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Density preservation of unlabeled diffusion in systems with infinitely many particles

By

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Abstract

We consider an unlabeled diffusion with infinitely many particles and prove that the dynamics preserves density at the capacity level, that is, does not change the density of the system over the time evolution of the process.

§ 1. Introduction

Let S be a configuration space over $\mathbb{R}^d$ for $d \in \mathbb{N}$. We endow S with the vague topology. Let μ be a random point field on $\mathbb{R}^d$ with infinitely many particles, and consider a μ-reversible diffusion (X, P) with state space S. Here $\mathsf{X} = \{\mathsf{X}_t\}$ is of the form $\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$ and $\mathsf{P} = \{\mathsf{P}_\mathsf{s}\}_{\mathsf{s} \in \mathsf{S}}$ is the diffusion measure.

Suppose that for μ-a.s. s, there exists a limit $\lim_{r \to \infty} \mathsf{s}(S_r)/r^d$, where $S_r = \{x \in \mathbb{R}^d\, ;\, |x| < r\}$, and let

$$\Phi(\mathsf{s}) = \lim_{r \to \infty} \frac{\mathsf{s}(S_r)}{r^d}.$$

This assumption holds, for example, if μ is translation invariant. Note that Φ is tail σ-field measurable random variable by definition [see (2.2) below]. For a fixed positive constant θ, we set $\mathsf{A}_\theta = \{\mathsf{s}\, ;\, \Phi(\mathsf{s}) = \theta\}$. Then, from the reversibility of (X, P),

$$\mathsf{P}_\mu\Big(\lim_{r \to \infty} \frac{\mathsf{X}_t(S_r)}{r^d} = \theta\Big) = \mu(\mathsf{A}_\theta) \text{ for any } t. \tag{1.1}$$

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The purpose of this paper is to refine (1.1) such that for q.e. $\mathsf{s} \in \mathsf{A}_\theta$,

$$\mathsf{P}_{\mathsf{s}}\Big(\lim_{r\to\infty}\frac{\mathsf{X}_t(S_r)}{r^d}=\theta \text{ for any } t\Big)=1.$$

We prove that an unlabeled diffusion starting on a set that is specified in terms of density does not change the density over the course of its time evolution. This property is useful for the study of the dynamics of infinite particle systems.

Note that the set A_θ is an element of the tail σ-field of S. The tail σ-field plays an important role in the study of the properties of unlabeled diffusions. Indeed, the tail σ-field contains global information about infinite particle systems. A typical example is the particle density, as mentioned above. We are particularly interested in the tail-preserving property of unlabeled diffusions, that is, whether an unlabeled diffusion starts on an element of the tail σ-field, then it stays on the set permanently. However, the tail σ-field is not topologically well behaved; for example, it is not countably determined in general even if the state space is countably determined. Consequently, it is hard to treat the tail σ-field directly. Conversely, if the tail σ-field is identified by particle densities, we can discuss the behavior of an unlabeled diffusion on the field by studying the density instead of the field itself. Then, in some cases the tail-preserving property follows from the preservation of density.

Our result is closely related to the ergodic decomposition of unlabeled diffusions. Because the space of an unlabeled diffusion is huge, it is an important and difficult problem to specify the topological support when infinitely many particles are in motion. Our result is a first step toward addressing this problem.

Density preservation is also important from the point of view of infinite-dimensional stochastic differential equations (ISDEs), because the tail preserving property implies the strong uniqueness of a solution of an ISDE. We consider interacting Brownian motions with infinitely many particles having an interaction potential Ψ. The dynamics is described by the ISDE

$$dX_t^i = dB_t^i - \frac{1}{2}\sum_{i\neq j}\nabla_x\Psi(X_t^i, X_t^j)dt,\ 1\le i<\infty. \tag{1.2}$$

Lang began to study (1.2) using Itô's calculus [3, 4]. In this work, he assumed that Ψ is C_0^3 or exponentially decaying. Lang's result therefore does not work if Ψ is a long-range potential, for example, logarithmic. This work was followed by Fritz [1], Tanemura [10], and others. Recently, Tsai [11] solved (1.2) for the case in which Ψ is logarithmic and $d = 1$, that is, Dyson's Brownian motion in infinite dimensions. This result can be applied to out-of-equilibrium initial conditions, then this is a strong way to study ISDEs. On the other hand, the Dirichlet form approach can also solve (1.2) under assumptions including long-range potentials. In fact, Osada [5] constructed an unlabeled diffusion of (1.2) whenever Ψ is logarithmic potential using this approach. Then, using this unlabeled diffusion, (1.2) was again solved using Dirichlet forms [7]. Furthermore, the sufficiency condition that an ISDE of the form given by (1.2) has a unique strong solution has been shown by Osada and Tanemura [9]. They identified the sufficient conditions in the context of a random point field. Their results guarantee that an ISDE in the form of (1.2) has a unique strong solution when a random point field is tail trivial.

In addition, they also discussed the strong uniqueness of a solution of an ISDE when a random point field is *not* tail trivial. In this case, the random point field has multiple tails. They proved that if a solution of an ISDE satisfies the absolute continuity condition with respect to the random point field conditioned by the tail σ- field, then strong uniqueness holds. That is, so long as a solution has the tail-preserving property, strong uniqueness holds. However, they could not exclude existence of a solution that does not satisfy this condition. Proving that there is no solution such that the tail-preserving condition is not satisfied remain an open question in [9].

