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Infinite Dimensional Stochastic Differential Equations for Dyson's Model

By

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Abstract

In this article we consider the infinite-dimensional Stochastic Differential Equation (SDE) corresponding to the bulk limit of Dyson's Brownian Motion (DBM), for all $\beta \geq 1$. We give a pathwise construction of the strong solution and prove the pathwise uniqueness, for an explicit and general class of initial conditions, including the lattice configuration $\{x_i\} = \mathbb{Z}$ and the sine process.

§ 1. Introduction

Here we study the well-posedness of the infinite-dimensional SDE,

$$X_i(t) = X_i(0) + B_i(t) + \beta \int_0^t \phi_i(\mathbf{X}(s))ds, \ i \in \mathbb{Z}, \tag{1.1}$$

where $\mathbf{X}(s) = (\ldots < X_0(s) < X_1(s) < \ldots)$ describes ordered particles on $\mathbb{R}$, $B_i(t)$, $i \in \mathbb{Z}$, denote independent standard Brownian motions, and the interaction $\phi_i(\mathbf{x})$ takes the form

$$\phi_i(\mathbf{x}) := \frac{1}{2} \lim_{k\to\infty} \sum_{j:|j-i|\leq k} \frac{1}{x_i - x_j}, \tag{1.2}$$

with $\beta \geq 1$ measuring its strength. The interest of such SDE arises from random matrix theory. Equation (1.1) represents the bulk limit of DBM, which describes the evolution of the eigenvalues of the symmetric and Hermitian random matrices with independent

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Brownian entries, for $\beta = 1, 2$. The results stated in this article are from the paper [7]. Here we omit some of the more technical proofs and refer to [7] for the details.

The difficulty of establishing the well-posedness of (1.1) lies in the long-range and singular nature of ϕ_i. Indeed, for a particle configuration $\mathbf{x}$ with a roughly uniform density, we have $\sum_{j:j\neq i} \frac{1}{|x_i - x_j|} = \infty$, so the only way (1.2) converges is by canceling two divergent series from $j < i$ and $j > i$. Alternatively, under the framework of [3, 4], the system (1.1) of SDEs formally has the logarithmic potential $-\beta \sum_{i<j} \log |x_i - x_j|$. However, due the logarithmic growth as $|x_i - x_j| \to \infty$, such a potential is still *ill-defined* even under a limiting procedure as in (1.2).

At $\beta = 1, 2, 4$, this challenge has been largely overcome thanks to the integrable structure of DBM. We refer to [2, 5, 6] and the references therein for developments in this direction. Here we attack the problem, for all $\beta \geq 1$, *without* referring to the integrable structure, whereby establishing the strong existence and pathwise uniqueness of (1.1); see Theorem 1.1. As our techniques do not refer to a specific equilibrium measure, Theorem 1.1 holds for an *explicit*, *out-of-equilibrium* configuration space $\mathcal{X}^{\mathrm{rg}}(\alpha, \rho, p)$, which, loosely speaking, consists of particle configurations with a roughly uniform density $\rho^{-1} > 0$. In particular, the space includes the lattice configuration $\{x_i\} = \mathbb{Z}$ and the sine process; see [7, Lemma 8.2].

The approach used here is further adopted to establish certain finite-to-infinite-dimensional convergences of (1.1). As a corollary, it is shown that the determinantal point process constructed in [2] coincides with the unique strong solution given by Theorem 1.1. See [7, Theorem 1.4, Corollary 1.6].

§ 1.1. Definitions and Statement of the Results

We begin by defining the spaces $\mathcal{X}(\alpha, \rho)$ and $\mathcal{X}^{\mathrm{rg}}(\alpha, \rho, p)$. This is done by considering their corresponding *gap configurations*. More explicitly, let $\mathcal{W} := \{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}} : x_i < x_{i+1}, \forall i \in \mathbb{Z}\}$ denote the Weyl chamber (of particle configurations), and let u denote the map into gap configurations:

$$u : \mathcal{W} \longrightarrow (0, \infty)^{\mathbb{L}}, \quad \mathbb{L} := \tfrac{1}{2} + \mathbb{Z}, \qquad u(\mathbf{x}) := (x_{a+1/2} - x_{a-1/2})_{a \in \mathbb{L}}, \tag{1.3}$$

which is made bijective by augmenting the zeroth particle coordinate, as

$$\widetilde{u} : \ \mathcal{W} \xrightarrow{\text{bijective}} \mathbb{R} \times (0, \infty)^{\mathbb{L}}, \quad \widetilde{u}(\mathbf{x}) := (x_0, u(\mathbf{x})). \tag{1.4}$$

For $\alpha \in (0, 1)$ and $\rho > 0$, we consider the following space of gap configurations

$$\mathcal{Y}(\alpha, \rho) := \big\{\mathbf{y} \in (0, \infty)^{\mathbb{L}} : |\mathbf{y}|_{\alpha,\rho} < \infty\big\}, \tag{1.5}$$

$$|\mathbf{y}|_{\alpha,\rho} := \sup_{m \in \mathbb{Z} \setminus \{0\}} \big\{\big|\Sigma_{(0,m)}(\mathbf{y}) - \rho\big| \ |m|^{\alpha}\big\}, \tag{1.6}$$

where $\Sigma_{\mathcal{I}}(\mathbf{y})$ denotes the average over a generic finite set $\mathcal{I}$:

$$\Sigma^p_{\mathcal{I}}(\mathbf{y}) := |\mathcal{I}|^{-1} \sum_{a\in\mathcal{I}} (y_a)^p, \quad \Sigma_{\mathcal{I}}(\mathbf{y}) := \Sigma^1_{\mathcal{I}}(\mathbf{y}), \tag{1.7}$$

with the convention $(i,j] = [j,i)$ (and similarly for (i,j), $[i,j]$, etc) and $\Sigma^p_{\emptyset}(\mathbf{y}) := 0$. We *define* $\mathcal{X}(\alpha,\rho) := u^{-1}(\mathcal{Y}(\alpha,\rho))$. That is, $\mathcal{X}(\alpha,\rho)$ consists of particle configurations whose corresponding gap processes satisfy (1.5). Similarly, for $p>1$, we define $\mathcal{X}^{\mathrm{rg}}(\alpha,\rho,p) := u^{-1}(\mathcal{Y}(\alpha,\rho)\cap\mathcal{R}(p))$, where

$$\mathcal{R}(p) := \Big\{ \mathbf{y} \in (0,\infty)^{\mathbb{L}} : \sup_{m\in\mathbb{Z}} \Sigma^p_{(0,m)}(\mathbf{y}) < \infty \Big\}. \tag{1.8}$$

We proceed to defining the process-valued analogs of $\mathcal{X}(\alpha,\rho)$ and $\mathcal{X}^{\mathrm{rg}}(\alpha,\rho,p)$. To simply notations, we often use $\mathbf{x}$ and $\mathbf{y}$, instead of $\mathbf{x}(\boldsymbol{\cdot})$ and $\mathbf{y}(\boldsymbol{\cdot})$, to denote processes. Let $\mathcal{W}_{\mathcal{T}} := \{\mathbf{x} \in C([0,\infty))^{\mathbb{Z}} : \mathbf{x}(t) \in \mathcal{W}, \forall t \geq 0\}$ denote the process-valued analog of $\mathcal{W}$. By abuse of notation, we let u and $\widetilde{u}$ act on $\mathcal{W}_{\mathcal{T}}$ by $u(\mathbf{x})(t) := u(\mathbf{x}(t))$ and by $\widetilde{u}(\mathbf{x})(t) := \widetilde{u}(\mathbf{x}(t))$. With $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$ and $\mathcal{R}_{\mathcal{T}}(p)$ denoting the analogs of $\mathcal{Y}(\alpha,\rho)$ and $\mathcal{R}(p)$ as follows

$$\mathcal{Y}_{\mathcal{T}}(\alpha,\rho) := \Big\{ \mathbf{y} \in C_+([0,\infty))^{\mathbb{L}} : \sup_{s\in[0,t]} |\mathbf{y}(s)|_{\alpha,\rho} < \infty, \ \forall t \geq 0 \Big\}, \tag{1.9}$$

$$\mathcal{R}_{\mathcal{T}}(p) := \Big\{ \mathbf{y} \in C_+([0,\infty))^{\mathbb{L}} : \sup_{s\in[0,t],m\in\mathbb{Z}} \Sigma^p_{(0,m)}(\mathbf{y}) < \infty, \forall t \geq 0 \Big\}, \tag{1.10}$$

$$\text{where } C_+([0,\infty)) := \{ y \in C([0,\infty)) : y(t) > 0, \forall t \geq 0 \}, \tag{1.11}$$

we define $\mathcal{X}_{\mathcal{T}}(\alpha,\rho) := \widetilde{u}^{-1}(C([0,\infty)) \times \mathcal{Y}_{\mathcal{T}}(\alpha,\rho))$ and $\mathcal{X}^{\mathrm{rg}}_{\mathcal{T}}(\alpha,\rho,p) := \widetilde{u}^{-1}(C([0,\infty)) \times (\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)\cap\mathcal{R}(p)))$.

Recall from [1, Definition 5.2.1, 5.3.2] the notions of strong solutions and pathwise uniqueness of SDE, which are readily generalized to infinite dimensions here. Let $\mathbf{B}(t) := (B_i(t))_{i\in\mathbb{Z}}$ denote the driving Brownian motion, with the canonical filtration $\mathscr{F}^{\mathbf{B}}_t := \sigma(\mathbf{B}(s) : s \in [0,t])$. *Hereafter, we fix $\beta \geq 1$, $\alpha \in (0,1)$, $\rho > 0$ and $p > 1$ unless otherwise stated.* The following is our main result.

Theorem 1.1. *Given any $\mathbf{x}^{in} \in \mathcal{X}(\alpha,\rho)$, there exists an $\mathcal{X}_{\mathcal{T}}(\alpha,\rho)$-valued, $\mathscr{F}^{\mathbf{B}}$-adapted solution $\mathbf{X}$ of (1.1) starting from $\mathbf{x}^{in}$. If, in addition, $\mathbf{x}^{in} \in \mathcal{X}^{rg}(\alpha,\rho,p)$, this solution $\mathbf{X}$ takes value in $\mathcal{X}^{rg}_{\mathcal{T}}(\alpha,\rho,p)$, and is the unique $\mathcal{X}^{rg}_{\mathcal{T}}(\alpha,\rho,p)$-valued solution in the pathwise sense.*

Remark 1.2. For any $\mathbf{x} \in \mathcal{X}_{\mathcal{T}}(\alpha,\rho)$, one easily verifies that $(\lim_{k\to\infty} \sum_{j:|i-j|\leq k} \frac{1}{x_i(t)-x_j(t)})$ converges uniformly in $t \in [0,t']$, for any fixed $i \in \mathbb{Z}$ and $t' < \infty$. Further, the limit $\phi_i(\mathbf{x}(t))$ takes values in $L^\infty_{\mathrm{loc}}([0,\infty))$, so in particular the r.h.s. of (1.1) is well-defined for $\mathcal{X}_{\mathcal{T}}(\alpha,\rho)$-valued processes.

The rest of this paper is outlined as follows. In Section 2 we present a proof of Theorem 1.1, which is detailed in Section 3–6. Among these, Section 3 settles the monotonicity (2.13) and well-posedness of certain finite-dimensional SDE, and Section 4–6 handle the relevant propositions as indicated in their titles.

§ 2. Proof of Theorem 1.1

Throughout this article we use lower-case English and Greek letters such as x, y, α, γ, u to denote deterministic variables or functions, among which i, j, k, ℓ, m, n denote integers, and a, b denote half integers. We use upper-case English letters such as X, Y, I, J to denote random variables, use the calligraphic font (e.g. $\mathcal{A}, \mathcal{I}$) to denote deterministic sets, and use the Fraktur font (e.g. $\mathfrak{A}, \mathfrak{I}$) to denote random sets. We let $c = c(t, k, \dots)$ denote a generic deterministic positive finite constant that depends only on the designated variables.

The first step is to reduce the equation of particles, (1.1), to the equation of the *gaps*. To this end, we consider the interaction of the gaps

$$\eta_a(\mathbf{y}) := \eta_a(u(\mathbf{x})) := \phi_{a+1/2}(\mathbf{x}) - \phi_{a-1/2}(\mathbf{x}), \tag{2.1}$$

$$= \tfrac{1}{y_a} - \psi_a(y_a, \mathbf{y}), \tag{2.2}$$

consisting of the (Bessel-type) repulsion terms $1/y_a$ and the compression terms ψ_a defined as

(2.3)

$$\psi_a : [0, \infty) \times (0, \infty)^{\mathbb{L}} \to [0, \infty), \quad \psi_a(y, \mathbf{z}) := \begin{cases} \dfrac{1}{2} \displaystyle\sum_{i:|i-a|>1} \dfrac{y}{z_{(a,i)}(y + z_{(a,i)})}, & \text{for } y > 0, \\ 0 & \text{, for } y = 0, \end{cases}$$

where $z_{\mathcal{I}} := \sum_{a \in \mathcal{I}} z_a$ and $(a, i) := (i, a)$ (as mentioned before). We have the following equation for $(X_0, \mathbf{Y}) := u(\mathbf{X})$:

$$X_0(t) = X_0(0) + B_0(t) + \beta \int_0^t \phi_0(\mathbf{Y}(s)) ds, \tag{2.4}$$

$$Y_a(t) = Y_a(0) + W_a(t) + \beta \int_0^t \eta_a(\mathbf{Y}(s)) ds, \quad a \in \mathbb{L}, \tag{2.5}$$

where $\mathbf{W}(t) := u(\mathbf{B}(t))$, and, by abuse of notation,

$$\phi_0(\mathbf{y}) := \phi_0(\widetilde{u}^{-1}(0, \mathbf{y})) = \sum_{i=1}^{\infty} \left(\frac{1}{2y_{(-i,0)}} - \frac{1}{2y_{(0,i)}} \right).$$

Clearly, (1.1) is equivalent to (2.4)–(2.5) through the bijection $\widetilde{u}$, and one easily obtains the following

Proposition 2.1.

