RIMS Kôkyûroku Bessatsu
B59 (2016), 227–236

Enlargement of subgraphs of infinite graphs by Bernoulli percolation : A summary

By

Kazuki Okamura*

§ 1. Introduction

This article is a summary of a preprint [O] *by the author. Some questions and related issues, which are not stated in* [O], *will also be briefly described.*

A connected graph is called transient (resp. recurrent) if simple random walk on it is transient (resp. recurrent). Benjamini, Gurel-Gurevich and Lyons [BGGL] showed the cerebrating result claiming that the trace of simple random walk on a transient graph is recurrent almost surely. If a connected subgraph of an infinite connected graph is transient, then the infinite connected graph is transient. Therefore the trace is somewhat "smaller" than the graph on which the simple random walk run. Now we consider the following questions : How "far" are a transient graph G and the trace of simple random walk on G? More generally, how "far" are G and a recurrent subgraph H of G? How "many" edges of G do we need to add to H in order that the enlargement of H becomes transient?

There are numerous choices of edges of G to be added to H. In this article we add infinitely many edges of G to H *randomly*. Specifically we add open edges of Bernoulli bond percolation on G to H and consider the probability that the enlargement of H is transient.

We also consider questions of this kind for some properties other than transience. Let G be an infinite connected graph and H be a subgraph of G. Let $\mathcal{P}$ be a property for subgraphs of G. Assume that G satisfies $\mathcal{P}$ and H does not. Let $\mathcal{U}(H)$ be the graph obtained by adding open edges of Bernoulli bond percolation on G to H, specifically, the graph consisting of the connected components of unions of H and Bernoulli percolation

Received January 30, 2016.Revised April 17, 2016.
2010 Mathematics Subject Classification(s): 60K35, 82B41, 82B43, 05C63, 05C80, 05C81
Key Words: *Bernoulli percolation, critical probability, transient graph.*
Supported by Grand-in-Aid for Research Activity Start-Up no. 15H06311.
*RIMS, Kyoto University, Kyoto 606-8502, Japan.
e-mail: `kazukio@kurims.kyoto-u.ac.jp`

containing H. (see Definition 2.1 below.) Let $\mathbb{P}_p$ be the Bernoulli measure on the space of configurations of Bernoulli bond percolation on G such that each edge of G is open with probability $p \in (0,1)$. Then we consider the probability that $\mathcal{U}(H)$ satisfies $\mathcal{P}$ under $\mathbb{P}_p$. Let $p_{c,1}(G,H,\mathcal{P})$ (resp. $p_{c,2}(G,H,\mathcal{P})$) be the infimum of p such that the probability is positive (resp. one). For example, if $\mathcal{P}$ is being infinite and H is a subgraph consisting a vertex of G and no edges, then $p_{c,1}(G,H,\mathcal{P}) = p_c(G)$ and $p_{c,2}(G,H,\mathcal{P}) = 1$.

We compare $p_{c,1}(G,H,\mathcal{P})$ and $p_{c,2}(G,H,\mathcal{P})$ with Hammersley's critical probability $p_c(G)$. We focus on the following cases that $\mathcal{P}$ is being a transient subgraph, having finitely many cut points, having no cut points, being a recurrent subset, and, being connected. We also consider the case that H is chosen *randomly*, specifically, H is the trace of simple random walk on G.

We state related results. Benjamini, Häggström and Schramm [BHS] considered questions of this kind from a motivation different from ours. They introduced the notion called "percolating everywhere" (See Subsection 3.4 for the definition) and considered whether the following holds or not : if we add Bernoulli percolation to a percolating everywhere subgraph H then the enlargement of H is connected and moreover $p_c(\text{enlargement of } H) < 1$, $\mathbb{P}_p$-a.s. for any p. This case can be described by our terminologies. Specifically, G is $\mathbb{Z}^d$, H is a percolating everywhere subgraph, and, $\mathcal{P}$ is being connected and moreover $p_c(\mathcal{U}(H)) < 1$. They showed that if $d = 2$ then $p_{c,2}(G,H,\mathcal{P}) = 0$ by using planer duality, and, conjectured that it also holds for all $d \geq 2$. Recently Benjamini and Tassion [BeTa] showed the conjecture for all $d \geq 2$ by a method different from [BHS].

