RIMS Kôkyûroku Bessatsu
B60 (2016), 33–104

Methods and techniques in wave equation analysis

By

Daoyuan Fang*

This informal note is based on the mini course I gave at RIMS in the conference of this proceeding. I tried to give an introduction of the main methods and techniques in wave equation analysis for PhD students and young researchers. It may be useful to help them entering this field.

We first presented the geometry or symmetries of the linear wave equation. From these symmetries and the dispersive property of the equation, through Noether's theorem, a symmetry usually yields a conservation law, we can get several fixed time or space-time estimates, which include basic energy estimates and various kinds of Strichartz's estimates of the linear wave equation. Then we tried to use these estimates to study the classical and low regularity wellposedness researches for nonlinear wave equations with Cauchy data respectively. Along the line of the model semilinear wave equations, we introduced the basic methods and techniques, which have been used in the study of the classical energy methods and modern Fourier analysis methods. Because of the length limit of this note, we can not give a self-contained treatment of contents. We refer the reader to the books of Alinhac [1], Hörmander [6], Selberg [14], Sogge [16], Tao [20], and so on. Lots of the basic material of this note come from these books and the recent research works of Wang and myself [3, 4].

§ 1. Linear wave equation analysis

Let us begin with the linear wave equations in $\mathbb{R}^{1+d}$,

$$\begin{cases} \Box u \equiv (\partial_t^2 - \Delta)u = 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \sum_{i=1}^d \partial_{x_i}^2$.

Received September 21, 2015. Revised May 6, 2016.
2010 Mathematics Subject Classification(s): Primary 35LxxEE35Q55. Secondary 42B25, 46E35.
This work is partially supported by NSF of China under Grants 11271322, 11331005
*Department of mathematics, Zhejiang University, Hangzhou 310017, China.
e-mail: dyf@zju.edu.cn

For simplicity, assume $\phi(x), \psi(x) \in \mathcal{S}$, where $\mathcal{S}$ is the set of rapidly decreasing functions.

§ 1.1. Geometry

1.1.1. Symmetries

Like all constant coefficient dispersive equations, the wave equation has a number of symmetries. It is invariant under the time translations: $u(t,x) \to u(t-t_0,x)$, the spatial translations: $u(t,x) \to u(t,x-x_0)$, the time reversal symmetry: $u(t,x) \to u(-t,x)$, the rotations in $\mathbb{R}^d$: $u(t,x) \to u(t,Ux)$ for all orthogonal matrices U, and the scaling: u is a solution if and only if u_λ is, where $u_\lambda(t,x) := u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$ for any $\lambda > 0$.

One can also check that the wave equation is invariant under the Lorentz transformations: $u(t,x) \in C^2(\mathbb{R}^{1+d})$ is a solution of (1.1) if and only if u_v is, where

$$u_v(t,x) := u\left(\frac{t - v\cdot x}{\sqrt{1-|v|^2}}, x - x_v + \frac{x_v - vt}{\sqrt{1-|v|^2}}\right),$$

with $v \in \mathbb{R}^d$, $|v| < 1$, and $x_v = (x \cdot \frac{v}{|v|})\frac{v}{|v|}$, which is the projection of x onto the line parallel to v, and under conformal transformations: $u \in C^2_{loc}(\Gamma_+)$ is a solution of (1.1) if and only if $\tilde{u}$ is, where $\tilde{u}$ is

$$\tilde{u}(t,x) := (t^2 - |x|^2)^{-\frac{d-1}{2}} u\left(\frac{t}{t^2-|x|^2}, \frac{x}{t^2-|x|^2}\right)$$

in the forward light cone $\Gamma_+ := \{(t,x) \in \mathbb{R}^{1+d} : t > |x|\}$.

Just as pointed out in Tao's book [20], symmetries can give guidance as to what type of techniques to use to deal with a problem; for instance, if one is trying to establish wellposedness in a data class which is invariant under a certain symmetry, the use of estimates and techniques which are invariant under that symmetry is strongly suggestted; one also can spend the symmetry by normalizing the solution, for instance in making the solution centered or concentrated at the origin or some other specified location in space, time, or frequency.

To handle nonlinear equations, the crucial point is that we need to have some efficient ways to control the size of linear problem in terms of the size of the initial data or forcing term if it exists. So the important thing is to choose a suitable function space whose norm quantifies the size of the solutions. The symmetries give us some inspirations, say to use the scaling symmetry to trade between the life span of the solutions and the size of their data, wellposedness and regularity and so on.

1.1.2. Invariant vector fields **Poincaré group** is the group of linear transformations preserving $\Box u = 0$. This group is generated by the set Γ: the usual derivatives

$\partial_t, \partial_i, i = 1, \cdots, d$; the spatial rotations $\Omega_{ij} = x^i\partial_j - x^j\partial_i, 1 \le i < j \le d$; the hyperbolic rotations $\Omega_{0j} = t\partial_j + x^j\partial_t, j = 1, \cdots, d$; and the scaling vector field $S = t\partial_t + x \cdot \nabla_x$.

All these vector fields commute with the wave operator except for S, that is

$$[\Box, \partial] = [\Box, \Omega_{ij}] = 0, [\Box, S] = 2\Box.$$

Lorentz group is the group of linear transformations preserving $\Box$. $\{\partial_i, \Omega_{jk}, 0 \le i \le d,\ 0 \le j < k \le d\}$ is its generator set.

All these invariant vector fields are denoted respectively by $\Gamma_i,\ i = 0, 1, \cdots, (d+2)(d+1)/2$.