(a) If $\mathbf{Y}$ *is an* $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*-valued solution of* (2.5)*, defining* $X_0 \in C([0,\infty))$ *by* (2.4)*, we have that* $\widetilde{u}^{-1}(X_0,\mathbf{Y})$ *is a* $\mathcal{X}_{\mathcal{T}}(\alpha,\rho)$*-valued solution of* (1.1)*. Further, if* $\mathbf{Y}$ *is* $(\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)\cap\mathcal{R}_{\mathcal{T}}(p))$*-valued, then* $\widetilde{u}^{-1}(X_0,\mathbf{Y})$ *is* $\mathcal{X}^{rg}_{\mathcal{T}}(\alpha,\rho,p)$*-valued; if* $\mathbf{Y}$ *is* $\mathscr{F}^{\mathbf{W}}$*-adapted, then* $\widetilde{u}^{-1}(X_0,\mathbf{Y})$ *is* $\mathscr{F}^{\mathbf{B}}$*-adapted.*

(b) Conversely, if $\mathbf{X}$ *is an* $\mathcal{X}_{\mathcal{T}}(\alpha,\rho)$*-valued solution of* (1.1)*, then* $u(\mathbf{X})$ *is a* $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*-valued solution of* (2.5)*. Further, if* $\mathbf{X}$ *is* $\mathcal{X}^{rg}_{\mathcal{T}}(\alpha,\rho,p)$*-valued, then* $u(\mathbf{X})$ *is* $(\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)\cap\mathcal{R}_{\mathcal{T}}(p))$*-valued; if* $\mathbf{X}$ *is* $\mathscr{F}^{\mathbf{B}}$*-adapted, then so is* $u(\mathbf{X})$*.*

With this proposition, it now suffices to prove

Proposition 2.2. *For any given* $\mathbf{y}^{in}\in\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*, there exists a* $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*-valued,* $\mathscr{F}^{\mathbf{W}}$*-adapted solution* $\mathbf{Y}$ *of* (2.5)*. Moreover, if* $\mathbf{y}^{in}\in\mathcal{R}(p)$*, then* $\mathbf{Y}\in\mathcal{R}_{\mathcal{T}}(p)$*, and* $\mathbf{Y}$ *is the unique* $(\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)\cap\mathcal{R}_{\mathcal{T}}(p))$*-valued solution in the pathwise sense.*

We establish Proposition 2.2 in two steps: the existence, as in Proposition 2.3, and the uniqueness, as in Proposition 2.4. Defining the partial orders

$$\mathbf{y}\le\mathbf{y}'\in[0,\infty]^{\mathbb{L}} \text{ if and only if } y_a\le y'_a,\ \forall a\in\mathbb{L}, \tag{2.6}$$

$$\mathbf{y}(\cdot)\le\mathbf{y}'(\cdot) \text{ if and only if } \mathbf{y}(t)\le\mathbf{y}'(t),\ \forall t\ge 0, \tag{2.7}$$

we call $\mathbf{Y}$ the greatest $\mathcal{S}$-valued solution of (2.5) if, for any $\mathcal{S}$-valued weak solution $\mathbf{Y}'$ defined on a common probability space with $\mathbf{Y}'(0)\le\mathbf{y}^{\mathrm{in}}$, we have $\mathbf{Y}'(\cdot)\le\mathbf{Y}(\cdot)$ almost surely.

Proposition 2.3 (existence)**.** *For any* $\mathbf{y}^{in}\in\mathcal{Y}(\alpha,\rho)$*, there exists a* $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*-valued,* $\mathscr{F}^{\mathbf{W}}$*-adapted solution* $\mathbf{Y}$ *of* (2.5) *starting form* $\mathbf{y}^{in}$*, which is the greatest* $\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$*-valued solution. Further, if* $\mathbf{y}^{in}\in\mathcal{R}(p)$*, then* $\mathbf{Y}\in\mathcal{R}_{\mathcal{T}}(p)$*.*

Proposition 2.4 (uniqueness)**.** *Let* $\mathbf{Y}^{up}$ *and* $\mathbf{Y}^{lw}$ *be* $(\mathcal{Y}_{\mathcal{T}}(\alpha,\rho)\cap\mathcal{R}_{\mathcal{T}}(p))$*-valued weak solutions of* (2.5) *defined on a common probability space, starting from a common initial condition* $\mathbf{y}^{in}$*. If* $\mathbf{Y}^{lw}(\cdot)\le\mathbf{Y}^{up}(\cdot)$ *almost surely, we have* $\mathbf{Y}^{lw}(\cdot)=\mathbf{Y}^{up}(\cdot)$ *almost surely.*

Indeed, Proposition 2.2 follows by combining Proposition 2.3–2.4. In particular, the pathwise uniqueness follows by applying Proposition 2.4 for $\mathbf{Y}^{\mathrm{up}}=\mathbf{Y}$ and $\mathbf{Y}^{\mathrm{lw}}=\mathbf{Y}'$, where $\mathbf{Y}$ is the greatest solution as in Proposition 2.3, and $\mathbf{Y}'$ is an arbitrary weak solution with $\mathbf{Y}'(0)=\mathbf{Y}(0)$.

Proposition 2.3 is established in two steps: by first considering the special case $\mathbf{y}^{\mathrm{in}}\in[\gamma,\infty)^{\mathbb{L}}$, $\gamma>0$, and then the general case $\mathbf{y}^{\mathrm{in}}\in\mathcal{Y}(\alpha,\rho)$. For the former case, we construct the solution of (2.5) by the following iteration scheme,

$$Y_a^{(0)}(t) = y_a^{\mathrm{in}} + W_a(t) + \beta \int_0^t \frac{1}{Y_a^{(0)}(s)} ds, \quad a \in \mathbb{L}, \tag{2.8a}$$

$$\begin{aligned} Y_a^{(n)}(t) &= y_a^{\mathrm{in}} + W_a(t) \\ &+ \beta \int_0^t \left(\frac{1}{Y_a^{(n)}(s)} - \psi_a(Y_a^{(n)}(s), \mathbf{Y}^{(n-1)}(s)) \right) ds, \quad a \in \mathbb{L},\ n \in \mathbb{Z}_{>0}. \end{aligned} \tag{2.8b}$$

That is, we let $Y_a^{(0)}$ be the Bessel process (driven by W_a), and for $n \geq 1$, we let $Y_a^{(n)}$ be the solution of the following *one-dimensional* SDE

$$Y(t) = Y(0) + W_a(t) + \beta \int_0^t \left(\frac{1}{Y(s)} - \psi_a(Y(s), \mathbf{Z}(s)) \right) ds, \tag{2.9}$$

for *given* $\mathbf{Z} = \mathbf{Y}^{(n-1)}$. Letting

$$\underline{\mathcal{Y}}(\gamma) := \Big\{ \mathbf{y} \in (0,\infty)^{\mathbb{L}} : \liminf_{|m|\to\infty} \Sigma_{(0,m)}(\mathbf{y}) \geq \gamma \Big\}, \tag{2.10}$$

$$\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma) := \Big\{ \mathbf{y}(\bullet) \in C_+([0,\infty))^{\mathbb{L}} : \liminf_{|m|\to\infty} \inf_{s\in[0,t]} \Sigma_{(0,m)}(\mathbf{y}(s)) \geq \gamma, \forall t \geq 0 \Big\}, \tag{2.11}$$

$$\underline{\mathcal{Y}} := \cup_{\gamma>0} \underline{\mathcal{Y}}(\gamma), \quad \underline{\mathcal{Y}}_{\mathcal{T}} := \cup_{\gamma>0} \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma), \tag{2.12}$$

in Section 5 we prove

Proposition 2.5. *Fix* $\gamma > 0$. *For any given* $\mathbf{y}^{in} \in [\gamma,\infty)^{\mathbb{L}}$, *there exists a* $\mathcal{Y}_{\mathcal{T}}(\gamma)$-*valued,* $\mathscr{F}^{\mathbf{W}}$-*adapted sequence* $\{\mathbf{Y}^{(n)}\}_{n\in\mathbb{Z}_{\geq 0}}$ *satisfying* (2.8). *Further, such a sequence is* decreasing, *i.e.*

$$\mathbf{Y}^{(0)}(\bullet) \geq \mathbf{Y}^{(1)}(\bullet) \geq \mathbf{Y}^{(2)}(\bullet) \geq \ldots, \tag{2.13}$$

almost surely. Defining the $\mathscr{F}^{\mathbf{W}}$-*adapted process* $Y_a^{(\infty)}(t) := \lim_{n\to\infty} Y_a^{(n)}(t)$, *we have that* $\mathbf{Y}^{(\infty)}$ *is the greatest* $\underline{\mathcal{Y}}_{\mathcal{T}}$-*valued solution of* (2.5). *If* $\mathbf{y}^{in} \in \mathcal{R}(p)$, *then* $\mathbf{Y}^{(\infty)} \in \mathcal{R}_{\mathcal{T}}(p)$.

For the general case $\mathbf{y}^{\mathrm{in}} \in \mathcal{Y}(\alpha,\rho)$, we consider the truncated initial condition $(\mathbf{y}^{\mathrm{in}} \vee \boldsymbol{\gamma}) := (y_a^{\mathrm{in}} \vee \gamma)_{a\in\mathbb{L}}$, $\gamma > 0$, and let $\mathbf{Y}^{\vee\gamma}$ be the $\underline{\mathcal{Y}}_{\mathcal{T}}$-valued solution starting from $(\mathbf{y}^{\mathrm{in}} \vee \boldsymbol{\gamma})$ given by Proposition 2.5. As $\mathbf{Y}^{\vee\gamma}$ is the greatest solution, for any decreasing $\{\gamma_1 > \gamma_2 > \ldots\}$, the sequence $\{\mathbf{Y}^{\vee\gamma_k}\}_k$ is decreasing. In Section 6, we prove

Proposition 2.6. *Let* $\mathbf{y}^{in} \in \mathcal{Y}(\alpha,\rho)$ *and* $\mathbf{Y}^{\vee\gamma} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ *be as in the preceding. Fix an arbitrary decreasing sequence* $1 \geq \gamma_1 > \gamma_2 > \ldots \to 0$. *Defining the* $\mathscr{F}^{\mathbf{W}}$-*adapted process* $Y_a(t) := \lim_{n\to\infty} Y_a^{\vee\gamma_n}(t)$, *we have that* $\mathbf{Y}$ *is the greatest* $\mathcal{X}_{\mathcal{T}}(\alpha,\rho)$-*valued solution of* (2.5).

As for Proposition 2.4, letting

$$E_{(i_1,i_2)}(t) := \sum_{a\in(i_1,i_2)} (Y^{\mathrm{up}}_a(t) - Y^{\mathrm{lw}}_a(t)), \tag{2.14}$$

with $\mathbf{Y}^{\mathrm{lw}}(\cdot) \le \mathbf{Y}^{\mathrm{up}}(\cdot)$, we have $\left|Y^{\mathrm{up}}_a(t) - Y^{\mathrm{lw}}_a(t)\right| \le E_{(i_1,i_2)]}(t) \le E_{(-\infty,\infty)}(t)$, $\forall a \in (i_1, i_2)$. With this, in Section 4 we prove

Proposition 2.7. *For any* $t > 0$, $\sup_{s\in[0,t]} E_{(-\infty,\infty)}(s) = 0$, *almost surely,*

from which Proposition 2.4 follows immediately.

§ 2.1. Outline of the Proof of Proposition 2.5–2.7

The key step of proving Proposition 2.5 is to establish the monotonicity (2.13) of $\{\mathbf{Y}^{(n)}\}_n$. This, as well as many other monotonicity results (e.g. that $\mathbf{Y}^{(\infty)}$ as in Proposition 2.5 is the greatest solution), are consequences of the following simple observation:

$$\psi_a(y, \mathbf{z}) \le \psi_a(y', \mathbf{z}), \text{ if } y \le y', \tag{2.15}$$

$$\psi_a(y, \mathbf{z}) \ge \psi_a(y, \mathbf{z}'), \text{ if } \mathbf{z} \le \mathbf{z}', \tag{2.16}$$

which is clear from (2.3). Equipped with the monotonicity of $\{\mathbf{Y}^{(n)}\}_n$, the next step is to take the limit $n \to \infty$ in (2.8b), and show that the r.h.s. converges to the appropriate limit. The major challenge here is to control $\int_0^t \frac{\beta}{Y^{(\infty)}_a(s)} ds$, which we achieve by showing

$$\inf_{s\in[0,t]} Y^{(\infty)}_a(s) > 0, \qquad \text{almost surely, for all } t \ge 0.$$

The main step of proving Proposition 2.6 is to show $\mathbf{Y} \in \mathcal{Y}_{\mathcal{T}}(\alpha, \rho)$. To this end, in Section 6, we partition $\mathbb{L}$ into certain mesoscopic intervals $\mathcal{A}_{b,k}$, $b \in \mathbb{L}$, (see (6.5)) and simultaneously estimate $\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}^{\vee\gamma_n}(s))$, $\forall n \in \mathbb{Z}_{>0}, b \in \mathbb{L}$. This yields that the mesoscopic average of $\mathbf{Y}(s)$ over $\mathcal{A}_{b,k}$ is at least $\frac{\rho}{2}$ (see Proposition 6.3). Using this as a 'seed', we estimate the global density $\Sigma_{(0,m)}(\mathbf{Y}(s))$, $|m| \gg 1$ to obtain $\mathbf{Y} \in \mathcal{Y}_{\mathcal{T}}(\alpha, \rho)$.

To prove Proposition 2.7, we use the following readily verified identity (see [7, Section 4] for a proof)

$$E_{(i_1,i_2)}(t) = E_{(i_1,i_2)}(t') + \beta \int_{t'}^{t} \big(L^+_{i_2}(s) - L^-_{i_2}(s) - L^+_{i_1}(s) + L^-_{i_1}(s)\big) ds, \ \forall t \ge t', \tag{2.17}$$

that describes $E_{(i_1,i_2)}(t)$ in terms of certain boundary interactions $L^{\pm}_i(s)$, defined as

$$L^{\pm}_i(s) := \frac{1}{2} \sum_{j\in(i,\pm\infty)} \frac{Y^{\mathrm{up}}_{(i,j)}(s) - Y^{\mathrm{lw}}_{(i,j)}(s)}{Y^{\mathrm{up}}_{(i,j)}(s) Y^{\mathrm{lw}}_{(i,j)}(s)}. \tag{2.18}$$

With $E_{(i_1,i_2)}(0) = 0$, equipped with (2.17), in Section 4, we prove Proposition 2.7 by showing $\int_0^t L^{\pm}_{i_k}(s)ds \to 0$, along some suitable subsequence $i_k \to \pm\infty$.

§ 3. Comparison and Monotonicity

A basic tool we use to leverage (2.15)–(2.16) into the monotonicity of $\{\mathbf{Y}^{(n)}\}_n$ is the following comparison principle for *deterministic*, one-dimensional integral equations. Let

$$\mathbf{y} \leq_{[t',t'']} \mathbf{y}' \text{ if and only if } \mathbf{y}(t) \leq \mathbf{y}(t), \forall t \in [t', t'']$$

denote the restriction of (2.7) onto $[t', t'']$.