Our result addresses this problem in part. We can demonstrate the strong uniqueness of an ISDE in a more general situation than considered in [9]. In particular, this general theory can be applied to an ISDE related to random matrices. One of the most important examples of this is Dyson's Brownian motion with infinitely many particles, which has a logarithmic interaction potential. Then we can show that the strong uniqueness of Dyson's Brownian motion with multiple tails holds as a corollary of our result, but we do not pursue this topic here.

Density preservation is also important from the point of view of finite particle approximations of ISDEs. We will demonstrate that a solution of a finite dimensional stochastic differential equation converges to that of the corresponding ISDE as the particle number goes to infinity. One of the key points of the proof in the finite particle approximation is the uniqueness of a solution of an ISDE in the limit. Therefore, we can employ the finite particle approximation of an ISDE associated with many random point fields if we can prove that the tail-preserving property holds for an unlabeled diffusion associated with the random point fields.

This paper is organized as follows. In Section 2, we describe our framework and the main results. In Section 3, we prove the main result.

§ 2. Set up and main results

We begin by defining a random point field and introducing an unlabeled diffusion. Set $\mathrm{S} = \mathbb{R}^d$ $(d \geq 1)$ and let S be a configuration space over S defined by

$$\mathsf{S} = \Big\{ \mathsf{s} = \sum_i \delta_{s_i} \,;\, s_i \in \mathrm{S} \text{ with } \mathsf{s} \text{ is a Radon measure} \Big\}.$$

A probability measure μ on S is called a *random point field*.

A symmetric function $\rho^n : \mathrm{S}^n \to \mathbb{C}$ is called the *n-correlation function* of μ with respect to the Lebesgue measure if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(x_1,\dots,x_n)\prod_{i=1}^{n} dx_i = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!}\, d\mu(\mathsf{s})$$

holds for any disjoint compact sets $A_i \in \mathcal{B}(S)$ and $k_i \in \mathbb{N}$ such that $\sum_{i=1}^m k_i = n$.

Next, we define the Dirichlet form associated with μ. We set $S_r = \{x \in \mathrm{S}\,;\, |x| \le r\}$ and let $\pi_r : \mathsf{S} \to \mathsf{S}$ be a mapping such that $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$. A function f on S is called *local* if there exists an $r \in \mathbb{N}$ such that f is $\sigma[\pi_r]$-measurable and called *smooth* if $\check{f}$ is smooth, where $\check{f}((s_i)_i)$ is a permutation invariant function in $(s_i)_i \in \bigcup_{n\in\mathbb{N}} \mathrm{S}^n \cup \mathrm{S}^{\mathbb{N}}$ such that $f(\mathsf{s}) = \check{f}((s_i)_i)$. Let $\mathcal{D}_\circ$ be the set of all of local smooth functions on S.

For $f, g \in \mathcal{D}_\circ$, we define a bilinear form as

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_i \nabla_{s_i}\check{f}(\mathbf{s}) \cdot \nabla_{s_i}\check{g}(\mathbf{s}),$$

where $\mathsf{s} = \sum_i \delta_{s_i}$ and $\mathbf{s} = (s_i)_i$. We use the notation $\mathbb{D}[f]$ for $\mathbb{D}[f,f]$. Define a bilinear form $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ on $L^2(\mathsf{S},\mu)$ as

$$\mathcal{E}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s})\, d\mu(\mathsf{s}) \text{ for } f,g \in \mathcal{D}_\circ^\mu,$$
$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S},\mu)\,;\, \mathcal{E}(f,f) < \infty\}.$$

We further assume that

(A1) ρ^n is locally bounded for each $n \in \mathbb{N}$; and
(A2) $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mathsf{S},\mu)$.

Let $(\mathcal{E},\mathcal{D})$ be the closure of $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ on $L^2(\mathsf{S},\mu)$. It is known that, given **(A1)** and **(A2)**, $(\mathcal{E},\mathcal{D})$ is a local, quasi-regular Dirichlet form [5]. In particular, there exists an associated S-valued diffusion $(\mathsf{X}, \{\mathsf{P_s}\}_{\mathsf{s}\in\mathsf{H}})$ with state space $\mathsf{H} \subset \mathsf{S}$ such that $\mu(\mathsf{H}) = 1$. This S-valued diffusion is called the *unlabeled diffusion*.

Throughout this paper, we assume the random point field μ has infinitely many particles with probability 1, that is,

$$\mu(\mathsf{S}_\infty) = 1, \qquad \text{where } \mathsf{S}_\infty = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(\mathrm{S}) = \infty\}. \tag{2.1}$$

In addition to (2.1), we assume the following:

(A3) $\mathrm{Cap}^\mu(\mathsf{S}_\infty^c) = 0$.

Recall that for a subset $\mathsf{A} \subset \mathsf{S}$, $\mathrm{Cap}^\mu(\mathsf{A})$ denotes the capacity of A with respect to the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S},\mu))$.

Remark. It is known that if each tagged particle of $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ does not explode, then **(A3)** holds [6]. We do not explain about non-explosion property in this paper. Refer to [6] for this.

For **(A3)**, we can regard $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ as $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}_\infty, \mu))$ and the associated S-valued diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ as being S_∞-valued.

Hereafter, we fix the Dirichlet space $(\mathcal{E}, \mathcal{D}, L^2(\mathsf{S}_\infty, \mu))$ and consider the unlabeled diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ associated with it. We use the general concepts of Dirichlet form theory (see [2]).

