RIMS Kôkyûroku Bessatsu
B59 (2016), 85–100

On an extension of the Brascamp-Lieb inequality

By

Yuu HARIYA*

Abstract

In this article, we survey the author's recent results on an extension of the Brascamp-Lieb inequality; revealing its connection with a solution to the Skorokhod embedding problem, we extend the inequality.

§ 1. Introduction

The Brascamp-Lieb moment inequality plays an important role in statistical mechanics, such as in the analysis of so-called $\nabla\phi$ interface models; see, e.g., [10, 9, 11]. It asserts that the centered moments of a Gaussian distribution perturbed by a convex potential do not exceed those of the Gaussian distribution. The main theme of this article is to give a link between the Brascamp-Lieb inequality and Skorokhod embedding.

Given a one-dimensional Brownian motion B and a probability measure μ on $\mathbb{R}$, the Skorokhod embedding problem is to find a stopping time T of B such that $B(T)$ follows μ. The problem was proposed by Skorokhod [20] and more than twenty solutions have been constructed since then; see the detailed survey [16] by Obłój.

In this article, we give a proof of the Brascamp-Lieb inequality based on the Skorokhod embedding of Bass [1]; as a by-product, error bounds for the inequality in terms of the variance are provided. The same reasoning also enables us to extend the inequality as well as its error bounds to a relatively wide class of nonconvex potentials in the case of one dimension; our result applies to double-well potentials. This article is a survey of [12] and [13, Appendix] with some complementary exposition.

Let Y be an n-dimensional *centered* Gaussian random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ with law ν. Let X be an n-dimensional random variable on

Received January 31, 2016. Revised June 8, 2016.
2010 Mathematics Subject Classification(s): Primary 82B31; Secondary 60E15, 60J65, 60G40.
Key Words: *Brascamp-Lieb inequality*; *Skorokhod embedding*; *nonconvex potential.*
*Mathematical Institute, Tohoku University, Sendai 980-8578, Japan.
e-mail: hariya@math.tohoku.ac.jp

$(\Omega, \mathcal{F}, P)$, whose law μ is given in the form

$$\mu(dx) = \frac{1}{Z} e^{-V(x)}\, \nu(dx) \tag{1.1}$$

with $V : \mathbb{R}^n \to \mathbb{R}$ a convex function, where

$$Z := \int_{\mathbb{R}^n} e^{-V(x)}\, \nu(dx) \in (0, \infty).$$

In what follows, we fix $v \in \mathbb{R}^n$ ($v \neq 0$) arbitrarily. For a one-dimensional random variable ξ with $E[\xi^2] < \infty$, we denote its variance by $\mathrm{var}(\xi)$: $\mathrm{var}(\xi) = E[(\xi - E[\xi])^2]$. We set $\mathsf{a} := \mathrm{var}(v \cdot Y)$. Here $a \cdot b$ denotes the inner product of $a, b \in \mathbb{R}^n$. We also set

$$\mathsf{p}(t; x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad t > 0,\ x \in \mathbb{R}.$$

One of the main results of this article is then stated as follows:

Theorem 1.1 (Theorem 1.1 of [12]). *For every convex function ψ on $\mathbb{R}$, we have the following (i) and (ii):*

(i) It holds that

$$E\left[\psi\left(v \cdot Y\right)\right] \geq E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right]. \tag{1.2}$$

More precisely, we have

$$\begin{aligned} E\left[\psi\left(v \cdot Y\right)\right] \geq& E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] \\ &+ \frac{1}{2} \int_{\mathbb{R}} \int_0^{\mathsf{a}^{-1}(\mathsf{a} - \mathrm{var}(v \cdot X))^2} \mathsf{p}\left(s; \sqrt{x^2 + \mathsf{a}}\right) ds\, \psi''(dx), \end{aligned} \tag{1.3}$$

where $\psi''(dx)$ denotes the second derivative of ψ in the sense of distribution.

(ii) For every $p > 1$, it holds that

$$E\left[\psi\left(v \cdot Y\right)\right] \leq E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] + C(\mathsf{a}, \psi, q) \left(\mathsf{a} - \mathrm{var}(v \cdot X)\right)^{\frac{1}{2p}}. \tag{1.4}$$

Here $C(\mathsf{a}, \psi, q) \in [0, \infty]$ is given by

$$C(\mathsf{a}, \psi, q) = (\mathsf{a}(1+q))^{\frac{1}{2q}} \int_{\mathbb{R}} \mathsf{p}\left(1; \frac{x}{\sqrt{\mathsf{a}(1+q)}}\right) \psi''(dx) \tag{1.5}$$

with q the conjugate of p: $p^{-1} + q^{-1} = 1$. Note that $\mathsf{a} - \mathrm{var}(v \cdot X) \geq 0$ by (1.2).

The above inequalities (1.2)–(1.4) are understood to hold as well in the case that both sides are infinity, due to Fubini's theorem utilized in the proof; see Subsection 2.1.

The inequality (1.2) is called the *Brascamp-Lieb inequality*; it was originally proven by Brascamp and Lieb [4, Theorem 5.1] in the case $\psi(x) = |x|^\alpha$, $\alpha \geq 1$, and then extended to general convex ψ's by Caffarelli [5, Corollary 6] based on the optimal transport between μ and ν.

As a corollary to Theorem 1.1 (ii), we have the following estimate, which we think is of interest itself; note that the right-hand side is not dependent on V.

