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Central limit theorem for finite and infinite dimensional diffusions in ergodic environments

By

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Abstract

In this article, we introduce a central limit theorem for the solution to a 1-dimensional stochastic heat equation with a random, ergodic non-linear term. Since it can be viewed as an infinite dimensional diffusion in random environment, we first summarize the ideas in the classical central limit theorem for finite dimensional diffusion in random environment, based on [8, §9]. Afterward, we illustrate the strategy to extend these ideas to stochastic heat equations, and explain the main differences between finite and infinite dimensional models. Due to our result, a central limit theorem in L^1 sense with respect to the randomness of the environment holds and the limit distribution is a centered Gaussian law, whose covariance operator is explicitly described. Moreover, it concentrates only on the space of constant functions.

§ 1. Diffusion Process in Ergodic Environment

The study on homogenization of diffusion processes in stationary, ergodic random environments begins from [9] and [14], where central limit theorem for diffusions driven by random self-adjoint, divergence-type operators is established. Later in [11], an invariance principle in a quenched sense is obtained for the case that all the coefficients are almost surely C^2-smooth. The book [8] gives a very good introduction to the finite dimensional classical results, where the main idea is to decompose the sample path of the diffusion into the sum of an additive functional of the "trajectory" of the environment, and a martingale with stationary increments. Since the evolution of the moving environment is a Markov process which satisfies a sector condition, the general strategy to prove a central limit theorem can be applied. This method is first developed in [7] for

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reversible Markov processes, and is extended to non-reversible processes with a sector condition in [12], [13] and [17].

In these works, only diffusions in finite-dimensional spaces $\mathbb{R}^d$ are considered. The aim of this paper is to extend the story to infinite dimensional diffusions, namely, the solutions to stochastic partial differential equations in random environment. Since the strategy in the infinite dimensional setting is parallel to the finite dimensional case, we first briefly introduce a general framework under which we can prove the central limit theorem of a (finite dimensional) diffusion process in random environment, based mainly on the contents of [8, §9].

Let $(\Sigma, \mathscr{F}, q)$ be a probability space. Suppose that $\{\tilde{V}(\sigma,x)\}_{x\in\mathbb{R}^d}$ and $\{\tilde{a}(\sigma,x)\}_{x\in\mathbb{R}^d}$ are real and $d\times d$ real matrix valued random fields over Σ, respectively. Assume that the realizations $\tilde{V}(\sigma,\cdot)$ and every entries of $\tilde{a}(\sigma,\cdot)$ belong to $C_b^2(\mathbb{R}^d)$, q-almost surely. Fix some $\sigma\in\Sigma$, we consider the diffusion process $X^\sigma(t)\in\mathbb{R}^d$ starting at some vector x, formally driven by a divergence-type operator

$$L^\sigma f(x) = \frac{1}{2} e^{\tilde{V}(\sigma,x)} \nabla_x \cdot \left(e^{-\tilde{V}(\sigma,x)} \tilde{a}(\sigma,x) \nabla_x f(x) \right), \tag{1.1}$$

where $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_d})$. In this section we prove a central limit theorem for $X^\sigma(t)/\sqrt{t}$ when the randomness of σ is taken into consideration.

Consider the space of all C^2 mappings from $\mathbb{R}^d$ to $\mathbb{R}^{1+d^2}$, which is the path space of the joint random field $(\tilde{V},\tilde{a})$. Notice that on $C^2(\mathbb{R}^d,\mathbb{R}^{1+d^2})$ there exists a natural semigroup of "shifts" $\{\tau_x; x\in\mathbb{R}^d\}$ defined by

$$\tau_x \circ \phi \triangleq \phi(\cdot + x), \quad \forall \phi \in C^2(\mathbb{R}^d, \mathbb{R}^{1+d^2}). \tag{1.2}$$

Obviously, $\tau_x\circ\tau_y = \tau_{x+y}$. Denoting by $P_{\tilde{V},\tilde{a}}$ the distribution determined by $(\tilde{V},\tilde{a})$ on its path space, we call the random field to be stationary and ergodic if and only if $P_{\tilde{V},\tilde{a}}$ is stationary and ergodic under $\{\tau_x; x\in\mathbb{R}^d\}$.

