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# The Hairer–Quastel universality result at stationarity

By

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## Abstract

We use the notion of energy solutions of the stochastic Burgers equation to give a short proof of the Hairer-Quastel universality result for a class of stationary weakly asymmetric stochastic PDEs.

## § 1. Introduction

Consider the stochastic PDE

$$\partial_t v = \Delta v + \varepsilon^{1/2} \partial_x F(v) + \partial_x \chi^\varepsilon \tag{1.1}$$

on $[0,\infty) \times \mathbb{T}_\varepsilon$ with $\mathbb{T}_\varepsilon = \mathbb{R}/(2\pi\varepsilon^{-1}\mathbb{Z})$, where $\chi^\varepsilon$ is a Gaussian noise that is white in time and spatially smooth. The Hairer–Quastel universality result [HQ15] states that there exist constants $c_1, c_2 \in \mathbb{R}$ such that the rescaled process $\varepsilon^{-1/2} v_{t\varepsilon^{-2}}((x - c_1 \varepsilon^{-1/2} t)\varepsilon^{-1})$ converges to the solution $u$ of the stochastic Burgers equation

$$\partial_t u = \Delta u + c_2 \partial_x u^2 + \partial_x \xi,$$

where $\xi$ is a space-time white noise. Here we give an alternative proof of this result, based on the concept of energy solutions [GJ13a, GJ13b, GP15a, GP15b]. Energy solutions formulate the equilibrium Burgers equation as a martingale problem and allow us to give a simpler proof than the one of [HQ15]. On the other hand our method only applies at stationarity and moreover we need an explicit control of the invariant measure.

Let us state the result more precisely. We modify (1.1) such that after rescaling $\tilde{u}^\varepsilon_t(x) = \varepsilon^{-1/2} v_{t\varepsilon^{-2}}(x\varepsilon^{-1})$ we have

$$\partial_t \tilde{u}^\varepsilon = \Delta \tilde{u}^\varepsilon + \varepsilon^{-1} \partial_x \Pi_0^N F(\varepsilon^{1/2} \tilde{u}^\varepsilon) + \partial_x \Pi_0^N \tilde{\xi}, \qquad \tilde{u}^\varepsilon_0 = \Pi_0^N \eta, \tag{1.2}$$

where $\tilde{\xi}$ is a space-time white noise on $[0,\infty) \times \mathbb{T}$ (where $\mathbb{T} = \mathbb{T}_1$) with variance 2, $\eta$ is a space white noise which is independent of $\tilde{\xi}$, $\Pi_0^N$ denotes the projection onto the Fourier modes $0 < |k| \leqslant N$, and we always link $N$ and $\varepsilon$ via

$$N = \pi/\varepsilon.$$

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**Theorem 1.1.** *Let $F$ be almost everywhere differentiable and assume that for all $\varepsilon > 0$ there is a unique solution $\tilde{u}^\varepsilon$ to (1.2) which does not blow up before $T > 0$. Assume also that $F, F' \in L^2(\nu)$ where $\nu$ is the standard normal distribution. Then $u^\varepsilon_t(x) := \tilde{u}^\varepsilon_t(x - \varepsilon^{-1/2} c_1(F) t)$, $(t, x) \in [0, T] \times \mathbb{T}$, converges in distribution to the unique stationary energy solution $u$ of*

$$\partial_t u = \Delta u + c_2(F) \partial_x u^2 + \partial_x \xi,$$

*where $\xi$ is a space-time white noise with variance 2 and for $U \sim \nu$ and $k \geqslant 0$ and $H_k$ the $k$-th Hermite polynomial*

$$c_k(F) = \frac{1}{k!} \mathbb{E}[F(U) H_k(U)].$$

*Remark.* If $F$ is even, then $c_1(F) = 0$ while $c_2(F) = 0$ if $F$ is odd.

*Remark.* Note that we introduced a second regularization in (1.2) compared to (1.1) which acts on $F(\varepsilon^{1/2} u^\varepsilon)$. The reason is that we need to keep track of the invariant measure and this second regularization allows us to write it down explicitly. For the moment we are unable to deal with the original equation (1.1). For simplicity here we only consider the mollification operator $\Pi_0^N$, but it is possible to extend everything to more general operators $\rho(\varepsilon |\partial_x|) u = \mathcal{F}^{-1}(\rho(\varepsilon \cdot) \mathcal{F} u)$, where $\mathcal{F}$ denotes the Fourier transform and $\rho$ is an even, compactly supported, bounded function which is continuous in a neighborhood of 0 and satisfies $\rho(0) = 1$. We should then modify the equation as

$$\partial_t \tilde{u}^\varepsilon = \Delta \tilde{u}^\varepsilon + \varepsilon^{-1} \partial_x \rho(\varepsilon \mathrm{D}) \rho(\varepsilon \mathrm{D}) F(\varepsilon^{1/2} \tilde{u}^\varepsilon) + \partial_x \rho(\varepsilon \mathrm{D}) \tilde{\xi}, \qquad \tilde{u}^\varepsilon_0 = \rho(\varepsilon \mathrm{D}) \eta,$$

to keep control of the invariant measure, see [FQ15].

*Remark.* While our result only applies in the stationary state, we have more freedom in choosing the nonlinearity $F$ than [HQ15] who require it to be an even polynomial. Also, the methods of this paper will extend without great difficulty to the (modified) equation on $[0, T] \times \mathbb{R}$.

**Notation** For $k \in \mathbb{Z}$ we write $e_k(x) = e^{ikx} / \sqrt{2\pi}$ for the $k$-th Fourier monomial, and for $u \in \mathscr{S}'$, the distributions on $\mathbb{T}$, we define $\hat{u}(k) = \mathcal{F} u(k) = \langle u, e_{-k} \rangle$. We use $\langle \cdot, \cdot \rangle$ to denote both the duality pairing in $\mathscr{S}' \times C^\infty(\mathbb{T}, \mathbb{C})$ and the inner product in $L^2(\mathbb{T})$, so since we want the notation to be consistent we will always consider the $L^2(\mathbb{T}, \mathbb{R})$ inner product and not that of $L^2(\mathbb{T}, \mathbb{C})$. That is, even for complex valued $f, g$ we set $\langle f, g \rangle = \int_{\mathbb{T}} f(x) g(x) \mathrm{d}x$ and do not take a complex conjugate. The Fourier projection operator $\Pi_0^N$ is given by

$$\Pi_0^N v = \sum_{0 < |k| \leqslant N} e_k \hat{v}(k).$$

## § 2. Preliminaries

Let us start by making some basic observations concerning the solution to (1.2).

**Galilean transformation** Recall that $\tilde{u}^\varepsilon$ solves

$$\partial_t \tilde{u}^\varepsilon = \Delta \tilde{u}^\varepsilon + \varepsilon^{-1}\partial_x \Pi_0^N F(\varepsilon^{1/2}\tilde{u}^\varepsilon) + \partial_x \Pi_0^N \tilde{\xi},$$

and that $u_t^\varepsilon(x) = \tilde{u}_t^\varepsilon(x - \varepsilon^{-1/2}c_1(F)t)$. We define the modified test function $\tilde{\varphi}_t(x) = \varphi(x + \varepsilon^{-1/2}c_1(F)t)$ and then $\langle u_t^\varepsilon, \varphi\rangle = \langle \tilde{u}_t^\varepsilon, \tilde{\varphi}_t\rangle$. The Itô–Wentzell formula gives

$$\begin{aligned} \mathrm{d}\langle u_t^\varepsilon, \varphi\rangle &= \langle \mathrm{d}\tilde{u}_t^\varepsilon, \tilde{\varphi}_t\rangle + \langle \tilde{u}_t^\varepsilon, \partial_t \tilde{\varphi}_t\rangle \mathrm{d}t \\ &= \langle \Delta \tilde{u}_t^\varepsilon, \tilde{\varphi}_t\rangle \mathrm{d}t + \langle \varepsilon^{-1}\partial_x \Pi_0^N F(\varepsilon^{1/2}\tilde{u}^\varepsilon), \tilde{\varphi}_t\rangle \mathrm{d}t + \langle \mathrm{d}\partial_x \tilde{M}_t^\varepsilon, \tilde{\varphi}_t\rangle \\ &\qquad + \langle \varepsilon^{-1/2}c_1(F)\tilde{u}_t^\varepsilon, \partial_x \tilde{\varphi}_t\rangle \mathrm{d}t, \end{aligned}$$