The commutator of two homogenous vector fields is a linear combination of homogeneous vector fields:

$$[\Gamma_i, \Gamma_j] = \sum c_{ijk}\Gamma_k.$$

The commutators of ∂_j with a homogeneous vector field is a linear combination of ∂_i,

$$[\Gamma_k, \partial_j] = \sum_{i=0}^{d} a_{ijk}\partial_i.$$

Consider the span of the homogeneous vector fields. If $(t, x) \in \mathbb{R}^{1+d} \setminus \{0\}$ is not on the light cone, these vector fields span the full tangent space at (t, x); if it is on the light cone, they only span the tangent space to the light cone $t^2 = |x|^2$.

§ 1.2. Explicit Solutions

From the invariance of the spatial rotations, we see the wave equation must have a radial solution $u(t, r) = u(t, |x|)$.

Consider the radial field $v(t, r) := \frac{1}{r}\partial_r u(t, r)$, it is easy to check

$$\Box_{d+2} v = 0 \ \text{ if and only if } \ \Box_d u = 0,$$

where we use $\Box_d$ to emphasize the d dimensional wave operator $\Box$. Thus in the radial case at least, it is possible to construct solutions to the $d+2$ dimensional equation out of the d dimensional case.

Making the Fourier transform in x,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$$

to the equation (1.1), one can write the solution as

$$u(t, x) = \cos(t\mathrm{D})\ \phi + \mathrm{D}^{-1}\sin(t\mathrm{D})\ \psi,\ \mathrm{D} = \sqrt{-\Delta}.$$

We call $E_+(x) := \frac{\sin(t\mathrm{D})}{\mathrm{D}}\delta(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\sin(t|\xi|)}{|\xi|}e^{ix\cdot\xi}d\xi$ the forward fundamental solution. It is a distribution solution of $\Box u = 0$ with initial data $(0,\delta)$. Then the solution of the Cauchy problem is expressed as

$$u(x,t) = \partial_t E_+ *_x \phi + E_+ *_x \psi.$$

From the superposition and Duharmel's principle, one only needs to find the fundamental solution.

Using Euler's identity, it can essentially be reduced to the study of the half-wave operator $e^{it\mathrm{D}}$.

Because E_+ is radial, thanks to the radial symmetry property of the equation, when d is odd, one can compute the Fourier transform of the distribution and show

$$\int_{\mathbb{R}^d} E_+(x)\varphi(x)dx = c_d(t^{-1}\partial_t)^{(d-3)/2}\left(t^{d-2}\int_{\mathrm{S}^{d-1}}\varphi(t\omega)d\sigma(\omega)\right),$$

for all test functions φ, where $d\sigma(\omega)$ is surface measure on the sphere S^{d-1}.

For the even dimensional case, one can use the method of descent to obtain the formula.

When $d(\geq 3)$ is odd, we arrive at

$$\begin{aligned}u(t,x) = c_d\Bigg[&\partial_t(t^{-1}\partial_t)^{(d-3)/2}\left(t^{d-2}\int_{y\in S^{d-1}}\phi(x+ty)d\sigma(y)\right)\\&+(t^{-1}\partial_t)^{(d-3)/2}\left(t^{d-2}\int_{y\in S^{d-1}}\psi(x+ty)d\sigma(y)\right)\Bigg].\end{aligned}$$

When d is even,

$$\begin{aligned}u(t,x) = c_d\Bigg[&\partial_t(t^{-1}\partial_t)^{(d-2)/2}\left(t^{d-1}\int_{|y|<1}\phi(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right)\\&+(t^{-1}\partial_t)^{(d-2)/2}\left(t^{d-1}\int_{|y|<1}\psi(x+ty)\frac{dy}{\sqrt{1-|y|^2}}\right)\Bigg].\end{aligned}$$

From the formula of the solution, we see the propagation speed is at most one in all directions. This is **finite speed of propagation** property of the solution. Furthermore, in odd dimensions, the value of u at a point $(t,x)(t>0)$ only depends on the values of the data ϕ,ψ in an infinitesimal neighborhood of the sphere $\{y : |y-x| = t\}$. A flash of light at the initial time propagates with unit speed and can only be seen on the forward light cone with vertex at the origin $\{(t,x) : t = |x|\}$. This is the **Huygens principle**.

For the even dimensions, a weaker version of Huygens principle holds. u at (t,x) depends on the values of ϕ,ψ in the ball $\{y : |y-x| \leq t\}$. A flash light at the origin will be visible to an observer at a point x_0 in space, at times $t \geq |x_0|$, and not just at $t = |x_0|$ as in odd dimensions (>1), although the intensity of the light will decay.

Remark.

From the representation of the solution , we know that the solution of homogeneous wave equation loses up to $d/2$ degrees of differentiability from time $t = 0$ to any time $t > 0$. That is, to ensure $u \in C^2(\mathbb{R}^d)$ for $t > 0$, we must assume $\phi \in C^{2+d/2}$, $\psi \in C^{1+d/2}$ if d is even, and $\phi \in C^{2+(d-1)/2}$, $\psi \in C^{1+(d-1)/2}$ if d is odd. This is because weak singularities at $t = 0$ propagate on forward light cones, thus they are interacting at time $t > 0$ to create stronger singularities.

From the following energy identity, we will know that there is not loss of L^2 differentiability from $t = 0$ to $t > 0$. That is, if the initial data have weak derivatives in L^2 up to some order k, then so does the solution $u(t, \cdot)$ for all times $t > 0$.

Dispersive equations usually do not preserve any L^p norm other than the L^2 norm.