Lemma 3.1. *Fixing $t' \leq t'' \in [0,\infty)$, we let $w \in C([t',t''])$, and let $f^{up}, f^{lw} \in C((0,\infty)\times[t',t''])$ be locally Lipschitz functions in the first variable. That is, given any compact $\mathcal{K} \subset (0,\infty)$, there exists $c(\mathcal{K}) > 0$ such that*

$$|f^{up}(y,t) - f^{up}(y',t)|,\ \left|f^{lw}(y,t) - f^{lw}(y',t)\right| \leq c(\mathcal{K})|y - y'|,$$

for all $y, y' \in \mathcal{K}$ and $t \in [t', t'']$. If $y^{up}, y^{lw} \in C_+([t',t''])$ solve the follows integral equations

$$y^{up}(t) = y^{up}(t') + (w(t) - w(t')) + \int_{t'}^{t} f^{up}(y^{up}(s), s)ds, \quad \forall\ t \in [t', t''], \tag{3.1}$$

$$y^{lw}(t) = y^{lw}(t') + (w(t) - w(t')) + \int_{t'}^{t} f^{lw}(y^{lw}(s), s)ds, \quad \forall\ t \in [t', t''], \tag{3.2}$$

and if $f^{lw}(y,\bullet) \leq_{[t',t'']} f^{up}(y,\bullet)$, $\forall y \in (0,\infty)$, and $y^{lw}(t') \leq y^{up}(t')$, then

$$y^{lw} \leq_{[t',t'']} y^{up}.$$

With $f^{\mathrm{up}}(\bullet, s)$ and $f^{\mathrm{lw}}(\bullet, s)$ being locally Lipschitz, Lemma 3.1 is proven by standard ODE arguments using Gronwall's inequality. We omit the proof.

Equipped with Lemma 3.1, we establish a backward lower-semicontinuouity for a generic process of the form (3.5). To this end, we consider $\mathbf{Q}^{t_1} := (Q^{t_1}_a)_{a\in\mathbb{L}} \in (C([t_1,\infty)) \cap C_+(t_1,\infty))^{\mathbb{L}}$,

$$Q^{t_1}_a(t) = W_a(t) - W_a(t_1) + \int_{t_1}^{t} \frac{\beta}{Q^{t_1}_a(s)} ds, \quad t \geq t_1, \tag{3.3}$$

the Bessel process starting from 0 at t_1, and let $Q^{t_1,t_2}_a := \sup_{t\in[t_1,t_2]} Q^{t_1}_a(t)$. Indeed, for $\mathbb{L}_1 := \frac{1}{2} + 2\mathbb{L}$ and $\mathbb{L}_2 := 1 + \mathbb{L}_1$, $\{Q^{t_1}_a(\bullet)\}_{a\in\mathbb{L}_i}$, $i = 1, 2$, are i.i.d. collections of processes. Hence, by the Law of Large Numbers, we have

$$\lim_{|m|\to\infty} \Sigma^p_{(0,m)}\big(\mathbf{Q}^{t_1,t_2}\big) = \mathbb{E}\big((Q^{t_1,t_2}_{1/2})^p\big) := q(t_2 - t_1, p) < \infty. \tag{3.4}$$

Hereafter, for generic processes $Y(\bullet)$ and $\mathbf{Y}(\bullet)$, we adopt the notations

$$\overline{Y}(t',t'') := \sup_{s\in[t',t'']} Y(s), \qquad \underline{Y}(t',t'') := \inf_{s\in[t',t'']} Y(s),$$

$\overline{\mathbf{Y}}(t',t'') := (\overline{Y}_a(t',t''))_{a\in\mathbb{L}}$ and $\underline{\mathbf{Y}}(t',t'') := (\underline{Y}_a(t',t''))_{a\in\mathbb{L}}$.

Lemma 3.2. *Let $a \in \mathbb{L}$, $Y^* \in C_+([0,\infty))$, $F \in L^1_{loc}([0,\infty))$, $\{\mathscr{G}_t\}_{t\geq 0}$ be a filtration such that Y^*, F and $\mathbf{W}$ are $\mathscr{G}$-adapted and that $\mathbf{W}$ is a Brownian motion with respect to $\mathscr{G}$. If $F \geq 0$ and if Y^* solves the equation*

$$Y^*(t) = Y^*(0) + W_a(t) + \beta \int_0^t \left(\frac{1}{Y(s)} - F(s)\right) ds, \quad a \in \mathbb{L}, \tag{3.5}$$

then, for all $t' \leq t'' \in [0,\infty)$, we have

$$Y^*(t'') - \underline{Y}^*(t',t'') = \sup_{s\in[t',t'']} (Y^*(t'') - Y^*(s)) \leq Q_a^{t'}(t''), \tag{3.6}$$

$$\sup_{s<t\in[t',t'']} (Y^*(t) - Y^*(s)) \leq Q_a^{t',t''} := \sup_{t\in[t',t'']} Q_a^{t'}(t), \tag{3.7}$$

almost surely.

Proof. To the end of showing (3.6), fixing $s_1 \in (t',t'')$, we consider the process $Y^{s_1} \in C_+([s_1,\infty))$ defined as

$$Y^{s_1}(t) = Y^*(s_1) + W_a(t) - W_a(s_1) + \beta \int_{s_1}^t \frac{1}{Y^{s_1}(s)} ds, \quad t \geq s_1, \tag{3.8}$$

which is a Bessel process starting from $Y^*(s_1)$ at time s_1. With Y^* and Y^{s_1} satisfying (3.5) and (3.8), applying Lemma 3.1 (for $[t',t''] = [s_1,t'']$, $y^{\mathrm{up}} = Y^{s_1}$, $y^{\mathrm{lw}} = Y^*$, $f^{\mathrm{up}}(y,s) = \beta/y$ and $f^{\mathrm{lw}}(y,s) = \beta(1/y - F(s))$), we obtain $Y^*(\bullet) \leq_{[s_1,t'']} Y^{s_1}(\bullet)$, and therefore, with $Y^*(s_1) = Y^{s_1}(s_1)$,

$$Y^*(t'') - Y^*(s_1) \leq Y^{s_1}(t'') - Y^{s_1}(s_1). \tag{3.9}$$

We next compare Y^{s_1} and $Q_a^{s_1}$. They solve the same equation, (3.3) and (3.8), with different initial conditions $Y^{s_1}(s_1) > 0 = Q_a^{s_1}(s_1)$. Hence, applying Lemma 3.1 for $(t',t'') = (s_1+\varepsilon,t'')$, $\varepsilon > 0$ (so that $Y^{s_1}, Q_a^{s_1} \in C_+([s_1+\varepsilon,t'']))$, conditioned on $\{Y^{s_1}(s_1+\varepsilon) \geq Q_a^{s_1}(s_1+\varepsilon)\}$, and then sending $\varepsilon \to 0$, we obtain $Q_a^{s_1} \leq_{[s_1,t'']} Y^{s_1}$ almost surely, thereby $\int_{s_1}^{t''} \frac{\beta}{Y^{s_1}(s)} ds \leq \int_{s_1}^{t''} \frac{\beta}{Q_a^{s_1}(s)} ds$. Plugging this in (3.3) and (3.8), we obtain

$$Y^{s_1}(t'') - Y^{s_1}(s_1) \leq Q_a^{s_1}(t'') - Q_a^{s_1}(s_1) = Q_a^{s_1}(t''). \tag{3.10}$$

Next, as $Q_a^{s_1}$ and $Q_a^{t'}$ solve the same equation on $[s_1,t'']$ with the initial conditions $Q_a^{s_1}(s_1) = 0 < Q_a^{t'}(s_1)$, by the preceding comparison argument we obtain $Q_a^{s_1}(t'') \leq$

$Q_a^{t'}(t'')$. Combining this with (3.9)–(3.10), we arrive at $Y^*(t'') - Y^*(s_1) \le Q_a^{t'}(t'')$. As this holds almost surely for each $s_1 \in (t', t'')$, taking the infimum over $s_1 \in (t', t'') \cap \mathbb{Q}$, using the continuity of $Y^*(\bullet)$, we conclude (3.6).

As for (3.7), taking the supremum over $t'' \in [t', \widetilde{t}''] \cap \mathbb{Q}$ in (3.6), using the continuity of $Y^*(\bullet)$, we obtain

$$\sup_{t'' \in [t', \widetilde{t}'']} \Big(Y^*(t'') - \inf_{s \in [t', \widetilde{t}'']} Y^*(s) \Big) = \sup_{s < t \in [t', \widetilde{t}'']} (Y^*(t) - Y^*(s)) \le \sup_{t'' \in [t', \widetilde{t}'']} Q_a^{t'}(t'') = Q_a^{t', \widetilde{t}''}.$$

□

In the following we state the well-posedness and comparison results of certain finite-dimensional SDE. We omit the proofs. They are modifications of standard techniques based on Lemma 3.1 and can be found in [7, Section 3].

Lemma 3.3. *Let $t' \ge 0$ and $F : [0, \infty) \times [t', \infty) \to \mathbb{R}$ be random, such that*

$s \mapsto F(y, s)$ is $C([t', \infty), \mathbb{R})$-valued and $\mathscr{F}^{\mathbf{W}}$-adapted for all $y \in [0, \infty)$,

$y \mapsto F(y, s)$ is Lipschitz, uniformly over $(y, s) \in [0, \infty) \times [t', t]$, for all $t \ge t'$,

$F(0, t) = 0$, for all $t \ge t'$.

Given any $(0, \infty)$-valued, $\mathscr{F}^{\mathbf{W}}_{t'}$-measurable Y^{in}, the equation

$$Y(t) = Y^{in} + (W_a(t) - W_a(t')) + \int_{t'}^{t} \Big(\frac{\beta}{Y(s)} + F(Y(s), s) \Big) ds \tag{3.11}$$

has a $C_+([t', \infty))$-valued, $\mathscr{F}^{\mathbf{W}}$-adapted solution starting from Y^{in} at t', which is the unique $C_+([t', \infty))$-valued solution in the pathwise sense.

Next we turn to the well-posedness and comparison principle of the equation (3.16) as follows, which is a finite-dimensional version of (2.5) with external forces. For $\mathcal{A} \subset \mathbb{R}$, we let $\psi_a^{\mathcal{A}}(y, \mathbf{z})$ and $\eta_a^{\mathcal{A}}(\mathbf{y})$ denote the restriction of $\psi_a(y, \mathbf{z})$ and $\eta_a(\mathbf{y})$ onto $[0, \infty) \times (0, \infty)^{\mathcal{A} \cap \mathbb{L}}$,

$$\psi_a^{\mathcal{A}}(y, \mathbf{z}) := \frac{1}{2} \sum_{i \in \mathcal{A}, |i-a|>1} \frac{y}{z_{(a,i)}(y + z_{(a,i)})}, \tag{3.12}$$

$$\eta_a^{\mathcal{A}}(\mathbf{y}) := \tfrac{1}{y_a} - \psi_a^{\mathcal{A}}(y_a, \mathbf{y}), \tag{3.13}$$

which indeed satisfy the following analog of (2.15)–(2.16),

$$\psi_a^{\mathcal{A}}(y, \mathbf{z}) \le \psi_a^{\mathcal{A}}(y', \mathbf{z}), \text{ for } y \le y', \tag{3.14}$$

$$\psi_a^{\mathcal{A}}(y, \mathbf{z}) \ge \psi_a^{\mathcal{A}}(y, \mathbf{z}'), \text{ for } \mathbf{z} \le \mathbf{z}'. \tag{3.15}$$

By abuse of notation, we let u and $\widetilde{u}$, defined as in (1.3)–(1.4), act on the space $\mathcal{W}^{[i_1,i_2]}$, whereby $\widetilde{u} : \mathcal{W}^{[i_1,i_2]} \to (0,\infty) \times (0,\infty)^{(i_1,i_2)\cap\mathbb{L}}$ is also a bijection. For $\mathcal{I} \subset \mathbb{L}$, we let

$$\mathbf{y} \leq^{\mathcal{I}} \mathbf{y}' \text{ if and only if } y_a \leq y'_a, \ \forall a \in \mathcal{I},$$
$$\mathbf{y} \leq^{\mathcal{I}}_{[t',t'']} \mathbf{y}' \text{ if and only if } y_a(t) \leq y'_a(t), \ \forall a \in \mathcal{I}, \ t \in [t',t'']$$

denote the restriction of (2.6)–(2.7) onto $\mathcal{I}$ and $[t',t'']$.

Lemma 3.4. *Let $i_1 \leq i_2 \in \mathbb{Z}$, $\mathcal{I} := (i_1,i_2) \cap \mathbb{L}$, $t' \geq 0$, $\mathbf{Z}^* \in C_+([t',\infty))^{\mathcal{I}}$ be $\mathscr{F}^{\mathbf{W}}$-adapted. For any $\mathscr{F}^{\mathbf{W}}_{t'}$-measurable $\mathbf{Y}^{in} \in (0,\infty)^{\mathcal{I}}$, the equation*

$$\begin{aligned} Y_a(t) &= Y^{in}_a(t') + (W_a(t) - W_a(t')) \\ &\quad + \beta \int_{t'}^{t} \big(\eta^{\mathcal{I}}_a(\mathbf{Y}(s)) + Y_a(s) Z^*_a(s)\big) ds, \quad t \geq t', \ a \in \mathcal{I} \end{aligned} \tag{3.16}$$

has a $C_+([t',\infty))^{\mathcal{I}}$-valued, $\mathscr{F}^{\mathbf{W}}$-adapted solution starting from $\mathbf{Y}^{in}$, which is the unique $C_+([t',\infty))^{\mathcal{I}}$-valued solution in the pathwise sense.

Lemma 3.5. *Fixing $t' < t'' \in [0,\infty)$, $I_1 < I_2 \in \mathbb{Z}$ (possibly random), we let $\mathfrak{I} := (I_1,I_2) \cap \mathbb{L}$, $\mathbf{Z}^{up}$ and $\mathbf{Z}^{lw} \in C([t',t''])^{\mathfrak{I}}$, and $\mathbf{Y}^{up}$, $\mathbf{Y}^{lw}$ be the $C_+([t',t''])^{\mathfrak{I}}$-valued solutions of* (3.16) *with the respective external forces $\mathbf{Z}^{up}$ and $\mathbf{Z}^{lw}$, i.e.*

$$\begin{aligned} Y^{up}_a(t) &= Y^{up}_a(t') + (W_a(t) - W_a(t')) \\ &\quad + \beta \int_{t'}^{t} \big(\eta^{\mathfrak{I}}_a(\mathbf{Y}^{up}(s)) + Y^{up}_a(s) Z^{up}_a(s)\big) ds, \quad t \in [t',t''], \ a \in \mathfrak{I}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} Y^{lw}_a(t) &= Y^{lw}_a(t') + (W_a(t) - W_a(t')) \\ &\quad + \beta \int_{t'}^{t} \big(\eta^{\mathfrak{I}}_a(\mathbf{Y}^{lw}(s)) + Y^{lw}_a(s) Z^{lw}_a(s)\big) ds, \quad t \in [t',t''], \ a \in \mathfrak{I}. \end{aligned} \tag{3.18}$$

If $\mathbf{Z}^{lw} \leq^{\mathfrak{I}}_{[t',t'']} \mathbf{Z}^{up}$, and $\mathbf{Y}^{lw}(t') \leq^{\mathfrak{I}} \mathbf{Y}^{up}(t')$, then

$$\mathbf{Y}^{lw} \leq^{\mathfrak{I}}_{[t',t'']} \mathbf{Y}^{up}, \qquad \textit{almost surely.} \tag{3.19}$$

Remark 3.6. Note that here we do *not* assume $W_a(\cdot)$ conditioned on (I_1,I_2) is a Brownian motion or even a martingale.

