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Extrema of logarithmically correlated random field: Known and new results

By

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Abstract

In this paper, we have two main purposes. Firstly, we describe known results on extrema of logarithmically correlated random fields such as the branching Brownian motion, the branching random walk, the two-dimensional discrete Gaussian free field, and cover times of the two-dimensional torus and the binary tree. Secondly, we announce a new result on extrema of local times for the simple random walk on the b-ary tree.

§ 1. Introduction

Logarithmically correlated random fields have been studied extensively since they appear naturally in relation to a number of different mathematical problems, some of which are mentioned below. There are several properties which are common to all such fields, and which are different from those of independent and identically distributed random variables. We will focus our attention to the branching Brownian motion, the branching random walk, the two-dimensional discrete Gaussian free field, and cover times for the two-dimensional torus and the binary tree. These models have fruitful connections with many subjects: partial differential equations (the Fisher-KPP equation) [21], spin glasses [19, 20], random multiplicative cascade measures [36], the theory of fixed points of smoothing transforms [33], the two-dimensional Liouville quantum gravity [32, 14].

We recently studied extrema of local times for the simple random walk on the b-ary tree [1]. Thanks to the Dynkin isomorphism [34, 35], the local times are closely related

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to the branching random walk, and exploring explicit connections between the two is the main motivation of the study.

In Section 2, we describe known results on logarithmically correlated random models. In Section 3, we announce a new result on local times for the simple random walk on the b-ary tree.

We give some notation which we use throughout the paper. Given functions f and g, we will write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. For random variables X and Y, "$X \overset{\text{law}}{=} Y$" means that the law of X is the same as that of Y. We will write δ_p to denote the Dirac measure at a point p. Fix a complete, separable metric space $\mathcal{X}$. Let $M_+(\mathcal{X})$ be the set of all non-negative Radon measures on $\mathcal{X}$ topologized with the vague topology. Since $M_+(\mathcal{X})$ is metrizable as a complete, separable metric space, we can consider convergence in law of random elements of $M_+(\mathcal{X})$. Given a sequence of random measures $(\mu_n)_{n\geq 1}$ and a random element μ in $M_+(\mathcal{X})$, we will write "$\mu_n \xrightarrow{\text{law}} \mu$ in $M_+(\mathcal{X})$" as $n \to \infty$ if for any continuous, non-negative function f on $\mathcal{X}$ with compact support, $\int_{\mathcal{X}} f d\mu_n$ weakly converges to $\int_{\mathcal{X}} f d\mu$ as $n \to \infty$. Given a random element ν of $M_+(\mathcal{X})$, we will write $\mathrm{PPP}(\nu)$ to denote a point process on $\mathcal{X}$ which, conditioned on ν, is the Poisson point process on $\mathcal{X}$ with intensity measure ν (that is, $\mathrm{PPP}(\nu)$ is a Cox process).

§ 2. Logarithmically correlated random fields

In this section, we describe known results on some logarithmically correlated random fields. We refer to [37] for a general description of this topic.

§ 2.1. Branching Brownian motion

In this subsection, we collect known results on extrema of the branching Brownian motion (BBM). We refer to [17] for an overview of BBM. BBM is a probabilistic model describing the growth of population of particles moving in a space. This model is defined as follows: At time 0, a particle starts at the origin and behaves like a standard Brownian motion on $\mathbb{R}$ up to time T which is distributed as an exponential distribution with mean 1 and independent of the Brownian motion. Let a be the position of the particle at time T. At time T, the particle splits into two particles. Independently, each of the particles starts at a and performs a standard Brownian motion on $\mathbb{R}$ up to an exponential time of parameter 1 and splits into two particles. Repeating this procedure, we obtain a BBM. Let $x_1(t), \dots, x_{n(t)}(t)$ be positions of the particles at time t. One of the striking features of the BBM is that the maximum of the BBM is related to a reaction-diffusion equation called the Fisher equation or the Kolmogorov-Petrovsky-Piscounov (KPP) equation: McKean [41] proved that the law of the maximum of the

BBM $u(t,x) := \mathbb{P}\left(\max_{1\le i\le n(t)} x_i(t) \le x\right)$ is the solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u, \quad u(0,x) = 1_{\{x\ge 0\}}.$$