§ 1.3. Fixed time estimates

1.3.1. Energy estimates Thanks to the time translation invariance, we can choose a multiplier $X = \partial_t u$, to multiply the equation (1.1) and integrate the resulting equation over the strip $S_T = \{(s, x) : 0 \leq s < T, x \in \mathbb{R}^d\}$ for any $T > 0$, and obtain

$$\int_{S_t} (\Box u)(\partial_t u) dx ds = E(t) - E(0),$$

where $E(t) = \frac{1}{2} \int |\partial u|^2(t, x) dx$ is the energy of u at time t.

For the equation (1.1), we have the energy identity $E(t) = E(0)$, or

$$\|\partial_t u(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{L^2}^2 = \|\nabla \phi\|_{L^2}^2 + \|\psi\|_{L^2}^2,$$

i. e., the energy of the solution u is constant in time. The solution in L^2 type Sobolev space is conserved, but one can find that it is dispersed over an increasing larger region as time progresses. If there is a force term, the identity will give us the energy inequality

$$\sqrt{E(t)} \leq \sqrt{E(0)} + \sqrt{2} \int_{S_t} \|\Box u\|_{L^2} ds.$$

To obtain the energy inequality, just as Alinhac said in [1], we always proceed in three steps:

Step 1 Establishing a differential identity. We always try to write the product $(\Box u)(\partial_t u)$ in divergence form

$$(\Box u)(\partial_t u) = \frac{1}{2} \partial_t [\sum (\partial_i u)^2 + (\partial_t u)^2] - \sum \partial_i [(\partial_i u)(\partial_t u)].$$

Step 2 Integration over the domain. Using the above expression in divergence form, we integrate over the strip S_t for $s < t$:

$$\int_{S_t} (\Box u)(\partial_t u) dx ds = E(t) - E(0).$$

To ensure the term appearing in the first step to be $\int_{\mathbb{R}^d}\sum\partial_i[(\partial_i u)(\partial_t u)]dx = 0$, we have to assume that u is decaying sufficiently fast when $|x|\to+\infty$ for fixed $s<t$.

Step 3 Handling the remainder term. Handling the term $\int_{S_t}(\Box u)(\partial_t u)dxds$, from the Cauchy-Schwartz inequality, we can get the required inequality.

Remark. If u is complex, it suffices to split u into real and imaginary parts.

Remark. The estimate for the L^2 norm of u itself is $\|u\|_{L^2} = O(t)$ as $t\to\infty$, which is sharp. In general, it fails to be $O(t^\theta)$ for any $\theta<1$.

Let $\hat{\psi}(\xi) = h(|\xi|)$, where $h(r) = r^{-d/2}(-\log r)^{-1}$ if $0<r<1/2$; 0 if $r\geq 1/2$. We have

$$\|u\|_{L^2}^2 = c_d t^2\int_0^{1/2}\frac{\sin^2(tr)}{(tr)^2}\cdot\frac{dr}{r\log^2 r}\geq c_d' t^2\int_0^{1/(2t)}\frac{dr}{r\log^2 r},$$

and we can show

$$\|u\|_{L^2}\geq\frac{Ct}{(\log t)^{1/2}}$$

as $t\to\infty$.

1.3.2. An improved energy inequality If we choose $X = \partial_t u e^{a(r-t)}$, where $a'(t) = (1+t)^{-1-\epsilon}$, as a multiplier of $\Box u = f$, we have

$$E^a(t) - E^a(0) + \frac{1}{2}\int_{S_t} e^a(1+|r-s|)^{-1-\epsilon}[\Sigma(T_i u)^2](s,x)dxds$$
$$\leq\int_{S_t} e^a|\Box u||\partial_t u|dxds,$$

where $r=|x|$,$\omega = x/r$, and $T_i u(t,x) = [\partial_i u+\omega_i\partial_t u](t,x)$. The modified energy of u at time t is

$$E^a(t) = \frac{1}{2}\int e^a[\Sigma(\partial_i u)^2+(\partial_t u)^2](t,x)dx.$$

Note that the difference between E^a and E is the presence of e^a in the integrand. It is easy to show that for all t, $E^a(t)\sim E(t)$ because a is bounded.

Thus, we have the following improved energy inequality

$$\sqrt{E(t)} + \left[\int_{S_t}(1+|r-s|)^{-1-\epsilon}[\Sigma(T_i u)^2](s,x)dxds\right]^{1/2}$$
$$\leq C_\epsilon[\sqrt{E(0)}+\int_{S_t}\|\Box u\|_{L^2}ds].$$

Remark. In fact, this is a weighted energy inequality. One can establish the differential identity as the first step (see [1]):

$$(\Box u)(\partial_t u)e^a = \frac{1}{2}\partial_t[e^a(\sum(\partial_i u)^2+(\partial_i u)^2)] - \sum\partial_i[e^a(\partial_i u)(\partial_t u)] + \frac{e^a}{2}Q,$$

where $Q = -(\partial_t a)[(\partial_i u)^2 + (\partial_t u)^2] + 2(\partial_t u)\sum(\partial_i a)(\partial_i u)$ is a sum of quadratic terms in $\partial_i u, \partial_t u$ with coefficients depending on $\partial_i a, \partial_t a$. If we choose $a'(s) = (1+|s|)^{-1-\epsilon}$, the function a is bounded, and we have $Q = a' \sum(T_i u)^2$.