§ 2. Framework

In this article a graph is a locally-finite simple graph. Denote the sets of vertices and edges of a graph X by $V(X)$ and $E(X)$, respectively. If we consider the d-dimensional integer lattice $\mathbb{Z}^d$ then it is the nearest-neighbor model.

Let G be an infinite connected graph. We consider Bernoulli *bond* percolation on G. Denote a configuration by $\omega = (\omega_e)_{e \in E(G)} \in \{0,1\}^{E(G)}$. We say that an edge e is open if $\omega_e = 1$, and, closed otherwise. We say that an event $A \subset \{0,1\}^{E(G)}$ is increasing (resp. decreasing) if the following holds : whenever $\omega = (\omega_e) \in A$ and $\omega'_e \geq \omega_e$ (resp. $\omega'_e \leq \omega_e$) for any $e \in E(G)$ then $\omega' \in A$. Let C_x be the open cluster containing $x \in V(G)$. We remark that $\{x\} \subset V(C_x)$ holds. By convention we often denote the set of vertices $V(C_x)$ by C_x. Let $p_c(G)$ be Hammersley's critical probability of G. That is, for some $x \in V(G)$,

$$p_c(G) = \inf \{p \in (0,1) : \mathbb{P}_p(|C_x| = +\infty) > 0\}.$$

Definition 2.1 (Enlargement of subgraph). Let H be a subgraph of G. Let

$\mathcal{U}(H) = \mathcal{U}_\omega(H)$ be a random subgraph of G such that

$$V(\mathcal{U}(H)) := \bigcup_{x \in V(H)} V(C_x) \text{ and } E(\mathcal{U}(H)) := E(H) \cup \left(\bigcup_{x \in V(H)} E(C_x) \right).$$

If H is connected then $\mathcal{U}(H)$ is also connected. If H consists of a single vertex x with no edges then $\mathcal{U}(H)$ is identical to C_x.

In this article a *property*, which is denoted by $\mathcal{P}$, is a subset of the class of subgraphs of G which is invariant under any graph automorphism of G. We consider a property which is well-defined *only* on a class of subgraphs of G and call the class the *scope* of the property. For example, being a transient subgraph is defined only for connected subgraphs of G, and, the scope of being transient is the class of connected subgraphs of G. We denote $X \in \mathcal{P}$ (resp. $X \notin \mathcal{P}$) if a subgraph X of G is in the scope of $\mathcal{P}$ and satisfies (resp. does not satisfy) $\mathcal{P}$.

Let $\mathcal{F}$ be the cylindrical σ-algebra on the configuration space $\{0,1\}^{E(G)}$.

Asuumption 2.2. We assume that G, a subgraph H of G, and, a property $\mathcal{P}$ satisfy the following :

(i) G, H and $\mathcal{U}(H)$ are in the scope of $\mathcal{P}$.

(ii) $G \in \mathcal{P}$ and $H \notin \mathcal{P}$.

(iii) The event that $\mathcal{U}(H) \in \mathcal{P}$ is $\mathcal{F}$-measurable and increasing.

If H is chosen according to a probability law $(\Omega', \mathcal{F}', \mathbb{P}')$, then we assume that (i) and (ii) above hold $\mathbb{P}'$-a.s., and, the event $\mathcal{U}(H) \in \mathcal{P}$ is $\mathcal{F}' \otimes \mathcal{F}$-measurable, and, increasing for $\mathbb{P}'$-a.s. $\mathcal{F}' \otimes \mathcal{F}$ denotes the product σ-algebra of $\mathcal{F}'$ and $\mathcal{F}$.

We remark that there is an example of a triplet $(G, H, \mathcal{P})$ such that $\mathcal{U}(H) \in \mathcal{P}$ is *not* $\mathcal{F}$-measurable. (See [O, Example 2.6].)

Definition 2.3 (A certain kind of critical probability)**.**

$$p_{c,1}(G, H, \mathcal{P}) := \inf \{p \in [0,1] : \mathbb{P}_p(\mathcal{U}(H) \in \mathcal{P}) > 0\}.$$

$$p_{c,2}(G, H, \mathcal{P}) := \inf \{p \in [0,1] : \mathbb{P}_p(\mathcal{U}(H) \in \mathcal{P}) = 1\}.$$

If H obeys a law $\mathbb{P}'$ then we define $p_{c,i}(G, H, \mathcal{P})$, $i = 1, 2$, by replacing $\mathbb{P}_p$ above with the product measure $\mathbb{P}' \otimes \mathbb{P}_p$ of $\mathbb{P}'$ and $\mathbb{P}_p$.