Lalley and Sellke [38] showed that there exists a positive constant α_{bbm} such that for all $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}\left(\max_{1\le i\le n(t)} x_i(t) \le m_t^{\mathrm{bbm}} + x\right) = \mathbb{E}\left[\exp\left\{-\alpha_{\mathrm{bbm}} Z_{\mathrm{bbm}} e^{-\sqrt{2}x}\right\}\right], \tag{2.1}$$

where

$$m_t^{\mathrm{bbm}} := \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t,$$

and Z_{bbm} is the limit of the so-called derivative martingale:

$$Z_{\mathrm{bbm}} := \lim_{t\to\infty}\sum_{i=1}^{n(t)}\left(\sqrt{2}t - x_i(t)\right)e^{-\sqrt{2}\left(\sqrt{2}t - x_i(t)\right)}. \tag{2.2}$$

(It is known that the limit on the right of (2.2) exists almost surely.) Thus, the limit law of the maximum of BBM is a Gumbel distribution with a random mean given by the derivative martingale. The derivative martingale is essentially determined by the positions of the particles at time $o(t)$ and conditionally on these positions, particles behave independently, which explains the appearance of the Gumbel distribution. Let $\omega_{\mathrm{bbm}}(x)$ be the right of (2.1). It is known that

$$1 - \omega_{\mathrm{bbm}}(x) \sim \alpha_{\mathrm{bbm}} x e^{-\sqrt{2}x} \quad \text{as } x \to \infty. \tag{2.3}$$

In [4, 5], Arguin, Bovier, and Kistler showed that the first branching time for any two extremal particles at time t is either less than r or larger than $t-r$ with probability tending to 1 as $t\to\infty$ and then $r\to\infty$, and that a point process encoding local extrema of positions of the particles converges to a Cox process. The convergence of the full extremal process of the BBM has been settled by [3, 6]: they proved that there exist independent and identically distributed point processes $\mathcal{D}^{(i)} := \sum_{j\ge 1}\delta_{d_j^{(i)}}, i \in \mathbb{N}$ such that as $t\to\infty$,

$$\sum_{i=1}^{n(t)}\delta_{x_i(t)-m_t^{bbm}} \xrightarrow{\mathrm{law}} \sum_{i,j}\delta_{x_i+d_j^{(i)}} \quad \text{in } M_+(\mathbb{R}), \tag{2.4}$$

where $x_i, i\in\mathbb{N}$ are atoms of the Cox process $\mathrm{PPP}\left(\alpha_{\mathrm{bbm}} Z_{\mathrm{bbm}}\sqrt{2}e^{-\sqrt{2}x}dx\right)$. We emphasize that the authors of [3, 6] gave an explicit description of the decoration point process $\mathcal{D}^{(1)}$. Recently, adding information about the "location" of particles in the genealogy tree, Bovier and Hartung [18] extended the convergence of the extremal process. We stress that the above results have been established for BBM with more general branching behaviour.

§ 2.2. Branching random walk

In this subsection, we describe known results on extrema of the branching random walk (BRW) and random multiplicative cascade measures. We refer to [42] for the former and to [9] and references therein for the latter. To simplify the description, we only consider a very simple version of the BRW: Consider the binary tree T with a distinguished point ρ called root. This is a graph with the vertex set $T = \cup_{k\geq 0} T_k$ and the edge set $E(T) := \{\{\rho, v\} : v \in T_1\} \cup \cup_{k\geq 1}\{\{v, vu\} : v \in T_k, u \in T_1\}$, where

$$T_0 := \{\rho\},\ T_k := \{(\overline{v}_1, \dots, \overline{v}_k) : \overline{v}_i \in \{0,1\}, 1 \leq i \leq k\},\ k \geq 1,$$

and for $v = (\overline{v}_1, \dots, \overline{v}_k) \in T_k$ and $u = (\overline{u}_1, \dots, \overline{u}_\ell) \in T_\ell$, we set

$$vu := (\overline{v}_1, \dots, \overline{v}_k, \overline{u}_1, \dots, \overline{u}_\ell).$$

Let $(Y_e)_{e\in E(T)}$ be a family of independent standard normal random variables. To each $v \in T_k, k \geq 1$, we assign

$$h_v^T := \sum_{i=1}^k Y_{e_i^v},$$

where $e_1^v, \dots, e_k^v$ are the edges on the unique path from ρ to v. We will call $(h_v^T)_{v\in T}$ a BRW on T.