Corollary 1.2. *It holds that*

$$\frac{E\left[|v \cdot X - E\left[v \cdot X\right]|\right]}{\mathrm{var}(v \cdot X)} \geq \frac{1}{\sqrt{2\pi \mathsf{a}}}.$$

For the proof, see Subsection 2.2.

Remark 1. *Similarly to the Brascamp-Lieb inequality* (1.2) *itself not yielding any useful bounds on the mean* $E[v \cdot X]$*, the inequalities* (1.3) *and* (1.4) *do not give any information on the variance other than* $\mathrm{var}(v \cdot X) \leq \mathsf{a}$*. It is known [4, Theorem 4.1] that if* $V \in C^2(\mathbb{R}^n)$*, then* $\mathrm{var}(v \cdot X)$ *admits the upper bound*

$$\int_{\mathbb{R}^n} v \cdot \left(\Sigma^{-1} + D^2 V(x)\right)^{-1} v \, \mu(dx),$$

not exceeding $\mathsf{a} \equiv v \cdot \Sigma v$*. Here* Σ *denotes the covariance matrix of the Gaussian measure* ν *and* $D^2 V$ *the Hessian of* V*. See also* (1) *of Remark 4 at the end of the next section.*

The rest of the article is organized as follows: We explain an idea of the proof of Theorem 1.1 in Section 2. The proof of (1.2) is detailed in Subsection 2.1 while in Subsection 2.2, we give an outline of the proof of (1.3) and (1.4); Section 2 is concluded with a remark on some related results deduced from our argument. In Section 3, we discuss an extension of the Brascamp-Lieb inequality and its error bounds to the case of nonconvex potentials when $n = 1$.

In the sequel every random variable and every stochastic process are assumed to be defined on the probability space $(\Omega, \mathcal{F}, P)$. For every real-valued function f on $\mathbb{R}$ and for every $x \in \mathbb{R}$, we denote respectively by $f'_+(x)$ and $f'_-(x)$ the right- and left-derivatives of f at x if they exist. For each $x, y \in \mathbb{R}$, we write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$ and $x^+ = x \vee 0$.

§ 2. Proof of Theorem 1.1

In this section we give an idea of the proof of Theorem 1.1. Note that Theorem 4.3 of [4] reduces the proof to the case $n = 1$, namely the density of the law $P \circ (v \cdot X)^{-1}$ with respect to the one-dimensional Gaussian measure $P\circ(v\cdot Y)^{-1}$ is log-concave. Therefore in what follows, we take the Gaussian measure ν in (1.1) as

$$\nu(dx) = \frac{1}{\sqrt{2\pi\mathsf{a}}}\exp\left(-\frac{x^2}{2\mathsf{a}}\right)dx, \quad x\in\mathbb{R},$$

and V as a convex function on $\mathbb{R}$. We accordingly write X and Y for $v\cdot X$ and $v\cdot Y$, respectively; that is, X is distributed as μ and Y as ν. We recall that the above-mentioned theorem is often referred to as Prékopa's theorem, which was originally proven by Prékopa [17] and then independently by Brascamp-Lieb [4] and Rinott [19].

§ 2.1. Proof of (1.2)

Because of its intimate connection with Section 3, we detail the proof of (1.2) following [12]. We define F_μ to be the distribution function of μ:

$$F_\mu(x) := \frac{1}{Z}\int_{-\infty}^{x} e^{-V(y)}\,\nu(dy), \quad x\in\mathbb{R}.$$

We set

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}y^2\right)dy, \quad x\in\mathbb{R},$$

and

$$g := F_\mu^{-1}\circ\Phi. \tag{2.1}$$

Here $F_\mu^{-1} : (0,1)\to\mathbb{R}$ is the inverse function of F_μ. By definition, it is clear that g is differentiable and strictly increasing. Moreover, by the convexity of V we have the

Lemma 2.1. *It holds that $g'(x)\le\sqrt{\mathsf{a}}$ for all $x\in\mathbb{R}$.*

Once this lemma is shown, the inequality (1.2) is straightforward from the Skorokhod embedding of Bass [1]. Let $\{W_t\}_{t\ge 0}$ be a standard one-dimensional Brownian motion. Notice that $g(W_1)$ follows μ by the definition of g.

Proof of (1.2). Applying Clark's formula (see, e.g., [15, Appendix E]) to $g(W_1)$ yields

$$g(W_1) - E\left[g(W_1)\right] = \int_0^1 a(s,W_s)\,dW_s \quad \text{a.s.},$$

where for $0\le s\le 1$ and $y\in\mathbb{R}$,

$$\begin{aligned} a(s,y) :=& \frac{\partial}{\partial y}E\left[g(y+W_{1-s})\right] \\ =& E\left[g'(y+W_{1-s})\right]. \end{aligned} \tag{2.2}$$

In (2.2), the second line follows from the boundedness of g. By time change due to Dambis-Dubins-Schwarz (see, e.g., [18, Theorem V.1.6]), there exists a Brownian motion $\{B(t)\}_{t\geq 0}$ such that a.s.,

$$\int_0^t a(s,W_s)\,dW_s = B\left(\int_0^t a(s,W_s)^2\,ds\right) \quad \text{for all } 0\leq t\leq 1.$$