Note that we can see in the deep inside of the light cone $\{r = t\}$, that is $\{|r-t| \geq C_0 t\}, C_0 > 0$, the factor is integrable, so

$$\int_{S_t \cap \{x:|r-t|\geq c_0 t\}} (1+|r-s|)^{-1-\epsilon}[\Sigma(T_i u)^2](s,x)dxds \leq C_\epsilon \max_{0\leq s\leq t} E(s).$$

There is no any improvement over the energy inequality in this region. In constract, in a thin strip around the light cone, say $|r-t| \leq C_0$, the special energy

$$\frac{1}{2}\int [\Sigma(T_i u)^2](s,x)dxds$$

is not just bounded, but also an L^2 function of t. To understand the reason, let $d = 3$, we remark that the special derivatives T_i are the "good" derivatives, since $L = \partial_t + \partial_r = \sum \omega_i T_i, \Omega/r = \omega \wedge T$, and $T_i = [\Omega/r \wedge \omega]_i + \omega_i L$, where $\Omega := \{\Omega_{ij} : 1 \leq i < j \leq 3\}$. From the representation of the solution u for free wave equation with data $(0, v)$, $v \in C_0^\infty$, we see that all derivatives of $u(t,x) = \frac{1}{r}F(r-t, \omega, 1/r)$ are less than C/t, while the spacial derivatives $T_i u$ satisfy $|T_i u| \leq C/t^2$.

1.3.3. A simple application to the uniqueness We also can establish the energy inequality in a domain. A direct application of the finite speed of the propagation and the energy inequality yields the proof of the uniqueness of smooth solutions for nonlinear equations.

Proposition 1.1. *Assume that $u \in C^2(\Gamma)$ solves*

$$\Box u = F(u, \partial u) \tag{1.2}$$

in the solid backward light cone

$$\Gamma = \{(t,x) : 0 \leq t < T, |x - x_0| < T - t\},$$

with base $B_0 = \{x : |x - x_0| < T\}$, and F is a given C^∞ function. Then u is uniquely determined by its data $u|_{B_0}$ and $\partial_t u|_{B_0}$.

If we consider the difference of the two solutions of the given problem (1.2) with the same data $w = u - v$, then the proof can be reduced to proving that the solution to the following linear equation of the form

$$\Box u = a(t,x)u + b(t,x) \cdot \partial u$$

with zero data should be identically zero, where a and b are $\mathbb{R}^{1+d}$-valued continuous functions.

To prove the reduced problem, we denote $B_t = \{x : |x - x_0| \leq T - \varepsilon - t\}$, and set

$$E(t) = \frac{1}{2}\int_{B_t} \left(|u(t,x)|^2 + |\partial u(t,x)|^2\right) dx.$$

Then, from the given equation and the equality $\operatorname{div}(u_t \nabla u) = \nabla u_t \cdot \nabla u + u_t \Delta u$, we have

$$\begin{aligned}
\frac{d}{dt}E(t) &= \int_{B_t} (uu_t + u_t u_{tt} + \nabla u \cdot \nabla u_t) \\
&- \frac{1}{2}\int_{\partial B_t} \left(|u(t,x)|^2 + |\partial u(t,x)|^2\right) d\sigma(x) \\
&= \int_{B_t} u_t[(1+a)u + b \cdot \partial u] dx \\
&+ \int_{B_t} \operatorname{div}(u_t \nabla u) dx - \frac{1}{2}\int_{\partial B_t} \left(|u(t,x)|^2 + |\partial u(t,x)|^2\right) d\sigma(x)
\end{aligned}$$

Note that the last line of the right hand side of the above equality can be controlled by $-\frac{1}{2}\int_{\partial B_t} |u(t,x)|^2 d\sigma(x) \leq 0$ from divergence theorem. From the uniformly boundedness of a and b, we have $\frac{d}{dt}E(t) \leq CE(t)$ for $0 \leq t \leq T - \varepsilon$, so we conclude the result from $E(0) = 0$.

1.3.4. Conformal energy inequality

The $1 + d$ dimensional Einstein universe is the product $\mathbb{R} \times \mathbb{S}^d$ with the pseudo-Riemannian metric $dT^2 - dX^2$. We consider the map

$$\Psi : \mathbb{R} \times \mathbb{S}^d \ni (T, X) \mapsto \frac{1}{\cos T + X_0}(\sin T, \vec{X}) \in \mathbb{R}^{1+d}.$$

Ψ is smooth (analytic), and is a conformal bijection on Minkowski space, with

$$dT^2 - dX^2 = (\cos T + X_0)^2(dt^2 - |d\vec{x}|^2),$$

when $\cos T + X_0 > 0$, $-\pi < T < \pi$.

The pushfoward of the time translation vector ∂_T on $\mathbb{R} \times \mathbb{S}^d$ under the conformal map to $\mathbb{R}^{1+d}$ is $\frac{1}{2}(1 + t^2 + |x|^2)\partial_t + t(x, \partial_x)$, which is conformal with coefficient $4t$.

Set

$$X(\partial) = \mathbf{X} \cdot \partial + 2t, \mathbf{X} = (1 + t^2 + |x|^2, 2tx_1, 2tx_2, 2tx_3),$$

and let m be the matrix $\operatorname{diag}(1, -1, -1, -1)$ and $\mathbf{1} = (1, 0, 0, 0)$.

By multiplying the Morawetz vector field $X(\partial)$ to the left hand side of wave equation (1.1), we have

$$X(\partial)u \cdot \Box u = \operatorname{div}_{t,x}(e_0, e'),$$

where

$$(e_0, e') = X(\partial)u \cdot m(\partial)\mathbf{X} - v^2\mathbf{1},$$

and $e_0 = \frac{1}{2}|\partial u|^2$, and $e' = -u_t \nabla_x u$.