If H is a single vertex and $\mathcal{P}$ is being an infinite graph then the definitions of $p_{c,1}(G, H, \mathcal{P})$ and $p_c(G)$ are identical and hence $p_{c,1}(G, H, \mathcal{P}) = p_c(G)$. It is easy to see that $p_{c,2}(G, H, \mathcal{P}) = 1$.

§ 3. Main results

We focus on each of the following properties : (i) being a transient subgraph, (ii) having finitely many cut points, or, having no cut points, (iii) being a recurrent subset, and, (iv) being a connected subgraph.

Scopes for properties (i) and (ii) are connected subgraphs of G, and, scopes for (iii) and (iv) are all subgraphs. We do not check whether $\{\mathcal{U}(H) \in \mathcal{P}\}$ is measurable. We refer readers to [O, Sections 3 to 6] for proofs of the following results. Approaches for proofs are different according to assertions. Here we give sketches of proofs of some of them.

§ 3.1. Being a transient subgraph

Theorem 3.1 (Extreme cases). *(i) There is a graph G such that*

$$0 < p_c(G) < p_{c,1}(G, H, \mathcal{P}) = 1.$$

(ii) There is a graph G such that for any infinite recurrent subgraph H of G,

$$p_{c,2}(G, H, \mathcal{P}) = 0.$$

For (i), we let G be the graph which is constructed as follows : Take $\mathbb{Z}^2$ and attach a transient tree T such that $p_c(T) = 1$ to the origin of $\mathbb{Z}^2$. This appears in Häggström and Mossel [HM, Section 6].

For (ii), we let G be an infinite connected line-graph in Benjamini and Gurel-Gurevich [BGG, Section 2]. In their paper, it is given as a graph having multi-lines, but we can construct a simple graph by adding a new vertex on each edge.

We give rough figures of the two graphs explaining them.

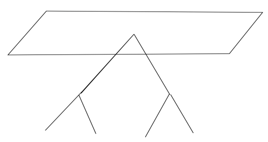

Figure 1. Graph for Theorem 3.1 (i)

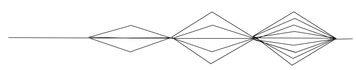

Figure 2. Graph for Theorem 3.1 (ii)

The graph G for Theorem 3.1 (ii) above has unbounded degrees. Now we consider a case that G has bounded degrees.

Theorem 3.2. *Let $G = \mathbb{Z}^d, d \geq 3$. Then for any $\epsilon > 0$ there is a recurrent subgraph H_ϵ such that $p_{c,2}(G, H_\epsilon, \mathcal{P}) \leq \epsilon$.*

The proof depends on a technical assertion [O, Lemma 2.1] and we need the assumption that ϵ is positive for it. We are not sure whether there is a recurrent subgraph H of $\mathbb{Z}^d$ such that

$$p_{c,2}(\mathbb{Z}^d, H, \mathcal{P}) = 0.$$

Theorem 3.3. *Let T be an infinite transient tree. Then*
(i) If T' is a recurrent subtree of T then

$$p_{c,1}(T, T', \mathcal{P}) = p_c(T).$$

(ii) If T' is an infinite recurrent subtree of T then

$$p_{c,2}(T, T', \mathcal{P}) = \sup\left\{p_c(H) : H \text{ is a transient subtree in } T \text{ and } E(H) \cap E(T') = \emptyset\right\}.$$

Let $\mathbb{T}_d$ be the d-regular tree, $d \geq 2$. The value $p_{c,2}$ depends on choices of a subgraph T' as the following example shows.