where $\tilde{M}_t^\varepsilon(x) = \int_0^t \Pi_0^N \tilde{\xi}(s,x)\mathrm{d}s$. Integrating the last term on the right hand side by parts, we get

$$\mathrm{d}\langle u_t^\varepsilon, \varphi\rangle = \langle \Delta u_t^\varepsilon, \varphi\rangle \mathrm{d}t + \langle \varepsilon^{-1}\partial_x \Pi_0^N F(\varepsilon^{1/2}u^\varepsilon), \varphi\rangle \mathrm{d}t - \varepsilon^{-1/2}c_1(F)\langle \partial_x u_t^\varepsilon, \varphi\rangle \mathrm{d}t + \langle \mathrm{d}\partial_x \tilde{M}_t^\varepsilon, \tilde{\varphi}_t\rangle.$$

The martingale term has quadratic variation

$$\mathrm{d}[\langle \partial_x \tilde{M}^\varepsilon, \tilde{\varphi}_t\rangle]_t = \mathrm{d}[\langle \tilde{M}^\varepsilon, \partial_x \tilde{\varphi}_t\rangle]_t = 2\|\Pi_0^N \partial_x \tilde{\varphi}_t\|_{L^2}^2 \mathrm{d}t = 2\|\Pi_0^N \partial_x \varphi\|_{L^2}^2 \mathrm{d}t,$$

which means that the process $\langle M_t^\varepsilon, \varphi\rangle := \langle \tilde{M}_t^\varepsilon, \tilde{\varphi}_t\rangle$ is of the form $M_t^\varepsilon = \int_0^t \Pi_0^N \xi(s,x)\mathrm{d}s$ for a new space-time white noise $\tilde{\xi}$ with variance 2. In conclusion, $u^\varepsilon$ solves

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-1}\partial_x \Pi_0^N (F(\varepsilon^{1/2}u^\varepsilon) - c_1(F)\varepsilon^{1/2}u^\varepsilon) + \partial_x \Pi_0^N \xi, \qquad u_0^\varepsilon = \Pi_0^N \eta, \tag{2.1}$$

so in other words by performing the change of variables $u_t^\varepsilon(x) = \tilde{u}_t^\varepsilon(x - \varepsilon^{-1/2}c_1(F)t)$ we replaced the function $F$ by $\tilde{F}(x) = F(x) - c_1(F)x$, and now it suffices to study equation (2.1).

**Invariant measure** Note that (2.1) actually is an SDE in the finite dimensional space $Y_N = \Pi_0^N L^2(\mathbb{T},\mathbb{R}) \simeq \mathbb{R}^{2N}$, so that we can apply Echeverria's criterion to show the stationarity of a given distribution. The natural candidate is $\mu^\varepsilon = \mathrm{law}(\Pi_0^N \eta)$, where $\eta$ is a space white noise, since we know that the dynamics of the regularized Ornstein-Uhlenbeck process

$$\partial_t X^\varepsilon = \Delta X^\varepsilon + \partial_x \Pi_0^N \xi$$

are invariant and even reversible under $\mu^\varepsilon$ and that for models in the KPZ universality class the asymmetric version often has the same invariant measure as the symmetric one. Let us write

$$B_F^\varepsilon(u) = \varepsilon^{-1}\partial_x \Pi_0^N (F(\varepsilon^{1/2}u) - c_1(F)\varepsilon^{1/2}u) =: \varepsilon^{-1}\partial_x \Pi_0^N \tilde{F}(\varepsilon^{1/2}u),$$

where $\tilde{F} = F - c_1(F)x$.

**Lemma 2.1.** *The vector field $B_F^\varepsilon : Y_N \to Y_N$ leaves the Gaussian measure $\mu^\varepsilon$ invariant. More precisely, if* D *denotes the gradient with respect to the Fourier monomials $(e_k)_{0<|k|\leqslant N}$ on $Y_N$, then*

$$\int_{Y_N} (B_F^\varepsilon(u) \cdot \mathrm{D}\Phi(u))\Psi(u)\mu^\varepsilon(\mathrm{d}u) = -\int_{Y_N} \Phi(u)B_F^\varepsilon(u) \cdot \mathrm{D}\Psi(u)\mu^\varepsilon(\mathrm{d}u)$$

*for all $\Phi, \Psi \in L^2(\mu^\varepsilon)$ with $B_F^\varepsilon \cdot \mathrm{D}\Phi, B_F^\varepsilon \cdot \mathrm{D}\Psi \in L^2(\mu^\varepsilon)$.*

*Proof.* In this proof it is more convenient to work with the orthonormal basis

$$\left\{\frac{1}{\sqrt{\pi}}\sin(k\cdot), \frac{1}{\sqrt{\pi}}\cos(k\cdot), 0 < k \leqslant N\right\}$$

of $Y_N$, rather than with Fourier monomials. We write $(\varphi_k)_{k=1,\ldots,2N}$ for an enumeration of these trigonometric functions. Then $B^\varepsilon_F \cdot \mathrm{D}$ can also be expressed in terms of the $(\varphi_k)$, and we have

$$\Phi(u) = f(\langle u, \varphi_1\rangle, \ldots, \langle u, \varphi_{2N}\rangle), \qquad \Psi(u) = g(\langle u, \varphi_1\rangle, \ldots, \langle u, \varphi_{2N}\rangle)$$

for some $f, g : \mathbb{R}^{2N} \to \mathbb{R}$. We assume that $f$ and $g$ are continuously differentiable, with polynomial growth of the first order derivatives. The general case then follows by an approximation argument (note that Hermite polynomials of linear combinations of $(\langle u, \varphi_k\rangle)_k$ form an orthogonal basis of $L^2(\mu^\varepsilon)$). Identifying $Y_N$ with $\mathbb{R}^{2N}$, we can write $\mu^\varepsilon(\mathrm{d}u) = \gamma_{2N}(u)\mathrm{d}u$, where $\gamma_{2N}$ is the density of a $2N$-dimensional standard normal variable. Integrating by parts we therefore have

$$\int_{Y_N} (B^\varepsilon_F(u)\cdot \mathrm{D}\Phi(u))\Psi(u)\mu^\varepsilon(\mathrm{d}u) = -\int_{Y_N} (B^\varepsilon_F(u)\cdot \mathrm{D}\Psi(u))\Phi(u)\mu^\varepsilon(\mathrm{d}u)$$

$$-\int_{Y_N}\sum_{k=1}^{2N}(\langle \partial_{\langle u,\varphi_k\rangle} B^\varepsilon_F(u), \varphi_k\rangle - \langle B^\varepsilon_F(u),\varphi_k\rangle\langle u,\varphi_k\rangle)\Psi(u)\Phi(u)\mu^\varepsilon(\mathrm{d}u) \tag{2.2}$$

and it suffices to show that the zero order differential operator terms on the right hand side vanish. For the first one of them we have

$$\begin{aligned}\sum_{k=1}^{2N}\langle \partial_{\langle u,\varphi_k\rangle} B^\varepsilon_F(u), \varphi_k\rangle &= \sum_{k=1}^{2N}\langle \partial_{\langle u,\varphi_k\rangle} \varepsilon^{-1}\partial_x \Pi_0^N \tilde F(\varepsilon^{1/2}u), \varphi_k\rangle \\ &= \sum_{k=1}^{2N}\langle \varepsilon^{-1/2}\partial_x(\Pi_0^N \tilde F'(\varepsilon^{1/2}u)\varphi_k), \varphi_k\rangle \\ &= -\sum_{k=1}^{2N}\langle \varepsilon^{-1/2}\Pi_0^N \tilde F'(\varepsilon^{1/2}u)\varphi_k, \partial_x\varphi_k\rangle \\ &= -\frac{\varepsilon^{-1/2}}{2}\langle \Pi_0^N\tilde F'(\varepsilon^{1/2}u), \partial_x\sum_{k=1}^{2N}\varphi_k^2\rangle,\end{aligned}$$