Given a stationary and ergodic field $(\tilde{V},\tilde{a})$ with C^2 paths, by taking Σ to be the path space $C^2(\mathbb{R}^d,\mathbb{R}^{1+d^2})$ and q to be $P_{\tilde{V},\tilde{a}}$, we can observe that (see [8, Remark 9.2, 9.4] for details) there exist a real random variables V and a matrix valued random variable a, such that

$$\left\{ \left(V(\tau_x\sigma), a(\tau_x\sigma) \right); x\in\mathbb{R}^d \right\} \tag{1.3}$$

has the identical distribution with $(\tilde{V},\tilde{a})$. Since the argument in this article depends only on the distribution of the diffusion, from now on we shall restrict our discussion to the random fields of the form (1.3).

In view of the argument above, we are dealing with a probability space $(\Sigma,\mathscr{F},q)$ where a semigroup of transformations $\{\tau_x; x\in\mathbb{R}^d\}$ are defined, such that q is stationary and ergodic under this semigroup. V and a are real and $d\times d$ real matrix valued random variables on Σ such that $\{V(\tau_x\sigma), a(\tau_x\sigma)\}$ has C^2 realizations. We also need to assume the positive definiteness of a in a uniform sense: there exists a positive constant c_0 such that $\langle x, a(\sigma)\cdot x\rangle_{\mathbb{R}^d} \geq c_0|x|^2$ for all $x \in \mathbb{R}^d$, and q-almost every $\sigma \in \Sigma$.

We introduce a formal derivative for random variables on Σ. Call a measurable function $f : \Sigma \to \mathbb{R}$ belongs to $C_b^1(\Sigma)$ if $f^\sigma(x) \triangleq f(\tau_x\sigma)$ defines a function in $C_b^1(\mathbb{R}^d)$ for all $\sigma \in \Sigma$, and let $D_i f(\sigma) = \partial_{x_i} f^\sigma(x)|_{x=0}$ for $i = 1, \ldots, d$. For $k \geq 2$, $C^k(\Sigma)$ can be defined similarly. By virtue of these notations, the Itô equation of $X^\sigma(t)$ reads

$$\begin{cases} dX^\sigma(t) = U(\tau_{X^\sigma(t)}\sigma)dt + c(\tau_{X^\sigma(t)}\sigma)dB_t, \\ X^\sigma(0) = x, \end{cases} \tag{1.4}$$

where B_t is a d-dimensional standard Brownian motion over some other probability space $(\Omega, \mathscr{F}, p)$, $U = (U_1, \ldots, U_d)$ is the random vector on Σ defined by

$$U_i(\sigma) = -\sum_{j=1}^d a_{ij}(\sigma) D_j V(\sigma) + \sum_{j=1}^d D_j a_{ij}(\sigma), \tag{1.5}$$

and c is the square root of the symmetric part a^s of a.

In view of (1.4), the evolution of $X^\sigma(t)$ depends on shifted environment $\tau_{X^\sigma(t)}\sigma$, which is the origin of the following definition. Let η_t be a stochastic process tracking the environment seen from $X^\sigma(t)$:

$$\eta_t = \tau_{X^\sigma(t)}\sigma \in \Sigma, \quad \forall t \geq 0. \tag{1.6}$$

It is a Σ-valued process over $(\Omega, \mathscr{F}, p)$.

Proposition 1.1 ([8, Proposition 9.7]). *$\{\eta_t\}$ is a Markov process. For bounded measurable functions f on Σ, its Markov semigroup P_t can be written as*

$$P_t f(\sigma) = P_t^\sigma f^\sigma(x)|_{x=0}, \tag{1.7}$$

where P_t^σ is the Markov semigroup of $X^\sigma(t)$ for fixed σ.

Proof. To specify the initial condition, here we write the solution of (1.4) as $X^{\sigma,x}(t)$. The uniqueness of the solution implies that $X^{\tau_y\sigma,x}(t) = X^{\sigma,x+y}(t) - y$. Hence if $P^\sigma(t; x, \cdot)$ stands for the transition probability of $X^{\sigma,x}(t)$, then ([8, Lemma 9.6])

$$P^{\tau_y\sigma}(t; x, \cdot) = P^\sigma(x + y, \cdot + y). \tag{1.8}$$

To complete the remaining proof, just use (1.8) and the Markov property of $X^\sigma(t)$ for fixed σ. □