Example 3.4. Let T be the graph obtained by attaching a vertex of $\mathbb{T}_3$ to a vertex of $\mathbb{T}_4$.
(i) If T' is a subgraph of $\mathbb{T}_3$ which is isomorphic to $(\mathbb{N}, \{\{n, n+1\} : n \in \mathbb{N}\})$, then

$$p_{c,2}(T, T', \mathcal{P}) = p_c(\mathbb{T}_3) = \frac{1}{2}.$$

(ii) If T' is a subgraph of $\mathbb{T}_4$ which is isomorphic to $(\mathbb{N}, \{\{n, n+1\} : n \in \mathbb{N}\})$, then

$$p_{c,2}(T, T', \mathcal{P}) = p_c(\mathbb{T}_4) = \frac{1}{3}.$$

We now deal with Cayley graphs of finitely generated groups. (See Woess [W, Section 1.E] for the definition.) In this article all results concerning Cayley graphs of groups do not depend on choices of a generating set. Denote by $B(x, n)$ the open ball with center x and radius n with respect to the graph metric. We say that a graph G has the degree of growth $d \in (0, +\infty)$ if for any vertex x of G,

$$0 < \liminf_{n \to \infty} \frac{|B(x, n)|}{n^d} \leq \limsup_{n \to \infty} \frac{|B(x, n)|}{n^d} < +\infty.$$

Theorem 3.5. *Let G be a Cayley graph of a finitely generated countable group with the degree of growth $d \geq 3$. Let H be the trace of simple random walk on G. Then*

$$p_{c,1}(G, H, \mathcal{P}) \geq p_c(G).$$

The key point is to show that $\mathcal{U}(H)$ has at most the degree of growth 2, if $p < p_c(G)$.

Let $G = \mathbb{Z}^d$, $d \geq 3$ and H be the trace of simple random walk on G. Then by using the transience of infinite cluster by Grimmett-Kesten-Zhang [GKZ],

$$p_{c,2}(G, H, \mathcal{P}) \leq p_c(G).$$

Theorem 3.6. *Let $G = \mathbb{Z}^d$, $d \geq 3$ and H be the trace of simple random walk on G. Then,*

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = p_c(G).$$

In [BeTa, Subsection 1.2], examining properties for simple random walk on $\mathcal{U}(H)$ is stated as an open question. The results in this subsection may give answers to the question.

§ 3.2. Having finitely many cut points, or, having no cut points

In this subsection we assume that G is a transient graph and H is a recurrent subgraph of G.

Definition 3.7 (cut point). We say that a vertex $x \in V(G)$ is a *cut point* if we remove an edge e containing x, then the graph splits into two *infinite* connected components.

Theorem 3.8. *Let G be a Cayley graph of a finitely generated countable group with the degree of growth $d \geq 5$. Let H be the trace of two-sided simple random walk on G. Then if $p < p_c(G)$, then $\mathcal{U}(H)$ has infinitely many cut points, $P^{o,o} \otimes \mathbb{P}_p$-a.s.*

The key point is to show that

$$P^{o,o} \otimes \mathbb{P}_p\left(o \text{ is a cut point of } \mathcal{U}(H)\right) > 0. \tag{3.1}$$

We give a rough sketch of the proof of (3.1). We first show there exists a vertex z such that two simple random walks starting at o and z respectively do not intersect with positive probability. Then we "make" vertices in a large box closed and show the two random walks do not return to the large box with positive probability. Finally we choose a path connecting the two traces in a suitable way.

Theorem 3.9. *Let $G = \mathbb{Z}^d, d \geq 3$. Let H be the trace of two-sided simple random walk on $\mathbb{Z}^d$. If $p > p_c(G)$ then $\mathcal{U}(H)$ has no cut points $P^{0,0} \otimes \mathbb{P}_p$-a.s.*

The following considers this problem *at the critical point* in high dimensions. It is pointed out by Itai Benjamini. (personal communication). A recent paper by Fitzner and van der Hofstad [FvdH, Theorem 1.4] claims that the decay rate for the two-point function $\mathbb{P}_{p_c(\mathbb{Z}^d)}(0 \leftrightarrow x)$ is $|x|^{2-d}$ as $|x| \to +\infty$. This plays an important role in the following.

Theorem 3.10 (at critical point). *Let $G = \mathbb{Z}^d, d \geq 3$. Let H be the trace of two-sided simple random walk on $\mathbb{Z}^d$, $d \geq 11$. Let $p = p_c$. Then $\mathcal{U}(H)$ has infinitely many cut points $P^{o,o} \otimes \mathbb{P}_{p_c(\mathbb{Z}^d)}$-a.s.*

§ 3.3. Being a recurrent subset

In this subsection we assume that G is a transient graph. We say that a subset A of $V(G)$ is a *recurrent subset* if for some $x \in V(G)$

$$P^x \left(S_n \in A \text{ i.o. } n\right) > 0.$$

Otherwise A is called a *transient subset.* This definition does not depend on choices of a vertex $x \in V(G)$.