and since $\sin(mx)^2 + \cos(mx)^2 = 1$ the sum of the squares of the $\varphi_k$ does not depend on $x$ so its derivative is 0. For the remaining term in (2.2) we get $\mu^\varepsilon$-almost surely

$$\begin{aligned}\sum_{k=1}^{2N}\langle B^\varepsilon_F(u),\varphi_k\rangle\langle u,\varphi_k\rangle &= \langle B^\varepsilon_F(u), u\rangle = \langle \varepsilon^{-1}\partial_x\Pi_0^N\tilde F(\varepsilon^{1/2}u), u\rangle \\ &= \varepsilon^{-1}\langle \partial_x \tilde F(\varepsilon^{1/2}u), \Pi_0^N u\rangle = -\varepsilon^{-1}\langle \tilde F(\varepsilon^{1/2}u), \partial_x\Pi_0^N u\rangle.\end{aligned}$$

Now observe that there exists $G$ with $G' = \tilde F$, and that under $\mu^\varepsilon$ we have $u = \Pi_0^N u$ almost surely, which yields

$$-\varepsilon^{-1}\langle \tilde F(\varepsilon^{1/2}u), \partial_x\Pi_0^N u\rangle = -\varepsilon^{-1}\langle G'(\varepsilon^{1/2}\Pi_0^N u), \partial_x \Pi_0^N u\rangle = -\varepsilon^{-3/2}\langle \partial_x G(\varepsilon\Pi_0^N u), 1\rangle = 0,$$

and therefore the proof is complete. □

The previous lemma, together with the reversibility of the Ornstein-Uhlenbeck dynamics under $\mu^\varepsilon$, implies that the Itô SDE (2.1) has $\mu^\varepsilon$ as invariant measure and that for $T > 0$ the time reversed process $\hat{u}^\varepsilon_t = \hat{u}^\varepsilon_{T-t}$ solves

$$\partial_t \hat{u}^\varepsilon = \Delta \hat{u}^\varepsilon - \varepsilon^{-1}\partial_x \tilde{F}(\varepsilon^{1/2}\Pi_0^N \hat{u}^\varepsilon) + \partial_x \Pi_0^N \hat{\xi} \tag{2.3}$$

with a time-reversed space-time white noise $\hat{\xi}$.

## § 3. Boltzmann-Gibbs principle

In the theory of interacting particle systems the phenomenon that local quantities of the microscopic fields can be replaced in time averages by simple functionals of the conserved quantities is called *Boltzmann–Gibbs principle.* In this section we investigate a similar phenomenon in order to control the antisymmetric drift term

$$\int_0^t \varepsilon^{-1}\partial_x \tilde{F}(\varepsilon^{1/2} u^\varepsilon_s(x))\mathrm{d}s \tag{3.1}$$

as $N \to +\infty$. Note that since $\varepsilon = \pi/N$ and $u^\varepsilon = \Pi_0^N u^\varepsilon$ we have $\mathbb{E}[(\varepsilon^{1/2}u^\varepsilon_s(x))^2] = 1$ for all $N$, and therefore the Gaussian random variables $(\varepsilon^{1/2}u^\varepsilon_s(x))_N$ stay bounded in $L^2$ for fixed $(s, x)$, but for large $N$ there will be wild fluctuations in $(s, x)$. We show that the quantity in (3.1) can be replaced by simpler expressions that are constant, linear, or quadratic in $u^\varepsilon$.

### § 3.1. A first computation

In the following we use $\eta$ to denote a generic space white noise and we write $\mu$ for its law, and $G \in C(\mathbb{R}, \mathbb{R})$ denotes a generic continuous function. A first interesting computation is to consider the random field $x \mapsto G(\varepsilon^{1/2}\Pi_0^N \eta(x))$ and to derive its chaos expansion in the variables $(\eta_k)_k$ where $\eta_k = \langle \eta, e_{-k}\rangle$ are the Fourier coordinates of $\eta$. To do so consider the standard Gaussian random variable

$$\eta^N(x) = \varepsilon^{1/2}\Pi_0^N \eta(x) = \varepsilon^{1/2} \sum_{0<|k|\leqslant N} e_k(x)\eta_k,$$

and observe that the chaos expansion in $L^2(\mathrm{law}(\eta^N(x)))$ yields

$$G(\eta^N(x)) = \sum_{n\geqslant 0} c_n(G) H_n(\eta^N(x)),$$

where $H_n$ is the $n$-th Hermite polynomial and

$$c_n(G) = \frac{1}{n!}\mathbb{E}[G(\eta^N(x))H_n(\eta^N(x))] = \frac{1}{n!}\int_{\mathbb{R}} G(x)H_n(x)\gamma(x)\mathrm{d}x,$$

where $\gamma$ is the standard Gaussian density. Since $H_n(x) = (-1)^n e^{x^2/2}\partial_x^n e^{-x^2/2}$, we get

$$c_n(G) = \frac{1}{n!}\int_{\mathbb{R}} G(x)(-1)^n \partial_x^n \gamma(x)\mathrm{d}x = \frac{\psi_G^{(n)}(0)}{n!},$$

where $\psi_G(\lambda) = \mathbb{E}[G(\lambda + \eta^N(x))]$.

Our next aim is to relate the Hermite polynomials of $\eta^N(x)$ with the Wick powers of the family $(\eta_k)_k$. To do so we observe that, on one hand the monomials $H_n(\eta^N(x))$ are the coefficients of the powers of $\lambda$ in $\exp(\lambda\eta^N(x) - \lambda^2/2)$, and on the other hand

$$\sum_n \frac{\lambda^n}{n!} H_n(\eta^N(x)) = \exp(\lambda\eta^N(x) - \lambda^2/2) = \exp\Big(\lambda\varepsilon^{1/2} \sum_{0<|k|\leqslant N} e_k(x)\eta_k - \frac{1}{4\pi} \sum_{0<|k|\leqslant N} (\lambda\varepsilon^{1/2})^2\Big).$$

Writing $[\![\cdot]\!]_n$ for the projection onto the $n$-th homogeneous chaos generated by $\eta$, we have

$$\exp\Big(\sum_{0<|k|\leqslant N} \mu_k\eta_k - \frac{1}{2}\sum_{0<|k|\leqslant N} \mu_k\mu_{-k}\Big) = \sum_{k_1\cdots k_n} \frac{\mu_{k_1}\cdots\mu_{k_n}}{n!}[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n,$$

where the sum on the right hand side and all the following sums in $k_1 \dots k_n$ are over $0 < |k_1|, \dots, |k_n| \leqslant N$. Setting $\mu_k = \varepsilon^{1/2}\lambda e_k(x)$ and identifying the coefficients for different powers of $\lambda$, we get

$$H_n(\varepsilon^{1/2}\Pi_0^N\eta(x)) = \varepsilon^{n/2} \sum_{k_1\cdots k_n} \frac{e^{i(k_1+\cdots+k_n)x}}{(2\pi)^{n/2}}[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n,$$

which can also be obtained by writing $H_n(\varepsilon^{1/2}\Pi_0^N\eta(x)) = [\![(\varepsilon^{1/2}\Pi_0^N\eta(x))^n]\!]_n$ and expanding the power $(\cdot)^n$ inside the projection. We can thus represent the function $G(\eta^N(x))$ as

$$G(\eta^N(x)) = \sum_{n\geqslant 0} c_n(G)H_n(\varepsilon^{1/2}\Pi_0^N\eta(x)) = \sum_{n\geqslant 0} c_n(G)\varepsilon^{n/2} \sum_{k_1,\dots,k_n} \frac{e^{i(k_1+\cdots+k_n)x}}{(2\pi)^{n/2}}[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n.$$