2.2.1. Extrema of the branching random walk

Due to [7, 2, 25], it is known that there exists a positive constant α_{brw} such that for all $\lambda \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{v\in T_n} \frac{1}{\sqrt{2}} h_v^T \leq m_n^{\mathrm{brw}} + \lambda\right) = \mathbb{E}\left[\exp\left\{-\alpha_{\mathrm{brw}} Z_{\mathrm{brw}} e^{-2\sqrt{\log 2}\lambda}\right\}\right], \tag{2.5}$$

where

$$m_n^{\mathrm{brw}} := \sqrt{\log 2}n - \frac{3}{4\sqrt{\log 2}} \log n, \tag{2.6}$$

and Z_{brw} is the limit of the derivative martingale:

$$Z_{\mathrm{brw}} := \lim_{n\to\infty} \sum_{v\in T_n} \left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h_v^T\right) e^{-2\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h_v^T\right)}. \tag{2.7}$$

(It is known that the limit on the right of (2.7) exists almost surely.) Let $\omega_{\mathrm{brw}}(\lambda)$ be the right of (2.5). It is known (see [2, Proposition 1.3] or [25, Proposition 3.1]) that

$$1 - \omega_{\mathrm{brw}}(\lambda) \sim \alpha_{\mathrm{brw}} \lambda e^{-2\sqrt{\log 2}\lambda} \quad \text{as } \lambda \to \infty. \tag{2.8}$$

We emphasize that Bachmann [7], Aïdékon [2], and Bramson, Ding, and Zeitouni [25] obtained results corresponding to (2.5) for more general branching random walks. Madaule [40] established the convergence of the full extremal process of the BRW similar to (2.4).

2.2.2. Random multiplicative cascade measures

Random multiplicative cascade measures (we will call cascade measures, for short) were introduced by Mandelbrot in the study of turbulence. According to a positive parameter β corresponding to the inverse temperature, cascade measures fall into three types: subcritical ($\beta < 1$), critical ($\beta = 1$), and supercritical ($\beta > 1$). To define the cascade measures, we give some notation. Let

$$\sigma(v) := \sum_{i=1}^{n} \frac{\overline{v}_i}{2^i}, \quad v = (\overline{v}_1, \ldots, \overline{v}_n) \in T_n \tag{2.9}$$

be a mapping of the leaves to a dyadic subset of $[0,1]$. For each $n \in \mathbb{N}$, we define random measures $Z_{n,\beta}$ on $[0,1]$ as follows:

$$Z_{n,\beta}(dx) := \begin{cases} 2^n \{m(\beta)\}^{-n} e^{-2\beta\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_{v(x)}\right)} dx, & \text{if } \beta < 1, \\ 2^n \left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_{v(x)}\right) e^{-2\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_{v(x)}\right)} dx, & \text{if } \beta = 1, \\ 2^n n^{\frac{3}{2}\beta} e^{-2\beta\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_{v(x)}\right)} dx, & \text{if } \beta > 1, \end{cases}$$

where dx is the Lebesgue measure on $[0,1]$, $m(\beta) := \mathbb{E}\left[\sum_{v \in T_1} e^{-2\beta\sqrt{\log 2}\left(\sqrt{\log 2} - \frac{1}{\sqrt{2}} h^T_v\right)}\right]$, and for each $x \in [0,1]$, $v(x)$ is the vertex in T_n such that $x \in [\sigma(v(x)), \sigma(v(x)) + 2^{-n}]$. Due to [36, 8] (see also [10, 9]), as $n \to \infty$,

$$Z_{n,\beta} \xrightarrow{\text{law}} \begin{cases} Z_{\infty,\beta} \text{ a.s.}, & \text{if } \beta \le 1, \\ Z_{\infty,\beta} \text{ in } M_+([0,1]), & \text{if } \beta > 1, \end{cases} \tag{2.10}$$

where given random finite Borel measures $\mu_n, n \ge 1$ and μ on $[0,1]$, we write "$\mu_n \xrightarrow{\text{law}} \mu$ a.s." as $n \to \infty$ if for any bounded, continuous function f on $[0,1]$, $\lim_{n\to\infty} \int_{[0,1]} f d\mu_n = \int_{[0,1]} f d\mu$ almost surely. The limit measures $Z_{\infty,\beta}$ are what we call cascade measures.