We regard a recurrent subset as a subgraph and consider the *induced subgraph* (See Diestel [D, Section 1.1] for this terminology) of the recurrent subset. In other words, if A is a recurrent subset of $V(G)$ then we consider the graph such that the set of vertices is A and the set of edges $\{\{x, y\} \in E(G) : x, y \in A\}$.

Theorem 3.11 (Extreme cases)**.** *(i) There is a graph G such that for any transient subset H of G,*

$$0 < p_c(G) < p_{c,1}(G, H, \mathcal{P}) = 1.$$

(ii) There is a graph G such that for any infinite transient subset H of G,

$$p_{c,2}(G, H, \mathcal{P}) = 0.$$

This assertion correspond to Theorem 3.1. Moreover, the two graphs in Theorem 3.1 (i) and (ii) give examples of graphs of this assertion, respectively.

Proposition 3.12. *Let $G = \mathbb{Z}^d$, $d \geq 3$. Then for any transient subset H of G*

$$p_{c,1}(G, H, \mathcal{P}) \leq p_c(G).$$

Theorem 3.13. *Let T be an infinite tree and H be a transient subset of T. Then, $p_{c,1}(T, H, \mathcal{P}) = 1$.*

Theorem 3.14. *Let G be a Cayley graph of a finitely generated countable group with polynomial growth $d \geq 3$. Let H be the trace of simple random walk on G. Then*
(i) If $d \geq 5$,

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = p_c(G).$$

(ii) If $d = 3, 4$,

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = 0.$$

Assertion (i) corresponds to Theorem 3.6. However, if $d = 3, 4$, the result is different from Theorem 3.6.

§ 3.4. Being a connected subgraph

We say that a subgraph H of G is *connected* if for any two vertices x and y of H there are vertices $x_0, \dots, x_n$ of H such that $x_0 = x$, $x_n = y$, and, $\{x_{i-1}, x_i\}$ is an edge of H for each i.

If H is *not* connected, then, $\mathcal{U}(H)$ can be disconnected. For example, if $(V(G), E(G)) = (\mathbb{Z}, \{n, n+1 : n \in \mathbb{Z}\})$ and $(V(H), E(H)) = (\mathbb{Z}, \emptyset)$, then,

$$\mathbb{P}_p(\mathcal{U}(H) \text{ is connected}) = 0, \quad p < 1.$$

We say that a subgraph H of G is *percolating everywhere* if $V(H) = V(G)$ and every connected component of H is infinite.

We introduce a notion concerning connectivity. For $A, B \subset V(G)$, we let

$$E(A, B) := \{\{y, z\} \in E(G) : y \in A, z \in B\}.$$

We say that G satisfies (TI) if for every $A, B \subset V(G)$ satisfy $V(G) = A \cup B$, $A \cap B = \emptyset$ and $|A| = |B| = +\infty$, then $E(A, B)$ is an infinite set. $\mathbb{Z}^d, d \geq 2$, satisfy (TI). On the other hand $\mathbb{T}_d, d \geq 2$, does not satisfy (TI). The trace of two-sided simple random walk on $\mathbb{Z}^d, d \geq 5$, does not satisfy (TI) a.s.

Theorem 3.15. *(i) If G satisfies (TI), then for any percolating everywhere subgraph H*

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}).$$

If the number of connected components of H is finite, then

$$p_{c,1}(G, H, \mathcal{P}) = p_{c,2}(G, H, \mathcal{P}) = 0.$$

(ii) If G does not satisfy (TI), then there is a percolating everywhere subgraph H such that

$$p_{c,1}(G, H, \mathcal{P}) = 0 \text{ and } p_{c,2}(G, H, \mathcal{P}) = 1.$$

[BeTa, Question 3 in Subsection 1.2] considers the question whether $p_{c,2}(G, H, \mathcal{P}) > 0$ or not for any amenable transitive graph G satisfying $p_c(G) < 1$. By Benjamini-Lyons-Peres-Schramm [BLPS, Theorem 13.8], any non-amenable graph G contains a subgraph H which is a spanning forest (i.e. percolating everywhere graph with no cycle) and $p_{c,1}(G, H, \mathcal{P}) > 0$.