If $\varphi \in C^\infty(\mathbb{T})$ is a test function, we get

$$\langle G(\eta^N), \varphi\rangle = \sum_{n\geqslant 0} c_n(G)\varepsilon^{n/2} \sum_{k_1,\dots,k_n} \frac{\hat{\varphi}(-k_1 - \cdots - k_n)}{(2\pi)^{(n-1)/2}}[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n. \tag{3.2}$$

So in particular the $q$-th Littlewood-Paley block of $G(\eta^N)$ is given by

$$\Delta_q G(\eta^N)(x) = \sum_{n\geqslant 0} c_n(G)\varepsilon^{n/2} \sum_{k_1,\dots,k_n} \theta_q(k_1 + \cdots + k_n)\frac{e^{i(k_1+\cdots+k_n)x}}{(2\pi)^{(n-1)/2}}[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n,$$

where $(\theta_q)_{q\geqslant -1}$ is a dyadic partition of unity, and

$$\begin{aligned}\mathbb{E}[|\Delta_q(G(\eta^N) - c_0(G))(x)|^2] &\leqslant \sum_{n\geqslant 1} c_n(G)^2 \frac{z_n\varepsilon^n}{(2\pi)^{n-1}} \sum_{k_1,\dots,k_n} \theta_q(k_1 + \cdots + k_n)^2 \\ &\lesssim \sum_{n\geqslant 1} c_n(G)^2 z_n \frac{\varepsilon^n(2N)^{n-1}}{(2\pi)^{n-1}}(2^q \wedge N) \lesssim \varepsilon \sum_{n\geqslant 1} c_n(G)^2 z_n(2^q \wedge N),\end{aligned}$$

where $z_n = \max_{k_1\dots k_n} \mathbb{E}[|[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n|^2] \leqslant n!$ is a combinatorial factor. We thus obtain

$$\mathbb{E}[\|\Delta_q(G(\eta^N) - c_0(G))\|^2_{L^2(\mathbb{T})}] \lesssim \min\{\varepsilon 2^q, 1\},$$

uniformly in $N$, and then

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \Delta_q(G(\varepsilon^{1/2}u_r^\varepsilon(x)) - c_0(G))\mathrm{d}r\right|^2\right] &\leqslant |t-s| \int_s^t \mathbb{E}[|\Delta_q(G(\varepsilon^{1/2}u_r^\varepsilon(x)) - c_0(G))|^2]\mathrm{d}r \\ &\lesssim |t-s|^2 \min\{\varepsilon 2^q, 1\},\end{aligned}$$

where in the last step we used that $\varepsilon^{1/2}u_r^\varepsilon$ has the same distribution as $\eta^N$, which easily implies the following result.

**Lemma 3.1.** *Assume that* $\mathbb{E}[|G(U)|^2] < \infty$ *for a standard normal variable* $U$*, and let* $c_0(G) = \mathbb{E}[G(U)]$*. Then*

$$\lim_{N\to\infty} \int_0^t G(\varepsilon^{1/2} u_s^\varepsilon(x))\mathrm{d}s = c_0(G)t,$$

*where the convergence is in* $C([0,T], H^{0-})$*. If* $c_0(G) = 0$*, then*

$$\varepsilon^{-1/2} \int_0^t G(\varepsilon^{1/2} u_s^\varepsilon(x))\mathrm{d}s$$

*is bounded in* $C([0,T], H^{-1/2-})$*.*

To analyse the case where $c_0(G) = 0$ we need a more refined argument which is provided by the technique of regularization by noise for controlled paths.

## § 3.2. Regularization by noise

Let us write $\mathscr{L}_0^\varepsilon$ for the generator of the mollified Ornstein–Uhlenbeck process

$$\partial_t X^\varepsilon = \Delta X^\varepsilon + \partial_x \Pi_0^N \xi.$$

The basic tool which allows us to control time integrals such as $\int_0^t G(\varepsilon^{1/2} u_s^\varepsilon(x))\mathrm{d}s$ is the following Itô trick. Define for $\Psi \in L^2(\mu^\varepsilon)$

$$\mathcal{E}^\varepsilon(\Psi) := \sum_{0<|k|\leqslant N} k^2 |\mathrm{D}_k \Psi|^2,$$

where $\mathrm{D}_k$ is the derivative in the direction $e_k$.

**Lemma 3.2** (Itô trick)**.**
*For* $\Psi \in \mathrm{dom}\,(\mathscr{L}_0^\varepsilon)$ *and* $T > 0$, $p \geqslant 1$ *we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathscr{L}_0^\varepsilon \Psi(u_s^\varepsilon)\mathrm{d}s\right|^p\right] \lesssim T^{p/2}\mathbb{E}[\mathcal{E}^\varepsilon(\Psi)^{p/2}].$$

The proof is given in [GJ13b, GP15b] and extends without difficulty to our setting, so we do not repeat the arguments here.

To apply the Itô trick we need to solve the Poisson equation. In our setting this can be done efficiently by using the chaos expansion (3.2). Recall that we wrote $\eta_k = \langle \eta, e_k\rangle$ for the Fourier coefficients of a truncated spatial white noise $\Pi_0^N \eta$ (which therefore has law $\mu^\varepsilon$), and that $[\![\cdot]\!]_n$ denotes the projection onto the $n$-th chaos. We need to compute $\mathscr{L}_0^\varepsilon [\![\eta_{k_1} \dots \eta_{k_n}]\!]_n$, as these are the random variables appearing in a general chaos expansion. Let us start by considering $\varphi \in Y_N = \Pi_0^N L^2(\mathbb{T}, \mathbb{R})$ with $\|\varphi\|_{L^2} = 1$ for which we have $[\![\langle\eta,\varphi\rangle^n]\!]_n = H_n(\langle\eta,\varphi\rangle)$, where $H_n$ is the $n$-th Hermite polynomial. Itô's formula gives

$$\mathrm{d}H_n(\langle X_t^\varepsilon,\varphi\rangle) = H_n'(\langle X_t^\varepsilon,\varphi\rangle)\langle X_t^\varepsilon, \Delta\varphi\rangle\mathrm{d}t + H_n''(\langle X_t^\varepsilon,\varphi\rangle)\langle \Pi_0^N\partial_x\varphi, \Pi_0^N\partial_x\varphi\rangle\mathrm{d}t + \mathrm{d}M_t,$$

with a square integrable martingale $M$. The Hermite polynomials satisfy $H_n' = nH_{n-1}$, so we get

$$\begin{aligned} &H_n'(\langle X_t^\varepsilon,\varphi\rangle)\langle X_t^\varepsilon, \Delta\varphi\rangle + H_n''(\langle X_t^\varepsilon,\varphi\rangle)\langle \Pi_0^N\partial_x\varphi, \Pi_0^N\partial_x\varphi\rangle \\ &\qquad = nH_{n-1}(\langle X_t^\varepsilon,\varphi\rangle)H_1(\langle X_t^\varepsilon,\Delta\varphi\rangle) - n(n-1)H_{n-2}(\langle X_t^\varepsilon,\varphi\rangle)\langle \Pi_0^N\varphi, \Pi_0^N\Delta\varphi\rangle. \end{aligned}$$