In the $\beta \le 1$ case, $Z_{\infty,\beta}$ is non-atomic almost surely [10, 9], and satisfies the following: for all $n \in \mathbb{N}$,

$$\left(Z_{\infty,\beta}\left([\sigma(v), \sigma(v) + 2^{-n}]\right)\right)_{v \in T_n} \overset{\text{law}}{=} \begin{cases} \left(\frac{e^{-2\beta\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_v\right)}}{m(\beta)^n} D^{(v)}_{\infty,\beta}\right)_{v \in T_n}, & \text{if } \beta < 1, \\ \left(e^{-2\sqrt{\log 2}\left(\sqrt{\log 2}n - \frac{1}{\sqrt{2}} h^T_v\right)} D^{(v)}_{\infty,1}\right)_{v \in T_n}, & \text{if } \beta = 1, \end{cases}$$

where $D^{(v)}_{\infty,\beta}, v \in T_n$ are independent copies of $Z_{\infty,\beta}([0,1])$. In particular, we have

$$Z_{\infty,\beta}([0,1]) \overset{\text{law}}{=} \begin{cases} \sum_{v\in T_1} \frac{e^{-2\beta\sqrt{\log 2}\left(\sqrt{\log 2}-\frac{1}{\sqrt{2}}h_v^T\right)}}{m(\beta)} D^{(v)}_{\infty,\beta}, & \text{if } \beta < 1, \\ \sum_{v\in T_1} e^{-2\sqrt{\log 2}\left(\sqrt{\log 2}-\frac{1}{\sqrt{2}}h_v^T\right)} D^{(v)}_{\infty,1}, & \text{if } \beta = 1. \end{cases} \tag{2.11}$$

Due to, for example, [12, Theorem 3], $Z_{\infty,\beta}([0,1])$ is the unique solution of the distributional equation (2.11) (called the fixed point of the smoothing transform) up to a multiplicative constant in the space of non-trivial finite non-negative random variables. Tails of $Z_{\infty,\beta}([0,1])$ are well studied. See, for example, [39, 26].

In the $\beta > 1$ case, Barral, Rhodes, and Vargas [8] showed that there exists a positive constant $c(\beta)$ such that

$$Z_{\infty,\beta} \overset{\text{law}}{=} c(\beta) T_{\frac{1}{\beta}}(Z_{\infty,1}),$$

where $T_{\frac{1}{\beta}}$ is a subordinator with the Laplace transform

$$\mathbb{E}\left[e^{-\lambda T_{\frac{1}{\beta}}(t)}\right] = e^{-t\lambda^{\frac{1}{\beta}}}, \quad t \geq 0,\ \lambda \geq 0,$$

and $T_{\frac{1}{\beta}}(Z_{\infty,1})$ is a random Borel measure on $[0,1]$ which satisfies

$$T_{\frac{1}{\beta}}(Z_{\infty,1})((a,b]) = T_{\frac{1}{\beta}}(Z_{\infty,1}([0,b])) - T_{\frac{1}{\beta}}(Z_{\infty,1}([0,a])), \quad 0 \leq a < b \leq 1,$$

where $T_{\frac{1}{\beta}}$ is independent of $Z_{\infty,1}$. In particular, $Z_{n,\beta}$ is atomic almost surely if $\beta > 1$.

One of the remarkable features of cascade measures is the so-called KPZ formula proved by Benjamini and Schramm [10] in the $\beta < 1$ case and Barral et al.[9] in the $\beta = 1$ case: Fix $\beta \leq 1$ and a deterministic, nonempty Borel set $K \subset [0,1]$. Let ξ_0 (ξ, respectively) be the Hausdorff dimension of K with respect to the Euclidean metric (with respect to the random metric d_β defined by $d_\beta(x,y) := Z_{\infty,\beta}([x,y]),\ 0 \leq x \leq y \leq 1$, respectively). Then, the following holds almost surely:

$$\xi_0 - \xi = \beta^2 \xi(1-\xi).$$

We note that the above results hold in more general settings.