For a non-decreasing function $f : [0, \infty) \to (0, \infty)$ that satisfy $\lim_{r\to\infty} f(r) = \infty$, we define random variables $\Phi_\pm(\mathsf{s}) : \mathsf{S}_\infty \to [0, \infty]$ as

$$\Phi_+(\mathsf{s}) = \limsup_{r\to\infty} \frac{\mathsf{s}(S_r)}{f(r)},$$
$$\Phi_-(\mathsf{s}) = \liminf_{r\to\infty} \frac{\mathsf{s}(S_r)}{f(r)}.$$

Let $\mathcal{T}(\mathsf{S}_\infty)$ be the tail σ-field given by

$$\mathcal{T}(\mathsf{S}_\infty) = \bigcap_{r\in\mathbb{N}} \sigma[\pi_r^c], \tag{2.2}$$

where $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$. For each $i \in \{+, -\}$, we define A_i as

$$\mathsf{A}_i = \{\mathsf{s}\,;\, \Phi_i(\mathsf{s}) = 1\}. \tag{2.3}$$

Note that $\mathsf{A}_i \in \mathcal{T}(\mathsf{S}_\infty)$, because $\Phi_\pm$ is $\mathcal{T}(\mathsf{S}_\infty)$-measurable.

Theorem 2.1. *With assumptions **(A1)**–**(A3)**, if f satisfies*

$$\lim_{r\to\infty} \frac{f(r+1)}{f(r)} = 1, \tag{2.4}$$

then the associated unlabeled diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ satisfies, for $i \in \{+, -\}$,

$$\mathsf{P}_{\mathsf{s}}(\tau_{\mathsf{A}_i} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_i, \tag{2.5}$$

Here, τ_{A} is the first exit time from A defined as

$$\tau_{\mathsf{A}} = \inf\{t > 0\,;\, \mathsf{X}_t \notin \mathsf{A}\}.$$

Recall that "q.e." in (2.5) is the abbreviation of "quasi-everywhere," which means that the equations holds with the exception of a set of zero capacity ([2, p.68]). Then (2.5) means that for fixed f and i, there exists $\mathsf{N}_{f,i}$ such that $\mathrm{Cap}^\mu(\mathsf{N}_{f,i}) = 0$ and $\mathsf{P}_{\mathsf{s}}(\tau_{\mathsf{A}_i} = \infty) = 1$ for any $\mathsf{s} \in \mathsf{A}_i \setminus \mathsf{N}_{f,i}$.

Remark. If f is a polynomial growth function, then it satisfies (2.4). Exponential growth functions do not.

Remark. Note that $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{H}})$ is μ-reversible by construction, and thus the following trivially holds:

$$\mathsf{P}_{\mu}(\mathsf{X}_t \in \mathsf{A}_i) = \mu(\mathsf{A}_i) \text{ for any } t \in [0,\infty). \tag{2.6}$$

Equation (2.6) implies that the probability of X_t being in A_i is invariant for each $t \in [0,\infty)$. It does not, however, provide information about the trajectory of the diffusion. In contrast, what we prove in Theorem 2.1 is that for q.e. $\mathsf{s} \in \mathsf{A}_i$,

$$\mathsf{P}_{\mathsf{s}}(\mathsf{X}_t \in \mathsf{A}_i \text{ for any } t \in [0,\infty)) = 1, \tag{2.7}$$

that is, A_i is an invariant set of the diffusion.

We next provide an application of Theorem 2.1. Fix a positive constant $\theta \in (0,\infty)$. Let A_θ represent all of the configurations with density θ given by

$$\mathsf{A}_\theta = \Big\{\mathsf{s}\,;\, \lim_{r\to\infty} \frac{\mathsf{s}(S_r)}{\mathrm{vol}(S_r)} = \theta\Big\}.$$

From Theorem 2.1 by choosing $f(r) = \theta\,\mathrm{vol}(S_r)$, we obtain the corollary that the associated unlabeled diffusion does not change its density over the time evolution:

Corollary 2.2. *Given assumptions **(A1)**–**(A3)**, for each $\theta \in (0,\infty)$ the associated unlabeled diffusion satisfies*

$$\mathsf{P}_{\mathsf{s}}(\tau_{\mathsf{A}_\theta} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_\theta.$$

§ 3. Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1. We begin by introducing cut off functions. Let $\mathfrak{u} : \mathrm{S}^{\mathbb{N}} \to \mathsf{S}_\infty$ be an unlabeled map defined as

$$\mathfrak{u}(\mathbf{s}) = \sum_{i\in\mathbb{N}} \delta_{s_i} \text{ for } \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathrm{S}^{\mathbb{N}}.$$

A mapping $\mathfrak{l} : \mathsf{S}_\infty \to \mathrm{S}^{\mathbb{N}}$ is called a labeled map if $\mathfrak{l}$ is measurable and $\mathfrak{u}\circ\mathfrak{l}$ is the identity.

We fix a non-decreasing sequence $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}} \subset \mathbb{N}$ and a label $\mathfrak{l} = (\mathfrak{l}_1, \mathfrak{l}_2, \dots)$ satisfying $|\mathfrak{l}_j(\mathsf{s})| \le |\mathfrak{l}_{j+1}(\mathsf{s})|$ for any $j \in \mathbb{N}$. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\rho(t) = \begin{cases} 1, & t \in (-\infty, 0], \\ 0, & t \in [1,\infty), \end{cases}$$

and let c_1 be a positive constant given by

$$c_1 := \sup_{x \in \mathbb{R}} |\rho\prime(x)| (< \infty).$$

We set

$$J_{r,\mathsf{s},+} = \{j\,;\, j > a_r, \mathfrak{l}_j(\mathsf{s}) \in S_r\}.$$

Then, for each $m \in \mathbb{N}$, we define $\chi_+^m[\mathbf{a}] : \mathsf{S}_\infty \to [0,1]$ as

$$\chi_+^m[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},+}^m(\mathsf{s}),$$

where

$$h_{\mathbf{a},+}^m(\mathsf{s}) = \frac{\log(d_{\mathbf{a},+}^m(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},+}^m(\mathsf{s}) = \sum_{r=m}^{\infty} \sum_{j \in J_{r,\mathsf{s},+}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