The projection onto the $n$-th chaos of the first term is explicitly given by

$$\begin{aligned}\llbracket H_{n-1}(\langle X_t^\varepsilon,\varphi\rangle)H_1(\langle X_t^\varepsilon,\Delta\varphi\rangle)\rrbracket_n &= \llbracket\llbracket\langle X_t^\varepsilon,\varphi\rangle^{n-1}\rrbracket_{n-1}\llbracket\langle X_t^\varepsilon,\Delta\varphi\rangle\rrbracket_1\rrbracket_n\\ &= \llbracket\langle X_t^\varepsilon,\varphi\rangle^{n-1}\rrbracket_{n-1}\llbracket\langle X_t^\varepsilon,\Delta\varphi\rangle\rrbracket_1\\ &\qquad -(n-1)\llbracket\langle X_t^\varepsilon,\varphi\rangle^{n-2}\rrbracket_{n-2}\langle\Pi_0^N\varphi,\Pi_0^N\Delta\varphi\rangle,\end{aligned}$$

which is obtained by contracting $\langle X_t^\varepsilon,\Delta\varphi\rangle$ with each of the $n-1$ variables $\langle X_t^\varepsilon,\varphi\rangle$ inside the projector $\llbracket\cdot\rrbracket_{n-1}$. Therefore, we have

$$\begin{aligned}\mathrm{d}H_n(\langle X_t^\varepsilon,\varphi\rangle) &= n\llbracket H_{n-1}(\langle X_t^\varepsilon,\varphi\rangle)H_1(\langle X_t^\varepsilon,\Delta\varphi\rangle)\rrbracket_n\mathrm{d}t+\mathrm{d}M_t\\ &= n\llbracket\langle X_t^\varepsilon,\varphi\rangle^{n-1}\langle X_t^\varepsilon,\Delta\varphi\rangle\rrbracket_n\mathrm{d}t+\mathrm{d}M_t,\end{aligned}$$

which shows that

$$\mathscr{L}_0^\varepsilon\llbracket\langle\eta,\varphi\rangle^n\rrbracket_n = n\llbracket\langle\eta,\varphi\rangle^{n-1}\langle\eta,\Delta\varphi\rangle\rrbracket_n.$$

So far we assumed $\|\varphi\|_{L^2}=1$, but actually this last formula is invariant under scaling so it extends to all $\varphi\in\Pi_0^N L^2(\mathbb{T},\mathbb{R})$, and then to $\varphi\in\Pi_0^N L^2(\mathbb{T},\mathbb{C})$, and for general products we obtain by polarization

$$\mathscr{L}_0^\varepsilon\llbracket\langle\eta,\varphi_1\rangle\dots\langle\eta,\varphi_n\rangle\rrbracket_n = \sum_{k=1}^n\llbracket\langle\eta,\varphi_1\rangle\dots\cancel{\langle\eta,\varphi_k\rangle}\dots\langle\eta,\varphi_n\rangle\langle\eta,\Delta\varphi_k\rangle\rrbracket_n\,.$$

So finally we deduce that

$$\mathscr{L}_0^\varepsilon\llbracket\eta_{k_1}\cdots\eta_{k_n}\rrbracket = -(k_1^2+\cdots+k_n^2)\llbracket\eta_{k_1}\cdots\eta_{k_n}\rrbracket \tag{3.3}$$

for all $0<|k_1|,\dots,|k_n|\leqslant N$. Combining that formula with (3.2), we obtain the following lemma.

**Lemma 3.3.** *Consider a function of the form $\Phi(\eta)=\langle G(\varepsilon^{1/2}\Pi_0^N\eta),\varphi\rangle$ and assume that $\mathbb{E}[G(U)]=0$, where $U$ is a standard normal variable, or that $\hat{\varphi}(0)=0$. Then the solution $\Psi$ to the Poisson equation $\mathscr{L}_0^\varepsilon\Psi=\Phi$ is explicitly given by*

$$\Psi(\eta) = -\sum_{n\geqslant1}c_n(G)\varepsilon^{n/2}\sum_{k_1\cdots k_n}\frac{\hat{\varphi}(-k_1-\cdots-k_n)}{(2\pi)^{(n-1)/2}}\frac{\llbracket\eta_{k_1}\cdots\eta_{k_n}\rrbracket_n}{(k_1^2+\cdots+k_n^2)},$$

*where the sum is over all $0<|k_1|,\dots,|k_n|\leqslant N$.*

*Remark.* Incidentally note that the solution can be represented as

$$\begin{aligned}\Psi(\eta) &= -\int_0^\infty\mathrm{d}t\sum_{n\geqslant1}c_n(G)\varepsilon^{n/2}\sum_{k_1\cdots k_n}e^{-(k_1^2+\cdots+k_n^2)t}\frac{e^{i(k_1+\cdots+k_n)x}}{(2\pi)^{n/2}}\llbracket\eta_{k_1}\cdots\eta_{k_n}\rrbracket_n\\ &= -\int_0^\infty\mathrm{d}tG(\varepsilon^{1/2}(e^{\Delta t}\Pi_0^N\eta)(x)).\end{aligned}$$

To apply the Itô trick we need to compute $\mathcal{E}(\Psi)=\sum_k k^2\mathrm{D}_{-k}\Psi\mathrm{D}_k\Psi$ for the solution $\Psi$ of the Poisson equation. For that purpose consider again $\varphi\in Y_N$ with $\|\varphi\|_{L^2}=1$ and $H_n(\langle\eta,\varphi\rangle)=\llbracket\langle\eta,\varphi\rangle^n\rrbracket_n$, for which we have

$$\mathrm{D}_kH_n(\langle\eta,\varphi\rangle) = H_n'(\langle\eta,\varphi\rangle)\langle e_k,\varphi\rangle = nH_{n-1}(\langle\eta,\varphi\rangle)\langle e_k,\varphi\rangle = n\llbracket\langle\eta,\varphi\rangle^{n-1}\rrbracket_{n-1}\langle e_k,\varphi\rangle,$$

so by polarization

$$\mathrm{D}_k[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n = \sum_j \mathbf{1}_{k_j=k}[\![\eta_{k_1}\cdots \not{\eta_{k_j}}\cdots\eta_{k_n}]\!]_{n-1}. \tag{3.4}$$

To prove the Boltzmann–Gibbs principle we need one more auxiliary result.

**Lemma 3.4.** *For all $M \leqslant N$, $\ell \in \mathbb{Z}$ and $0 \leqslant s < t < \infty$ we have the estimate*

$$\mathbb{E}\left[\left|\int_s^t \langle \partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \lesssim \ell^2 |t-s|^2 M.$$

*Proof.* We simply bound

$$\mathbb{E}\left[\left|\int_s^t \langle \partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \leqslant |t-s| \int_s^t \mathbb{E}[|\langle \partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle|^2]\mathrm{d}r,$$

and since we can replace $(\Pi_0^M u_r^\varepsilon)^2$ by $(\Pi_0^M u_r^\varepsilon)^2 - \mathbb{E}[(\Pi_0^M u_r^\varepsilon)^2]$, the integrand is given by

$$\begin{aligned}\mathbb{E}[|\langle \partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle|^2] &= \ell^2 \int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' \mathbb{E}[[\![(\Pi_0^M u_r^\varepsilon(x))^2]\!]_2 [\![(\Pi_0^M u_r^\varepsilon(x'))^2]\!]_2] \\ &\lesssim \ell^2 \int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' |\mathbb{E}[\Pi_0^M u_r^\varepsilon(x)\Pi_0^M u_r^\varepsilon(x')]|^2.\end{aligned}$$

The expectation on the right hand side can be explicitly computed as

$$\begin{aligned}|\mathbb{E}[\Pi_0^M u_r^\varepsilon(x)\Pi_0^M u_r^\varepsilon(x')]| &= \Big|\sum_{0<|k|\leqslant M} e^{ik(x-x')}\Big| = \Big|\frac{\cos(M(x-x')) - \cos((M+1)(x-x'))}{1-\cos(x-x')} - 1\Big| \\ &\leqslant \min\{2M, C|x-x'|^{-1}\},\end{aligned}$$

for some constant $C < +\infty$, for which

$$\int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' \min\{2M, C|x-x'|^{-1}\}^2 \mathrm{d}x \lesssim 2M,$$

and therefore the claim follows. □

**Proposition 3.5** (Boltzmann–Gibbs principle)**.**
*Let $G, G' \in L^2(\nu)$, where $\nu$ denotes the law of a standard normal variable. Then for all $\ell \in \mathbb{Z}$ and $0 \leqslant s < t \leqslant s+1$ and all $\kappa > 0$*

$$\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2}u_r^\varepsilon) - \varepsilon^{-1/2}c_1(G)\partial_x\Pi_0^N u_r^\varepsilon, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \lesssim |t-s|^{3/2-\kappa}\ell^2 \int_{\mathbb{R}} |G'(x)|^2\nu(\mathrm{d}x)$$