Integration of the above identity yields

$$\frac{d}{dt}E(t) = \int X(\partial)u \cdot \Box u dx$$

for the energy $E(t) = \int e_0 dx$, and the right hand side is bounded by

$$\|(1+t+|\cdot|)^{-1}X(\partial)u(t,\cdot)\|_{L^2}\|(1+t+|\cdot|)\Box u(t,\cdot)\|_{L^2}.$$

From

$$\|(1+t+|\cdot|)^{-1}X(\partial)u(t,\cdot)\|_{L^2} \le C\sqrt{E(t)},$$

we have

$$\sqrt{E(t)} \le C\sqrt{E(0)} + C\int_0^t \|(1+t+|\cdot|)\Box u(t,\cdot)\|_{L^2} ds.$$

Observing $E(t) \approx \sum_{|\alpha|\le 1} \|\Gamma^\alpha u(t,\cdot)\|^2_{L^2}$, and the commutation relations between $\Box$ and the invariant vector fields Γ, one can get the conformal energy inequality

$$\begin{aligned}\sum_{|\alpha|\le M+1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2} \le C &\sum_{|\alpha|\le M+1} \|\Gamma^\alpha u(0,\cdot)\|_{L^2} \\ &+ C\sum_{|\alpha|\le M+1} \int_0^t \|(1+s+|\cdot|)\Gamma^\alpha \Box u(s,\cdot)\|_{L^2} ds \end{aligned} \tag{1.3}$$

for all $t > 0$ and all $u \in C^\infty([0,\infty)\times\mathbb{R}^3)$ with compact support in x for each t.

Let $d = 3$. If we use the multiplier $Xu = \partial_r u + \frac{u}{r}$, we can obtain the following Morawetz inequality

$$4\pi\int_0^t u^2(0,s)ds + \int_{S_t} r^{-1}[\sum(\partial_i u)^2 - (\partial_r u)^2]dxds \le 4E(t).$$

This inequality can only be used when it is coupled with an energy inequality which yields a control of $E(t)$.

Remark. We also have

$$\begin{aligned}&(t\partial_t u + x\cdot\nabla_x u + u)(\Box u + \varphi_\kappa(u)) \\ &= \mathrm{div}_{t,x}(tQ + \partial_t uu, -tP) - 4\Phi_\kappa(u) + u\varphi_\kappa(u),\end{aligned}$$

where $Q = \frac{1}{2}|u'|^2 + \Phi_\kappa(u) + t^{-1}\partial_t ux\cdot\nabla_x u$,

$$P = (\frac{1}{2}|\partial_t u|^2 - \frac{1}{2}|\nabla_x u|^2 - \Phi_\kappa(u))x/t + (t^{-1}u + \partial_t u + t^{-1}x\cdot\nabla_x u)\nabla_x u,$$

and $\Phi_\kappa(u)=\int_0^u \varphi_\kappa(\tau)d\tau$.

From

$$2(\Box u)(\partial_r u)=\partial_t[2(\partial_t u)(\partial_r u)]+\sum \partial_i\omega_i[\sum(\partial_j u)^2-(\partial_t u)^2]-2(\partial_i u)(\partial_r u)+Q,$$

$$Q=\frac{2}{r}[\sum(\partial_i u)^2-(\partial_r u)^2]+\frac{2}{r}[(\partial_t u)^2-\sum(\partial_i u)^2],$$

we see Q is only almost nonnegative, and is the sum of the quadratic terms in ∂u containing two different terms. The first term is nonnegative, and the second term is an expression with no special sign. Note that

$$\begin{aligned} au(\Box u)=&\,\partial_t[au\partial_t u-\frac{1}{2}(\partial_t a)u^2]\\ &-\sum\partial_i[au\partial_i u-\frac{1}{2}(\partial_i a)u^2]-a[(\partial_t u)^2-\sum(\partial_i u)^2]+\frac{u^2}{2}\Box a,\end{aligned}$$

and if we consider $2(\Box u)(\partial_r u+\frac{u}{r})$, we can have the bad quadratic terms cancelled.

1.3.5. Decay estimates

Conservation law can give global bounds on a solution $u(t)$ of a nonlinear wave equation that are either uniform in time, or grow at some controlled rate. Usually such bounds can be sufficient to obtain global existence of the solution. However, there is no much information about the asymptotic behavior of the solutions as $t\to\infty$.

There are several ways to get the decay estimate. Usually one can use the following approaches:

- Morawetz inequalities approach, which is based on the monotonicity formulae method.
- Conformal energy methods , which provide decay of the solution in an L^p.
- Sobolev embedding, or the vector field method, which establishes decay of the solution in an L^∞ sense.