Similarly, we set

$$J_{r,\mathsf{s},-} = \{j\,;\, j < a_r, \mathfrak{l}_j(\mathsf{s}) \in S_r^c\},$$

and for each $m \in \mathbb{N}$, define $\chi_-^m[\mathbf{a}] : \mathsf{S}_\infty \to [0,1]$ as

$$\chi_-^m[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},-}^m(\mathsf{s}),$$

where

$$h_{\mathbf{a},-}^m(\mathsf{s}) = \frac{\log(d_{\mathbf{a},-}^m(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},-}^m(\mathsf{s}) = \sum_{r=m}^{\infty} \sum_{j \in J_{r,\mathsf{s},-}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

In addition, we prepare maps approximating $\chi_i^m[\mathbf{a}]$ for $i \in \{+,-\}$. Let

$$\chi_i^{m,s}[\mathbf{a}](\mathsf{s}) = \rho \circ h_{\mathbf{a},i}^{m,s}(\mathsf{s}),$$

where

$$h_{\mathbf{a},i}^{m,s}(\mathsf{s}) = \frac{\log(d_{\mathbf{a},i}^{m,s}(\mathsf{s}) + 1)}{\log 2}, \quad d_{\mathbf{a},i}^{m,s}(\mathsf{s}) = \sum_{r=m}^{s} \sum_{j \in J_{r,\mathsf{s},i}} (r - |\mathfrak{l}_j(\mathsf{s})|)^2.$$

Clearly, $\chi_+^{m,s}[\mathbf{a}]$ is thus $\sigma[\pi_s]$-measurable. By the definition of S_∞, $\chi_-^{m,s}[\mathbf{a}]$ is $\sigma[\pi_{s+1}]$-measurable. Therefore, for each $i \in \{+,-\}$,

$$\chi_i^{m,s}[\mathbf{a}] \in \mathcal{D}_\circ.$$

Furthermore, it is easily deduced that $\lim_{s\to\infty} \chi_i^{m,s}[\mathbf{a}] = \chi_i^m[\mathbf{a}]$ in $L^2(\mathsf{S}_\infty, \mu)$.

Lemma 3.1. *Recall that $c_1 = \sup_{x\in\mathbb{R}} |\rho\prime(x)| < \infty$. For each $i \in \{+,-\}$ and each $m, s \in \mathbb{N}$,*

$$\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) \leq \frac{2c_1^2}{(\log 2)^2} \frac{d_{\mathbf{a},i}^{m,s}(\mathsf{s})}{(d_{\mathbf{a},i}^{m,s}(\mathsf{s})+1)^2}. \tag{3.1}$$

In particular, there exists a positive constant c_2 independent of m, $\mathbf{a}$, i, and s such that

$$\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) \leq c_2. \tag{3.2}$$

Proof. Easy calculation yields (3.1). In fact, we have

$$\begin{aligned}\mathbb{D}[\chi_i^{m,s}[\mathbf{a}]](\mathsf{s}) &= \frac{1}{2}\sum_{r=m}^{s}\sum_{j\in J_{r,\mathsf{s},i}} \left\{ \frac{\rho\prime(h_{\mathbf{a},i}^{m,s}(\mathsf{s}))}{\log 2} \frac{2(r-|\mathfrak{l}_j(\mathsf{s})|)}{d_{\mathbf{a},i}^{m,s}(\mathsf{s})+1} \right\}^2 \\ &\leq \frac{2c_1^2}{(\log 2)^2} \cdot \frac{d_{\mathbf{a},i}^{m,s}(\mathsf{s})}{(d_{\mathbf{a},i}^{m,s}(\mathsf{s})+1)^2}.\end{aligned}$$

Equation (3.2) then follows from (3.1) immediately. □

For a given non-decreasing sequence $\mathbf{a} = \{a_r\}_{r\in\mathbb{N}}$, we set

$$\begin{aligned}\mathsf{S}_+^m[\mathbf{a}] &= \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \mathsf{s}(S_r) \leq a_r \text{ for any } r \geq m\},\\ \mathsf{S}_-^m[\mathbf{a}] &= \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \mathsf{s}(S_r) \geq a_r \text{ for any } r \geq m\}.\end{aligned}$$

Clearly, the $\mathsf{S}_\pm^m[\mathbf{a}]$ are non-decreasing sets with respect to m. For given $\mathbf{a}$, we define new sequences $\mathbf{a}^\pm = \{a_{r\pm 1}\}_{r\in\mathbb{N}}$. We use the bilinear form $\mathcal{E}_1$ given by $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(\mathsf{S}_\infty,\mu)}$ for $u, v \in \mathcal{D}$. Below, $||u||_{\mathcal{E}_1}$ denotes the norm with respect to $\mathcal{E}_1(u,u)$. Note that $(\mathcal{E}_1, \mathcal{D})$ is a Hilbert space.