*uniformly in $N \in \mathbb{N}$, and for all $M \leqslant N/2$*

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2}u_r^\varepsilon) - \varepsilon^{-1/2}c_1(G)\partial_x\Pi_0^N u_r^\varepsilon - c_2(G)\partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \\ \lesssim |t-s|\ell^2(M^{-1} + \varepsilon\log^2 N)\int_{\mathbb{R}} |G'(x)|^2\nu(\mathrm{d}x).\end{aligned}$$

*Proof.* We first show the second bound. Towards this end note that $\partial_x(\Pi_0^M \eta)^2 = \Pi_0^N \partial_x(\Pi_0^M \eta)^2$ for $M \leqslant N/2$ and that by Lemma 3.3 the solution $\Psi$ to

$$\mathscr{L}_0^\varepsilon \Psi(\eta) = -\varepsilon^{-1}\langle G(\varepsilon^{1/2}\Pi_0^N \eta) - c_1(G)\varepsilon^{1/2}\Pi_0^N \eta - c_2(G)(\varepsilon^{1/2}\Pi_0^M \eta)^2, \partial_x \Pi_0^N e_{-\ell}\rangle$$

is given by

$$\Psi(\eta) = c_2(G)\sum_{k_1,k_2} \mathbf{1}_{|k_1|\vee|k_2|>M}\mathbf{1}_{0<|\ell|\leqslant N}(i\ell)\frac{\mathbf{1}_{k_1+k_2=\ell}}{(2\pi)^{1/2}}\frac{[\![\eta_{k_1}\eta_{k_2}]\!]_2}{(k_1^2+k_2^2)}$$
$$+\sum_{n\geqslant 3} c_n(G)\varepsilon^{n/2-1}\sum_{k_1\cdots k_n}\mathbf{1}_{0<|\ell|\leqslant N}(i\ell)\frac{\mathbf{1}_{k_1+\cdots+k_n=\ell}}{(2\pi)^{(n-1)/2}}\frac{[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n}{(k_1^2+\cdots+k_n^2)},$$

where it is understood that all sums sums in $k_i$ are over $0<|k_i|\leqslant N$. Therefore (3.4) yields for $0<|\ell|\leqslant N$

$$\mathrm{D}_k\Psi(\eta) = c_2(G)2\sum_{k_1}\mathbf{1}_{|k|\vee|k_1|>M}i\ell\frac{\mathbf{1}_{k+k_1=\ell}}{(2\pi)^{1/2}}\frac{[\![\eta_{k_1}]\!]_1}{(k^2+k_1^2)}$$
$$+\sum_{n\geqslant 2} c_{n+1}(G)\varepsilon^{(n-1)/2}(n+1)\sum_{k_1\cdots k_n} i\ell\frac{\mathbf{1}_{k+k_1+\cdots+k_n=\ell}}{(2\pi)^{n/2}}\frac{[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n}{(k^2+k_1^2+\cdots+k_n^2)}.$$

Applying the Itô trick we get

$$\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x\Pi_0^N G(\varepsilon^{1/2}u_r^\varepsilon) - \varepsilon^{-1/2}c_1(G)\partial_x\Pi_0^N u_r^\varepsilon - c_2(G)\partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle \mathrm{d}r\right|^2\right]$$
$$\lesssim |t-s|\sum_{0<|k|\leqslant N} k^2\mathbb{E}[|\mathrm{D}_k\Psi|^2]$$
$$= |t-s|\sum_{0<|k|\leqslant N} k^2 c_2(G)^2 2^2\ell^2\sum_{k_1}\mathbf{1}_{|k|\vee|k_1|>M}\frac{\mathbf{1}_{k+k_1=\ell}}{2\pi}\frac{\mathbb{E}[|[\![\eta_{k_1}]\!]_1|^2]}{(k^2+k_1^2)^2}$$
$$+|t-s|\sum_{0<|k|\leqslant N} k^2\sum_{n\geqslant 2} c_{n+1}(G)^2\varepsilon^{(n+1)-2}(n+1)^2\ell^2$$
$$\times\sum_{k_1\cdots k_n}\frac{\mathbf{1}_{k+k_1+\cdots+k_n=\ell}}{(2\pi)^n}\frac{\mathbb{E}[|[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n|^2]}{(k^2+k_1^2+\cdots+k_n^2)^2}$$
$$= |t-s|\sum_{n\geqslant 1} A_n,$$

where the $(A_n)$ are implicitly defined by the equation. Now $\mathbb{E}[|[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n|^2] \leqslant n!$ for all $k_1,\ldots,k_n$, so that

$$A_1 \lesssim \sum_{0<|k|,|k_1|\leqslant N} k^2 c_2(G)^2\ell^2\mathbf{1}_{k+k_1=\ell}\frac{\mathbf{1}_{|k|\vee|k_1|>M}}{(k^2+k_1^2)^2} \leqslant \sum_{0<|k|,|k_1|\leqslant N} c_2(G)^2\ell^2\mathbf{1}_{k+k_1=\ell}\frac{\mathbf{1}_{|k|\vee|k_1|>M}}{k^2+k_1^2}$$
$$\lesssim c_2(G)^2\ell^2\sum_{0<|k|<\infty}\frac{\mathbf{1}_{\ell\neq k}\mathbf{1}_{|k|\vee|\ell-k|>M}}{k^2+(\ell-k)^2} \leqslant c_2(G)^2\ell^2\sum_{0<|k|<\infty}\left(\frac{\mathbf{1}_{\ell\neq k}}{M^2+(\ell-k)^2}+\frac{\mathbf{1}_{\ell\neq k}}{k^2+M^2}\right)$$
$$\lesssim c_2(G)^2\ell^2 M^{-1},$$

while for $n > 1$

$$
\begin{aligned}
A_n &= \sum_{0<|k|\leqslant N} k^2 c_{n+1}(G)^2 \varepsilon^{n-1} (n+1)^2 \ell^2 \sum_{k_1\cdots k_n} \frac{\mathbf{1}_{k+k_1+\cdots+k_n=\ell}}{(2\pi)^n} \frac{\mathbb{E}[|[\![\eta_{k_1}\cdots\eta_{k_n}]\!]_n|^2]}{(k^2+k_1^2+\cdots+k_n^2)^2} \\
&\leqslant \frac{\varepsilon^{n-1}}{(2\pi)^n} \ell^2 (n+1)^2 c_{n+1}(G)^2 n! \sum_{0<|k|,|k_1|,\ldots,|k_n|\leqslant N} k^2 \frac{\mathbf{1}_{k+k_1+\cdots+k_n=\ell}}{(k^2+k_1^2+\cdots+k_n^2)^2} \\
&\leqslant \frac{\varepsilon^{n-1}}{(2\pi)^n} \ell^2 (n+1)^2 c_{n+1}(G)^2 n! \sum_{0<|k_1|,\ldots,|k_n|\leqslant N} \frac{1}{k_1^2+\cdots+k_n^2} \\
&\leqslant \frac{\varepsilon^{n-1}}{(2\pi)^n} \ell^2 (n+1)^2 c_{n+1}(G)^2 n! \sum_{0<|k_1|,\ldots,|k_n|\leqslant N} \frac{1}{k_1^2+k_2^2} \\
&= \frac{\varepsilon^{n-1}}{(2\pi)^n} \ell^2 (n+1)^2 c_{n+1}(G)^2 n! (2N)^{n-2} \sum_{0<|k_1|,|k_2|\leqslant N} \frac{1}{k_1^2+k_2^2} \lesssim \varepsilon \ell^2 (n+1)^2 c_{n+1}(G)^2 n! \log^2 N.
\end{aligned}
$$

The sum over $n$ is bounded by

$$
\sum_{n=2}^{\infty} c_{n+1}(G)^2 n! (n+1)^2 = \sum_{n=1}^{\infty} n c_n(G)^2 n! \lesssim \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x),
$$

so that overall we get

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2} u_r^\varepsilon) - \varepsilon^{-1/2} c_1(G) \partial_x \Pi_0^N u_r^\varepsilon - c_2(G)\partial_x(\Pi_0^M u_r^\varepsilon)^2, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \\
&\qquad \lesssim |t-s| \ell^2 (M^{-1} + \varepsilon \log^2 N) \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x), \qquad (3.5)
\end{aligned}
$$

which is our second claimed bound.