For example, from the following Sobolev inequality, L^2 norms of products ∂_x^α of the usual derivatives applied to u can be used to obtain a control of $|u(t,x)|$. That is for $s>\frac{d}{2}$, and $u\in\mathcal{S}(\mathbb{R}^d)$

$$\|u\|_{L^\infty}\le C_{d,s}\sum_{|\alpha|\le s}\|\partial_x^\alpha u\|_{L^2}.$$

L^2 norms of products Γ^k of the invariant vector fields Γ, which generate the Poincaré group, applied to u can be used to obtain a weighted control of $|u(t,x)|$, that is, the Klainerman-Sobolev inequality

$$|u(x,t)| \le C(1+|t|+|x|)^{-\frac{d-1}{2}}(1+|t-|x||)^{-1/2} \sum_{0\le|\alpha|\le\frac{d+2}{2}} \|\Gamma^\alpha u(\cdot,t)\|_{L^2}.$$

From the respresentation of the solution we see: if $\phi \in C^{[d/2]+k}(\mathbb{R}^d)$, $\psi \in C^{[d/2]+k-1}$, and suppϕ, supp$\psi \subset \{x : |x| < R\}$, then when d is odd, $u(t,x) = 0$, unless $|t-|x|| < R$, and $u(t,x) = O((1+t)^{-\frac{d-1}{2}})$, when d is even, $u = 0$ if $|x| > t+R$, and

$$u(t,x) = O((1+t)^{-\frac{d-1}{2}}(1+|t-|x||)^{-\frac{d-1}{2}}).$$

Combining the energy estimate with the Klainerman-Sobolev inequality, we can get

$$|\partial u(t,x)| \le C(1+t)^{-\frac{d-1}{2}}(1+|t-|x||)^{-\frac{1}{2}}.$$

From the support property of the solution and the fundamental theorem of calculus, we have $|u(t,x)| \le O(1+t)^{-\frac{d-1}{2}}$ for odd d, and

$$|u(t,x)| \le (1+t)^{-\frac{d-1}{2}}(1+|t-|x||)^{\frac{1}{2}}$$

for even d, which is worse than the above ones.

In fact, if one only considers the angular derivatives Ω and the spatial ones ∂_x, for $f \in C^\infty(\mathbb{R}^3)$, from the local Sobolev estimates we have

$$\|f\|_{L^\infty(R<|x|<R+1)} \le CR^{-1} \sum_{|\alpha|+|\beta|\le 2} \|\Omega^\alpha \partial_x^\beta f\|_{L^2(R-1<|x|<R+2)} \tag{1.4}$$

for $R > 2$. If one uses the vector fields $Z = \{\partial, S, \Omega\}$, for $f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$, one has the weighted Sobolev estimates (due to Hidano and Yokoyama [5])

$$|x|^{1/2}\langle t-|x|\rangle|\partial f(t,x)| \le C \sum_{|\beta|\le 1} \left[\|Z^\beta \partial f(t,\cdot)\|_{L^2} + \langle t-r\rangle \|Z^\beta \partial^2 f(t,\cdot)\|_{L^2}\right]. \tag{1.5}$$

In the case $d = 3$, The Cauchy problem of $\Box w = F$ with vanishing initial data at $t = 0$ has a simple formula

$$\begin{aligned} w(t,x) &= \frac{1}{4\pi}\int_0^t \int_{S^2} (t-s)F(s, x-(t-s)y)d\sigma(y)ds \\ &= \frac{1}{4\pi}\int_{|y|<t} F(t-|y|, x-y)\frac{dy}{|y|}. \end{aligned}$$

By using it, we have the following Hörmander's inequality

$$(1+t+|x|)|w| \le C \sum_{|\alpha|\le 2} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\alpha F(s,y)| \frac{dyds}{(1+s+|y|)} \tag{1.6}$$

for $F \in C^2([0,\infty) \times \mathbb{R}^3)$.

The following is obtained by Keel-Smith-Sogge

$$(t+|x|)|w(t,x)| \leq C \sum_{|\alpha|\leq 3} \int_0^t \int_{\mathbb{R}^3} |Z^\alpha F(s,y)| \frac{dyds}{(1+|y|)} \tag{1.7}$$

for $F \in C^2([0,\infty) \times \mathbb{R}^3)$. Compared with Hörmander's inequality, this inequality is stronger in the sense that it involves a smaller collection of vector fields, but it is weaker mainly in the sense that its right side involves the denominator $1+|y|$, as opposed to $1+s+|y|$.

This inequality is useful in studying systems of wave equations involving multiple speeds.

§ 1.4. Spacetime Estimates

1.4.1. Littlewood-Paley decomposition

Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \chi \leq 1$, be a radial cut off function such that $\chi(\xi) = 1$, if $|\xi| \leq 1/2$; and 0 if $|\xi| \geq 1$. Denote $S_j = \mathcal{F}^{-1}\chi(2^{-j}\cdot)\mathcal{F}$ for $j \in \mathbb{N}$. One has $S_j f \to f$ for every $f \in \mathcal{S}'$. Let $\Delta_0 f = \mathcal{F}^{-1}\beta_0(\xi)\hat{f}$, $\Delta_j f = \mathcal{F}^{-1}\beta_j(\xi)\hat{f}$, where $\beta_0(\xi) = \chi(\xi)$, $\beta_j(\xi) = \chi(2^{-j}\xi) - \chi(2^{1-j}\xi)$ for $j \in \mathbb{N}_+$. It is easy to know that $\sum_0^\infty \beta_j = 1$.

The following is the elementary properties of the Littlewood-Paley decomposition:

- $\mathrm{supp}\widehat{\Delta_j f} \subset \{2^{j-1} \leq |\xi| \leq 2^j\}$. $\Delta_j f = 2^{jd}\phi(2^j\cdot) * f$, where $\hat{\phi} = \chi(\xi) - \chi(2\xi)$, $\Delta_j f \perp \Delta_k f = 0$ if $|j-k| \geq 2$, $\Delta_j f \in C^\infty$ if $f \in L^p, 1 \leq p \leq \infty$, and $\|\Delta_j f\| \leq \|\phi\|_{L^1}\|f\|_{L^p}$.