§ 4. Questions

(1) Barlow and Taylor [BaTa] introduced the notion of a *discrete Hausdorff dimension* on $\mathbb{Z}^d$. [BaTa] shows the discrete Hausdorff dimension of the trace of simple

random walk on $\mathbb{Z}^d, d \ge 3$, is almost surely 2. It might be interesting to consider the case that $G = \mathbb{Z}^d$, H is the trace, and $\mathcal{P}$ is a property concerning the value of the dimension.

(2) Grimmett [G, Theorem 8.92] states that $p \mapsto \mathbb{P}_p(|C_0| = +\infty)$ is infinitely differentiable on $(p_c(\mathbb{Z}^d), 1)$. Fix $(G, H, \mathcal{P})$. Then consider regularity of $p \mapsto \mathbb{P}_p(\mathcal{U}(H) \in \mathcal{P})$ on $(p_{c,1}(G, H, \mathcal{P}), p_{c,2}(G, H, \mathcal{P}))$.

(3) For any p_1, p_2, p_3 with $0 < p_1 < p_2 < p_3 < 1$, is there $(G, H, \mathcal{P})$ such that

$$(p_{c,1}(G, H, \mathcal{P}), p_{c,2}(G, H, \mathcal{P}), p_c(G)) = (p_1, p_2, p_3), (p_1, p_3, p_2), \text{ or, } (p_2, p_3, p_1)?$$

Is there an example such that $0 < p_{c,1}(G, H, \mathcal{P}) < p_c(G)$?

(4) We can consider this issue for *site* percolation in a manner similar to the bond case.

(5) We may consider an analogous problem for continuum percolation : Let H be a deterministic or random subsets of $\mathbb{R}^d$ and add the Gilbert disk model or more generally the Poisson Boolean model (See Meester-Roy [MR]) to H.

Acknowledgement. The author thanks H. Duminil-Copin for notifying me of the reference [BeTa].

References

[BaTa] M. T. Barlow and S. J. Taylor, Defining fractal subsets of $\mathbb{Z}^d$, *Proc. London Math. Soc.* (3) 64 (1992) 125-152.

[BGG] I. Benjamini and O. Gurel-Gurevich, Almost sure recurrence of the simple random walk path, preprint, available at arXiv math 0508270.

[BGGL] I. Benjamini, O. Gurel-Gurevich, and R. Lyons, Recurrence of random walk traces, *Ann. Probab.* 35 (2007) 732-738.

[BHS] I. Benjamini, O. Häggström and O. Schramm, On the effect of adding ε-Bernoulli percolation to everywhere percolating subgraphs of $\mathbb{Z}^d$, *J. Math. Phys.* 41 (2000) 1294-1297.

[BLPS] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, Uniform spanning forests, *Ann. Probab.* 29 (2001) 1-65.

[BeTa] I. Benjamini and V. Tassion, Homogenization via sprinkling, arXiv 1505.06069.

[D] R. Diestel, *Graph theory*, Fourth edition, Springer, 2010.

[FvdH] R. Fitzner and R. van der Hofstad, Nearest-neighbor percolation function is continuous for $d > 10$: Extended version, arXiv 1506.07977.

[G] G. R. Grimmett, *Percolation*, 2nd edition, Springer, 1999.

[GKZ] G. R. Grimmett, H. Kesten and Y. Zhang, Random walk on the infinite cluster of the percolation model, *Probab. Theory Related Fields* 96 (1993) 33-44.

[HM] O. Häggström and E. Mossel. Nearest-neighbor walks with low predictability profile and percolation in $2 + \epsilon$ dimensions, *Ann. Probab.* 26 (1998) 1212-1231.

[MR] R. Meester and R. Roy, *Continuum Percolation.* Cambridge Univ. Press, 1996.

[O] K. Okamura, Enlargement of subgraphs of infinite graphs by Bernoulli percolation, arXiv:1506.05868.

[W] W. Woess, *Random walks an infinite graphs and groups*, Cambridge Univ. Press, 2000.