Now consider the probability measure $\pi(d\sigma) = Z^{-1}e^{-V(\sigma)}q(d\sigma)$ on Σ, where Z is the normalization constant. Due to a direct calculation based on (1.1) and (1.7) (see [8, Proposition 9.8 and §9.7]), P_t extends to a strongly continuous semigroup of contractions on $L^2(\Sigma,\pi)$, and its generator $(D(\mathcal{L}),\mathcal{L})$ satisfies that

$$\mathcal{L}f = \frac{1}{2}e^V\nabla\cdot\left(e^{-V}a\nabla f\right), \quad \forall f\in C_b^2(\Sigma), \tag{1.9}$$

where $\nabla=(D_1,\ldots,D_d)$. Due to [4, Proposition 1.3.3], $C_b^2(\Sigma)$ forms a core of $\mathcal{L}$, i.e. $\mathcal{L}$ coincides with the closure of its own restriction on $C_b^2(\Sigma)$. The stationarity and ergodicity of q yields that for all $f\in C_b^2(\Sigma)$,

$$\int_\Sigma -\mathcal{L}f(\sigma)\cdot g(\sigma)\pi(d\sigma) = \frac{1}{2}\int_\Sigma \langle\nabla f(\sigma),\nabla g(\sigma)\rangle_{\mathbb{R}^d}\,\pi(d\sigma). \tag{1.10}$$

Since $C_b^2(\Sigma)$ is a core, we know from (1.10) that π is invariant and ergodic. Moreover, (1.10) also allows us to extend the definition of the gradient operator ∇ to $D(\mathcal{L})$. Indeed for $f\in D(\mathcal{L})$, we can pick an approximating sequence $f_k\in C_b^2(\Sigma)$ such that $\mathcal{L}f_k$ and f_k converges in $L^2(\Sigma,\pi)$ to $\mathcal{L}f$ and f respectively. Due to (1.10), D_if can be defined as the $L^2(\Sigma,\pi)$-limit of D_if_k. Let ∇f then be $(D_1f,\ldots,D_df)$.

Now we can complete the decomposition mentioned at the beginning of this section. The definition of η_t allows us to rewrite the first equation in (1.4) as

$$X^\sigma(t)-X^\sigma(0)=\int_0^t U(\eta_s)ds+\int_0^t c(\eta_s)dB_s. \tag{1.11}$$

Here $\int_0^t c(\eta_s)dB_s$ is a martingale with stationary increments and $\int_0^t U(\eta_s)ds$ is an additive functional of the stationary, ergodic Markov process η_t.

To complete the proof of the central limit theorem, we need to consider the resolvent equation for all $\lambda>0$ and every component of U, written as

$$(\lambda-\mathcal{L})\chi^{i,\lambda}=U_i. \tag{1.12}$$

Notice that $C_b^2(\Sigma)$ is a core of $\mathcal{L}$ and $\chi^{i,\lambda}$ belongs to the domain of $\mathcal{L}$, we can apply Itô formula to $\chi^{i,\lambda}$ to get (see [8, Proposition 9.13])

$$X_i^\sigma(t)-X_i^\sigma(0)=R_t^{i,\lambda}+\int_0^t c(\eta_s)\left(\nabla\chi^{i,\lambda}(\eta_s)+e_i\right)dB_s, \tag{1.13}$$

where e_i is the i-th orthonormal vector in $\mathbb{R}^d$, and the reminder $R_t^{i,\lambda}$ is

$$R_t^{i,\lambda}=\chi^{i,\lambda}(\eta_0)-\chi^{i,\lambda}(\eta_t)+\lambda\int_0^t\chi^{i,\lambda}(\eta_s)ds, \quad \lambda>0, i=1,\ldots,d. \tag{1.14}$$

We want to take limit when $\lambda\downarrow 0$ and then $t\to\infty$ under the time scale $t^{-\frac{1}{2}}$. A sector condition is sufficient to prove the vanishment of the remainder.

Proposition 1.2 ([8, Corollary 9.10]). *There exists a finite constant C depending only on V and a, such that*

$$\langle \mathcal{L}f, g\rangle_\pi^2 \le C\langle -\mathcal{L}f, f\rangle_\pi \langle -\mathcal{L}g, g\rangle_\pi, \quad \forall f, g \in D(\mathcal{L}). \tag{1.15}$$

We omit the proof for this proposition because the idea is quite different in the infinite dimensional case. Under the sector condition, the argument in [8, Proposition 2.8] implies that

$$\lim_{t\to\infty} \lim_{\lambda\downarrow 0} E_\pi \left[\frac{1}{t}\left(R_t^{i,\lambda}\right)^2\right] = 0. \tag{1.16}$$

Finally, applying the central limit theorem for stationary, ergodic martingales (see, e.g., [18]), we obtain the following central limit theorem.