Set

$$T := \int_0^1 a(s,W_s)^2\,ds. \tag{2.3}$$

We know from [1] that T is a stopping time in the natural filtration of B. Moreover, by (2.2) and Lemma 2.1, we have $T\leq \mathsf{a}$ a.s. Let $\{L_t^x\}_{t\geq 0, x\in\mathbb{R}}$ denote the local time process of B. By Tanaka's formula we have for every $x\in\mathbb{R}$,

$$E\left[(B(\mathsf{a})-x)^+\right] = E\left[(B(T)-x)^+\right] + \frac{1}{2}E\left[L_{\mathsf{a}}^x - L_T^x\right], \tag{2.4}$$

$$E\left[(x-B(\mathsf{a}))^+\right] = E\left[(x-B(T))^+\right] + \frac{1}{2}E\left[L_{\mathsf{a}}^x - L_T^x\right]. \tag{2.5}$$

From (2.4) and (2.5), we obtain for every convex ψ,

$$E\left[\psi\left(B(\mathsf{a})\right)\right] = E\left[\psi\left(B(T)\right)\right] + \frac{1}{2}\int_{\mathbb{R}} E\left[L_{\mathsf{a}}^x - L_T^x\right]\psi''(dx). \tag{2.6}$$

Indeed, by Fubini's theorem,

$$\begin{aligned}
&\int_{[0,\infty)} E\left[(B(\mathsf{a})-x)^+\right]\psi''(dx) + \int_{(-\infty,0)} E\left[(x-B(\mathsf{a}))^+\right]\psi''(dx)\\
&= E\left[\psi\left(B(\mathsf{a})\right) - \psi'_-(0)B(\mathsf{a}) - \psi(0)\right]\\
&= E\left[\psi\left(B(\mathsf{a})\right)\right] - \psi(0),
\end{aligned}$$

which, by (2.4), (2.5) and $E\left[B(T)\right]=0$, equals the right-hand side of (2.6) with $\psi(0)$ subtracted. Hence (2.6) holds. As ψ'' is nonnegative and $T\leq\mathsf{a}$ a.s., it follows immediately from (2.6) that

$$E\left[\psi\left(B(\mathsf{a})\right)\right] \geq E\left[\psi\left(B(T)\right)\right]. \tag{2.7}$$

The proof ends by noting identities in law:

$$B(T) = g(W_1) - E\left[g(W_1)\right] \overset{(d)}{=} X - E\left[X\right] \tag{2.8}$$

and $B(\mathsf{a}) \overset{(d)}{=} Y$. □

Remark 2. (1) *For any convex ψ such that the process $\int_0^t \psi'_-(B(s))\,dB(s)$, $0 \le t \le \mathsf{a}$, is a martingale, the identity* (2.6) *is immediate from the Itô-Tanaka formula.*
(2) *For any convex ψ such that $E[|\psi(B(\mathsf{a}))|] < \infty$ (i.e., $E[\psi(B(\mathsf{a}))] < \infty$ as ψ is bounded from below by a linear function), the inequality* (2.7) *follows readily from the optional sampling theorem applied to the submartingale $\{\psi(B(t))\}_{0\le t\le \mathsf{a}}$.*

We proceed to the proof of Lemma 2.1. Since convex functions remain convex under scaling, it suffices to show the assertion with $\mathsf{a} = 1$; however, we give a proof retaining a for later use. In the sequel we write $\sigma = \sqrt{\mathsf{a}}$ for notational simplicity.

Lemma 2.2. *It holds that for all $x \in \mathbb{R}$,*

$$\sigma F'_\mu(x) \ge \Phi'\left(\frac{x}{\sigma} + \sigma V'_-(x)\right).$$

Proof. Since $V(y) - V(x) \ge V'_-(x)(y-x)$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{F'_\mu(x)} &= \int_{\mathbb{R}} \exp\left(-\frac{y^2}{2\sigma^2} - V(y)\right) dy \times \exp\left(\frac{x^2}{2\sigma^2} + V(x)\right) \\
&\le \exp\left(\frac{x^2}{2\sigma^2}\right) \int_{\mathbb{R}} \exp\left\{-\frac{y^2}{2\sigma^2} - V'_-(x)(y-x)\right\} dy \\
&= \exp\left\{\frac{1}{2}\left(\frac{x}{\sigma} + \sigma V'_-(x)\right)^2\right\} \times \sqrt{2\pi}\sigma
\end{aligned}$$

as claimed. □

The proof of Lemma 2.1 follows readily from the above lemma.

Proof of Lemma 2.1. Since

$$g'(x) = \frac{\Phi'(x)}{F'_\mu \circ F_\mu^{-1}(\Phi(x))}$$

by the definition (2.1) of g, the assertion of the lemma is equivalent to

$$G(\xi) := \sigma F'_\mu \circ F_\mu^{-1}(\xi) - \Phi' \circ \Phi^{-1}(\xi) \ge 0 \quad \text{for all } \xi \in (0,1). \tag{2.9}$$