§ 4. Uniqueness, Proof of Proposition 2.7

Fix $\mathbf{y}^{\mathrm{in}} \in \mathcal{Y}((\alpha,\rho) \cap \mathcal{R}(p))$ and $\mathbf{Y}^{\mathrm{lw}}(\cdot) \leq \mathbf{Y}^{\mathrm{up}}(\cdot) \in (\mathcal{Y}_{\mathcal{T}}(\alpha,\rho) \cap \mathcal{R}_{\mathcal{T}}(p))$ as in Proposition 2.7. Recall that $E_{\mathcal{I}}(s) := \sum_{a\in\mathcal{I}} (Y^{\mathrm{up}}_a(s) - Y^{\mathrm{lw}}_a(s))$ and that $L^{\pm}_i(t)$ is defined as in (2.18).

With (2.17), proving Proposition 2.7 amounts to controlling $\int_0^t L_i^{\pm}(s)ds$ for suitable i. Fixing $m \in \mathbb{Z}_{>0}$ (which will be sent to ∞ later), for $i \in [\pm m, \pm 2m]$ we decompose $L_i^{\pm}(s)$ into the long-range interaction

$$\widetilde{L}^{\pm}_{i,m}(s) := \frac{1}{2} \sum_{j\in(\pm 3m,\pm\infty)} \frac{Y^{\mathrm{up}}_{(i,j)}(s) - Y^{\mathrm{lw}}_{(i,j)}(s)}{Y^{\mathrm{up}}_{(i,j)}(s) Y^{\mathrm{lw}}_{(i,j)}(s)}, \tag{4.1}$$

and the short-range interaction

$$L^{\pm}_{i,m}(s) := \frac{1}{2} \sum_{j\in(i,\pm 3m]} \frac{Y^{\mathrm{up}}_{(i,j)}(s) - Y^{\mathrm{lw}}_{(i,j)}(s)}{Y^{\mathrm{up}}_{(i,j)}(s) Y^{\mathrm{lw}}_{(i,j)}(s)}. \tag{4.2}$$

Our goal is to establish certain bounds the long- and short-range interactions, as stated in Lemma 4.1 in the following. It is not hard to show (see [7, proof of Proposition 2.7]) that Lemma 4.1 implies Proposition 2.7. We omit the proof of this implication and prove only Lemma 4.1.

Lemma 4.1.

(a) For any $t \geq 0$, we have

$$\widetilde{L}^{\pm}_{m} := \sup_{s\in[0,t]} \sup_{i\in[\pm m,\pm 2m]} \left\{ \widetilde{L}^{\pm}_{i,m}(s) \right\} \to 0, \qquad \textit{almost surely.} \tag{4.3}$$

(b) For any $t' < t'' \in [0,\infty)$ such that $q(t''-t',1) < \frac{\rho}{2}$, we have

$$\lim_{m\to\infty} \inf_{i\in[\pm m,\pm 2m]} \int_{t'}^{t''} L^{\pm}_{i,m}(s)ds = 0, \qquad \textit{almost surely.} \tag{4.4}$$

proof of Part (a). With $\mathbf{Y}^{\mathrm{up}}, \mathbf{Y}^{\mathrm{lw}} \in \mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$, we have

$$\sup_{s\in[0,t],j\in\mathbb{Z}} \left\{ \Sigma_{(0,j)}(\mathbf{Y}^{\mathrm{up}}(s) - \mathbf{Y}^{\mathrm{lw}}(s))|j|^{\alpha} \right\} =: N < \infty. \tag{4.5}$$

The desired result follows by using (4.5) and $\mathbf{Y}^{\mathrm{lw}} \in \mathcal{Y}_{\mathcal{T}}(\alpha,\rho) \subset \underline{\mathcal{Y}}_{\mathcal{T}}(\rho)$ to control the numerator and denominator of the expression (4.1), respectively. $\square$

Before proceeding to proving Lemma 4.1(b), we remark that, unlike (4.3), it is impossible to obtain a bound on the short-range interaction $L^{\pm}_{i,m}(s)$ that is uniform in $i \in [\pm m, \pm 2m]$. This is so because $\mathbf{Y}^{\mathrm{lw}} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\alpha,\rho)$ does *not* imply $\mathbf{Y}^{\mathrm{lw}}_{(i,j)}(s) > |j-i|/c$ for *small* $|j-i|$, and similarly (4.5) does not imply $Y^{\mathrm{up}}_{(i,j)}(s) - Y^{\mathrm{lw}}_{(i,j)}(s) \leq c|i-j|^{1-\alpha}$ for small $|j-i|$.

To prove Lemma 4.1, we first construct certain 'good' index set $\mathfrak{G}^{\pm}_{m,k} \neq \emptyset$, such that $L^{\pm}_{i,m}(s)$ is controlled for $i \in \mathfrak{G}^{\pm}_{m,k}$. To construct $\mathfrak{G}^{\pm}_{m,k}$, letting $p' \in (1,\infty)$ denote the Hölder conjugates of p, i.e. $1/p + 1/p' = 1$, for fixed $s \in [0, \infty)$ and $m \in \mathbb{Z}_{>0}$, we consider the set

$$\mathfrak{A}_m(s) := \left\{ a \in \mathbb{L} : |Y_a^{\mathrm{up}}(s) - Y_a^{\mathrm{lw}}(s)| \geq |m|^{-\alpha/(3p')} \right\} \tag{4.6}$$

of 'bad' indices, where the corresponding terms in the numerator of (4.2) may be large at time s. For $\mathcal{A} \subset \mathbb{L}$, $i, i' \in \mathbb{Z}$, let

$$g_{(i,i')}(\mathcal{A}) := \sup_{j \in (i,i']} \frac{|(i,j) \cap \mathcal{A}|}{|j-i|}$$

denote the maximal cumulative occurrence frequency of $\mathcal{A}$ when searching to the right (when $i' > i$) or left (when $i' < i$) over the interval (i, i'), starting from i. Consider the set

$$\mathfrak{I}_m^{\pm}(s) := \left\{ i \in [\pm m, \pm 3m] \cap \mathbb{Z} : g_{(i,\pm 3m)}(\mathfrak{A}_m(s)) > m^{-\alpha/3} \right\} \tag{4.7}$$

of 'bad' indices, where the occurrence of $\mathfrak{A}_m(s)$ may be large over the interval $(\pm m, \pm 3m)$. The sets $\mathfrak{A}_m(s)$ and $\mathfrak{I}_m^{\pm}(s)$ are constructed for a fixed s. We now fix $t' < t''$ as in Lemma 4.1(b), let $\mathcal{T}_k := \{t' + \frac{(t''-t')\ell}{k}\}_{\ell=1}^{k}$, and consider the set

$$\mathfrak{N}_{m,k}^{\pm} := \left\{ i \in \mathbb{Z} : \frac{1}{k} \sum_{s \in \mathcal{T}_k} \mathbf{1}\{ i \in \mathfrak{I}_m^{\pm}(s) \} \leq m^{-\alpha/(3p')} \right\}, \tag{4.8}$$

consisting of 'good' indices i such that $\{\mathfrak{I}_m^{\pm}(s) \ni i\}$ occurs rarely alone the discrete samples $s \in \mathcal{T}_k$ of time. The set $\mathfrak{N}_{m,k}^{\pm}$ is constructed for bounding the numerator in the expression (4.2). As for the denominator, we consider

$$h_{(i,j)}(\mathbf{y}) := \inf_{i' \in (i,j] \cap \mathbb{Z}} \Sigma_{(i,i')}(\mathbf{y}), \tag{4.9}$$

and define

$$\mathfrak{G}_{m,k}^{\pm} := \{ i \in [\pm m, \pm 2m] \cap \mathbb{Z} : i \in \mathfrak{N}_{m,k}^{\pm}, h_{(i,\pm 3m)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t', t'')) \geq \tfrac{\rho}{3} \}. \tag{4.10}$$

Let $L_{i,m}^{\pm,k} := \frac{t''-t'}{k} \sum_{s \in \mathcal{T}_k} L_{i,m}^{\pm}(s)$ denote the k-th discrete approximation of $\int_{t'}^{t''} L_{i,m}^{\pm}(s) ds$. Having constructed $\mathfrak{G}_{m,k}^{\pm}$, we proceed to establishing a bound on $L_{i,m}^{\pm,k}$ for $i \in \mathfrak{G}_{m,k}^{\pm}$. Let $P := \sup_{m \in \mathbb{Z}} \Sigma_{(0,m)}^{p}(\overline{\mathbf{Y}}^{\mathrm{up}}(t', t''))$, which is almost surely finite as $\mathbf{Y}^{\mathrm{up}} \in \mathcal{R}_{\mathcal{T}}(p)$.

Lemma 4.2. *For all $m, k \in \mathbb{Z}_{>0}$, there exists $c = c(t''-t', \rho, p) < \infty$ such that*

$$L_{i_*,m}^{\pm,k} \leq (1 + P^{1/p}) \frac{c \log m}{m^{\alpha/(3p')}}, \quad \forall i_* \in \mathfrak{G}_{m,k}^{\pm}. \tag{4.11}$$

Proof. Fixing $k, m \in \mathbb{Z}_{>0}$ and $i_* \in \mathfrak{N}^{\pm}_{m,k}$, we let $c < \infty$ denote a generic constant depending only on $t''-t', \rho, p$. We begin by bounding the expression $L^{\pm}_{i_*,m}(s)$, for $s \in \mathcal{T}_k$, to which end we consider separately the two cases *i)* $\{\mathfrak{I}^{\pm}_m(s) \not\ni i_*\}$; and *ii)* $\{\mathfrak{I}^{\pm}_m(s) \ni i_*\}$.

i) In (4.2), using $\mathbf{Y}^{\mathrm{lw}}(s) \leq \mathbf{Y}^{\mathrm{up}}(s)$ and $h_{(i_*,\pm 3m)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')) \geq \frac{\rho}{3}$, we bound the denominator from below by $(|j-i_*|\rho/3)^2$. As for the numerator, we divide $Y^{\mathrm{up}}_{(i_*,j)}(s) - Y^{\mathrm{lw}}_{(i_*,j)}(s) = \sum_{a\in(i,j)}(Y^{\mathrm{up}}_a(s) - Y^{\mathrm{lw}}_a(s))$ into two sums subject to the constraints $\{a \notin \mathfrak{A}_m(s)\}$ and $\{a \in \mathfrak{A}_m(s)\}$. The former sum, by (4.6), is bounded by $m^{-\alpha/(3p')}|j-i_*|$. As for the latter, we apply the Hölder inequality to obtain

$$\begin{aligned}
&\sum_{a\in(i_*,j)} \big(|Y^{\mathrm{up}}_a(s) - Y^{\mathrm{lw}}_a(s)|\big)(\mathbf{1}\{a \in \mathfrak{A}_m(s)\}) \\
&\quad\leq \Big(\sum_{a\in(i_*,j)} Y^{\mathrm{up}}_a(s)^p\Big)^{1/p}\Big(\sum_{a\in(i_*,j)} \mathbf{1}\{a \in \mathfrak{A}_m(s)\}\Big)^{1/p'} \\
&\quad\leq (|j-i_*|P)^{1/p}\big(g_{(i_*,\pm 3m)}(\mathfrak{A}_m(s))\ |j-i_*|\big)^{1/p'}.
\end{aligned}$$

With $i_* \notin \mathfrak{I}^{\pm}_m(s)$, we have $g_{(i_*,\pm 3m)}(\mathfrak{A}_m(s)) \leq m^{-\alpha/3}$, so the last expression is further bounded by $cP^{1/p}m^{-\alpha/(3p')}|j-i_*|$. Combining the preceding bounds yields

$$\begin{aligned}
L^{\pm}_{i_*,m}(s) &\leq c(1+P^{1/p})m^{-\alpha/(3p')} \sum_{j\in(i_*,\pm 3m]} \frac{|j-i_*|}{|j-i_*|^2} \\
&\leq c(1+P^{1/p})m^{-\alpha/(3p')}\log m.
\end{aligned} \tag{4.12}$$

ii) Using $\mathbf{Y}^{\mathrm{up}}(s) \geq \mathbf{Y}^{\mathrm{lw}}(s)$ in (4.2), we bound the j-th term by $1/Y^{\mathrm{lw}}_{(i_*,j)}(s)$. This, with $h_{(i_*,\pm 3m)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')) \geq \frac{\rho}{3}$, is further bounded by $(|j-i_*|\rho/3)^{-1}$. Consequently,

$$L^{\pm}_{i_*,m}(s) \leq c\log(m+1). \tag{4.13}$$

Although the bound (4.13) is undesired ($\to \infty$ as $m \to \infty$), the corresponding case $\{s \in \mathcal{T}_k : \mathfrak{I}^{\pm}_m(s) \ni i_*\}$ occurs at low frequency $\leq m^{-\alpha/(3p')}$. Hence

$$\frac{t''-t'}{k}\sum_{s\in\mathcal{T}_k} \mathbf{1}\{\mathfrak{I}^{\pm}_m(s) \ni i_*\}L^{\pm}_{i_*,m}(s) \leq c\log(m+1)m^{-\alpha/(3p')}. \tag{4.14}$$

Averaging (4.12) over $s \in \mathcal{T}_k$ for $\{s \in \mathcal{T}_k : \mathfrak{I}^{\pm}_m(s) \not\ni i_*\}$, and combining the result with (4.14), we conclude (4.11). $\square$

Next, we show that $\mathfrak{G}^{\pm}_{m,k}$ is nonempty for all large enough m.

Lemma 4.3. *We have* $\liminf_{m\to\infty}\big(\inf_{k\in\mathbb{Z}_{>0}} |\mathfrak{G}^{\pm}_{m,k}|\big) \geq 1$, *almost surely.*

With $\mathfrak{G}^{\pm}_{m,k}$ defined as in (4.10), proving $|\mathfrak{G}^{\pm}_{m,k}| \geq 1$ requires finding $i \in [\pm m, \pm 2m)$ such that $h_{(i,\pm 3m)}(\mathbf{y}) \geq \frac{\rho}{3}$ for $\mathbf{y} = \underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')$. This is conveniently reduced to estimating $\Sigma_{[\pm m,j)}(\mathbf{y})$, $j \in [\pm 2m, \pm 3m]$, by the following lemma, which is proven by a simple graphical argument as in [7, Proof of Lemma 4.4].

Lemma 4.4. *Let $\mathbf{y} \in [0,\infty]^{\mathbb{L}}$, $i_1^+ < i_2^+ \leq i_3^+$ and $i_3^- \leq i_2^- < i_1^-$, where $i_1^\pm, i_2^\pm \in \mathbb{Z}$ and $i_3^\pm \in \mathbb{Z} \cup \{\pm\infty\}$. If, for some $\gamma \in (0,\infty)$,*

$$\Sigma_{(i_1^\pm, i)}(\mathbf{y}) > \gamma, \quad \forall\, i \in [i_2^\pm, i_3^\pm] \cap \mathbb{Z}, \tag{4.15}$$

then there exists $i_*^\pm \in [i_1^\pm, i_2^\pm) \cap \mathbb{L}$ such that $h_{(i_*^\pm, i_3^\pm)}(\mathbf{y}) \geq \gamma$.