Theorem 1.3 ([8, Theorem 9.15]). *Under the settings in this section, $X^\sigma(t)/\sqrt{t}$ satisfies the central limit theorem in L^1 sense with respect to the environment, i.e., for all $f \in C_b(\mathbb{R}^d)$,*

$$\lim_{t\to\infty} E_q \left| E_p \left[f\left(\frac{X^\sigma(t)}{\sqrt{t}}\right)\right] - \int_{\mathbb{R}^d} f(y)\phi_{\bar{a}}(y)dy \right| = 0, \tag{1.17}$$

where $\phi_{\bar{a}}$ is the density function of a d-dimensional Gaussian distribution with covariance matrix $\bar{a}$ determined by

$$\bar{a}_{ij} = \lim_{\lambda\downarrow 0} E_\pi \left\langle a^s(e_i + \nabla\chi^{i,\lambda}), e_j + \nabla\chi^{j,\lambda}\right\rangle_{\mathbb{R}^d}. \tag{1.18}$$

§ 2. Stochastic Heat Equation in Ergodic Environment

In this section, we extend the method introduced in §1 to stochastic partial differential equations. Consider a 1-dimensional stochastic heat equation with a random nonlinear term. We choose a suitable shift semigroup to describe the ergodicity of the nonlinear term, with which the general idea of environment process can be applied. Basic properties of the environment process which holds under this setting are summarized in Proposition 2.3. After that, the homogenization results of its solution are studied under an additional assumption that the nonlinear term in the equation can be decomposed into a gradient and a divergence-free vector field. For homogenizations of finite dimensional diffusion in divergence-free environment, we refer to [10].

To state our model, we first list some notations here. Denote by $(H, \langle\cdot,\cdot\rangle_H)$ the Hilbert space consisting of all square integrable real functions on $[0,1]$. Set $\varphi_0(x) \equiv 1$ and $\varphi_i(x) = \sqrt{2}\cos(\pi i x)$ for $i \ge 1$. φ_i's are smooth and form a complete orthonormal system of H. Denote by E the Banach space of all continuous real functions on $[0,1]$

equipped with the uniform topology. Let E_0 be the subset of E consisting of all elements u such that $u(0) = 0$. The standard 1-dimensional Brownian motion on $[0, 1]$ deduces the classical Winer measure μ_0 on E_0.

Similarly to §1, we pick two complete probability spaces $(\Sigma, \mathscr{A}, q)$ and $(\Omega, \mathscr{F}, p)$. With a little abuse of symbols, let $(W_t, \mathscr{F}_t)_{t\geq 0}$ be a standard cylindrical Brownian motion defined on $(\Omega, \mathscr{F}, p)$. Fix $\sigma \in \Sigma$ and let $u^\sigma(t) = u^\sigma(t, \cdot)$ be the solution to the following stochastic partial differential equation

$$\begin{cases} \partial_t u^\sigma(t,x) = \frac{1}{2}\partial_x^2 u^\sigma(t,x) - U(\sigma, u^\sigma(t)) + \dot{W}(t,x), & t > 0, x \in (0,1); \\ \partial_x u^\sigma(t,x)|_{x=0} = \partial_x u^\sigma(t,x)|_{x=1} = 0, & t > 0; \\ u^\sigma(0,x) = v(x), & x \in [0,1], \end{cases} \tag{2.1}$$

where $U : \Sigma \times H \to H$ is an H-valued random field, acting as the environment in which the equation lives. To make sure that we can apply the classical theories on stochastic calculus to define a concrete function-valued process with (2.1), we need to assume (see [5] and [6]):

(A1) $U(\sigma, \cdot)$ is bounded and Lipschitz continuous, uniformly in q-almost all σ.

Under **(A1)**, the unique solution to (2.1) is α-Hölder continuous in t with $\alpha < \frac{1}{4}$, and β-Hölder continuous in x with $\beta < \frac{1}{2}$, q-almost surely. Therefore,

$$u(t) \triangleq u^{\cdot}(t, \cdot) \tag{2.2}$$

defines an E-valued stochastic process on the product space $(\Sigma \times \Omega, \mathscr{A} \otimes \mathscr{F})$. To specify the initial condition, we sometimes write $u^v(t)$ and $u^{v,\sigma}(t,x)$ instead. Let $\mathscr{C}(E)$ be the collection of all linear combinations of functionals on E with the following forms:

$$F(v) = \sin(\langle v, \varphi_j \rangle_H) \text{ or } F(v) = \cos(\langle v, \varphi_j \rangle_H). \tag{2.3}$$

The classical Itô formula implies that the generator corresponding to (2.1) reads

$$\mathcal{K}^\sigma F(u) = \left\langle \frac{1}{2}\partial_x^2 DF(u), u \right\rangle_H - \langle DF(u), U(\sigma, u)\rangle_H + \frac{1}{2}\mathrm{Tr}[D^2 F(u)], \tag{2.4}$$

for all $F \in \mathscr{C}(E)$.