First note that

$$G(0+) = \lim_{\xi\to 0+} G(\xi) = 0, \quad G(1-) = \lim_{\xi\to 1-} G(\xi) = 0 \tag{2.10}$$

because both $F'_\mu \circ F_\mu^{-1}$ and $\Phi' \circ \Phi^{-1}$ are zero at $\xi = 0+$ and $\xi = 1-$. Next, G is both right- and left-differentiable because F'_μ is so and F_μ^{-1} is monotone. Now we suppose that G has a local minimum at some $\xi_0 \in (0,1)$. Then $G'_-(\xi_0) \le 0 \le G'_+(\xi_0)$. Since

$$\begin{aligned}
G'_\pm(\xi) &= \sigma \frac{(F'_\mu)'_\pm}{F'_\mu} \circ F_\mu^{-1}(\xi) + \Phi^{-1}(\xi) \\
&= -\left(\frac{x}{\sigma} + \sigma V'_\pm(x)\right)\Big|_{x=F_\mu^{-1}(\xi)} + \Phi^{-1}(\xi),
\end{aligned}$$

we have

$$\left(\frac{x}{\sigma}+\sigma V_+'(x)\right)\Big|_{x=F_\mu^{-1}(\xi_0)} \le \Phi^{-1}(\xi_0) \le \left(\frac{x}{\sigma}+\sigma V_-'(x)\right)\Big|_{x=F_\mu^{-1}(\xi_0)},$$

which entails that by the convexity of V,

$$\Phi^{-1}(\xi_0) = \left(\frac{x}{\sigma}+\sigma V_-'(x)\right)\Big|_{x=F_\mu^{-1}(\xi_0)}.$$

Hence by Lemma 2.2,

$$G(\xi_0) = \left\{\sigma F_\mu'(x) - \Phi'\left(\frac{x}{\sigma}+\sigma V_-'(x)\right)\right\}\Big|_{x=F_\mu^{-1}(\xi_0)} \ge 0.$$

This observation together with (2.10), leads to (2.9) and concludes the proof. □

§ 2.2. Proofs of (1.3), (1.4) and Corollary 1.2

We start this subsection with an outline of the proof of (1.3) and (1.4) in Theorem 1.1. These inequalities are immediate from the identity (2.6) and Proposition 2.3 of [12]. As the proof proceeds in the same way as that of the proposition, we put its statement in a slightly general setting. Let $\beta = \{\beta(t)\}_{t\ge 0}$ be a standard one-dimensional Brownian motion, $\{l_t^x\}_{t\ge 0, x\in\mathbb{R}}$ its local time process and S a stopping time in the natural filtration of β.

Proposition 2.3. *Suppose that there exists a positive real* b *such that* $S \le \mathsf{b}$ *a.s. Then it holds that for all* $x \in \mathbb{R}$,

$$E\left[l_{\mathsf{b}}^x - l_S^x\right] \ge \int_0^{\mathsf{b}^{-1}(\mathsf{b}-E[S])^2} \mathsf{p}\left(s; \sqrt{x^2+\mathsf{b}}\right) ds$$

and

$$E\left[l_{\mathsf{b}}^x - l_S^x\right] \le 2\left(\mathsf{b}(1+q)\right)^{\frac{1}{2q}} \mathsf{p}\left(1; \frac{x}{\sqrt{\mathsf{b}(1+q)}}\right)\left(\mathsf{b}-E[S]\right)^{\frac{1}{2p}}$$

for every $p > 1$ *with* q *its conjugate.*

Upon noting the expression

$$E\left[l_{\mathsf{b}}^x - l_S^x\right] = E\left[E\left[l_t^z\right]\Big|_{(t,z)=(\mathsf{b}-S, x-\beta(S))}\right] \tag{2.11}$$

thanks to the strong Markov property of Brownian motion, the proof of the above proposition makes use of the following expressions for $E\left[l_t^z\right]$, $t > 0, z \in \mathbb{R}$:

$$\begin{aligned} E\left[l_t^z\right] &= \int_0^t \mathsf{p}(s; z)\, ds \\ &= 2\int_0^\infty (y-|z|)^+ \mathsf{p}(t; y)\, dy \\ &= 2\int_0^\infty \left(\sqrt{t}\, y - |z|\right)^+ \mathsf{p}(1; y)\, dy, \end{aligned} \tag{2.12}$$

and Wald's identity as well: $E\left[\beta(S)^2\right] = E[S]$. For more details of the proof, refer to [12, Subsection 2.2].

We turn to the proof of Corollary 1.2. Note that when $\psi''(\mathbb{R}) < \infty$, we may let $p \to \infty$ in (1.4) to obtain

$$E\left[\psi\left(v \cdot Y\right)\right] \le E\left[\psi\left(v \cdot X - E\left[v \cdot X\right]\right)\right] + \frac{1}{\sqrt{2\pi}}\psi''(\mathbb{R})\left(\mathsf{a} - \operatorname{var}(v \cdot X)\right)^{\frac{1}{2}}. \tag{2.13}$$

Proof of Corollary 1.2. Taking $\psi(x) = |x|$ in (2.13) and noting $E\left[|v \cdot Y|\right] = \sqrt{2\mathsf{a}/\pi}$, we have

$$E\left[|v \cdot X - E\left[v \cdot X\right]|\right] \ge \sqrt{\frac{2}{\pi}}\left\{\sqrt{\mathsf{a}} - \sqrt{\mathsf{a} - \operatorname{var}(v \cdot X)}\right\},$$

from which the assertion follows readily. □

Remark 3. *A more direct derivation of the identity* (2.13) *is possible. By* (2.12), *we have for any $t \ge 0$ and $z \in \mathbb{R}$,*

$$E\left[l_t^z\right] \le \int_0^t \mathsf{p}(s,0)\,ds = \sqrt{\frac{2t}{\pi}}.$$