§ 2.3. Two-dimensional discrete Gaussian free field

In this subsection, we collect known results on extrema of the two-dimensional discrete Gaussian free field (DGFF). We refer to [42] for an overview of this model. Set $V_n := [0,n]^2 \cap \mathbb{Z}^2$. Let ∂V_n be the inner vertex boundary of V_n. The two-dimensional DGFF is a family of centered Gaussian random variables $h^{V_n} = \{h^{V_n}_x : x \in V_n\}$ with covariance

$$\mathbb{E}\left[h^{V_n}_x h^{V_n}_y\right] = E_x\left[\sum_{i=0}^{H_{\partial V_n}-1} 1_{\{S_i = y\}}\right],$$

where $S = (S_i, i \geq 0, P_x, x \in V_n)$ is the discrete-time simple random walk on V_n and H_A is the hitting time of a subset $A \subset V_n$ by S. Thanks to the so-called Gibbs-Markov property of h^{V_n}, the DGFF can be approximated by a BRW. This elegant idea goes back to [15], and was revisited by [16, 23]. Bramson, Ding, and Zeitouni [24] proved that there exist a positive constant α_{gff} and an almost surely positive random variable Z_{gff} such that for all $\lambda \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{x\in V_n} h_x^{V_n} \leq m_n^{\mathrm{gff}} + \lambda\right) = \mathbb{E}\left[\exp\left\{-\alpha_{\mathrm{gff}} Z_{\mathrm{gff}} e^{-\sqrt{2\pi}\lambda}\right\}\right], \tag{2.12}$$

where

$$m_n^{\mathrm{gff}} := 2\sqrt{\frac{2}{\pi}}\log n - \frac{3}{4}\sqrt{\frac{2}{\pi}}\log(\log n).$$

Let $\omega_{\mathrm{gff}}(\lambda)$ be the right of (2.12). It is known (see [24, Proposition 4.1]) that

$$1 - \omega_{\mathrm{gff}}(\lambda) \sim \alpha_{\mathrm{gff}} \lambda e^{-\sqrt{2\pi}\lambda} \quad \text{as } \lambda \to \infty. \tag{2.13}$$

Ding and Zeitouni [31] studied the geometry of the set of vertices with values close to m_n^{gff}. Biskup and Louidor [13] considered the point process on $[0,1]^2 \times \mathbb{R}$

$$\eta_{n,r} := \sum_{x\in V_n} \delta_{\left(\frac{x}{n}, h_x^{V_n} - m_n^{\mathrm{gff}}\right)} 1_{\{h_x^{V_n} = \max_{y\in\Lambda_r(x)} h_y^{V_n}\}}, \quad r > 0,\ n \in \mathbb{N},$$

where $\Lambda_r(x) := \{y \in \mathbb{Z}^2 : |y - x|_1 \leq r\}$ and showed that there exists a random Borel measure Z_∞^{gff} on $[0,1]^2$ such that for any $(r_n)_{n\geq 1}$ with $\lim_{n\to\infty} r_n = \infty$ and $\lim_{n\to\infty} r_n/n = 0$, as $n \to \infty$,

$$\eta_{n,r_n} \xrightarrow{\text{law}} \mathrm{PPP}\left(Z_\infty^{\mathrm{gff}}(dx) \otimes e^{-\sqrt{2\pi}h} dh\right) \quad \text{in } M_+\left([0,1]^2 \times \mathbb{R}\right). \tag{2.14}$$

They investigated properties of the limiting measure Z_∞^{gff} and revealed that it is a version of the derivative martingale associated with the continuum Gaussian free field. In [14], they discuss a possible connection between extrema of the DGFF and the so-called critical Liouville quantum gravity.

§ 2.4. Cover times

In this subsection, we describe results on cover times for the planar Brownian motion by Belius and Kistler [11] and the simple random walk on the binary tree due to Ding and Zeitouni [30].

2.4.1. Cover time for the planar Brownian motion

Consider the two-dimensional torus $\mathbb{T} := (\mathbb{R}/\mathbb{Z})^2$. Let $B(x,r)$ be the closed ball of radius r in $\mathbb{T}$ centered at x. For $\varepsilon > 0$, we define the ε-cover time by

$$C_\varepsilon := \sup_{x\in\mathbb{T}} H_{B(x,\varepsilon)},$$

where H_A is the hitting time of a subset $A \subset \mathbb{T}$ by the standard Brownian motion on $\mathbb{T}$. Let P_x be the law of the Brownian motion started at $x \in \mathbb{T}$. Belius and Kistler [11] studied the ε-cover time by analyzing the number of crossings of consecutive annuli by the Brownian motion ("traversal process", in their word) which enabled them to apply techniques developed in the study of the BBM. A key of the relation with the BBM is a hierarchical structure of the traversal process (see [11, Figure 6.1]) and this idea goes back to [27, 28]. Belius and Kistler [11] proved that for all $\delta > 0$ and $x \in \mathbb{T}$, the following holds with P_x-probability tending to 1 as $\varepsilon \to 0$:

$$2\log\varepsilon^{-1} - (1+\delta)\log(\log\varepsilon^{-1}) \leq \frac{C_\varepsilon}{\frac{1}{\pi}\log\varepsilon^{-1}} \leq 2\log\varepsilon^{-1} - (1-\delta)\log(\log\varepsilon^{-1}).$$

Further properties such as the tightness and convergence in law are still unknown.