Lemma 3.2. *For each $i \in \{+,-\}$ and each $m \in \mathbb{N}$, the following hold:*
(i) $\chi_i^m[\mathbf{a}] = 1$ on $\mathsf{S}_i^m[\mathbf{a}]$ and $\chi_i^m[\mathbf{a}] = 0$ on $(\mathsf{S}_i^m[\mathbf{a}^i])^c$.
(ii)

$$\lim_{s\to\infty} \chi_i^{m,s}[\mathbf{a}] = \chi_i^m[\mathbf{a}] \,\text{weakly in}\, (\mathcal{E}_1, \mathcal{D}). \tag{3.3}$$

(iii) $\chi_i^m[\mathbf{a}] \in \mathcal{D}$.

Proof. From the definition of $\chi_+^m[\mathbf{a}]$ and $\chi_-^m[\mathbf{a}]$, we obtain (i) immediately.

Equation (3.2) implies that $\sup_{s\in\mathbb{N}} ||\chi_i^{m,s}[\mathbf{a}]||_{\mathcal{E}_1} \leq \sqrt{1+c_2}$. Using this together with the $L^2(\mu)$-convergence of $\chi_i^{m,s}[\mathbf{a}]$, we obtain (ii)

Clearly, (iii) follows from (ii). □

Lemma 3.3. *For each $i \in \{+,-\}$, $\{\chi_i^m[\mathbf{a}]\}_{m\in\mathbb{N}}$ is a Cauchy sequences in $\mathcal{E}_1$.*

Proof. We prove only the case in which $i = +$; the $(-)$–case can be demonstrated similarly.

Let δ be a constant satisfying $0 < \delta < 1$. We define subsets $\mathsf{S}_1^{M,\delta}$ and $\mathsf{S}_2^{M,\delta}$ for each $M \in \mathbb{N}$ as

$$\mathsf{S}_1^{M,\delta} = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, d_{\mathbf{a},+}^M(\mathsf{s}) < \delta\},$$
$$\mathsf{S}_2^{M,\delta} = \{\mathsf{s} \in \mathsf{S}_\infty \,;\, \delta \le d_{\mathbf{a},+}^M(\mathsf{s}) < \infty\}.$$

We can and do take M sufficiently large that

$$\mu(\mathsf{S}_2^{M,\delta}) \le \delta. \tag{3.4}$$

From (3.3), we have

$$\begin{aligned} &||\chi_+^l[\mathbf{a}] - \chi_+^m[\mathbf{a}]||_{\mathcal{E}_1}^2 \\ &\le \liminf_{s\to\infty} ||\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]||_{\mathcal{E}_1}^2 \\ &= \liminf_{s\to\infty} \Big\{ \int_{\mathsf{S}_\infty} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{\mathsf{S}_\infty} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \Big\}. \end{aligned} \tag{3.5}$$

We set $\mathsf{S}^{M,\delta} = \mathsf{S}_1^{M,\delta} + \mathsf{S}_2^{M,\delta}$ and

$$\mathsf{S}^{l,m,s} = \{\mathsf{s}\,;\, d_{\mathbf{a},+}^{l,s}(\mathsf{s}) < 1 \text{ or } d_{\mathbf{a},+}^{m,s}(\mathsf{s}) < 1\}.$$

Clearly,

$$\lim_{s\to\infty} \mu((\mathsf{S}^{M,\delta})^c \cap \mathsf{S}^{l,m,s}) = 0. \tag{3.6}$$

By the definition of $\chi_+^{m,s}[\mathbf{a}]$,

$$\chi_+^{l,s}[\mathbf{a}] = \chi_+^{m,s}[\mathbf{a}] \text{ on } (\mathsf{S}^{l,m,s})^c \text{ for } l, m \in \mathbb{N}.$$

From this and (3.2), we have

$$\begin{aligned} &\int_{(\mathsf{S}^{M,\delta})^c} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{(\mathsf{S}^{M,\delta})^c} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \\ &= \int_{(\mathsf{S}^{M,\delta})^c \cap \mathsf{S}^{l,m,s}} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{(\mathsf{S}^{M,\delta})^c \cap \mathsf{S}^{l,m,s}} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \\ &\le (1 + 4c_2)\mu((\mathsf{S}^{M,\delta})^c \cap \mathsf{S}^{l,m,s}). \end{aligned}$$

Combining this and (3.6), we conclude

$$\lim_{s\to\infty} \{ \int_{(\mathsf{S}^{M,\delta})^c} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{(\mathsf{S}^{M,\delta})^c} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \} = 0. \tag{3.7}$$

By virtue of Lipschitz continuity, there exists a positive constant c_3 such that

$$|\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]| \le c_3 |d_{\mathbf{a},+}^{l,s}(\mathsf{s}) - d_{\mathbf{a},+}^{m,s}(\mathsf{s})|. \tag{3.8}$$

Note that $d_{\mathbf{a},+}^{m}(\mathsf{s}) \le d_{\mathbf{a},+}^{M}(\mathsf{s})$ for $m \ge M$ and $d_{\mathbf{a},+}^{m,s}(\mathsf{s}) \le d_{\mathbf{a},+}^{m}(\mathsf{s})$. Then, for $s \ge m \ge M$,

$$d_{\mathbf{a},+}^{m,s}(\mathsf{s}) < \delta \text{ on } \mathsf{S}_1^{M,\delta}. \tag{3.9}$$

Therefore, for each $l, m \ge M$, we have from (3.1), (3.8), and (3.9),

$$\begin{aligned} &\int_{\mathsf{S}_1^{M,\delta}} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{\mathsf{S}_1^{M,\delta}} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \\ &\quad < c_3^2\delta^2 + \frac{8c_1^2}{(\log 2)^2}\delta. \end{aligned} \tag{3.10}$$