Therefore under the assumption of Proposition 2.3, we have by (2.11) *and Jensen's inequality,*

$$\begin{aligned} E\left[l_{\mathsf{b}}^x - l_S^x\right] &\le \sqrt{\frac{2}{\pi}}E\left[\sqrt{\mathsf{b} - S}\right] \\ &\le \sqrt{\frac{2}{\pi}}\sqrt{\mathsf{b} - E[S]} \end{aligned}$$

for all $x \in \mathbb{R}$. Combining this estimate with the identity (2.6), *we obtain*

$$E\left[\psi\left(B(\mathsf{a})\right)\right] \le E\left[\psi\left(B(T)\right)\right] + \frac{1}{2}\psi''(\mathbb{R}) \times \sqrt{\frac{2}{\pi}}\sqrt{\mathsf{a} - E[T]},$$

which is nothing but (2.13) *because of Wald's identity and the equivalence* (2.8) *in law.*

We conclude this section with a remark on the stopping time T.

Remark 4. (1) *By* (2.2) *we may write $a(s, W_s) = E\left[g'(W_1)|W_s\right]$ a.s. Hence by the definition* (2.3) *of T, and by Jensen's inequality and Fubini's theorem, we have*

$$\begin{aligned} E[T] &\le E\left[\int_0^1 E\left[g'(W_1)^2\middle|W_s\right]ds\right] \\ &= E\left[g'(W_1)^2\right], \end{aligned}$$

and

$$
\begin{aligned}
E[T] &\geq E\left[\int_0^1 E\left[g'(W_1)|W_s\right] ds\right]^2 \\
&= E\left[g'(W_1)\right]^2 .
\end{aligned}
$$

Therefore by Wald's identity and (2.8), *we obtain the following upper and lower bounds on* $\mathrm{var}(X)$*:*

$$
E\left[g'(W_1)\right]^2 \leq \mathrm{var}(X) \leq E\left[g'(W_1)^2\right].
$$

Recalling that $g(W_1)$ *is distributed as* μ*, we rewrite the rightmost side as*

$$
\int_{\mathbb{R}} \left(g' \circ g^{-1}\right)^2 (x)\, \mu(dx).
$$

In view of Remark 1, it is plausible that in the case $V \in C^2(\mathbb{R})$*,*

$$
\left(g' \circ g^{-1}\right)^2 (x) \leq \frac{\mathsf{a}}{1 + \mathsf{a}V''(x)}
$$

for all $x \in \mathbb{R}$*, however, we have not had a proof yet; we note that both sides agree when* V *is a quadratic function.*
(2) *Let* $\beta = \{\beta(t)\}_{t\geq 0}$ *be a standard one-dimensional Brownian motion and* τ_{R} *denote Root's solution to the Skorokhod embedding problem that embeds the law of* $X - E[X]$ *into* β*:* $\beta(\tau_{\mathsf{R}}) \overset{(d)}{=} X - E[X]$*. Since* τ_{R} *is of* minimal residual expectation*, it follows that* τ_{R} *is also bounded from above by* a*; indeed, if we let* τ_{B} *be Bass' solution embedding the same law into* β*, namely* $\tau_{\mathsf{B}} \overset{(d)}{=} T$*, then we have*

$$
E\left[(\tau_{\mathsf{R}} - t)^+\right] \leq E\left[(\tau_{\mathsf{B}} - t)^+\right] \quad \textit{for all } t \geq 0,
$$

and hence $\tau_{\mathsf{R}} \leq \mathsf{a}$ *a.s. This fact indicates that the Brascamp-Lieb inequality* (1.2) *can also be proven based on Root's solution. For the construction of embedding due to D.H. Root and the notion of minimal residual expectation, see [14, Section 5.1] and references therein. In addition, the boundedness of Root's solution as noted above in the Brascamp-Lieb framework gives an answer to the question raised in [8, Section 7] as to when Root's* barrier *is bounded. If* V *is in* $C^2(\mathbb{R})$*, the convexity condition on* V *can also be relaxed; see the next section for details.*

§ 3. Extension of the Brascamp-Lieb inequality to nonconvex potentials

In this section we take $n = 1$ and continue our discussion in [12, Appendix] as to an extension of the Brascamp-Lieb inequality (1.2) to the case of nonconvex potentials; we explore conditions on the potential function V under which the inequality (1.2) remains true. Recently $\nabla\phi$ interface models with nonconvex potentials have been studied with great interest; see e.g., [2, 7, 3, 6]. Although our exposition here is restricted to one dimension, we think that it would be beneficial to that study. This section is based on [13, Appendix].