Fix some $\beta < \frac{1}{2}$ and denote by E^β the class of β-Hölder continuous functions on $[0, 1]$. Since $u^\sigma(t, \cdot) \in E^\beta$, we take $C(E^\beta, H)$ to be the path space of U, and denote by P_U the distribution of U on this path space. The central matter remains to determine is the group of "shifts", under which P_U is stationary and ergodic. A natural way is to proceed along the idea in the finite dimensional case. Consider a set of transformations acting on $C(E^\beta, H)$ indexed by $v \in E^\beta$ such that:

$$\tau_v \circ \phi(\cdot) = \phi(\cdot + v), \quad \forall \phi \in C(E^\beta, H), \tag{2.5}$$

and assume that P_U is invariant and ergodic under $\left\{\tau_v; v \in E^\beta\right\}$. In analogy with §1, U is then generated by some random variable $\mathbf{U}$ on Σ in the form $U(\sigma, u) = \mathbf{U}(\tau_u\sigma)$; however, when trying to define the environment process, the absence of shift-invariance of Laplacian stops us from getting the Markov property. More precisely, with $P^\sigma(t; v, \cdot)$ standing for the transition probability of $u^\sigma(t)$, we have

$$P^{\tau_u\sigma}(t; v, \cdot) \neq P^\sigma(t; v + u, \cdot + u), \tag{2.6}$$

so that Proposition 1.1 fails to hold now. Hence we need to adopt fewer shifts here. Assume that

(A2) P_U is stationary and ergodic under $\{\tau_c; c \in \mathbb{R}\}$, where τ_c is defined by

$$\tau_v \circ \phi(\cdot) = \phi(\cdot + c\mathbf{1}), \quad \forall \phi \in C(E^\beta, H), \tag{2.7}$$

and $\mathbf{1}$ is the constant function on $[0,1]$ such that $\mathbf{1}(x) \equiv 1$.

As already observed in §1, with **(A1)** and **(A2)**, without loss of generality we can work under the following framework: $\{\tau_c; c \in \mathbb{R}\}$ is a group of measurable transformations on $(\Sigma, \mathscr{F})$, such that for all c_1, $c_2 \in \mathbb{R}$, $\tau_{c_1} \circ \tau_{c_2} = \tau_{c_1+c_2}$; q is a probability measure on $(\Sigma, \mathscr{F})$ which is stationary and ergodic under $\{\tau_c; c \in \mathbb{R}\}$.

Since the τ's are indexed by $\mathbb{R}$, given $\sigma \in \Sigma$ and $v = v(\cdot) \in E$, to apply the idea of "environment seen from $u(\cdot)$", we have to shift σ at every site x by $u(x)$ to get a *series of environments* $(\tau_{u(x)}\sigma)_{x\in[0,1]}$. Henceforth define

$$\Xi = \left\{\xi = \xi(\cdot) = \tau_{u(\cdot)}\sigma | \sigma \in \Sigma, u \in E\right\}. \tag{2.8}$$

Now the environment process associated with (2.1) can be defined as follows.

Definition 2.1. The environment process of (2.1) is a stochastic process ξ_t defined on the product space $(\Sigma \times \Omega, \mathscr{A} \otimes \mathscr{F})$ with values in Ξ:

$$\xi_t = \left\{\tau_{u^\sigma(t,x)}\sigma; x \in [0,1]\right\} \in \Xi, \quad \forall t \geq 0. \tag{2.9}$$

The next lemma shows the exact necessity and advantage of adopting **(A2)**. It plays a central role in proving the Markov property of ξ_t.

Lemma 2.2 (cf. [20, Lemma 2.2])**.** *There exists a subset $\Xi_0 \subseteq \Xi$ such that the environment process ξ_t never falls into Ξ_0:*

$$P(\exists t > 0, \xi_t \in \Xi_0) = 0, \tag{2.10}$$

and $\tau_{u(\cdot)}\sigma = \tau_{u'(\cdot)}\sigma' \in \Xi \backslash \Xi_0$ implies that $v - v'$ is a constant on $[0,1]$.