Proof of Lemma 4.3. Fixing $m, k \in \mathbb{Z}_{>0}$, to simplify notations, we omit the dependence on m, k of the index sets (e.g. $\mathfrak{N}^\pm := \mathfrak{N}^\pm_{m,k}$) and let $\widetilde{Y}_a^\pm := \underline{Y}_a^{\mathrm{lw}}(t', t'')\mathbf{1}\{a \in \mathfrak{N}^\pm \pm \frac{1}{2}\}$. We show

$$\Sigma_{(\pm m, \pm j)}(\widetilde{\mathbf{Y}}^\pm) > \tfrac{\varrho}{3},\ \forall j \in [\pm 2m, \pm 3m], \quad \forall \text{ large enough } m. \tag{4.16}$$

This, by Lemma 4.4 for $(i_1^\pm, i_2^\pm, i_3^\pm) = (\pm m, \pm 2m, \pm 3m)$, implies the existence of $I^\pm \in [\pm m, \pm 2m) \cap \mathbb{Z}$ such that $h_{(I^\pm, \pm 3m)}(\widetilde{\mathbf{Y}}^\pm) \geq \frac{\varrho}{3}$. For such $I^\pm$, we have $h_{(I^\pm, \pm 3m)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t', t'')) \geq \frac{\varrho}{3}$ and $\widetilde{Y}^\pm_{(I^\pm, I^\pm \pm 1)} \geq \frac{\varrho}{3} > 0$. The later implies $I^\pm \in \mathfrak{N}^\pm$, and therefore $I^\pm \in \mathfrak{G}^\pm$. Hence, it suffices to prove (4.16).

To the end of showing (4.16), with $\widetilde{\mathbf{Y}}^\pm$ defined as in the preceding, we begin by estimating $|(\mathfrak{N}^\pm)^c|$. To this end, as $\mathfrak{N}^\pm$ is defined in terms of $\mathfrak{A}(s)$ and $\mathfrak{I}^\pm(s)$, we first establish bounds on $|\mathfrak{A}(s) \cap (\pm m, \pm 3m)|$ and $|\mathfrak{I}^\pm(s)|$. Fixing $s \in [t', t'']$, with N as in (4.5) and $\mathfrak{A}(s)$ as in (4.6), we have

$$|\mathfrak{A}(s) \cap (\pm m, \pm 3m)| \leq |\mathfrak{A}(s) \cap (0, \pm 3m)| \leq \tfrac{(3m)^{1-\alpha} N}{m^{-\alpha/(3p')}} \leq (3m)^{1-\frac{2\alpha}{3}} N. \tag{4.17}$$

Proceeding to bounding $|\mathfrak{I}^\pm_m(s)|$, we require the following inequality: for any finite $\mathcal{A} \subset \mathbb{L}$, $n \in \mathbb{Z}_{>0}$, we have

$$|\mathcal{I}_n^\pm| \leq n|\mathcal{A}|, \qquad \text{where } \mathcal{I}_n^\pm := \big\{ i \in \mathbb{Z} : g_{(i, \pm\infty)}(\mathcal{A}) > n^{-1} \big\} \subset \mathbb{L}. \tag{4.18}$$

To prove this inequality, we image a pile of n particles at each site of $\mathcal{A}$, and topple the particles to the left (for $+$) or right (for $-$) in any order, so that each sites of $\mathbb{L}$ contains at most one particle. Letting $\mathcal{A}_n^\pm \subset \mathbb{L}$ denote the resulting set of particles, we clearly have $\mathcal{I}_n^\pm \subset (\mathcal{A}_n^\pm \mp \frac{1}{2})$ and $|\mathcal{A}_n^\pm| = n|\mathcal{A}|$, thereby concluding (4.18). Now, with $\mathfrak{I}^\pm(s)$ as in (4.7), combining (4.17) and (4.18) for $\mathcal{A} = \mathfrak{A}(s) \cap (\pm m, \pm 3m)$ and $n = \lceil m^{\alpha/3} \rceil$, we arrive at

$$|\mathfrak{I}^\pm(s)| \leq \lceil m^{\alpha/3} \rceil |\mathfrak{A}(s) \cap (\pm m, \pm 3m)| \leq 6Nm^{1-\frac{\alpha}{3}}.$$

Now, with $\mathfrak{N}^\pm$ as in (4.8), we have $\mathbf{1}\{i \notin \mathfrak{N}^\pm\} \leq m^{\frac{\alpha}{3p'}} \frac{1}{k} \sum_{s \in \mathcal{T}_k} \mathbf{1}\{i \in \mathfrak{I}^\pm(s)\}$. Summing both sides over $i \in \mathbb{Z}$, we arrive at

$$|(\mathfrak{N}^\pm)^c| \leq \frac{1}{k} \sum_{s \in \mathcal{T}_k} |\mathfrak{I}^\pm(s)| m^{\frac{\alpha}{3p'}} \leq 6Nm^{1-\alpha'}, \tag{4.19}$$

where $\alpha' := \frac{\alpha}{3}(1-\frac{1}{p'}) > 0$.

We proceed to proving (4.16). Fix $j \in [\pm 2m, \pm 3m]$. With $\widetilde{\mathbf{Y}}^{\pm}$ defined as in the proceeding, we have

$$\Sigma_{(\pm m,j)}(\widetilde{\mathbf{Y}}^{\pm}) = \Sigma_{(\pm m,j)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')) - \frac{1}{|j\mp m|}\sum_{(\pm m,j)} \underline{Y}^{\mathrm{lw}}_a(t',t'')\mathbf{1}\{a \in (\mathfrak{N}^{\pm})^c \pm \tfrac{1}{2}\}.$$

For the last term, with $\frac{1}{|j\mp m|}|(\mathfrak{N}^{\pm})^c| \leq 6Nm^{-\alpha'}$ (by (4.19)) and $\mathbf{Y}^{\mathrm{lw}} \in \mathcal{R}_{\mathcal{T}}(p)$, we have

$$\frac{1}{|j\pm m|}\sum_{(\pm m,j)} \underline{Y}^{\mathrm{lw}}_a(t',t'')\mathbf{1}\{a \in (\mathfrak{N}^{\pm})^c \pm \tfrac{1}{2}\} \xrightarrow{m\to\infty} 0, \quad \text{uniformly in } j \in [\pm 2m, \pm 3m].$$

Consequently, to prove (4.16), we may and shall replace $\widetilde{\mathbf{Y}}^{\pm}$ with $\underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')$. Applying the continuity estimate (3.6) for $Y^* = Y^{\mathrm{lw}}_a$, we have

$$\Sigma_{(\pm m,j)}(\underline{\mathbf{Y}}^{\mathrm{lw}}(t',t'')) \geq \Sigma_{(\pm m,j)}(\mathbf{Y}^{\mathrm{lw}}(t'')) - \Sigma_{(\pm m,j)}(\mathbf{Q}^{t'}(t'')).$$

With $\mathbf{Y}^{\mathrm{lw}} \in \mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$, the first term on the r.h.s. converges to ρ as $m\to\infty$, uniformly in $j \in [\pm 2m, \pm 3m]$. With $q(t''-t',1) \leq \frac{\rho}{2}$, by (3.4), the last term contributes $\geq -\frac{\rho}{2}$ as $m\to\infty$. Combining the preceding we conclude (4.16). □

Proof of Lemma 4.1(b). By Lemma 4.2–4.3, we have that

$$\inf_{i\in[\pm m,\pm 2m]} L^{\pm,k}_{i,m} \leq (1+P^{1/p})cm^{-\alpha/(3p')}\log(1+m).$$

Since the constant c does not depend on k, upon letting $k\to\infty$, by the continuity of $Y^{\mathrm{up}}_a(\cdot)$ and $Y^{\mathrm{lw}}_a(\cdot)$, the l.h.s. tends to $(\inf_{i\in[\pm m,\pm 2m]}\int_{t'}^{t''} L^{\pm}_{m,i}(s)ds)$. Consequently, further letting $m\to\infty$, we complete the proof. □

§ 5. Existence, Proof of Proposition 2.5

We begin by establishing the monotonicity (2.13). Recall the definition of $\mathcal{Y}(\gamma)$ and $\mathcal{Y}_{\mathcal{T}}(\gamma)$ from (2.10)–(2.11).

Proposition 5.1. *Fixing $\mathbf{y}^{in}, \mathbf{z}^{in} \in \mathcal{Y}(\gamma)$, $\gamma > 0$, we let $\{\mathbf{Y}^{(i)}\}_{i=0}^n$ and $\{\mathbf{Z}^{(i)}\}_{i=0}^n$ be $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued sequences satisfying* (2.8)*, with $\mathbf{Y}^{(i)}(0) = \mathbf{y}^{in}$ and $\mathbf{Z}^{(i)}(0) = \mathbf{z}^{in}$, $i = 0,\ldots,n$.*

(a) The sequence $\{\mathbf{Y}^{(i)}\}_{i=0}^n$ is decreasing, $\mathbf{Y}^{(0)}(\cdot) \geq \ldots \geq \mathbf{Y}^{(n)}(\cdot)$.

(b) If $\mathbf{y}^{in} \geq \mathbf{z}^{in}$, we have $\mathbf{Y}^{(i)}(\cdot) \geq \mathbf{Z}^{(i)}(\cdot)$, for $i = 0,\ldots,n$.

Proof. With ψ_a defined as in (2.3), it is easily to show that

$$y \mapsto \psi_a(y, \mathbf{z}(s)) \text{ is uniform Lipschitz over } [0,\infty) \times [0,t], \quad \forall t \geq 0,\ \forall \mathbf{z} \in \mathcal{Y}_{\mathcal{T}}. \tag{5.1}$$

We now prove $\mathbf{Y}^{(i-1)}(\boldsymbol{\cdot}) \geq \mathbf{Y}^{(i)}(\boldsymbol{\cdot})$ by induction on i. For $i = 1$, by (5.1), we have that $-\psi_a(\boldsymbol{\cdot}, \mathbf{Y}^{(0)}(t))$ is uniformly Lipschitz. With $-\psi_a(y_a, \mathbf{Y}^{(0)}(t)) \leq 0$, and $Y_a^{(0)}$ and $Y_a^{(1)}$ solving the respective equations (2.8a) and (2.8b), applying Lemma 3.1 for $y^{\text{up}} = Y_a^{(0)}$ and $y^{\text{lw}} = Y_a^{(1)}$, we conclude $\mathbf{Y}^{(0)}(\boldsymbol{\cdot}) \geq \mathbf{Y}^{(1)}(\boldsymbol{\cdot})$. Assuming $\mathbf{Y}^{(i-1)}(\boldsymbol{\cdot}) \geq \mathbf{Y}^{(i)}(\boldsymbol{\cdot})$, $i > 1$, by (2.16) we have $-\psi_a(y, \mathbf{Y}^{(i-1)}(t)) \geq -\psi_a(y, \mathbf{Y}^{(i)}(t))$. With $Y_a^{(i)}$ and $Y_a^{(i+1)}$ solving (2.8b), applying Lemma 3.1 for $y^{\text{up}} = Y_a^{(i)}$ and $y^{\text{lw}} = Y_a^{(i+1)}$ we conclude $\mathbf{Y}^{(i)}(\boldsymbol{\cdot}) \geq \mathbf{Y}^{(i+1)}(\boldsymbol{\cdot})$. This completes the proof of (a).

As for (b), the case $i = 0$ follows directly by applying Lemma 3.1. For $i > 0$, by (2.16), we have that

$$\mathbf{Z}^{(i)}(\boldsymbol{\cdot}) \leq \mathbf{Y}^{(i)}(\boldsymbol{\cdot}) \text{ implies } -\psi_a(y, \mathbf{Z}^{(i)}(s)) \leq -\psi_a(y, \mathbf{Y}^{(i)}(s)),$$

so, by induction, the case $i > 0$ follows by the preceding comparison argument. □

Now, fixing $\gamma > 0$ and $\mathbf{y}^{\text{in}} \in [\gamma, \infty)^{\mathbb{L}}$ as in Proposition 2.5, We consider first the special case of *equally spaced* initial condition, $\mathbf{z}^{\text{in}} := \boldsymbol{\gamma} = (\ldots, \gamma, \gamma, \ldots)$, and construct the corresponding iteration sequence $\{\mathbf{Z}^{(n)}\}_{n \in \mathbb{Z}_{\geq 0}}$. For $n = 0$, $\mathbf{Z}^{(0)}$ is the $\mathscr{F}^{\mathbf{W}}$-adapted Bessel process (as in (2.8a)) starting at $\boldsymbol{\gamma}$. Recalling $\underline{\mathcal{Y}}(\gamma)$ and $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ are defined as in (2.10)–(2.11), we check that $\mathbf{Z}^{(0)}$ is $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued.

Lemma 5.2. *We have* $\mathbf{Z}^{(0)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$.

Proof. Fix arbitrary $t \geq 0$. With $Z_a^{(0)}$ satisfying (2.8a) and $Z_a^{(0)}(0) = \gamma$, averaging (2.8a) over $a \in (0, m)$ using $W_a(t) = B_{a+1/2}(t) - B_{a-1/2}(t)$, we obtain

$$\inf_{s \in [0,t]} \Big\{ \Sigma_{(0,m)}\Big(\mathbf{Z}^{(0)}(s)\Big) \Big\} - \gamma \geq - \sup_{s \in [0,t]} |m|^{-1} |B_m(s) - B_0(s)|.$$

Upon letting $|m| \to \infty$, the r.h.s. tends to zero, whereby $\mathbf{Z}^{(0)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ follows. □

For $n > 0$, we construct the $\mathscr{F}^{\mathbf{W}}$-adapted, $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued process $\mathbf{Z}^{(n)}$ by induction on n, using Lemma 3.3. That is, fixing $n > 0$, for each $a \in \mathbb{L}$, we let $Z_a^{(n)}$ be the unique solution of (3.11) for $F(y, s) = -\psi_a(y, \mathbf{Z}^{(n-1)}(s))$, assuming $\mathbf{Z}^{(n-1)}$ is the $\mathscr{F}^{\mathbf{W}}$-adapted, $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued process satisfying (2.8). For Lemma 3.3 to apply, we indeed have that $F(0, s) = 0$, that $F(y, s)$ is $\mathscr{F}^{\mathbf{W}}$-adapted (since $\mathbf{Z}^{(n-1)}(s)$ is), and that $F(\boldsymbol{\cdot}, s)$ is uniformly Lipschitz, by (5.1). This yields the unique $\mathscr{F}^{\mathbf{W}}$-adapted, $C_+([0,\infty))^{\mathbb{L}}$-valued process $\mathbf{Z}^{(n)}$.

To complete the construction, we show that $\mathbf{Z}^{(n)}$ is also $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued. To this end, we first establish the *shift-invariance* of $\mathbf{Z}^{(n)}$. We say $\mathbf{Z} : [0, \infty) \to [0, \infty)^{\mathbb{L}}$ is shift-invariant if $\mathbf{Z}(\boldsymbol{\cdot}) \stackrel{\text{distr}}{=} (Z_{a+i}(\boldsymbol{\cdot}))_{a \in \mathbb{L}} := \theta_i(\mathbf{Z}(\boldsymbol{\cdot}))$, $\forall i \in \mathbb{Z}$.