From (3.2) and (3.4), we deduce that

$$\begin{aligned} &\int_{\mathsf{S}_2^{M,\delta}} |\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]|^2 \, d\mu(\mathsf{s}) + \int_{\mathsf{S}_2^{M,\delta}} \mathbb{D}[\chi_+^{l,s}[\mathbf{a}] - \chi_+^{m,s}[\mathbf{a}]] \, d\mu(\mathsf{s}) \\ &\quad \le \mu(\mathsf{S}_2^{M,\delta})(1 + 4c_2) \\ &\quad \le \delta(1 + 4c_2). \end{aligned} \tag{3.11}$$

Combining (3.5), (3.7), (3.10), and (3.11), we conclude that for any δ satisfying $0 < \delta < 1$, there exists $M \in \mathbb{N}$ such that for any $l, m \ge M$,

$$||\chi_+^{l}[\mathbf{a}] - \chi_+^{m}[\mathbf{a}]||_{\mathcal{E}_1} < \left\{c_3^2\delta^2 + \frac{8c_1^2}{(\log 2)^2}\delta + \delta(1 + 4c_2)\right\}^{1/2}.$$

Hence, $\{\chi_+^m[\mathbf{a}]\}_{m\in\mathbb{N}}$ is a Cauchy sequences in $\mathcal{E}_1$. □

For a subset $\mathsf{B} \subset \mathsf{S}_\infty$, an element $e_{\mathsf{B}} \in \mathcal{D}$ is called the 1-equilibrium potential of B if $\tilde{e_{\mathsf{B}}} = 1$ q.e. on B and $\mathcal{E}_1(e_{\mathsf{B}}, v) \ge 0$ for any $v \in \mathcal{D}$ satisfying $\tilde{v} \ge 0$ q.e. on B. Here, $\tilde{u}$ is a quasi-continuous μ-version of $u \in \mathcal{D}$.

Lemma 3.4. *Take $i \in \{+, -\}$ and set*

$$\mathsf{S}_i^{\mathbf{a}} = \bigcup_{m\in\mathbb{N}} \mathsf{S}_i^m[\mathbf{a}], \quad \mathsf{S}_i^{\mathbf{a}^i} = \bigcup_{m\in\mathbb{N}} \mathsf{S}_i^m[\mathbf{a}^i].$$

Assume that

$$\mu((\mathsf{S}_i^{\mathbf{a}})^c \cap \mathsf{S}_i^{\mathbf{a}^i}) = 0. \tag{3.12}$$

Then

$$\lim_{m\to\infty} \chi_i^m[\mathbf{a}] = e_{\mathsf{S}_i^{\mathbf{a}}} \text{ in } \mathcal{E}_1, \tag{3.13}$$

$$1 - \lim_{m\to\infty} \chi_i^m[\mathbf{a}] = e_{(\mathsf{S}_i^{\mathbf{a}})^c} \text{ in } \mathcal{E}_1. \tag{3.14}$$

Furthermore, we have

$$\tilde{e}_{\mathsf{S}_i^{\mathbf{a}}} = 0 \text{ for q.e. } \mathsf{s} \in (\mathsf{S}_i^{\mathbf{a}})^c, \tag{3.15}$$

$$\tilde{e}_{(\mathsf{S}_i^{\mathbf{a}})^c} = 0 \text{ for q.e. } \mathsf{s} \in \mathsf{S}_i^{\mathbf{a}}. \tag{3.16}$$

Proof. We give a proof only for $i = +$; The $(-)$–case can be proved similarly.

First, there exists a $u \in \mathcal{D}$ such that $\lim_{m\to\infty} \chi_+^m[\mathbf{a}] = u$ in $\mathcal{E}_1$ from Lemma 3.3. To show (3.13), it is enough to prove that $\tilde{u} = 1$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$ and $\mathcal{E}_1(u,v) \geq 0$ for any $v \in \mathcal{D}$ with $\tilde{v} \geq 0$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$.

From Lemma 3.2 (i), we have $u = 1$ μ-a.e. on $\mathsf{S}_+^{\mathbf{a}}$. Here, we use the monotonicity of $\mathsf{S}_+^m[\mathbf{a}]$. Therefore, we can take $\tilde{u}$ as a version of u such that $\tilde{u} = 1$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$.

Next, we take $v \in \mathcal{D}$ such that $\tilde{v} \geq 0$ q.e. on $\mathsf{S}_+^{\mathbf{a}}$. We use the result that $u = 1$ μ-a.e. on $\mathsf{S}_+^{\mathbf{a}}$ to obtain

$$\int_{\mathsf{S}_+^{\mathbf{a}} \cap \mathsf{S}_+^{\mathbf{a}^+}} u(\mathsf{s})v(\mathsf{s})\, d\mu(\mathsf{s}) \geq 0. \tag{3.17}$$

We have $u = 0$ μ-a.e. on $(\mathsf{S}_+^{\mathbf{a}^+})^c$ from Lemma 3.2 (i) and the monotonicity. From this and (3.17), we deduce

$$\begin{aligned}\int_{\mathsf{S}_\infty} u(\mathsf{s})v(\mathsf{s})\, d\mu(\mathsf{s}) &= \int_{\mathsf{S}_+^{\mathbf{a}^+}} u(\mathsf{s})v(\mathsf{s})\, d\mu(\mathsf{s}) \\ &= \Big\{ \int_{\mathsf{S}_+^{\mathbf{a}} \cap \mathsf{S}_+^{\mathbf{a}^+}} + \int_{(\mathsf{S}_+^{\mathbf{a}})^c \cap \mathsf{S}_+^{\mathbf{a}^+}} \Big\} u(\mathsf{s})v(\mathsf{s})\, d\mu(\mathsf{s}) \\ &\geq 0.\end{aligned} \tag{3.18}$$

Here we have used the fact that the second term in the second line in (3.18) vanishes because of (3.12).