Proof. Assume that for some u, $u' \in E$ and σ, $\sigma' \in \Sigma$ we have $\tau_{v(x)}\sigma = \tau_{v'(x)}\sigma'$. To prove the lemma, only to notice that $\tau_{\Delta(x,y)}\sigma = \sigma$ for all x, $y \in \mathbb{R}$, where $\Delta(x,y) = v(x) - v'(x) - v(y) + v'(y)$. The fact that Δ is continuous and the semigroup property of τ's imply that either $\Delta(x,y) \equiv 0$, or $\tau_c\sigma = \sigma$ for all $c \in \mathbb{R}$. By virtue of the ergodicity of q, $\Xi_0 \triangleq \left\{\xi = \tau_{v(\cdot)}\sigma | v \in E, \tau_c\sigma = \sigma, \text{ for all } c \in \mathbb{R}\right\}$ satisfies (2.10). $\square$

From the proof of Lemma 2.2 it is easy to see that $u(\cdot) - u(0) = u'(\cdot) - u'(0) \in E_0$. Therefore for $\xi \in \Xi \backslash \Xi_0$ there is a unique $v_\xi \in E_0$ such that $\xi = \tau_{v_\xi(\cdot)}[\xi(0)]$. By fixing $v_\xi = 0$ for $\xi \in \Xi_0$, $\xi \mapsto (v_\xi, \xi(0))$ can be viewed as an isomorphic map between Ξ and $E_0 \times \Sigma$. We can thus equip Ξ with the natural σ-field $\mathscr{G} = \mathscr{B}(E_0) \otimes \mathscr{A}$.

As seen in §1, the key to prove the Markov property of ξ_t is the invariance of transition probabilities under the shifts. Lemma 2.2 implies that for any given random field U satisfying **(A2)**, there exists an H-valued random variable $\mathbf{U}$ on $(\Xi, \mathscr{G})$, such that $\mathbf{U}(\tau_{u(\cdot)}\sigma)$ and $U(\sigma, u)$ have the identical distribution. Since the homogenization result only relies on the distribution of $u(t)$, we assume w.l.o.g. that $U(\sigma, u)$ is generated by $\mathbf{U}(\tau_{u(\cdot)}\sigma)$. Thanks to this observation, recall that $P^\sigma(t; v, \cdot)$ is the transition probability of $u^\sigma(t)$, Lemma 2.2 together with the discussion after it show that

$$P^\sigma(t; v, \cdot) = P^{\tau_{-v(0)}[\xi(0)]}(t; v, \cdot) = P^{\xi(0)}(t; v_\xi, \cdot - v(0)\mathbf{1}). \tag{2.11}$$

We summarize the properties of the environment process ξ_t holding under **(A1)** and **(A2)** in next proposition. Denote by $\mathcal{P}_t^\sigma$ the Markov semigroup determined by $u^\sigma(t)$ for fixed σ. For a function $\mathbf{F}$ on Ξ, let $F^\sigma(u) = \mathbf{F}(\tau_{u(\cdot)}\sigma)$ for each σ and denote by $\mathscr{C}(\Xi)$ the following function class on Ξ:

$$\mathscr{C}(\Xi) = \left\{\mathbf{F} : \Xi \to \mathbb{R} | F^\sigma \in \mathscr{C}(E), \forall \sigma \in \Sigma\right\}. \tag{2.12}$$

For $\mathbf{F} \in \mathscr{C}(\Xi)$, let $\mathcal{D}\mathbf{F}(\tau_{u(\cdot)}\sigma) \triangleq DF^\sigma(u) = DF^\sigma(u, \cdot) \in H$, where D stands for the Fréchet derivative operator in u.

Proposition 2.3 (cf. [20, Proposition 2.4, 2.5])**.** *Assume* **(A1)** *and* **(A2)**, *then* $\{\xi_t; t \geq 0\}$ *is a Markov process. Its Markov semigroup* $\mathcal{P}_t$ *satisfies*

$$\mathcal{P}_t\mathbf{F}(\xi) = \mathcal{P}_t^{\xi(0)} F^{\xi(0)}(v_\xi) \tag{2.13}$$

for every bounded measurable function $\mathbf{F}$ *on* Ξ. *If we suppose further that* $\mathcal{P}_t$ *admits an invariant probability measure* π *on* $(\Xi, \mathscr{G})$, *then* $\mathcal{P}_t$ *extends to a strongly continuous semigroup on* $L^2(\Xi, \pi)$, *with generator* $\mathcal{K}$ *satisfying that*

$$\mathcal{K}\mathbf{F}(\xi) = \frac{1}{2}\langle \partial_x^2[\mathcal{D}\mathbf{F}(\xi)], v_\xi\rangle_H - \langle \mathcal{D}\mathbf{F}(\xi), \mathbf{U}(\xi)\rangle_H + \frac{1}{2}\mathrm{Tr}[\mathcal{D}^2\mathbf{F}(\xi)], \tag{2.14}$$

for all $\mathbf{F} \in \mathscr{C}(\Xi)$. *Finally,* $\mathscr{C}(\Xi)$ *forms a core of* $\mathcal{K}$.