Lemma 5.3. *The processes $\mathbf{Z}^{(0)}, \ldots, \mathbf{Z}^{(n)}$, constructed in the preceding, are shift-invariant.*

Proof. This follows from the shift-invariance of the equation (2.5). See [7, proof of Lemma 5.2] for a complete proof. $\square$

Equipped with Lemma 5.3, we proceed to showing $\mathbf{Z}^{(n)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$. To this end, letting $\eta_{(i_1,i_2)}(\mathbf{y}) := \sum_{a\in(i_1,i_2)} \eta_a(\mathbf{y})$ (where $\eta_a(\mathbf{y})$ is defined as in (2.2)), we will use the following readily verified identity (c.f. (2.1)) in the proof of Lemma 5.4:

$$\eta_{(i_1,i_2)}(\mathbf{y}) = \eta^{\mathrm{up}}_{(i_1,i_2)}(\mathbf{y}) - \eta^{\mathrm{lw}}_{(i_1,i_2)}(\mathbf{y}), \tag{5.2}$$

where $i_- := (i_1 \wedge i_2) < i_+ := (i_1 \vee i_2)$ and

$$\eta^{\mathrm{up}}_{(i_1,i_2)}(\mathbf{y}) := \sum_{i\in(i_1,i_2]} \frac{1}{2y_{(i_1,i)}} + \sum_{i\in(i_2,i_1]} \frac{1}{2y_{(i_2,i)}}, \tag{5.3}$$

$$\eta^{\mathrm{lw}}_{(i_1,i_2)}(\mathbf{y}) := \widetilde{\eta}^{\mathrm{lw},+}_{(i_1,i_2)}(y_{(i_1,i_2)}, \mathbf{y}) + \widetilde{\eta}^{\mathrm{lw},-}_{(i_1,i_2)}(y_{(i_1,i_2)}, \mathbf{y}), \tag{5.4}$$

$$\widetilde{\eta}^{\mathrm{lw},\pm}_{(i_1,i_2)}(z, \mathbf{y}) := \sum_{i'\in(i_\pm,\pm\infty)} \frac{z}{2(z + y_{(i,i')})y_{(i,i')}}. \tag{5.5}$$

Note that the expressions $\eta_{(i_1,i_2)}(\mathbf{y})$, $\eta^{\mathrm{up}}_{(i_1,i_2)}(\mathbf{y})$ and $\eta^{\mathrm{lw}}_{(i_1,i_2)}(\mathbf{y})$ are well-defined for all $\mathbf{y} \in \underline{\mathcal{Y}}(\gamma)$.

Lemma 5.4. *Let $\mathbf{Z}^{(0)}, \ldots, \mathbf{Z}^{(n)}$, with $\{\mathbf{Z}^{(i)}\}_{i=0}^{n-1} \subset \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ and $\mathbf{Z}^{(n)} \in C_+([0,\infty))^{\mathbb{L}}$, be as in the proceeding, we have $\mathbf{Z}^{(n)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$.*

Proof. Let $V^n_{(i_1,i_2)}(s) := \Sigma_{(i_1,i_2)}(\mathbf{Z}^{(n)}(s))$. With $\mathbf{Z}^{(n)} \in C_+([0,\infty))^{\mathbb{L}}$, fixing $t \geq 0$, it suffices to show $(\liminf_{|m|\to\infty} \underline{V^n_{(0,m)}}(0,t)) \geq \gamma$. We achieve this in two steps by showing

i) $\quad \lim_{|m|\to\infty} V^n_{(0,m)}(s') \geq \gamma$ almost surely, for each fixed $s' \in [0,t]$;

ii) $\quad \liminf_{|m|\to\infty} \underline{V^n_{(0,m)}}(t) \geq \gamma$ almost surely.

i) Fixing $s' \in [0,t]$, we begin by deriving a lower bound on $V^n_{(0,m)}(s')$. With $\mathbf{Z}^{(n-1)}(\cdot) \geq \mathbf{Z}^{(n)}(\cdot)$ (by Proposition 5.1), by (2.15) we have

$$\tfrac{1}{Z^{(n)}_a(s)} - \psi_a(Z^{(n)}_a(s), \mathbf{Z}^{(n-1)}_a(s)) \geq \tfrac{1}{Z^{(n-1)}_a(s)} - \psi_a(Z^{(n-1)}_a(s), \mathbf{Z}^{(n-1)}_a(s)) = \eta_a(\mathbf{Z}^{(n-1)}(s)).$$

Inserting this into (2.8b), summing the result over $a \in (0,m)$, and dividing both sides by $|m|$, with $\sum_{a\in(0,m)} W_a(s') = B_m(s') - B_0(s')$, $Z^{(n)}_a(0) = \gamma$ and (5.2), we have

$$V^n_{(0,m)}(s') \geq \gamma + |m|^{-1}(B_m(s') - B_0(s')) - \frac{\beta}{|m|} \int_0^{s'} \eta^{\mathrm{lw}}_{(0,m)}(\mathbf{Z}^{(n-1)}(s))ds. \tag{5.6}$$

As $\lim_{|m|\to\infty}(|m|^{-1}(B_m(s') - B_0(s'))) = 0$ almost surely, it clearly suffices to show

$$\int_0^{s'} |m|^{-1}\eta^{\mathrm{lw}}_{(0,m)}(\mathbf{Z}^{(n-1)}(s))ds \longrightarrow 0 \ \text{ almost surely, as } |m| \to \infty. \tag{5.7}$$

With $\{Z_a^{(n)}(s')\}_{a\in\mathbb{L}}$ being shift-invariant (by Lemma 5.3) and having a finite first moment (since $\mathbf{Z}^{(n)}(s') \le \mathbf{Z}^{(0)}(s')$), by the Birkhoff–Khinchin ergodic theorem, we have that $V^n_{(0,m)}(s')$ converges almost surely (to a possibly random limit) as $|m| \to \infty$. Using this, we further reduce showing (5.7) to showing

$$\int_0^{s'} |m|^{-1}\eta^{\mathrm{lw}}_{(0,m)}(\mathbf{Z}^{(n-1)}(s))ds \Longrightarrow 0, \ \text{ as } |m| \to \infty, \tag{5.8}$$

where $\Rightarrow$ denotes convergence in law.

We proceed to showing (5.8). This, with (5.4), amounts to estimating $\eta^{\mathrm{lw},\pm}_{\mathcal{I}}(\mathbf{y}) := \widetilde{\eta}^{\mathrm{lw},\pm}_{\mathcal{I}}(y_{\mathcal{I}}, \mathbf{y})$, for $\mathcal{I} := (0,m)$ and $\mathbf{y} = \mathbf{Z}^{(n-1)}(s)$. With $Z_a^{(n-1)}$ satisfying (2.8b), by (3.7) we have that $Z_a^{(n-1)}(s') \le Z_a^{(n-1)}(0) + Q_a^{0,s'} = \gamma + Q_a^{0,s'}$. Combining this with (3.4), we have

$$N := \sup\Big\{\overline{V^{n-1}_{(0,m)}}(0,s') : m \in \mathbb{Z}\Big\} < \infty.$$

With $\mathbf{Z}^{(n-1)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$, we have $D := \inf\Big\{\underline{V^{n-1}_{(i,0)}}(0,s'), i \ne 0\Big\} > 0$. With

$$\eta^{\mathrm{lw},-}_{(0,|m|)}(\mathbf{y}) = \sum_{i=1}^{\infty} \frac{y_{(0,|m|)}}{y_{(-i,|m|)}y_{(-i,0)}}, \quad \eta^{\mathrm{lw},+}_{(0,-|m|)}(\mathbf{y}) = \sum_{i=1}^{\infty} \frac{y_{(-|m|,0)}}{y_{(0,i)}y_{(-|m|,i)}},$$

so by the preceding bounds we then have

$$\int_0^{s'} |m|^{-1}\eta^{\mathrm{lw},\mp}_{(0,\pm|m|)}\Big(\mathbf{Z}^{(n-1)}(s)\Big)ds \le s'\sum_{i=1}^{\infty}\frac{N}{2(iD+|m|N)iD} \xrightarrow{\text{a.s.}} 0, \ \text{ as } |m| \to \infty. \tag{5.9}$$

Next, using the shift-invariance of $\mathbf{Z}^{(n-1)}$, we have

$$\eta^{\mathrm{lw},\mp}_{(0,\pm|m|)}\Big(\mathbf{Z}^{(n-1)}(s)\Big) \overset{\text{distr}}{=} \eta^{\mathrm{lw},\mp}_{(0,\pm|m|)}\Big(\theta_{\mp|m|}(\mathbf{Z}^{(n-1)}(s))\Big) = \eta^{\mathrm{lw},\mp}_{(0,\mp|m|)}\Big(\mathbf{Z}^{(n-1)}(s)\Big).$$

Combining this with (5.9) yields

$$\int_0^{s'} |m|^{-1}\eta^{\mathrm{lw},\mp}_{(0,\mp|m|)}\Big(\mathbf{Z}^{(n-1)}(s)\Big)ds \Longrightarrow 0, \qquad \text{as } |m| \to \infty.$$

From this and (5.9) we conclude (5.8), thereby completing the proof of (i).

$ii)$ With (i), this is achieved by a continuity estimate based on (3.6). To this end, partition $[0,t]$ into j_* equally spaced subintervals $0 = t_0 < \ldots < t_{j_*} = t$. For each

$a \in \mathbb{L}$, with $Z_a^{(n)}$ satisfying (2.8b), we apply (3.6) for $Y^* = Z_a^{(n)}$. Averaging the result over $a \in (0, m)$, we obtain

$$\underline{V^n_{(0,m)}}(t_{j-1}, t_j) \geq V^n_{(0,m)}(t_j) - \Sigma_{(0,m)}\big(\mathbf{Q}^{t_j}(t_{j-1})\big). \tag{5.10}$$

Letting $|m| \to \infty$, by (i) and (3.4), we have

$$\liminf_{|m|\to\infty} \underline{V^n_{(0,m)}}(t_{j-1}, t_j) \geq \gamma - q(t/j_*, 1).$$

Combining this for $j = 1, \ldots, j_*$, using the readily verified inequality

$$\liminf_{|m|\to\infty} \underline{f_m}(0,t) \geq \min_{j=1}^{j_*} \Big\{ \liminf_{|m|\to\infty} \underline{f_m}(t_{j-1}, t_j) \Big\}, \quad f_m(\,\boldsymbol{\cdot}\,) : [0,\infty) \to \mathbb{R},$$

we thus conclude $(\liminf_{|m|\to\infty} \underline{V^n_{(0,m)}}(0,t)) \geq \gamma - q(t/j^*, 1)$, almost surely. With j^* being arbitrary, the proof is completed upon letting $j^* \to \infty$, (whence $q(t/j^*, 1) \to 0$). $\square$

Having constructed the iteration sequence $\{\mathbf{Z}^{(n)}\}_n$ for $\mathbf{z}^{\text{in}} = \boldsymbol{\gamma}$, with $\mathbf{Z}^{(n)}(\,\boldsymbol{\cdot}\,) \geq \mathbf{Z}^{(n+1)}(\,\boldsymbol{\cdot}\,)$ (by Proposition 5.1), we let $Z_a^{(\infty)}(t) := \lim_{n\to\infty} Z_a^{(n)}(t) \geq 0$ denote the limiting process. We next establish a lower bound on the average spacing of $\mathbf{Z}^{(\infty)}$.

Lemma 5.5. *We have $\mathbf{Z}^{(\infty)} \in \underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma)$ almost surely, where*

$$\underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma) := \Big\{ \mathbf{y}(\,\boldsymbol{\cdot}\,) : [0,\infty) \to [0,\infty)^{\mathbb{L}} : \liminf_{|m|\to\infty} \inf_{s\in[0,t]} \Sigma_{(0,m)}(\mathbf{y}(s)) \geq \gamma, \forall t \geq 0 \Big\}. \tag{5.11}$$

Proof. Fixing $t \geq 0$, we let $V_{\mathcal{I}}^\infty(s) := \Sigma_{\mathcal{I}}(\mathbf{Z}^{(\infty)}(s))$, and recall that $V_{\mathcal{I}}^n(s) := \Sigma_{\mathcal{I}}(\mathbf{Z}^{(n)}(s))$. As already mentioned in the proof of Lemma 5.4, since $\mathbf{Z}^{(n)}$ is shift-invariant for $n \in \mathbb{Z}_{>0}$ and (hence) for $n = \infty$, and since each $Z_a^{(n)}$ as a finite mean for $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ (because $\mathbf{Z}^{(\infty)}(s) \leq \mathbf{Z}^{(0)}(s)$), by the Birkhoff–Khinchin ergodic theorem, the limits

$$V^n(s) := \lim_{|m|\to\infty} V^n_{(0,m)}(s), \quad V^\infty(s) := \lim_{|m|\to\infty} V^\infty_{(0,m)}(s),$$

exists almost surely.

As in the proof of Lemma 5.4, we proceed by first proving $V^\infty(s) \geq \gamma$ almost surely, for any *fixed* $s \in [0,t]$. With $Z_a^{(\infty)}(s) \leq Z_a^{(n)}(s) \leq Z_a^{(0)}(s) \leq \gamma + Q_a^{0,s}$, we have that $\{V^n_{(0,m)}(s)\}_{m\in\mathbb{Z}}$ is uniformly integrable, for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Consequently, we have

$$\mathbb{E}(V^n(s)) = \lim_{|m|\to\infty} \mathbb{E}\Big(V^n_{(0,m)}(s)\Big) = \lim_{|m|\to\infty} \mathbb{E}\Big(\Sigma_{(0,m)}(\mathbf{Z}^{(n)}(s))\Big) = \mathbb{E}(Z^{(n)}_{1/2}(s)),$$

$\forall n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. With $Z^{(n)}_{1/2}(s) \searrow Z^{(\infty)}_{1/2}(s)$, we thus conclude $\mathbb{E}(V^n(s)) \to \mathbb{E}(V^\infty(s))$. Combining this with $V^n(s) \geq V^\infty(s) \geq 0$ (as $\mathbf{Z}^{(n)}(s) \geq \mathbf{Z}^\infty(s) \geq \mathbf{0}$), we further obtain that $V^n(s) \to V^\infty(s)$ almost surely. By Lemma 5.4, $V^\infty(s) \geq \gamma$ almost surely, so $V^\infty(s) \geq \gamma$ almost surely.

Now, letting $n \to \infty$ in (5.10), we obtain

$$\underline{V^\infty_{(0,m)}}(t_{j-1}, t_j) \geq \underline{V^\infty_{(0,m)}}(t_j) - \Sigma_{(0,m)}(\mathbf{Q}^{t_i}(t_{j-1})).$$

With this and $V^\infty(t_j) \geq \gamma$, the proof is completed by following the same continuity argument as in the proof of Lemma 5.4(ii). □

Now, we turn to the initial condition $\mathbf{y}^{\text{in}} \in [\gamma, \infty)^{\mathbb{L}}$ and construct the corresponding iteration sequence and limiting process.