We retain the notation of the previous section. In what follows we let $V:\mathbb{R}\to\mathbb{R}$ be in $C^2(\mathbb{R})$. We are interested in the case that $\{x\in\mathbb{R};\, V''(x)<0\}\neq\emptyset$. We assume that V satisfies

$$V(x)\geq ax+b \quad \text{for all } x\in\mathbb{R}, \tag{3.1}$$

for some reals a and b, so that

$$Z=E\left[e^{-V(Y)}\right]<\infty.$$

As in the proof of Lemma 2.1 given in Subsection 2.1, we denote by σ^2 with $\sigma>0$, instead of a, the variance of the centered Gaussian random variable Y. We set

$$\mathcal{D}_V:=\{x\in\mathbb{R};\, V''(x)\leq 0\}.$$

The aim of this section is to give a proof of

Theorem 3.1. *Suppose that*

$$\inf_{x\in\mathcal{D}_V}\left\{\frac{1}{2}\sigma^2V'(x)^2+xV'(x)-V(x)\right\}\geq\log Z. \tag{A}$$

Then it holds that for any convex function ψ on $\mathbb{R}$,

$$E\left[\psi(Y)\right]\geq E\left[\psi\left(X-E[X]\right)\right]; \tag{3.2}$$

moreover, it also holds that

$$\begin{aligned} E\left[\psi(Y)\right]\geq & E\left[\psi(X-E[X])\right]\\ &+\frac{1}{2}\int_{\mathbb{R}}\int_0^{\sigma^{-2}\left(\sigma^2-\mathrm{var}(X)\right)^2}\mathsf{p}\left(s;\sqrt{x^2+\sigma^2}\right)ds\,\psi''(dx),\end{aligned} \tag{3.3}$$

and for every $p>1$,

$$E\left[\psi(Y)\right]\leq E\left[\psi\left(X-E[X]\right)\right]+C(\sigma^2,\psi,q)\left(\sigma^2-\mathrm{var}(X)\right)^{\frac{1}{2p}}, \tag{3.4}$$

where $C(\sigma^2,\psi,q)$ is given by (1.5) *with $\mathsf{a}=\sigma^2$ and q the conjugate of p. In particular, these inequalities* (3.2)–(3.4) *hold true if*

$$\inf_{x\in\mathcal{D}_V}\left\{-\frac{x^2}{2\sigma^2}-V(x)\right\}\geq\log Z. \tag{A$'$}$$

We give two examples.

Example 3.2 (double-well potentials). Consider the potential V of the form

$$V(x) = \frac{1}{2}\alpha^2 x^4 - \frac{1}{2}\beta x^2, \quad x \in \mathbb{R},$$

for $\alpha, \beta > 0$. Take $\sigma = 1$ for simplicity. Then the left-hand side of (A′) is calculated as

$$\frac{\beta(5\beta - 6)}{72\alpha^2} \wedge 0, \tag{3.5}$$

which tends to 0 as $\alpha \to \infty$. On the other hand, as

$$Z = \frac{1}{\sqrt{2\pi\alpha}} \int_{\mathbb{R}} \exp\left(\frac{\beta - 1}{2\alpha} y^2 - \frac{1}{2} y^4\right) dy \tag{3.6}$$

by change of variables, it is clear that the right-hand side of (A′) diverges to $-\infty$ as $\alpha \to \infty$. Therefore even if $\beta \gg 1$, the condition (A′) is fulfilled by taking α sufficiently large, and hence the inequalities (3.2)–(3.4) hold for such a pair of α and β by Theorem 3.1. We shall see that one of the sufficient conditions is given by

$$\alpha \geq (\beta - 1) \vee 2,$$

namely (A′) is satisfied if $\alpha \geq \beta - 1$ and $\alpha \geq 2$. By $(\beta - 1)/\alpha \leq 1$ and (3.6),

$$\begin{aligned} Z &\leq \frac{2}{\sqrt{2\pi\alpha}} \int_0^\infty \exp\left(\frac{1}{2} y^2 - \frac{1}{2} y^4\right) dy \\ &= \frac{1}{\sqrt{2\pi\alpha}} e^{1/8} \int_{-1/2}^\infty \exp\left(-\frac{1}{2} z^2\right) \frac{dz}{\sqrt{z + 1/2}}. \end{aligned} \tag{3.7}$$

Here we changed variables with $y^2 = z + 1/2$ for the equality. Noting that $1/\sqrt{z + 1/2} \leq \sqrt{2/3} \times z$ for $z \geq 1$, we bound the integral in (3.7) from above by

$$\int_{-1/2}^1 \frac{dz}{\sqrt{z + 1/2}} + \sqrt{\frac{2}{3}} \int_1^\infty z \exp\left(-\frac{1}{2} z^2\right) dz = \sqrt{6} + \sqrt{\frac{2}{3e}}.$$

Combining this estimate with (3.7) and noting that $\sqrt{6} < 2.5$, $\sqrt{2/(3e)} < 0.5$ and $1/\sqrt{2\pi} < 0.4$, we have

$$Z < 1.2 \times \frac{e^{1/8}}{\sqrt{\alpha}}.$$

On the other hand, as $\beta(5\beta - 6) \geq -9/5$ for any $\beta > 0$, (3.5) is bounded from below by

$$-\frac{1}{40\alpha^2}.$$

Therefore the assumption (A$'$) is fulfilled if

$$\sqrt{\alpha}\exp\left(-\frac{1}{40\alpha^2}-\frac{1}{8}\right)\geq 1.2,$$

which is the case when $\alpha \geq 2$ since the left-hand side is not less than

$$\sqrt{2}\left(1-\frac{1}{40\times 2^2}-\frac{1}{8}\right)=\frac{139}{160}\sqrt{2}=1.22\cdots.$$