2.4.2. Cover time for the simple random walk on the binary tree

Recall the notation in Section 2.2. Let $T_{\leq n}$ be the binary tree of depth n given by

$$T_{\leq n} := \cup_{k=0}^{n} T_k.$$

We define the cover time for the simple random walk on $T_{\leq n}$ by

$$\tau_{\mathrm{cov}}^{n} := \max_{v \in T_{\leq n}} H_v,$$

where H_v is the hitting time of v by the simple random walk on $T_{\leq n}$ started at the root. Ding and Zeitouni [30] showed that there exist positive constants $c_1, c_2 \in (0,\infty)$ such that the following holds with probability tending to 1 as $n \to \infty$:

$$\sqrt{2\log 2}\,n - \frac{\log n}{\sqrt{2\log 2}} - c_1(\log(\log n))^8 \leq \sqrt{\frac{\tau_{\mathrm{cov}}^{n}}{|E_n|}} \leq \sqrt{2\log 2}\,n - \frac{\log n}{\sqrt{2\log 2}} + c_2(\log(\log n))^8, \tag{2.15}$$

where $|E_n|$ is the total number of edges in $T_{\leq n}$. A key step of the proof is comparison between densities of local times and Gaussian random variables [30, Lemma 2.7]. Bramson and Zeitouni [22, Theorem 1.2] established a tightness result on the cover time, but more detailed questions are still open.

§ 3. Model and results

In this section, we describe new results, to appear in [1], on extrema of local times for the simple random walk on the b-ary tree. To simplify the description, we only consider the $b = 2$ case.

§ 3.1. Local times for the simple random walk on the binary tree

In this subsection, we give our setting. We will use the notation in Sections 2.2 and 2.4.2. Let $X = (X_t, t \geq 0, P_x, x \in T_{\leq n})$ be the continuous-time simple random walk on $T_{\leq n}$ with exponential holding times of parameter 1. We define the local time by

$$L_t^n(v) := \frac{1}{\deg(v)} \int_0^t 1_{\{X_s = v\}} ds, \quad t > 0, \ v \in T_{\leq n},$$

where $\deg(v)$ is the degree of v, and the inverse local time by

$$\tau(t) := \inf\{s \geq 0 : L_s^n(\rho) > t\}, \quad t > 0.$$

We consider local times $\left(L^n_{\tau(t_n)}(v)\right)_{v \in T_n}$, where we will assume that $(t_n)_{n\geq 1}$ satisfies
(3.1)

$$\text{there exist } \theta \in [0,\infty], c_* > 0 \text{ such that } \lim_{n\to\infty} \frac{\sqrt{t_n}}{n} = \theta \text{ and } t_n \geq c_* n \log n \text{ for all } n \geq 1.$$

Due to (2.15) and [29, Lemma 2.1], it is known that $\tau^n_{\mathrm{cov}}/2^n n^2$ and $\tau(t_n)/2^n t_n$ converge in probability to some deterministic positive constants as $n \to \infty$. Thus, if $(t_n)_{n\geq 1}$ satisfies the assumption (3.1), then

$$\lim_{n\to\infty} \frac{\tau(t_n)}{\tau^n_{\mathrm{cov}}} = \begin{cases} 0 & \text{if } \theta = 0, \\ \text{a positive finite constant} & \text{if } \theta \in (0,\infty), \\ \infty & \text{if } \theta = \infty \end{cases} \quad \text{in probability.} \tag{3.2}$$

Remark 3.1. Roughly speaking, the local time process $L^n_{\tau(t_n)}(v_i), 1 \leq i \leq n$, when viewed as a process indexed by the vertices of a path $\rho = v_0, v_1, \ldots, v_n = v$ from the root to a leaf $v \in T_n$, is a Markov chain (zero-dimensional squared Bessel process). If one considers the collection of the processes indexed by each leaf, one gets a collection of *branching* Markov chains. We study the maximum of this field and the cover time "question" mentioned in Section 2.4.2 is a question about the minimum of this field.