To get the first bound, we take $M \simeq |t-s|^{-1/2}$ in (3.5) (which requires $N > |t-s|^{-1/2}$), and combine this with Lemma 3.4 to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2} u_r^\varepsilon) - \varepsilon^{-1/2} c_1(G) \partial_x \Pi_0^N u_r^\varepsilon, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \\
&\lesssim |t-s| \ell^2 (M^{-1} + \varepsilon \log^2 N + |t-s| M) \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x) \lesssim |t-s|^{3/2-\kappa} \ell^2 \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x).
\end{aligned}
$$

If $N \leqslant |t-s|^{-1/2}$ we use another estimate: as in the proof of Lemma 3.4 we have

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2}u_r^\varepsilon) - \varepsilon^{-1/2}c_1(G)\partial_x \Pi_0^N u_r^\varepsilon, e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \\
&\leqslant |t-s|^2 \mathbb{E}[|\langle \varepsilon^{-1}\partial_x \Pi_0^N G(\varepsilon^{1/2}u_0^\varepsilon) - \varepsilon^{-1/2}c_1(G)\partial_x \Pi_0^N u_0^\varepsilon, e_{-\ell}\rangle|^2] \\
&\lesssim |t-s|^2 \sum_{n\geqslant 2} \ell^2 \varepsilon^{-2} c_n(G)^2 \int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' \mathbb{E}[H_n(\varepsilon^{1/2}u_0^\varepsilon(x))H_n(\varepsilon^{1/2}u_0^\varepsilon(x'))], \\
&\lesssim |t-s|^2 \sum_{n\geqslant 2} \ell^2 \varepsilon^{-2} n! c_n(G)^2 \int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' |\mathbb{E}[\varepsilon^{1/2}u_0^\varepsilon(x)\varepsilon^{1/2}u_0^\varepsilon(x')]^n| \\
&\lesssim |t-s|^2 \sum_{n\geqslant 2} \ell^2 \varepsilon^{n-2} n! c_n(G)^2 \int_{\mathbb{T}} \mathrm{d}x \int_{\mathbb{T}} \mathrm{d}x' \min\{2N, C|x-x'|^{-1}\}^n \\
&\lesssim |t-s|^2 \sum_{n\geqslant 2} \ell^2 \varepsilon^{n-2} n! c_n(G)^2 (2N)^{n-1} \lesssim |t-s|^2 \sum_{n\geqslant 2} \ell^2 \varepsilon^{-1} c_n(G)^2 n! \\
&\lesssim \ell^2 |t-s|^{3/2} \int_{\mathbb{R}} |G'(x)|^2 \nu(\mathrm{d}x),
\end{aligned}
$$

where in the last step we used that $|t-s|^{-1/2}N^{-1} \geqslant 1$. □

## § 4. The invariance principle

We now have all the tools to prove the convergence of $(u^\varepsilon)$ to an energy solution of the stochastic Burgers equation. We proceed in two steps. First we establish the tightness of $(u^\varepsilon)$, and in a second step we show that every weak limit is an energy solution. Using the uniqueness of energy solutions, we therefore obtain the convergence of $(u^\varepsilon)$.

**Tightness** Let $(u^\varepsilon)$ solve (2.1) and write $\tilde{F}(x) = F(x) - c_1(F)x$. To prove the tightness of $(u^\varepsilon)$ it suffices to show that for all $\ell \in \mathbb{Z}$ the complex-valued process $(\langle u^\varepsilon, e_{-\ell}\rangle)$ is tight and satisfies a polynomial bound in $\ell$, uniformly in $\varepsilon$. We decompose $\langle u_t^\varepsilon, e_{-\ell}\rangle$ as

$$
\begin{aligned}
\langle u_t^\varepsilon, e_{-\ell}\rangle &= \langle u_0^\varepsilon, e_{-\ell}\rangle + \int_0^t \langle u_s^\varepsilon, \Delta e_{-\ell}\rangle \mathrm{d}s - \int_0^t \langle \varepsilon^{-1}\Pi_0^N \tilde{F}(\varepsilon^{1/2}u_s^\varepsilon), \partial_x e_{-\ell}\rangle \mathrm{d}s \\
&\quad - \int_0^t \langle \Pi_0^N \xi_s, \partial_x e_{-\ell}\rangle \mathrm{d}s \\
&=: \langle u_0^\varepsilon, e_{-\ell}\rangle + \langle S_t^\varepsilon, e_{-\ell}\rangle + \langle A_t^\varepsilon, e_{-\ell}\rangle + \langle M_t^\varepsilon, e_{-\ell}\rangle, \qquad (4.1)
\end{aligned}
$$

where $S^\varepsilon$, $A^\varepsilon$, $M^\varepsilon$ stand for symmetric, antisymmetric and martingale part, respectively, and we show tightness for each term on the right hand side separately. The convergence of $\langle u_t^\varepsilon, e_{-\ell}\rangle$ at a fixed time (in particular $t=0$) follows from the fact that the law of $u_t^\varepsilon$ is that of $\mu^\varepsilon$ for all $t$, and $(\mu^\varepsilon)$ obviously converges to the law of the white noise as $\varepsilon \to 0$. The linear term is tight because

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_s^t \langle u_r^\varepsilon, \Delta e_\ell\rangle \mathrm{d}r\right|^p\right] &\leqslant |t-s|^{p-1} \int_s^t \mathbb{E}[|\langle u_r^\varepsilon, \ell^2 e_\ell\rangle|^p] \mathrm{d}r \\
&\lesssim |t-s|^{p-1} \int_s^t \mathbb{E}[|\langle u_r^\varepsilon, \ell^2 e_\ell\rangle|^2]^{p/2} \mathrm{d}r = |t-s|^p |\ell|^{2p}.
\end{aligned}
$$

For all $\varepsilon > 0$ the martingale term is a mollified space–time white noise, so its convergence is immediate.

Only the nonlinear contribution to the dynamics is nontrivial to control. Here we use the Boltzmann–Gibbs principle stated in Proposition 3.5 to get

$$\mathbb{E}\left[\left|\int_s^t \langle \varepsilon^{-1}\Pi_0^N \tilde{F}(\varepsilon^{1/2}u_r^\varepsilon), \partial_x e_{-\ell}\rangle \mathrm{d}r\right|^2\right] \lesssim |t-s|^{3/2-\kappa}\ell^2 \int_{\mathbb{R}} |F'(x)|^2 \nu(\mathrm{d}x).$$

This bound gives readily tightness in $C([0,T],\mathbb{C})$ and also that any limit point has zero quadratic variation.