Next we consider $\mathcal{E}(u,v)$. Let $\{v_m\}_{m=1}^\infty \subset \mathcal{D}_\circ$ such that $\lim_{m\to\infty} v_m = v$ in $\mathcal{E}_1$. Recall that $\lim_{m\to\infty} \chi_+^m[\mathbf{a}] = u$ in $\mathcal{E}_1$. Then

$$\begin{aligned}\mathcal{E}(u,v)^2 &\leq \mathcal{E}(u,u)\mathcal{E}(v,v) \\ &= \lim_{m\to\infty} \mathcal{E}(\chi_+^m[\mathbf{a}], \chi_+^m[\mathbf{a}])\mathcal{E}(v,v) \\ &\leq \lim_{m\to\infty} \liminf_{s\to\infty} \mathcal{E}(\chi_+^{m,s}[\mathbf{a}], \chi_+^{m,s}[\mathbf{a}])\mathcal{E}(v,v).\end{aligned} \tag{3.19}$$

We set

$$\mathsf{S}_+^{m,s}[\mathbf{a}] = \{\mathsf{s} \in \mathsf{S}_\infty\, ;\, \mathsf{s}(S_r) \leq a_r \text{ for any } r \text{ satisfying } s \geq r \geq m\}.$$

Note that $\mathsf{S}_+^{m,s}[\mathbf{a}]$ is a non-increasing set with respect to s. Because $\chi_+^{m,s}[\mathbf{a}]$ is constant on $\mathsf{S}_+^{m,s}[\mathbf{a}] \cup (\mathsf{S}_+^{m,s}[\mathbf{a}^+])^c$ by definition,

$$\mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s}) = 0 \text{ on } \mathsf{S}_+^{m,s}[\mathbf{a}] \cup (\mathsf{S}_+^{m,s}[\mathbf{a}^+])^c.$$

From this, we have

$$\mathcal{E}(\chi_+^{m,s}[\mathbf{a}],\chi_+^{m,s}[\mathbf{a}]) = \int_{\mathsf{S}} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\,d\mu(\mathsf{s})$$

$$= \int_{(\mathsf{S}_+^{m,s}[\mathbf{a}])^c\cap\mathsf{S}_+^{m,s}[\mathbf{a}^+]} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\,d\mu(\mathsf{s}). \tag{3.20}$$

From (3.2) and $(\mathsf{S}_+^{m,s}[\mathbf{a}])^c \subset (\mathsf{S}_+^{m}[\mathbf{a}])^c$,

$$\int_{(\mathsf{S}_+^{m,s}[\mathbf{a}])^c\cap\mathsf{S}_+^{m,s}[\mathbf{a}^+]} \mathbb{D}[\chi_+^{m,s}[\mathbf{a}]](\mathsf{s})\,d\mu(\mathsf{s}) \le c_2\mu((\mathsf{S}_+^{m,s}[\mathbf{a}])^c\cap\mathsf{S}_+^{m,s}[\mathbf{a}^+])$$

$$\le c_2\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c\cap\mathsf{S}_+^{m,s}[\mathbf{a}^+]).$$

Combining this and (3.20), we have

$$\liminf_{s\to\infty}\mathcal{E}(\chi_+^{m,s}[\mathbf{a}],\chi_+^{m,s}[\mathbf{a}]) \le c_2\liminf_{s\to\infty}\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c\cap\mathsf{S}_+^{m,s}[\mathbf{a}^+]) \tag{3.21}$$

$$= c_2\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c\cap\mathsf{S}_+^{m}[\mathbf{a}^+]).$$

We use the monotonicity of $\mathsf{S}_+^{m,s}[\mathbf{a}^+]$ in the last line. From (3.19), (3.21), and the monotonicity of $\mathsf{S}_+^{m}[\mathbf{a}]$ with respect to m, we have

$$\begin{aligned}\mathcal{E}(u,v)^2 &\le \lim_{m\to\infty} c_2\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c\cap\mathsf{S}_+^{m}[\mathbf{a}^+])\mathcal{E}(v,v)\\ &\le \lim_{m\to\infty} c_2\mu((\mathsf{S}_+^{m}[\mathbf{a}])^c\cap\mathsf{S}_+^{\mathbf{a}^+})\mathcal{E}(v,v)\\ &= c_2\mu((\mathsf{S}_+^{\mathbf{a}})^c\cap\mathsf{S}_+^{\mathbf{a}^+})\mathcal{E}(v,v).\end{aligned}$$

Consequently, we find that $\mathcal{E}(u,v)=0$ by virtue of (3.12). Combining this and (3.18), we have $\mathcal{E}_1(u,v)\ge 0$. Then we conclude $u = e_{\mathsf{S}_+^{\mathbf{a}}}$. Equation (3.15) is clear because we have $u=0$ μ-a.e. on $(\mathsf{S}_i^{\mathbf{a}})^c$ from the discussion above.