Lemma 5.6. *Let $\mathbf{y}^{in} \in [\gamma, \infty)^{\mathbb{L}}$ be as in the preceding. There exists a $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued, $\mathscr{F}^{\mathbf{W}}$-adapted, decreasing sequence $\{\mathbf{Y}^n\}_{n \in \mathbb{Z}_{\geq 0}}$ satisfying* (2.8)*. Further, with $Y^{(\infty)}_a(t) := \lim_{n\to\infty} Y^{(n)}_a(t) \geq 0$ denoting the limiting process, we have $\mathbf{Y}^{(\infty)} \in \underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma)$.*

Proof. To construct such a sequence $\{\mathbf{Y}^n\}_n$, as seen from the proceeding construction of $\{\mathbf{Z}^n\}_n$, it suffices to show $\mathbf{Y}^{(n)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$. This follows directly by induction on n using Proposition 5.1, which assures $\mathbf{Y}^{(n)}(\bullet) \geq \mathbf{Z}^{(n)}(\bullet)$. Letting $n \to \infty$ in the previous inequality, we obtain $\mathbf{Y}^{(\infty)}(\bullet) \geq \mathbf{Z}^{(\infty)}(\bullet)$, thereby concluding $\mathbf{Y}^{(\infty)} \in \underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma)$. □

Proof of Proposition 2.5.

Let $\mathbf{Y}'$ be a generic $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ solution to (1.1) with $\mathbf{Y}'(0) \leq \mathbf{y}^{\text{in}}$. A simple comparison argument similar to the proof of Proposition 5.1 shows that $\mathbf{Y}' \leq \mathbf{Y}^{(\infty)}$. Further, if $\mathbf{y}^{\text{in}} \in \mathcal{R}_{\mathcal{T}}(p)$, one concludes $\mathbf{Y}^{(\infty)} \in \mathcal{R}_{\mathcal{T}}(p)$ by comparing $\mathbf{Y}^{(\infty)}$ to the Bessel process $\mathbf{Y}^{(0)}$ using Lemma 3.1.

With these, it now suffices to show that $\mathbf{Y}^{(\infty)}$ is in fact a $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$-valued solution. To this end, fixing $t \geq 0$ and $a_* \in \mathbb{L}$, we show

$$\underline{Y}^{(\infty)}_{a_*}(0, t) > 0, \quad \text{almost surely.} \tag{5.12}$$

With (5.12) and $\mathbf{Y}^{(\infty)} \in \underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma)$, by letting $n \to \infty$ in (2.8), it is not hard to see that $\mathbf{Y}^{(\infty)} \in \underline{\mathcal{Y}}_{\mathcal{T}}(\gamma)$ and that $\mathbf{Y}^{(\infty)}$ solves (2.5). We give a complete proof of (5.12) and refer to [7, proof of Proposition 5.6] for the rest of the details.

Fixing $a_* \in \mathbb{Z}$ and $t \in [0, \infty)$, without lost of generality we assume t is small enough such that $q(t, 1) < \gamma/2$, since the general case follows by partition $[0, t]$ into small enough subintervals. With $Y^{(n)}_a$ solving an equation of the type (3.5), applying (3.6) for $Y^* = Y^{(n)}_a$, we obtain

$$\Sigma_{(a_*,m)}(\underline{\mathbf{Y}}^{(n)}(0, t)) \geq \Sigma_{(a_*,m)}(\mathbf{Y}^{(n)}(t)) - \Sigma_{(a_*,m)}(\mathbf{Q}^0(t)).$$

Sending $n \to \infty$ and $|m| \to \infty$ in order, with $\mathbf{Y}^{(\infty)}(t) \in \underline{\mathcal{Y}}'_{\mathcal{T}}(\gamma)$ and (3.4), we obtain

$$\liminf_{|m|\to\infty} \Big\{ \Sigma_{(a_*,m)}(\underline{\mathbf{Y}}^{(\infty)}(0,t)) \Big\} \geq \gamma - q(t,1) > \tfrac{\gamma}{2}.$$

From this we obtain some random $I^{\pm} \in (a_*, \pm\infty) \cap \mathbb{Z}$ such that $\Sigma_{[a_*,i)}(\underline{\mathbf{Y}}^{(\infty)}(0,t)) > \frac{\gamma}{2}$, $\forall i \in (-\infty, I^-] \cup [I^+, \infty)$. Combining this with Lemma 4.4 for $(i_1^{\pm}, i_2^{\pm}, i_3^{\pm}) = (a_* \pm \frac{1}{2}, I^{\pm}, \pm\infty)$ we $J^{\pm} \in (a_*, \pm\infty) \cap \mathbb{Z}$ such that

$$h_{(J^{\pm},\pm\infty)}(\underline{\mathbf{Y}}^{(\infty)}(0,t)) \geq \tfrac{\gamma}{2}, \tag{5.13}$$

where $h_{\mathcal{I}}(\mathbf{y})$ is as in (4.9).

Equipped with (5.13), we proceed to truncating the equation (2.8) at the finite window $\mathfrak{J} := (J^-, J^+)$. To this end, we express (2.8) as a system of finite-dimensional equations with external forces (i.e. (3.16)), as

$$\begin{aligned} Y_a^{(n)}(t) &= Y_a^{(n)}(0) + W_a(t) \\ &\quad + \beta \int_0^t \big(\eta_a^{\mathfrak{J}}(Y_a^{(n)}(s), \mathbf{Y}^{(n)}(s)) + Y_a^{(n)}(s) Z_a^{**}(s) \big) ds, \quad \forall\, a \in \mathfrak{J}, \end{aligned} \tag{5.14}$$

where the external force $Z_a^{**}(s) := z_a^{**,\mathfrak{J}}(\mathbf{Y}^{(n)}(s), \mathbf{Y}_a^{(n-1)}(s))$ takes the form

$$z_a^{**,\mathcal{A}}(\mathbf{y}, \mathbf{y}') := \tfrac{1}{y} \Big(\psi_a^{\mathcal{A}}(y, \mathbf{y}) - \psi_a^{\mathcal{A}}(y, \mathbf{y}') - \psi_a^{\mathcal{A}^c}(y, \mathbf{y}') \Big).$$

With $\{\mathbf{Y}^{(n)}\}$ being decreasing, by (3.15) we have

$$\psi_a^{\mathcal{A}}(y, \mathbf{Y}^{(n)}(s)) - \psi_a^{\mathcal{A}}(y, \mathbf{Y}^{(n-1)}(s)) \geq 0, \qquad \psi_a^{\mathcal{A}^c}(y, \mathbf{Y}^{(n-1)}(s)) \leq \psi_a^{\mathcal{A}^c}(y, \mathbf{Y}^{(n)}(s)),$$

so $Z_a^{**}(s) \geq -\psi_a^{\mathcal{A}^c}(\mathbf{Y}_a^{(n)}(s), \mathbf{Y}^{(n)}(s))$. Further, with $\psi_a^{\mathcal{A}^c}(y, \mathbf{y})$ as in (3.12), we have $\psi_a^{\mathfrak{J}^c}(y, \mathbf{y})_a(\mathbf{y}) \geq -\frac{1}{2} \sum_{\sigma=\pm} \sum_{i=1}^{\infty} (y_{(J^\sigma, J^\sigma + \sigma i)})^{-2}$. Using (5.13), we thus conclude

$$Z_a^{**}(s) \geq z_a^{2,\mathfrak{J}}(\mathbf{Y}^{[n]}(s)) \geq -\sum_{i=1}^{\infty} (i\gamma/2)^{-2} =: c^* > -\infty.$$

With this, letting $\mathbf{Y}^{(i_1,i_2)}$ be the $C_+([0,\infty))^{(i_1,i_2)\cap\mathbb{L}}$-valued solution of (3.16) for $Z_a^*(s) = c^*$, by Lemma 3.5 we have $\mathbf{Y}^{[n]} \geq_{[0,t]}^{\mathfrak{J}} \mathbf{Y}^{\mathfrak{J}} \in C_+([0,\infty))^{\mathfrak{J}}$. As $a_* \in \mathfrak{J}$, letting $n \to \infty$, we conclude (5.12). □

§ 6. Existence: Proof of Proposition 2.6

Fixing $\mathbf{y}^{\text{in}} \in \mathcal{Y}(\alpha, \rho)$ and a sequence $1 \geq \gamma_1 \geq \gamma_2 \geq \ldots \to 0$, we let $\mathbf{Y}^{\vee\gamma_n}$ be as in Proposition 2.6. Our goal is to show that $\mathbf{Y} := \lim_{n\to\infty} \mathbf{Y}^{\vee\gamma_n}$ is a $\mathcal{Y}_{\mathcal{T}}(\alpha, \rho)$-valued solution of (2.5). The key to the proof is the following estimate

$$\sup_{s\in[0,t]} \Big\{ \sup_{m,n\in\mathbb{Z}} \big| \Sigma_{(0,m)}(\mathbf{Y}^{\vee\gamma_n}(s) - \mathbf{Y}^{\vee\gamma_n}(0)) \big| \, |m|^{\alpha} \Big\} < \infty. \tag{6.1}$$

Indeed, letting $n \to \infty$ we obtain

$$\sup_{s\in[0,t]} \Big\{ \sup_{m,n\in\mathbb{Z}} \big|\Sigma_{(0,m)}\big(\mathbf{Y}^{\vee\gamma_n}(s) - \mathbf{y}^{\text{in}}\big)\big| \, |m|^{\alpha} \Big\} < \infty.$$

With $\mathbf{y}^{\text{in}} \in \mathcal{Y}(\alpha,\rho)$, this implies $\mathbf{Y} \in \mathcal{Y}_{\mathcal{T}}(\alpha,\rho)$. Further, as each of $\mathbf{Y}^{\vee\gamma_n}$ satisfies the equation (2.5), it is not hard (given (6.1)) to pass the equation to the limit $n \to \infty$, whereby showing that $\mathbf{Y}$ solves (2.5). That $\mathbf{Y}$ is the greatest solution follows directly from comparison and the fact that each $\mathbf{Y}^{\vee\gamma_n}$ is the greatest solution.

The proof of (6.1) is divided into few steps as follows. The first step is to obtain lower bounds on $\sum_{\mathcal{A}^{\mathbb{L}}_{b,k}}(\mathbf{Y}^{\vee\gamma_n}(s))$, the averaged spacing of $\mathbf{Y}^{\vee\gamma_n}(s)$ over a certain partition $\{\mathcal{A}^{\mathbb{L}}_{b,k}\}_b$ of $\mathbb{L}$ constructed in the following. To the end of estimating $\sum_{\mathcal{A}^{\mathbb{L}}_{b,k}}(\mathbf{Y}^{\vee\gamma_n}(s))$, we will frequently use the following lemma.

Lemma 6.1. *Let $\mathbf{Y}^*$ be a $\underline{\mathcal{Y}}_{\mathcal{T}}$-valued solution of (2.5), $\mathcal{K} \subset \mathcal{I} \subset \mathcal{K}' \subset \mathbb{L}$ be nested intervals, and $s' < s'' \in [0,t]$. We have that*

$$\begin{aligned}\Sigma_{\mathcal{K}}(\mathbf{Y}^*(s'')) \geq & \Sigma_{\mathcal{K}}(\mathbf{Y}^*(s')) - \frac{\beta}{|\mathcal{K}\cap\mathbb{L}|}\int_{s'}^{s''} \eta^{lw}_{\mathcal{I}}(\mathbf{Y}^*(s))ds \\ & - \widehat{B}^{\mathcal{K}'}_{\mathcal{K}}(t) - \lambda^{\mathcal{K}'}_{\mathcal{K}}(\mathbf{Q}^{s',s''}),\end{aligned} \tag{6.2}$$

$$\begin{aligned}\Sigma_{\mathcal{K}}(\mathbf{Y}^*(s'')) \leq & \Sigma_{\mathcal{K}}(\mathbf{Y}^*(s')) + \frac{\beta}{|\mathcal{K}\cap\mathbb{L}|}\int_{s'}^{s''} \eta^{up}_{\mathcal{I}}(\mathbf{Y}^*(s))ds \\ & + \widehat{B}^{\mathcal{K}'}_{\mathcal{K}}(t) + \lambda^{\mathcal{K}'}_{\mathcal{K}}(\mathbf{Y}^*(s')),\end{aligned} \tag{6.3}$$

where

$$\widehat{B}^{\mathcal{K}'}_{\mathcal{K}}(t) := \frac{4}{|\mathcal{K}\cap\mathbb{Z}|} \sup_{j\in\overline{\mathcal{K}'}\setminus\mathcal{K}} \overline{|B_j|}(0,t), \quad \lambda^{\mathcal{K}'}_{\mathcal{K}}(\mathbf{y}) := \frac{1}{|\mathcal{K}\cap\mathbb{Z}|}\sum_{a\in\mathcal{K}'\setminus\mathcal{K}} y_a, \tag{6.4}$$

and $\overline{\mathcal{K}'}$ denotes the closure of $\mathcal{K}'$.

Proof. With $\mathbf{Y}^*$ satisfying (2.5), this lemma follows from (5.2) and elementary manipulations. See [7, proof of Lemma 6.1] for a complete proof. □

We now define the partition $\{\mathcal{A}^{\mathbb{L}}_{b,k}\}_b$. Let $m_i := \lfloor i^{1/\alpha} \rfloor$, for $i \geq 0$, and $m_i := -m_{|i|}$ for $i < 0$. For any fixed $k \in \mathbb{Z}_{>0}$, we construct a partition $\{\mathcal{A}^{\mathbb{L}}_{b,k}\}_{b\in\mathbb{L}}$ of $\mathbb{L}$ by letting $\widetilde{m}^k_i := m_{ki}$,

$$\mathcal{A}_{b,k} := (\widetilde{m}^k_{b-1/2}, \widetilde{m}^k_{b+1/2}), \quad \mathcal{A}^{\mathbb{L}}_{b,k} := \mathcal{A}_{b,k} \cap \mathbb{L}. \tag{6.5}$$

This partition is constructed so that $|\mathcal{A}^{\mathbb{L}}_{k,b}| \sim k(\widetilde{m}^k_{|b|+1/2})^{1-\alpha}$. More precisely, with $|\mathcal{A}^{\mathbb{L}}_{b,k}| = \widetilde{m}^k_{|b|+1/2} - \widetilde{m}^k_{|b|-1/2}$ and $\lfloor y-x \rfloor \leq \lfloor y \rfloor - \lfloor x \rfloor \leq \lceil y-x \rceil$, $\forall x \leq y \in [0,\infty)$, we have

$$|\mathcal{A}^{\mathbb{L}}_{b,k}| \geq \lfloor \tfrac{k^{\frac{1}{\alpha}}}{2\alpha}|b|^{\frac{1-\alpha}{\alpha}} \rfloor \geq \lfloor \tfrac{k}{\alpha 2^{1/\alpha}}(m_{k(|b|+1/2)})^{1-\alpha} \rfloor, \tag{6.6}$$

$$|\mathcal{A}^{\mathbb{L}}_{b,k}| \leq \lceil \tfrac{k^{\frac{1}{\alpha}}}{\alpha}(|b|+\tfrac{1}{2})^{\frac{1-\alpha}{\alpha}} \rceil \leq \lceil \tfrac{k}{\alpha 2^{1/\alpha}}(m_{k(|b|+1/2)}+1)^{1-\alpha} \rceil. \tag{6.7}$$

With (6.6)–(6.7) and $\Sigma_{\mathcal{A}_{b,k}}(\mathbf{y}) = \frac{1}{|\mathcal{A}^{\mathbb{L}}_{b,k}|}\big|\sum_{a\in(0,\widetilde{m}^k_{b+1/2})}(\mathbf{y}) - \sum_{a\in(0,\widetilde{m}^k_{b-1/2})}(\mathbf{y})\big|$, we have

$$\big|\Sigma_{(0,m)}(\mathbf{y}) - \rho\big| \leq \tfrac{1}{|\mathcal{A}^{\mathbb{L}}_{b,k}|}|y|_{\alpha,\rho}\Big(|\widetilde{m}^k_{b+1/2}|^{1-\alpha} + |\widetilde{m}^k_{b-1/2}|^{1-\alpha}\Big) \leq \tfrac{c}{k}|y|_{\alpha,\rho}, \tag{6.8}$$

where $|\mathbf{y}|_{\alpha,\rho}$ is defined as in (1.6). Hereafter, we assume $k \in \mathbb{Z}_{>0}$ is large enough so that $\{\mathcal{A}_{b,k}\}_b$ is nondegenerated: i.e. $\mathcal{A}_{b,k} \neq \emptyset$, $\forall b \in \mathbb{L}$. Recall $\eta^{\mathrm{up}}_{\mathcal{I}}(\mathbf{y})$ and $\eta^{\mathrm{lw}}_{\mathcal{I}}(\mathbf{y})$ are defined as in (5.3)–(5.4).