Similarly we have for the time reversed process $\hat{u}_t^\varepsilon = u_{T-t}^\varepsilon$

$$\begin{aligned}\langle \hat{u}_t^\varepsilon, e_{-\ell}\rangle &= \langle \hat{u}_0^\varepsilon, e_{-\ell}\rangle + \int_0^t \langle \hat{u}_s^\varepsilon, \Delta e_{-\ell}\rangle \mathrm{d}s + \int_0^t \langle \varepsilon^{-1}\Pi_0^N \tilde{F}(\varepsilon^{1/2}\hat{u}_s^\varepsilon), \partial_x e_{-\ell}\rangle \mathrm{d}s \\ &\quad - \int_0^t \langle \Pi_0^N \hat{\xi}_s, \partial_x e_{-\ell}\rangle \mathrm{d}s \\ &=: \langle \hat{u}_0^\varepsilon, e_{-\ell}\rangle + \langle \hat{S}_t^\varepsilon, e_{-\ell}\rangle + \langle \hat{A}_t^\varepsilon, e_{-\ell}\rangle + \langle \hat{M}_t^\varepsilon, e_{-\ell}\rangle, \end{aligned} \tag{4.2}$$

and the same arguments as before show that each term on the right hand side is tight in $C([0,T],\mathbb{C})$, satisfies a uniform polynomial bound, and that any limit point of $\langle \hat{A}^\varepsilon, e_{-\ell}\rangle$ has zero quadratic variation. Since we have suitable moment bounds for each term, we actually get the joint tightness:

**Lemma 4.1.** *Consider the decomposition (4.1), (4.2). Then the tuple*

$$(u_0^\varepsilon, \hat{u}_0^\varepsilon, S^\varepsilon, \hat{S}^\varepsilon, A^\varepsilon, \hat{A}^\varepsilon, M^\varepsilon, \hat{M}^\varepsilon)$$

*is tight in* $(\mathscr{S}')^2 \times C([0,T],\mathscr{S}')^6$. *For every weak limit* $(u_0, \hat{u}_0, S, \hat{S}, \mathcal{A}, \hat{\mathcal{A}}, M, \hat{M})$ *and any* $\varphi \in C^\infty(\mathbb{T})$ *the processes* $\langle \mathcal{A}, \varphi\rangle$ *and* $\langle \hat{\mathcal{A}}, \varphi\rangle$ *have zero quadratic variation and satisfy* $\hat{\mathcal{A}}_t = -(\mathcal{A}_T - \mathcal{A}_{T-t})$. *Moreover,* $u_t = u_0 + S_t + A_t + M_t$, $t \in [0,T]$, *is for every fixed time a spatial white noise.*

**Convergence** Recall the definition of energy solutions to the stochastic Burgers equation [GJ13b]:

**Definition 4.2.** (Controlled process)

Denote with $\mathcal{Q}$ the space of continuous stochastic processes $(u, \mathcal{A})$ on $[0,T]$ with values in $\mathscr{S}'$ such that

i) the law of $u_t$ is the white noise $\mu$ for all $t \in [0,T]$;

ii) For any test function $\varphi \in \mathscr{S}$ the process $t \mapsto \langle \mathcal{A}_t, \varphi\rangle$ is almost surely of zero quadratic variation, $\langle \mathcal{A}_0, \varphi\rangle = 0$ and the pair $(\langle u, \varphi\rangle, \langle \mathcal{A}, \varphi\rangle)$ satisfies the equation

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle u_s, \Delta\varphi\rangle \mathrm{d}s + \langle \mathcal{A}_t, \varphi\rangle - \langle M_t, \partial_x\varphi\rangle \tag{4.3}$$

where $(\langle M_t, \partial_x\varphi\rangle)_{0\leqslant t\leqslant T}$ is a martingale with respect to the filtration generated by $(u, \mathcal{A})$ with quadratic variation $[\langle M_\cdot, \partial_x\varphi\rangle]_t = 2t\|\partial_x\varphi\|^2_{L^2(\mathbb{T})}$;

iii) the reversed processes $\hat{u}_t = u_{T-t}$, $\hat{\mathcal{A}}_t = -(\hat{\mathcal{A}}_T - \mathcal{A}_{T-t})$ satisfy the same equation with respect to their own filtration (the backward filtration of $(u, \mathcal{A})$).

The pair $(u, \mathcal{A})$ is called *controlled* since for $\mathcal{A} \equiv 0$ we simply get the Ornstein–Uhlenbeck process, so in general $u$ is a "zero quadratic variation perturbation" of that process. Using the Itô trick, it is not hard to show that for controlled processes the Burgers nonlinearity is well defined:

**Lemma 4.3** ([GJ13b], Lemma 1)**.**
*Assume that $(u, \mathcal{A}) \in \mathcal{Q}$ and set for $M \in \mathbb{N}$*

$$\langle \mathcal{B}_t^M, \varphi \rangle = -\int_0^t \langle (\Pi_0^M u_s)^2, \partial_x \varphi \rangle \mathrm{d}s.$$

*Then $(\mathcal{B}_t^M)$ converges in probability in $C([0,T], \mathscr{S}')$ and we denote the limit by*

$$\langle \int_0^t \partial_x u_s^2 \mathrm{d}s, \varphi \rangle.$$

A controlled process $(u, \mathcal{A})$ is a solution to the stochastic Burgers equation

$$\partial_t u = \Delta u + c \partial_x u^2 + \partial_x \xi$$

if $\mathcal{A} = c \int_0^t \partial_x u_s^2 \mathrm{d}s$. According to [GP15b, Theorem 2], there is a unique energy solution. The following theorem thus implies our main result, Theorem 1.1.

**Theorem 4.4.** *Let $(u, \mathcal{A})$ be as in Lemma 4.1. Then $(u, \mathcal{A}) \in \mathcal{Q}$ and $u$ is the unique energy solution to*

$$\partial_t u = \Delta u + c_2(F) \partial_x u^2 + \partial_x \xi.$$

*Proof.* The tuple $(u_0^\varepsilon, \hat{u}_0^\varepsilon, S^\varepsilon, \hat{S}^\varepsilon, A^\varepsilon, \hat{A}^\varepsilon, M^\varepsilon, \hat{M}^\varepsilon)$ converges along a subsequence $\varepsilon_n \to 0$, but to simplify notation we still denote this subsequence by the same symbol. Since $(u_0^\varepsilon, S^\varepsilon, A^\varepsilon, M^\varepsilon)$ converges jointly and for every fixed $\varepsilon$ the process $u^\varepsilon$ solves (1.2), we get for $\varphi \in C^\infty(\mathbb{T})$

$$\langle u_t, \varphi \rangle = \langle u_0, \varphi \rangle + \langle S_t, \varphi \rangle + \langle \mathcal{A}_t, \varphi \rangle + \langle M_t, \varphi \rangle,$$

and since $\langle S_t^\varepsilon, \varphi \rangle = \int_0^t \langle u_s^\varepsilon, \Delta \varphi \rangle \mathrm{d}s$ also $\langle S_t, \varphi \rangle = \int_0^t \langle u_s, \Delta \varphi \rangle \mathrm{d}s$. The same argument works for the backward process, so that $(u, \mathcal{A}) \in \mathcal{Q}$. It remains to show that $\mathcal{A} = c_2(F) \partial_x u^2$, which follows from the Boltzmann–Gibbs principle, Proposition 3.5. For all $\varepsilon > 0$ and $M \leqslant N/2 = \pi/(2\varepsilon)$,

$$\mathbb{E}\left[\left|\int_s^t \langle A_r^\varepsilon - c_2(F) \partial_x (\Pi_0^M u_r^\varepsilon)^2, e_{-\ell} \rangle \mathrm{d}r\right|^2\right] \lesssim |t-s| \ell^2 (M^{-1} + \varepsilon \log^2 N) \int_{\mathbb{R}} |F'(x)|^2 \nu(\mathrm{d}x),$$

so by Fatou's lemma

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t \langle \mathcal{A} - c_2(F) \partial_x (\Pi_0^M u_r)^2, e_{-\ell} \rangle \mathrm{d}r\right|^2\right] &\leqslant \liminf_{\varepsilon \to 0} \mathbb{E}\left[\left|\int_s^t \langle A_r^\varepsilon - c_2(F) \partial_x (\Pi_0^M u_r^\varepsilon)^2, e_{-\ell} \rangle \mathrm{d}r\right|^2\right] \\ &\lesssim |t-s| \ell^2 M^{-1} \int_{\mathbb{R}} |F'(x)|^2 \nu(\mathrm{d}x).\end{aligned}$$

It now suffices to send $M \to \infty$. □

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