Finally, (3.14) and (3.16) are deduced easily from (3.13) and (3.15). □

Theorem 2.1 follows from Lemma 3.4 with an appropriate choice of $\mathbf{a}$. For small $\varepsilon>0$, we set

$$\mathbf{a}_\varepsilon = \{(f(r)(1-\varepsilon))\}_{r\in\mathbb{N}},\quad \mathbf{b}_\varepsilon = \{(f(r)(1+\varepsilon))\}_{r\in\mathbb{N}}, \tag{3.22}$$

as a non-decreasing sequence. We further set

$$\mathsf{A}_\varepsilon = \{\mathsf{s}\,;\,\Phi_+(\mathsf{s})<1-\varepsilon\},\ \mathsf{B}_\varepsilon = \{\mathsf{s}\,;\,\Phi_+(\mathsf{s})>1+\varepsilon\},$$
$$\mathsf{C}_\varepsilon = \{\mathsf{s}\,;\,\Phi_-(\mathsf{s})<1-\varepsilon\},\ \mathsf{D}_\varepsilon = \{\mathsf{s}\,;\,\Phi_-(\mathsf{s})>1+\varepsilon\}.$$

Lemma 3.5. *With* $\mathsf{A}_\pm$ *as in* (2.3)*, the following hold:*

$$\mathsf{A}_\varepsilon \subset \bigcup_{m\in\mathbb{N}}\mathsf{S}_+^{m}[\mathbf{a}_\varepsilon],\ \mathsf{B}_\varepsilon \subset \Big(\bigcup_{m\in\mathbb{N}}\mathsf{S}_+^{m}[\mathbf{b}_\varepsilon]\Big)^c, \tag{3.23}$$

$$\mathsf{C}_\varepsilon \subset \Big(\bigcup_{m\in\mathbb{N}}\mathsf{S}_-^{m}[\mathbf{a}_\varepsilon]\Big)^c,\ \mathsf{D}_\varepsilon \subset \bigcup_{m\in\mathbb{N}}\mathsf{S}_-^{m}[\mathbf{b}_\varepsilon], \tag{3.24}$$

and

$$\mathsf{A}_+ \subset \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{a}_\varepsilon]\Big)^c \cap \bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{b}_\varepsilon], \tag{3.25}$$

$$\mathsf{A}_- \subset \bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{a}_\varepsilon] \cap \Big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_-^m[\mathbf{b}_\varepsilon]\Big)^c. \tag{3.26}$$

Proof. The first inclusion relation in (3.23) is obvious by

$$\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{a}_\varepsilon] = \bigcup_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) \le f(r)(1-\varepsilon), {}^\forall r \ge m\}.$$

The second inclusion relation in (3.23) follows from

$$\begin{aligned}\big(\bigcup_{m\in\mathbb{N}} \mathsf{S}_+^m[\mathbf{b}_\varepsilon]\big)^c &= \big(\bigcup_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) \le f(r)(1+\varepsilon), {}^\forall r \ge m\}\big)^c \\ &= \bigcap_{m\in\mathbb{N}} \{\mathsf{s}\,;\, \mathsf{s}(S_r) > f(r)(1+\varepsilon), {}^\exists r \ge m\}.\end{aligned}$$

Equations (3.24), (3.25), and (3.26) can be checked in a similar way. □

Proof of Theorem 2.1. We use Lemma 3.4 for the non-decreasing sequence (3.22). For (2.4), we can take arbitrary small $\varepsilon > 0$ in (3.22), which yields (3.12). In fact, let $\mathsf{S}_+^{\mathbf{a}_\varepsilon}$ and $\mathsf{S}_+^{\mathbf{a}_\varepsilon^+}$ be as in Lemma 3.4, then we have

$$(\mathsf{S}_+^{\mathbf{a}_\varepsilon})^c = \bigcap_{m\in\mathbb{N}} \{\mathsf{s}(S_r) > f(r)(1-\varepsilon), {}^\exists r \ge m\} \subset \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) \ge 1-\varepsilon\}, \tag{3.27}$$

and

$$\mathsf{S}_+^{\mathbf{a}_\varepsilon^+} = \bigcup_{m\in\mathbb{N}} \{\mathsf{s}(S_r) \le f(r+1)(1-\varepsilon), {}^\forall r \ge m\} \subset \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) \le 1-\varepsilon\}. \tag{3.28}$$

Combining (3.27) and (3.28), we obtain

$$(\mathsf{S}_+^{\mathbf{a}_\varepsilon})^c \cap \mathsf{S}_+^{\mathbf{a}_\varepsilon^+} \subset \{\mathsf{s}\,;\, \Phi_+(\mathsf{s}) = 1-\varepsilon\}.$$

From this, (3.12) is satisfied for $\mathbf{a} = \mathbf{a}_\varepsilon$ with an arbitrary small $\varepsilon > 0$.

Therefore, we combine Lemma 3.4 with Lemma 3.5 to obtain

$$\mathsf{P}_\mathsf{s}(\tau_{\mathsf{A}_\varepsilon^c} = \infty) = \mathsf{P}_\mathsf{s}(\tau_{\mathsf{B}_\varepsilon^c} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_+, \tag{3.29}$$

$$\mathsf{P}_\mathsf{s}(\tau_{\mathsf{C}_\varepsilon^c} = \infty) = \mathsf{P}_\mathsf{s}(\tau_{\mathsf{D}_\varepsilon^c} = \infty) = 1 \text{ for q.e. } \mathsf{s} \in \mathsf{A}_-. \tag{3.30}$$

Here we have used the fact that for a nearly Borel set A, $p^1_{\mathsf{S}_\infty\backslash\mathsf{A}}(\cdot) = \mathsf{E}_\cdot[e^{-\tau_\mathsf{A}}]$ is a quasi-continuous μ-version of $e_{\mathsf{S}_\infty\backslash\mathsf{A}}$. Because (3.29) and (3.30) hold for arbitrarily small ε, we arrive at (2.5). □

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