Proof. The Markov property can be proved easily with (2.11). The remains can be proved along the ideas in [1], [2] and [3]. $\square$

In §1 we see that the invariant and ergodic probability measure can be identified, provided that the drift is gradient-type. Now we apply the same method to $u(t)$, for a slightly wider class of drifts. Assume that $\mathbf{U} = \mathcal{D}\mathbf{V} + \mathbf{B}$, where $(\mathbf{V}, \mathbf{B}) : \Xi \to \mathbb{R} \times H$ is measurable and $\mathcal{D}\mathbf{V}(\tau_{u(\cdot)}\sigma) = DV^\sigma(u)$. Recalling that μ_0 is the probability measure on E_0 determined by standard Brownian motion, we assume further that
(A3) $\mathbf{V}$ is uniformly bounded, V^σ is Fréchet differentiable for every σ, and

$$\int_{E_0} e^{-2\mathbf{V}(\tau_{u(\cdot)}\sigma)} \left\langle DF(u), \mathbf{B}(\tau_{u(\cdot)}\sigma)\right\rangle_H \mu_0(du) = 0, \quad \forall F \in \mathscr{C}(E). \tag{2.15}$$

From **(A3)**, $\mathbf{U}$ can be divided into a gradient-type field $\mathcal{D}\mathbf{V}$ which gives the invariant measure, and a divergence-free field $\mathbf{B}$ which preserves the stationarity. Indeed we can obtain the following proposition using the ideas in [5] and [15].

Proposition 2.4 ([20, Proposition 2.5]). *Assume* **(A1)**–**(A3)**, *then* $\{\mathcal{P}_t\}$ *admits an invariant probability measure*

$$\pi(d\xi) = Z^{-1}\exp(-2\mathbf{V}(\xi))\mu_0(dv_\xi) \otimes q(d\xi(0)), \tag{2.16}$$

where Z is the normalization constant. Moreover, π is the unique invariant measure so that ξ_t is ergodic under π.

Due to Proposition 2.3 and 2.4, ξ_t is a stationary, ergodic Markov process with infinitesimal generator $\mathcal{K}$. As already observed in §1, to prove a central limit theorem for $u(t)/\sqrt{t}$, it is sufficient to check the sector condition of $\mathcal{K}$.

Proposition 2.5 ([20, Proposition 2.6]). *Assume* **(A1)**–**(A3)**. *There exists a finite constant C depending only on* $\mathbf{V}$ *and* $\mathbf{B}$, *such that*

$$\langle \mathcal{K}\mathbf{F}, \mathbf{G}\rangle_\pi^2 \le C\langle -\mathcal{K}\mathbf{F}, \mathbf{F}\rangle_\pi \langle -\mathcal{K}\mathbf{G}, \mathbf{G}\rangle_\pi, \quad \forall \mathbf{F}, \mathbf{G} \in \mathscr{C}(\Xi). \tag{2.17}$$

Proof. Instead of $\mathcal{K}$, it is sufficient to consider its anti-symmetric part $\mathcal{K}_a$ for the left-hand side in (2.17). Thanks to **(A3)**, we can recenter it as

$$\langle \mathcal{K}_a\mathbf{F}, \mathbf{G}\rangle_\pi = E_\pi\left[\langle \mathcal{D}\mathbf{F}, \mathbf{B}\rangle_H (\mathbf{G} - \mathbf{G}_0)\right],$$

where $\mathbf{G}_0(\xi) = \langle G^{\xi(0)}\rangle_{\mu_0}$. The inequality (2.17) can be obtained by applying Schwarz inequality and the Poincaré inequality (see, e.g., [16]) for μ_0. $\square$

Now we are at the position to proof a central limit theorem for $u^\sigma(t)$. Pick the test function φ_i and the weak from of (2.1) reads

$$\langle u^\sigma(t), \varphi_i\rangle_H - \langle u^\sigma(0), \varphi_i\rangle_H = \int_0^t \mathbf{U}^i(\xi_r)dr + \langle W_t, \varphi\rangle_H, \tag{2.18}$$

where $\mathbf{U}^i(\xi) = \frac{1}{2}\left\langle v_\xi, \partial_x^2 \varphi_i \right\rangle_H - \langle \mathbf{U}(\xi), \varphi_i \rangle_H$ is the drift in (2.1) in the direction of φ_i. Here we need to consider the resolvent equations associated with $\mathcal{K}$, written as

$$(\lambda - \mathcal{K})\chi^{i,\lambda} = \mathbf{U}^i, \quad \lambda > 0. \tag{2.19}$$

