], \mathcal{M}_+)$,

$$\underline{\lim_{N\to\infty}} \frac{1}{N} \log \mathbb{P}_\gamma^N((\pi_\cdot^N, J_\cdot^N, K_\cdot^N) \in \mathcal{G}) \;\geq\; - \inf_{(\pi, J, K) \in \mathcal{G}} \mathcal{I}_T(\pi, J, K) \;,$$

We do not give the definition of the rate functional $\mathcal{I}_T$ here. See [4] for the precise definition of $\mathcal{I}_T$.

Remark. The concavity assumptions on B and D enable us to simplify the proof of the lower bound. The monotonicity assumptions on B and D are only used in the proof of the uniqueness of the weak solutions for the singular perturbations of the hydrodynamic equation. See [4] and [12] for more discussions on these assumptions.

§ 3.2. Static large deviations

We examine in this subsection large deviations for the stationary states. The result is shown in the forthcoming paper [7]. We also continue to use notation defined in Subsection 3.1.

Let S be the set of all classical solutions of the semilinear elliptic equation:

$$(1/2)\Delta\rho + F(\rho) \;=\; 0 \;\text{ on }\; \mathbb{T}\,, \tag{3.4}$$

where $F(\rho) = B(\rho) - D(\rho)$. Classical solution means a function $\rho : \mathbb{T} \to [0,1]$ in $C^2(\mathbb{T})$ which satisfies the equation (3.4) for any $u \in \mathbb{T}$. We also define by $\mathcal{M}_{sol}$ the set of all absolutely continuous measures whose density is a classical solution of (3.4):

$$\mathcal{M}_{sol} \;:=\; \{\bar\vartheta \in \mathcal{M}_+ : \bar\vartheta(du) = \bar\rho(u)du,\; \bar\rho \in S\}\,.$$

Let $\pi^N : X_N \to \mathcal{M}_+$ be the function which associates to a configuration η the positive measure obtained by assigning mass N^{-1} to each particle of η,

$$\pi^N(\eta) \;=\; \frac{1}{N}\sum_{x\in\mathbb{T}_N} \eta(x)\delta_{x/N}\,,$$

where δ_u stands for the Dirac measure which has a point mass at $u \in \mathbb{T}$.

Since the jump rate $c(\eta)$ is strictly positive, the Markov process η^N_t, $t \geq 0$, is irreducible. Therefore there exists a unique stationary probability measure under the dynamics. We denote it by μ^N. We also introduce the probability measure on $\mathcal{M}_+$ defined by $\mathcal{P}^N := \mu^N \circ (\pi^N)^{-1}$.

Following [8, Chapter 6], we now define the static large deviation rate functional. Assume that there exist density profiles $\bar\rho_1, \cdots, \bar\rho_l$, $l > 1$, such that any classical solution $\bar\rho : \mathbb{T} \to [0,1]$ of the equation (3.4) can be given by $\bar\rho(u) = \bar\rho_i(u - u_0)$ for some $1 \leq i \leq l$ and $u_0 \in \mathbb{T}$. In other words, it is equivalent to the following:

$$\mathcal{M}_{sol} \;=\; \{\bar\rho_i(\cdot - v)du : 1 \leq i \leq l, v \in \mathbb{T}\}\,.$$

For each $1 \leq i \leq l$, let $\mathcal{M}_i$ be the subset of $\mathcal{M}_{sol}$ given by $\mathcal{M}_i = \{\bar\rho_i(\cdot - v)du : v \in \mathbb{T}\}$.

Recall the definitions of the functionals I_{dyn} and I_{ini} (see (3.1) and (3.2) in Subsection 3.1). For measurable function $\gamma : \mathbb{T} \to [0,1]$, define the function $I_T(\cdot|\gamma) : D([0,T], M_+) \to [0,\infty]$ by (3.3).

For each $1 \le i \le l$, we define the functional $V_i : \mathcal{M}_+ \to [0, \infty]$ by

$$V_i(\vartheta) = \inf\{I_T(\pi|\bar{\rho}) : T > 0, \bar{\rho}(u)du \in \mathcal{M}_i, \pi \in D([0,T], \mathcal{M}_+) \text{ and } \pi_T = \vartheta\} ,$$

which is the minimal cost that creates the measure ϑ from the set $\mathcal{M}_i$. For each $1 \le i, j \le l$ with $i \ne j$, let $\bar{\vartheta}_i(du) = \bar{\rho}_i(u)du$ and $v_{ij} = V_i(\bar{\vartheta}_j)$.

To define the static large deviation rate function, we need to recall some notation introduced in [8, Chapter 6]. Let L be a finite set and let W be a subset of L. A graph consisting of arrows $m \to n$ ($m \in L \backslash W$, $n \in L$, $n \ne m$) is called a W-graph if it satisfies the following conditions:

i) Every point $m \in L \backslash W$ is the initial point of exactly one arrow,

ii) There are not closed cycles in the graph.

We denote by $G(W)$ the set of all W-graphs. If a graph W is given by the singleton-set $\{i\}$, then we simply denote $G(\{i\})$ by $G(i)$. We regard $L := \{1, \cdots, l\}$ as a graph with weights v_{ij} and, for each $1 \le i \le l$, consider the number

$$w_i = \min_{g \in G(i)} \sum_{(m \to n) \in g} v_{mn} ,$$

where the sum is taken over all arrows in g.

Let $w = \min_{1 \le i \le l} \{w_i\}$. For each $1 \le i \le l$, we define the functions $W_i, W : \mathcal{M}_+ \to [0, \infty]$ by

$$\begin{aligned} W_i(\vartheta) &= w_i - w + V_i(\vartheta) , \\ W(\vartheta) &= \min_{1 \le k \le l} W_i(\vartheta) . \end{aligned}$$

The following theorem asserts that the sequence of stationary measures satisfies the large deviation principle with the rate functional W.

Theorem 3.4. *Assume that the functions B and D are concave on $[0, 1]$. The sequence of measures $\{\mathcal{P}^N; N \ge 1\}$ satisfies a large deviation principle on $\mathcal{M}_+$ with speed N and the rate function W. Namely, for each closed set $\mathcal{C} \subset \mathcal{M}_+$ and each open set $\mathcal{O} \subset \mathcal{M}_+$,*

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \log \mathcal{P}^N(\mathcal{C}) \le - \inf_{\vartheta \in \mathcal{C}} W(\vartheta) ,$$

$$\underline{\lim}_{N \to \infty} \frac{1}{N} \log \mathcal{P}^N(\mathcal{O}) \ge - \inf_{\vartheta \in \mathcal{O}} W(\vartheta) .$$

Moreover, the rate functional W is bounded on $\mathcal{M}_{+,1}$, lower semicontinuous and has compact level sets.

Remark. We finally mention several results related to large deviations works. The large deviation principle for the empirical measure is used to give a mathematical formulation of "Macroscopic fluctuation theory" developed by Bertini et al. [1]. For more details, see [1] and references therein. The motivation of the study of large deviations for the Glauber-Kawasaki process comes from the one for the stationary nonequilibrium states since the Glauber-Kawasaki process is one of important examples in the theory. For the interesting phenomena of this model and physical discussions, see [9, 2, 3].

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