Example 3.3 (potential with oscillation). We take $\sigma = 1$ as well in this example. For a given positive real γ, consider

$$V(x)=\frac{1}{2}x^2-\gamma\cos x, \quad x\in\mathbb{R}.$$

We let $\gamma > 1$ so that $\{V'' < 0\} \neq \emptyset$. We shall see that the assumption (A) is fulfilled if $\gamma \leq 2$. Observe first that the left-hand side of (A) is equal to

$$\left(\pi-\sqrt{\gamma^2-1}+\arctan\sqrt{\gamma^2-1}\right)^2-\frac{\gamma^2+1}{2} \tag{3.8}$$

if γ is such that $2x+\gamma\sin x>0$ for all $x>0$ and that $2\pi-\sqrt{\gamma^2-1}+\arctan\sqrt{\gamma^2-1}$ is nonnegative. Note that these two requirements are satisfied when $\gamma \leq 2$. The infimum of (3.8) over $1<\gamma\leq 2$, is attained at $\gamma = 2$ and its value $\left(4\pi/3-\sqrt{3}\right)^2-5/2$ is greater than 2. On the other hand, as

$$\begin{aligned} Z &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2}\exp\left(\gamma\cos x\right)dx \\ &\leq \frac{e^{\gamma}}{\sqrt{2}}, \end{aligned}$$

the right-hand side of (A) is less than 2. Therefore the assumption (A) is fulfilled, and hence by Theorem 3.1 we have (3.2)–(3.4) when $\gamma \leq 2$.

Remark 5. (1) *As for Example 3.2, the left-hand side of* (A) *is equal to*

$$\frac{\beta^2(8\beta-9)}{216\alpha^2}\wedge 0,$$

from which we may draw a sharper condition on α and β.
(2) *In Example 3.3, the upper bound 2 on γ cannot be improved significantly. To see this, we bound the partition function Z from below in such a way that, as $|\sin x|\leq|x|$ for any $x\in\mathbb{R}$,*

$$\begin{aligned} Z &= \frac{e^{\gamma}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2}\exp\left(-2\gamma\sin^2\frac{x}{2}\right)dx \\ &\geq \frac{e^{\gamma}}{\sqrt{2+\gamma}}, \end{aligned}$$

from which we see that (A) *fails even for* $\gamma = 5/2$.
(3) *Theorem 3.1 applies to asymmetric potentials as well. For example, take* $V(x) = x^2/2 + \gamma \sin x$, $x \in \mathbb{R}$, *with a real* γ *such that* $|\gamma| > 1$. *Then it can be checked that when* $\sigma = 1$, *a sufficient condition for* (A) *is* $|\gamma| \le \sqrt{2}$.

We proceed to the proof of Theorem 3.1. In what follows we denote

$$U_V(x) = \frac{1}{2}\sigma^2 V'(x)^2 + xV'(x) - V(x), \quad x \in \mathbb{R}.$$

As in the previous section, we denote by F_μ the distribution function of μ:

$$F_\mu(x) = \frac{1}{Z}\int_{-\infty}^{x} e^{-V(y)}\,\nu(dy), \quad x \in \mathbb{R},$$

and set $g = F_\mu^{-1} \circ \Phi$ with F_μ^{-1} the inverse function of F_μ and Φ the standard normal cumulative distribution function.

Lemma 3.4. *Suppose that for all* $x \in \mathbb{R}$,

$$U_V(x) \ge \log Z. \tag{3.9}$$

Then the inequalities (3.2)–(3.4) *hold for any convex function* ψ *on* $\mathbb{R}$.

To prove the lemma, it suffices to show that

$$g'(x) \le \sigma \quad \text{for all } x \in \mathbb{R}, \tag{3.10}$$

in view of the proof of Theorem 1.1. Indeed, if we have (3.10), then we see that Bass' solution that embeds the law of $X - E[X]$ into a given Brownian motion is bounded from above by σ^2, from which (3.2) follows readily; as to the validity of (3.3) and (3.4), observe that the only assumption in Proposition 2.3 is the boundedness of the stopping time S.

Proof of Lemma 3.4. The proof of (3.10) proceeds along the same lines as in the proof of Lemma 2.1. If we define the function G as in (2.9), then $G(0+) = G(1-) = 0$ because $F'_\mu \circ F_\mu^{-1}(0+) = F'_\mu \circ F_\mu^{-1}(1-) = 0$ by (3.1). Provided that G has a local minimum at some $\xi_0 \in (0,1)$, we have

$$\begin{aligned} G(\xi_0) &= \left\{\sigma F'_\mu(x) - \Phi'\left(\frac{x}{\sigma} + \sigma V'(x)\right)\right\}\Big|_{x=F_\mu^{-1}(\xi_0)} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2} - V(x)\right)\left\{\frac{1}{Z} - \exp\left(-U_V(x)\right)\right\}\Big|_{x=F_\mu^{-1}(\xi_0)}, \end{aligned}$$

which is nonnegative by the assumption. This shows (3.10) and concludes the proof. □