- $\mathrm{supp}\widehat{S_j f} \subset \{|\xi| \leq 2^j\}$. $S_j f = 2^{jd}\psi(2^j\cdot) * f$, where $\hat{\psi} = \chi$, $S_j f \in C^\infty$ if $f \in L^p, 1 \leq p \leq \infty$, and $\|S_j f\| \leq \|\psi\|_{L^1}\|f\|_{L^p}$.

If $f \in H^s$, $s \in \mathbb{R}$, we have

$$\|f\|_{H^s}^2 \approx \sum_0^\infty 2^{2js}\|\Delta_j f\|_{L^2}^2,$$

which follows from $\|\Delta_j f\|_{H^s} \approx 2^{js}\|\Delta_j f\|_{L^2}$. It is easy to show that if $\{f_j\}$ is a sequence in L^2 such that for some $R > 0$, $\mathrm{supp}\hat{f}_j \subset \{R^{-1}2^j \leq |\xi| \leq R2^j\}$ and $\sum 2^{2js}\|f_j\|_{L^2}^2 < \infty$, then $\sum f_j$ converges to f in H^s, and

$$\|f\|_{H^s}^2 \approx \sum_0^\infty 2^{2js}\|f_j\|_{L^2}^2.$$

The important thing is that we have the following result.

Theorem 1.2. *Let $s > 0$. Assume $f_j \in L^2$ satisfies $\mathrm{supp}\hat{f}_j \subset \{|\xi| \leq R2^j\}$ for some $R \geq 1$ and that*

$$\sum_0^\infty 2^{2js} \|f_j\|_{L^2}^2 < \infty.$$

Then $\sum f_j$ converges to f in H^s, and

$$\|f\|_{H^s}^2 \leq C_{s,R} \sum_0^\infty 2^{2js} \|f_j\|_{L^2}^2.$$

Remark. If $R = 2^q$, then the constant $C_{s,R}$ is of the form $C_s 2^{2qs}$.

Using the machinery just developed, for $s \geq 0$, and $f, g \in H^s \cap L^\infty$, we can prove the inequality

$$\|fg\|_{H^s} \leq C_s(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}).$$

Recall that the spectrum of f is the support of its Fourier transform. From Young's inequality, we have the following Bernstein's Lemma.

Lemma 1.3. *Assume that the spectrum of $f \in L^p$, $1 \leq p \leq \infty$, is contained in the ball $|\xi| \leq 2^j$. Then*

$$\|\partial^\alpha f\|_{L^p} \leq C_\alpha 2^{j|\alpha|} \|f\|_{L^p}$$

for any multi-index α. Moreover, if the spectrum is contained in $2^{-j} \leq |\xi| \leq 2^j$, then

$$C_k^{-1} 2^{jk} \|f\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C_k 2^{jk} \|f\|_{L^p}$$

for any $k \in \mathbb{N}$.

By using this lemma, one can prove the Moser inequality in the following.

Theorem 1.4. *Assume that $F : \mathbb{R}^d \to \mathbb{R}$ is C^∞ and $F(0) = 0$. Then for all $s \geq 0$ and all $\mathbb{R}^d$-valued $f \in H^s \cap L^\infty$, there is a continuous function $\gamma = \gamma_s : \mathbb{R} \to \mathbb{R}$ such that*

$$\|F(f)\|_{H^s} \leq \gamma(\|f\|_{L^\infty}) \|f\|_{H^s}.$$

1.4.2. Spacetime energy inequality

Let $d = 3$. We introduce the following inequality obtained by M. Keel, H. Smith and C. Sogge in [9] :

$$\|\langle x\rangle^{-1/2} \partial u\|_{L_t^2 L_x^2} \lesssim (\ln(2+T))^{1/2} (\|\partial u(0)\|_2 + \|\Box u\|_{L_t^1 L_x^2}) \ .$$

Remark. From the following estimate

$$\begin{aligned}\int_{S_t\cap\{|x|\geq\alpha t\}}(1+|x|^2)^{-1/2}|\partial u|^2dxds &\leq C\int_0^t\frac{ds}{1+s}\int|\partial u|^2dx\\ &\leq C(\max_{0\leq s\leq t}E(s)\log(2+t)),\end{aligned}$$

we know that there is no improvement compared with the standard energy inequality in the region $S_t\cap\{|x|\geq\alpha t\}$. But this estimate provides an improved behavior of all derivatives of the solution in regions $\{|x|\leq\alpha t,\alpha<1\}$, while we already know that the improved energy inequality provides an improved behavior of some special derivatives of solution close to the boundary of the light cone, that is in the regions $\{|x|\leq\alpha t,\alpha>0\}$.

In contrast with the preceding energy inequality, the proof of it does not involve a multiplier method and integrating by parts. In the regions $|x|>t$, it follows from

$$(1+t)^{-1/2}\|\partial u\|_{L^2(\{(s,x):0\leq s\leq t\})}\leq C\|\partial u(0,\cdot)\|_{L^2(\mathbb{R}^3)}+C\int_0^t\|\Box u(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds,$$

which easily follows from the standard fixed time energy estimate. On the other set $|x|\leq t$, we need to make a dyadic decomposition for each piece and the following estimate, which follows from the local space-time energy estimate

$$\|\partial u\|_{L^2_tL^2_x(B_1)}\lesssim\|\partial u(0)\|_2+\|\Box u\|_{L^1_tL^2_x}\tag{1.8}$$

and a scaling argument,

$$\||x|^{-1/2}\partial u\|_{L^2(\{(s,x):0\leq s\leq t,|x|\in[2^j,2^{j+1}]\})}\lesssim\|\partial u(0)\|_2+\|\Box u\|_{L^1_tL^2_x}.\tag{1.9}$$

By squaring the above and summing up over the dyadic pieces with $2^j<2t$, we conclude the above together with (1.8).