Compared with the results in finite dimensional case, Proposition 2.3, 2.5 and (2.18) allow us to apply the method introduced in §1 and conclude that $\langle u^\sigma(t), \varphi_i \rangle_H / \sqrt{t}$ converges weakly to a 1-d central Gaussian measure with covariance

$$a_i^2 = \lim_{\lambda \downarrow 0} E_\pi \|\varphi_i + \mathcal{D}\chi^{i,\lambda}\|_H. \tag{2.20}$$

Noticing that for $i \geq 1$, $\chi^i(\xi) \triangleq -\langle v_\xi, \varphi_i \rangle_H$ is well-defined in $L^2(\Xi, \pi)$ and satisfies $-\mathcal{K}\chi^i = \mathbf{U}^i$, it is not hard to observe that $a^i = 0$. Hence under the diffusive time scaling, the limit distribution concentrates only on the direction of constant functions. In summary, we have the following central limit theorem.

Theorem 2.6 ([20, Theorem 1.1]). *Assume that* **(A1)**–**(A3)** *holds.* $u(t)/\sqrt{t}$ *satisfies the central limit theorem in the following sense: for any* $F \in C_b(E)$ *we have*

$$\lim_{t \to \infty} E_q \left| E_p \left[F\left(\frac{u(t)}{\sqrt{t}} \right) \right] - \int_{\mathbb{R}} F(\mathbf{1} \cdot y) \Phi_{a_0}(y) dy \right| = 0, \tag{2.21}$$

where a_0 *is the constant determined by* (2.20) *with* $i = 0$, *and* Φ_{a_0} *is the probability density function of a 1-dimensional central Gaussian law with covariance* a_0^2.

Remark. Furthermore we can prove that Φ_{a_0} never degenerate: there exists a strictly positive constant C such that $C \leq a^2 \leq 1$.

In accordance with the finite dimensional model, periodic case are included as a concrete example of our model.

Example 2.7. Take $(\Sigma, \mathscr{A}, q)$ to be the finite interval $[0, 1]$ equipped with the Lebesgue measure. Pick a measurable function V on $[0, 1] \times \mathbb{R}$ satisfying that $V(x, \cdot) \in C^1(\mathbb{R})$, $V(x, y) = V(x, y + 1)$ for all $x \in [0, 1], y \in \mathbb{R}$. Suppose that $u^{\sigma, v+\sigma}(t, x)$ is the solution to (2.1) with the initial condition $v + \sigma$ and the random field

$$U(\sigma, u) \triangleq \frac{d}{dy} V(\cdot, u(\cdot) + \sigma). \tag{2.22}$$

Easy to see that $u^{\sigma, v+\sigma} - \sigma$ solves a stochastic heat equation with a periodic nonlinear term of gradient type, written as

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2}\partial_x^2 u(t, x) - \frac{d}{dy} V(x, u(t, x)) + \dot{W}(t, x), & t > 0, x \in (0, 1), \\ \partial_x u(t, 0) = \partial_x u(t, 1) = 0, & t > 0, \\ u(0, x) = v(x), & x \in [0, 1]. \end{cases} \tag{2.23}$$

If both V and $\frac{d}{dy}V$ are uniformly bounded, then **(A1)** to **(A3)** are fulfilled, so that central limit theorem stated in Theorem 2.6 holds for the solution to (2.23). Indeed, we can moreover prove an invariance principle. Suppose that μ is infinite measure on E such that $v(0)$ follows the Lebesgue measure on $\mathbb{R}$ and $v(\cdot) - v(0)$ is a standard 1-d Brownian motion.

Corollary 2.8 ([19, Theorem 1.2]). *Fix some $T > 0$ and suppose ν is a probability measure on E which is absolutely continuous with respect to μ. Under initial distribution ν, the E-valued process $\{\epsilon u(\epsilon^{-2}t), t \in [0,T]\}$ converges weakly to a Gaussian process $\{a_0 B_t \cdot \mathbf{1}, t \in [0,T]\}$ as $\epsilon \downarrow 0$, where B_t is a 1-dimensional Brownian motion on $[0,T]$ and a_0 is the constant appeared in Theorem 2.6.*

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