Using the above lemma, we prove Theorem 3.1

Proof of Theorem 3.1. Since the assumption (A′) implies (A) due to the fact that

$$\begin{aligned} U_V(x) &= \frac{1}{2}\left(\sigma V'(x) + \frac{x}{\sigma}\right)^2 - \frac{x^2}{2\sigma^2} - V(x) \\ &\geq -\frac{x^2}{2\sigma^2} - V(x) \end{aligned}$$

for all $x \in \mathbb{R}$, the latter assertion follows as soon as we have proven the former. To this end, take an arbitrary $x_0 \in \mathbb{R}\backslash\mathcal{D}_V$, namely let x_0 be such that $V''(x_0) > 0$. First we suppose that

$$V(x) > V'(x_0)(x - x_0) + V(x_0)$$

for all $x \in \mathbb{R}$ but x_0. Then as seen in the proof of Lemma 2.2,

$$\begin{aligned} Z &= \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\sigma^2} - V(x)\right) dx \\ &\leq \frac{1}{\sqrt{2\pi}\sigma}\exp\left(x_0 V'(x_0) - V(x_0)\right)\int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\sigma^2} - V'(x_0)x\right) dx \\ &= \exp\left(U_V(x_0)\right), \end{aligned}$$

hence the inequality (3.9) holds for $x = x_0$. Next we suppose that

$$V(x_1) = V'(x_0)(x_1 - x_0) + V(x_0)$$

for some $x_1 \neq x_0$, say, $x_1 > x_0$. Let $x_2 \in [x_0, x_1]$ be a maximal point of the function

$$f(x) := V(x) - V'(x_0)(x - x_0) - V(x_0), \quad x_0 \leq x \leq x_1.$$

Then it is clear that $f'(x_2) = 0$ and $f''(x_2) \leq 0$; indeed, if either of them were not the case, it would contradict the assumption that x_2 is a maximal point. Therefore we have

$$V'(x_0) = V'(x_2) \tag{3.11}$$

and $x_2 \in \mathcal{D}_V$. Moreover, since

$$f(x_2) = V(x_2) - V'(x_0)(x_2 - x_0) - V(x_0) \geq f(x_0) = 0,$$

it also holds that by (3.11),

$$x_0 V'(x_0) - V(x_0) \geq x_2 V'(x_2) - V(x_2).$$

Combining this inequality with (3.11) yields

$$\begin{aligned} U_V(x_0) &\geq U_V(x_2) \\ &\geq \log Z, \end{aligned}$$

where the second line is due to $x_2 \in \mathcal{D}_V$ and the assumption (A). Consequently, (3.9) holds for all $x \in \mathbb{R}\backslash\mathcal{D}_V$, and hence for all $x \in \mathbb{R}$ by (A). Therefore we have the theorem thanks to Lemma 3.4. □

References

[1] R.F. Bass, Skorokhod imbedding via stochastic integrals, in Séminaire de Probabilités, XVII, 221–224, Lect. Notes in Math. **986**, Springer, Berlin, 1983.

[2] M. Biskup, R. Kotecký, Phase coexistence of gradient Gibbs states, Probab. Theory Relat. Fields **139** (2007), 1–39.

[3] M. Biskup, H. Spohn, Scaling limit for a class of gradient fields with nonconvex potentials, Ann. Probab. **39** (2011), 224–251.

[4] H.J. Brascamp, E.H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Funct. Anal. **22** (1976), 366–389.

[5] L.A. Caffarelli, Monotonicity properties of optimal transportation and the FKG and related inequalities, Commun. Math. Phys. **214** (2000), 547–563; Erratum, Commun. Math. Phys. **225** (2002), 449–450.

[6] C. Cotar, J.-D. Deuschel, Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential, Ann. Inst. H. Poincaré Probab. Statist. **48** (2012), 819–853.

[7] C. Cotar, J.-D. Deuschel, S. Müller, Strict convexity of the free energy for a class of non-convex gradient models, Commun. Math. Phys. **286** (2009), 359–376.

[8] A.M.G. Cox, J. Wang, Root's barrier: construction, optimality and applications to variance options, Ann. Appl. Probab. **23** (2013), 859–894.

[9] J.-D. Deuschel, G. Giacomin, D. Ioffe, Large deviations and concentration properties for $\nabla\varphi$ interface models, Probab. Theory Relat. Fields **117** (2000), 49–111.

[10] T. Funaki, H. Spohn, Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model, Commun. Math. Phys. **185** (1997), 1–36.

[11] G. Giacomin, S. Olla, H. Spohn, Equilibrium fluctuations for $\nabla\varphi$ interface model, Ann. Probab. **29** (2001), 1138–1172.

[12] Y. Hariya, A connection of the Brascamp-Lieb inequality with Skorokhod embedding, Electron. Commun. Probab. **19** (2014), no. 61, 1–12.

[13] Y. Hariya, A variational representation and Prékopa's theorem for Wiener functionals, arXiv:1505.02479 [math.PR] (2015).

[14] D. Hobson, The Skorokhod embedding problem and model-independent bounds for option prices, in Paris-Princeton Lectures on Mathematical Finance 2010, 267–318, Lecture Notes in Math. **2003**, Springer, Berlin, 2011.

[15] I. Karatzas, S.E. Shreve, Methods of Mathematical Finance, Springer, New York, 1998.

[16] J. Obłój, The Skorokhod embedding problem and its offspring, Probab. Surv. **1** (2004), 321–392.

[17] A. Prékopa, On logarithmic concave measures and functions, Acta Sci. Math. (Szeged) **34** (1973), 335–343.

[18] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, 3rd ed., Springer, Berlin, 1999.

[19] Y. Rinott, On convexity of measures, Ann. Probab. **4** (1976), 1020–1026.
[20] A.V. Skorokhod, Issledovaniya po teorii sluchaĭnykh protsessov (Stokhasticheskie differentsial'nye uravneniya i predel'nye teoremy dlya protsessov Markova), Izdat. Kiev. Univ., Kiev, 1961.