§ 3.2. Results on extrema of the local times

In this subsection, we describe our results on extrema of the local times. The following is an analogue to (2.1), (2.5), and (2.12):

Theorem 3.2. *For all $\lambda \in \mathbb{R}$ and $(t_n)_{n\geq 1}$ which satisfies (3.1),*

$$\lim_{n\to\infty} P_\rho\left(\max_{v\in T_n} \sqrt{L^n_{\tau(t_n)}(v)} \leq \sqrt{t_n} + a_n(t_n) + \lambda\right) = \mathbb{E}\left[e^{-\alpha_{brw}\beta(\theta) Z_{brw} e^{-2\sqrt{\log 2}\lambda}}\right],$$

where α_{brw} and Z_{brw} are the same as the ones in (2.5), $\beta(\theta) := \sqrt{\frac{\theta+1}{\theta+\sqrt{\log 2}}}$, and

$$a_n(t) := \sqrt{\log 2}n - \frac{3}{4\sqrt{\log 2}} \log n - \frac{1}{4\sqrt{\log 2}} \log\left(\frac{\sqrt{t}+n}{\sqrt{t}}\right), \quad t > 0.$$

Comparing Theorem 3.2 with (2.5), when $\theta = \infty$ (that is, when $\tau(t_n)$ is much larger than the cover time due to (3.2)), one can see that the limiting law of the maximum of local times are the same as that of the BRW on T and that $a_n(t_n) = m_n^{\text{brw}}$, where m_n^{brw} is the one in (2.6). This similarity is plausible in view of the Dynkin isomorphism [34, 35] which relates local times with the BRW. However, when θ is finite, we have a difference from the BRW since $\beta(\theta) \neq 1$, $a_n(t_n) \neq m_n^{\text{brw}}$. The term $\beta(\theta)$ comes from the Radon-Nikodym derivative of the law of a zero-dimensional squared Bessel process with respect to that of a one-dimensional squared Bessel process (recall Remark 3.1). Let $\omega_{\text{loc}}(\lambda)$ be the limiting law in Theorem 3.2. We further show that

$$1 - \omega_{\text{loc}}(\lambda) \sim \alpha_{\text{brw}} \beta(\theta) \lambda e^{-2\sqrt{\log 2}\lambda} \quad \text{as } \lambda \to \infty.$$

Thus, the tail of the limiting law ω_{loc} has an asymptotic behavior similar to those of the BBM, the BRW, and the two-dimensional DGFF, cf. (2.3), (2.8), (2.13).

We have a stronger result on extrema of the local times. Recall (2.9). We consider the point process on $[0,1] \times \mathbb{R}$

$$\Xi_{n,t}^{(m)} := \sum_{u \in T_{n-m}} \delta_{\left(\sigma\left(\arg\max_u L^n_{\tau(t)}\right),\ \max_{v \in T_m^u} \sqrt{L^n_{\tau(t)}(v)} - \sqrt{t} - a_n(t)\right)}, \quad t > 0,\ 0 \leq m \leq n,$$

where for each $u \in T_{n-m}$, we set $T_m^u := \{uw : w \in T_m\}$, and $\arg\max_u L^n_{\tau(t)}$ is the maximizer on T_m^u, that is, the vertex $v_* \in T_m^u$ such that $L^n_{\tau(t)}(v_*) = \max_{v \in T_m^u} L^n_{\tau(t)}(v)$. We now state the main result of [1]:

Theorem 3.3. *For all $(r_n)_{n \geq 1}$ with $\lim_{n\to\infty} r_n = \infty$ and $\lim_{n\to\infty} r_n/n = 0$ and $(t_n)_{n\geq 1}$ which satisfies (3.1), as $n \to \infty$,*

$$\Xi_{n,t_n}^{(r_n)} \xrightarrow{law} PPP\left(\alpha_{brw}\beta(\theta) Z_{\infty,1}(dx) \otimes 2\sqrt{\log 2}\, e^{-2\sqrt{\log 2}h} dh\right) \quad \text{in } M_+([0,1] \times \mathbb{R}),$$

where α_{brw} and $\beta(\theta)$ are the same as the ones in Theorem 3.2, and $Z_{\infty,1}$ is the critical random multiplicative cascade measure in (2.10).

Theorem 3.3 is an analogue of (2.14) for the two-dimensional DGFF and [5, Theorem 2] for the BBM. Our results are inspired by [18].

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