Recalling the definition of $q(t,1)$ from (3.4), we begin by establishing the following preliminary estimate.

Proposition 6.2. *Fix $t < \infty$ and let $\tau < \infty$ be such that $q(\tau,1) = \frac{\rho}{400}$. For any $t_* \in [0, t-\tau]$, if there exists some random $K_* \in \mathbb{Z}_{>0}$ such that*

$$\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}(t_*)) > \tfrac{3\rho}{4}, \quad \forall b \in \mathbb{L},\ k \geq K_*, \tag{6.9}$$

then there exists some other random $K \in \mathbb{Z}_{>0}$ such that

$$\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}(s)) \geq \tfrac{\rho}{2}, \quad \forall s \in [t_*, t_*+\tau],\ b \in \mathbb{L},\ k \geq K_* \vee K. \tag{6.10}$$

Proof. The proof is fairly technical. Here we give a sketch of the proof, and refer to [7, proof of Proposition 6.2] for the details.

Fixing arbitrary $n \in \mathbb{Z}_{>0}$, we let $S_{b,k} := \inf\{s \geq t_* : \Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}^{\vee\gamma_n}(s)) < \frac{\rho}{2}\}$ and $T_k := (t_*+\tau) \wedge (\inf_{b\in\mathbb{L}} S_{b,k})$. With $\mathbf{Y}(t) := \lim_{n\to\infty} \mathbf{Y}^{\vee\gamma_n}(t)$, proving (6.10) amount to constructing $K \in \mathbb{Z}_{>0}$ such that $T_k = t_*+\tau$, for all $k \geq K$. However, as T_k involves infinitely many $\mathcal{A}_{b,k}$, $b \in \mathbb{L}$, it is not even clear, a-priori, whether $T_k > t_*$. We circumvent this problem by truncating $\{\mathcal{A}_{b,k}\}_b$ as follows. Consider the $\underline{\mathcal{Y}}_{\mathcal{T}}(\gamma_n)$-valued solution $\mathbf{Z}_n$ of (2.5) starting from $(\ldots,\gamma_n,\gamma_n,\ldots)$, given by Proposition 2.5. With $\mathbf{Z}_n$ being shift-invariant (by Lemma 5.3), we have $\lim_{|m|\to\infty}\Sigma_{(0,m)}(\mathbf{Z}_n) = Z_n > 0$. Hence, given arbitrarily large $\ell \in \mathbb{Z}$, there exists $M' \in [\ell,\infty)\cap\mathbb{Z}$ such that so $\Sigma_{(\pm\ell,\pm m)}(\underline{\mathbf{Z}}(0,t)) > \frac{Z_n}{2} := Z'_n$ for all $|m| \geq M'$. Now, applying Lemma 4.4 for $(i^\pm_1, i^\pm_2, i^\pm_3) = (\pm\ell, \pm M', \pm\infty)$, we obtain $M_\pm \in [\pm\ell, \pm\infty)\cap\mathbb{Z}$ such that

$$h_{(M_\pm,\pm\infty)}(\underline{\mathbf{Y}}^{\vee\gamma_n}(0,t)) \geq h_{(M_\pm,\pm\infty)}(\underline{\mathbf{Z}}(0,t)) \geq Z'_n > 0. \tag{6.11}$$

With this, we then consider the truncated partition

$$\{b \in \mathbb{L} : \mathcal{A}_{b,k} \subset (M_-, M_+)\} =: (J_-, J_+)\cap\mathbb{L}, \quad J_- \leq J_+ \in \mathbb{Z}, \tag{6.12}$$

and define the analog of T_k as $\widetilde{T}_k := (t_* + \tau) \wedge (\inf_{b\in(J_-,J_+)} S_{b,k})$. Instead of proving $T_k = t_* + \tau$, we prove $\widetilde{T}_k = t_* + \tau$ for all large enough ℓ (which yields $T_k = t_* + \tau$ upon letting $\ell \to \infty$).

Suppose the contrary: $\widetilde{T}_k < t_* + \tau$. As each of $Y_a^{\vee\gamma_n}(s)$ is continuous, we must have $\Sigma_{\mathcal{A}_{B_*,k}}(\mathbf{Y}^{\vee\gamma_n}(\widetilde{T}_k)) = \frac{\varrho}{2}$, for some $\mathcal{A}_{B_*,k}$ within the truncation (6.12). Now, letting $\mathcal{K} = \mathcal{A}_{B_*,k}$ and $\mathcal{K}' = \mathcal{A}_{B_*-1,k} \cup \mathcal{A}_{B_*,k} \cup \mathcal{A}_{B_*+1,k}$, we then applying (6.2) for this $(\mathcal{K}, \mathcal{K}')$ (for some $\mathcal{I}$ to be specified latter) to derive a contradiction. This is done by showing that the last three terms in (6.2) are made arbitrarily small by choosing $K \in \mathbb{Z}_{>0}$ large enough and τ small enough. By (6.6)–(6.7), we have

$$|\mathcal{A}^{\mathbb{L}}_{b\pm1,k}|/|\mathcal{A}^{\mathbb{L}}_{b,k}| \le 16, \quad \forall b \in \mathbb{L},\ k \in \mathbb{Z}_{>0}. \tag{6.13}$$

Using this and (6.6), it is standard to show that such that the last two terms in (6.2) can be made arbitrarily small, $\forall k \ge K$, for some large enough $K \in \mathbb{Z}_{>0}$ and $q(\tau,1)$ small enough. See [7, proof of (6.18), (6.19)] for details. Next, turning to bounding the interaction term in (6.6), we use the continuity of each $Y_a^{\vee\gamma_n}(s)$ to obtain that $\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}^{\vee\gamma_n}(\widetilde{T}_k)) \ge \frac{\varrho}{2}$, for all $\mathcal{A}_{b,k}$ within the truncation (6.12). Further applying the continuity estimate (3.6) for $Y^* = Y_a^{\vee\gamma_n}$, it is not hard to show that, by choosing $K \in \mathbb{Z}_{>0}$ large enough and τ small enough, we have

$$\Sigma_{\mathcal{A}_{b,k}}(\underline{\mathbf{Y}}^{\vee\gamma_n}(t_*, \widetilde{T}_k)) \ge \tfrac{\varrho}{3}, \quad \forall \mathcal{A}_{b,k} \in \text{ the truncation (6.12)}. \tag{6.14}$$

For $|B_* - M^\pm| > 2$ (i.e. $\mathcal{A}_{B_*,k}$ sitting in the 'interior' of the truncation (6.12)), combining (6.14) with Lemma 4.4, we obtain some $I^\pm_{B_*} \in \mathcal{A}_{B_*\pm1,k}$ such that the interaction from gaps within $(J_-, J_+) \setminus (I^-_{B_*}, I^+_{B_*})$ is well under control. Taking into consideration the case $|B_* - M^\pm| < 2$, we let

$$I_+ := \begin{cases} I^+_{B_*+1}, \text{ if } B_* + 2 < J_+, \\ M_+\ \text{, otherwise,} \end{cases} \qquad I_- := \begin{cases} I^-_{B_*-1}, \text{ if } B_* - 2 > J_-, \\ M_-\ \text{, otherwise,} \end{cases}$$

and let $\mathcal{I} := (I_-, I_+)$. As mentioned in the preceding, the interaction from $(J_-, J_+) \setminus (I^-_{B_*}, I^+_{B_*})$ is well under control, so it remains to control the interaction from $(J_-, J_+)^c$. We do this by using (6.11). Even though this control seems to deteriorate when Z'_n is small, with Z'_n being independent of ℓ, we can always compensate this damage by letting $\ell \to \infty$. This is seen by considering the two cases where $\mathcal{A}_{B_*,k}$ is far from or close to the 'boundaries' $M_\pm$. For the former case the influence of $(J_-, J_+)^c$ on $\mathcal{A}_{B_*,k}$ decays as $|J_+ - J_-| \to \infty$, as the influence is 'mediated' by all the gaps within (J_-, J_+). For the latter case the $\frac{1}{|\mathcal{I}\cap\mathbb{Z}|}$ factor multiplying the interaction term decays because $\lim_{|b|\to\infty} |\mathcal{A}_{b,k}| = \infty$. □

Equipped with Proposition 6.2, we proceed to proving the following uniform density estimate.

Proposition 6.3. *For any $t \geq 0$, there exists some $K \in \mathbb{Z}_{>0}$, such that*

$$\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}(s)) \geq \tfrac{\rho}{2}, \quad \forall s \in [0,t],\ b \in \mathbb{L}, n \in \mathbb{Z}_{>0},\ k \geq K. \tag{6.15}$$

Proof. By (6.8) we have $\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}(0)) \geq \rho - ck^{-1}$. Hence for all large enough k: $k \geq k_0 = k_0(\rho, \mathbf{y}^{\text{in}})$, we have $\Sigma_{\mathcal{A}_{b,k}}(\mathbf{Y}(0)) > \frac{3\rho}{4}$, $\forall b \in \mathbb{L}$. With this and τ as in Proposition 6.2, applying Proposition 6.2 for $t_* = 0$ and $K_* = k_0$, we conclude (6.15) if $t \leq \tau$. To progress to $t > \tau$, we show that, actually, $\Sigma_{\mathcal{A}_{b,k\ell}}(\mathbf{Y}(\tau)) > \frac{3\rho}{4}$, for k further chosen large enough. Recall that we prove Proposition 6.2 by making the last three terms in (6.2) arbitrarily small by choosing K large enough and τ small enough. Upon letting K further larger (but keep τ fixed), we have that the interaction term and the Brownian term becomes smaller, but the last term $\widetilde{Q}^k_b$ may stay bounded away from zero, because the estimate (6.13) does not improve as $k \to \infty$. This problem is resolved by changing $k \mapsto k\ell$, which corresponds to grouping ℓ consecutive intervals of $\{\mathcal{A}_{b,k}\}_b$ to form a new, coarser, partition $\{\mathcal{A}_{b,k\ell}\}_b$. Fixing arbitrary $\mathcal{A}_{b,k\ell}$, we let $\mathcal{A}_\pm := \mathcal{A}_{\ell(b\pm 1/2)\pm 1/2,k}$ denote the neighboring 'small' intervals, and form the spliced interval $\widetilde{\mathcal{A}}' := \mathcal{A}_- \cup \mathcal{A}_{b,k\ell} \cup \mathcal{A}_+$. Let $\mathcal{I}'$ be such that $\mathcal{A}_{b,k\ell} \subset \mathcal{I}' \subset \widetilde{\mathcal{A}}'$. With such interval $\mathcal{I}'$ replacing $\mathcal{I}$, we have that $|\mathcal{A}^{\mathbb{L}}_\pm|/|\mathcal{A}^{\mathbb{L}}_{b,k\ell}| \to 0$ as $\ell \to \infty$, uniformly in $b \in \mathbb{L}$. With this improvement, we obtain that $\Sigma_{\mathcal{A}_{b,k\ell_1}}(\mathbf{Y}(\tau)) > \frac{3\rho}{4}$, for all $n \in \mathbb{Z}_{>0}$, $k \geq k_0 \vee K$ and some $\ell_1 = \ell_1(\rho)$, which then allows us to apply Proposition 6.2 for $K_* = \ell_1(k_0 \vee K)$ and $t_* = \tau$. Iterating the preceding procedure $i_* := \lceil t/\tau \rceil$ times, we conclude (6.15). □

Proof of (6.1). Here we give a sketch of the proof, and refer to [7, proof of Lemma 6.4] for the details.

Without lost of generality we assume $q(t,1) \leq \frac{\rho}{400}$. Combining Proposition 6.3 and the continuity estimate (3.6), we obtain

$$\mathcal{A}_{b,k}(Y^{\vee\gamma_n}(s)) \geq \tfrac{\rho}{3}, \quad \forall b \in \mathbb{Z},\ k \geq K,\ s \in [0,t], \tag{6.16}$$

for some $K \in \mathbb{Z}_{>0}$. Fix $m \in \mathbb{Z}$ and let b_* be such that $m \in (\widetilde{m}^K_{b_*-1/2}, \widetilde{m}^K_{b_*+1/2}]$. Combining (6.16) with Lemma 4.4, we obtain $I^\pm_{b,k} \in \mathcal{A}_{b,k}$ such that the interaction from outside of $(I^-_{b,k}, I^+_{b,k})$ is well under control. Using this in (6.2)–(6.3) yields the desired estimate of $|\Sigma_{\mathfrak{A}}(\mathbf{Y}^{\vee\gamma_n}(s)) - \Sigma_{\mathfrak{A}}(\mathbf{Y}^{\vee\gamma_n}(0))|$, for $\mathfrak{A}$ of the form $\mathfrak{A} = (I^-_{\frac{1}{2},K}, I^+_{b,K})$. This estimate is turned into estimate for $\mathcal{A} = (0,m)$ by comparing the difference of $\Sigma_{\mathcal{A}}(\mathbf{Y}^{\vee\gamma_n}(s))$ for $\mathcal{A} = (I^-_{\frac{1}{2},K}, I^+_{b_*,K})$ and for $\mathcal{A} = (0,m)$ at the 'boundaries' $\mathcal{A}_{1/2,K}$ and $\mathcal{A}_{b_*,K}$. □

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