1.4.3. Strichartz Estimates Before we show the Strichartz estimates, we investigate how the waves for the equation (1.1) disperse. Considering the wave $\phi(t,x)=Ae^{i(\xi\cdot x-\tau t)}$, which moves to the right with speed $\tau/|\xi|$, we find

$$\tau^2-|\xi|^2=0,$$

which should hold for parameters τ and $|\xi|$. Such a relationship is called dispersion relation.

For dispersive equations, usually waves with different frequencies will propagate with different speeds, causing the initial wave profile to distort as it propagates. From dispersion relation for the wave equation, we see that different frequencies move in different directions, but not in different speeds. So waves governed by homogeneous wave equation (1.1) propagate without change of shape or dissipation.

From above we also can see that the dispersion of the wave equation is with finite speed of propagation, and we see the concentration of solutions along light cones.

The decay estimates are useful in the long time asymptotic theory of nonlinear dispersive equations, especially, when the dimension d is large and the initial data have good integrabilities. However, in many situations, the initial data is only assumed to lie in an L^2 type Sobolev. Fortunately, by combining the above dispersive estimates with some duality arguments, one can obtain an extremely useful set of estimates: **Strichartz estimates**

$$\begin{aligned}&\|u\|_{L^qL^r(I\times\mathbb{R}^d)}+\|u\|_{C\dot{H}^s(I\times\mathbb{R}^d)}+\|\partial_t u\|_{C\dot{H}^{s-1}(I\times\mathbb{R}^d)}\\ &\lesssim\left(\|\phi\|_{\dot{H}^s(\mathbb{R}^d)}+\|\psi\|_{\dot{H}^{s-1}}+\|F\|_{L^{\tilde{q}'}L^{\tilde{r}'}(I\times\mathbb{R}^d)}\right)\end{aligned}$$

whenever $s\geq 0, 2\leq q,\tilde{q}\leq\infty$ and $2\leq r,\tilde{r}<\infty$ obey the scaling condition

$$\frac{1}{q}+\frac{d}{r}=\frac{d}{2}-s=\frac{1}{\tilde{q}'}+\frac{d}{\tilde{r}'}-2$$

and the wave admissibility conditions

$$\frac{1}{q}+\frac{d-1}{2r},\ \frac{1}{\tilde{q}}+\frac{d-1}{2\tilde{r}'}\leq\frac{d-1}{4}.$$

Let $d=3$, $q=r=4$, $s=1/2$. We have the original Strichartz estimate proved by Strichartz in 70's.

$$\|u\|_{L^4(\mathbb{R}^{1+3}_+)}\lesssim\left(\|\phi\|_{\dot{H}^{1/2}(\mathbb{R}^3)}+\|\psi\|_{\dot{H}^{-1/2}}+\|F\|_{L^{4/3}(\mathbb{R}^{1+3}_+)}\right).$$

Remark. From above inequality, we can say that, in some way, this is the L^p version of the sup-norm decay estimate, or the inequality (1.7) yielded $1/(t+|x|)$ decay for the solutions of the inhomogeneous wave equation in $\mathbb{R}^{1+3}_+$ with appropriate forcing terms. But this rate of point-wise dispersion is not quite enough to imply that the solution belongs to $L^4(\mathbb{R}^{1+3}_+)$. Other estimates, such as Klainerman-Sobolev inequality would yield this, but all of these involve a loss of derivative and decay assumptions on the derivatives, unlike the Strichartz estimate in above. For these reason, Strichartz estimates are much more useful in dealing with equations like $\Box u=F(u)$.

The H^s energy estimate is a special case of the Strichartz estimates. This estimate shows the feature of gaining a full degree of regularity. The force term F only has $s-1$ degrees of regularity, but the final solution u has s degrees of regularity.

The estimates obtained from Strichartz lose fewer derivatives than the fixed time estimates one would get from the Sobolev embedding, but the latter compensate for this by improving the time and space integrability.

Like energy estimates, the Strichartz estimate only controls the homogeneous Sobolev norm.

We move on to the general Strichartz estimates. It is easy to show that for proving the homogenous estimate

$$\|u\|_{L_t^q L_x^r} \leq C(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}}),$$

we only need to prove

$$\|e^{itD}\phi\|_{L_t^q L_x^r} \leq C\|\phi\|_{\dot{H}^s} \tag{1.10}$$

for $\phi \in \mathcal{S}$.

Before showing the proof, let us first discuss necessary conditions. By duality, it is not difficult to know it is equivalent to

$$\|\int_{\mathbb{R}} e^{i(t-s)D} F ds\|_{L_t^q L_x^r} \leq C\|D^{2s}F\|_{L_t^{q'} L_x^{r'}},$$

and the (time) translation invariance implies

$$q \geq q' \Rightarrow q \geq 2$$

(see Hörmander 1960's). From the Knapp example, we know

$$\hat{\phi}(\xi) = \chi(1 < \xi_1 < 2, |\xi'| \leq \epsilon), \epsilon \ll 1,$$

$$\Rightarrow \frac{2}{q} \leq (d-1)(\frac{1}{2} - \frac{1}{r}).$$

Brief proof of (1.10) At first, we prove it for frequency-localized ϕ, and then obtain the general case using the Littlewood-Paly theory. Fix a radial cut off function $\beta \in C_0^\infty$ supported away from zero, and consider the truncated cone operator

$$T\phi(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|} \beta(\xi)\hat{\phi}(\xi) d\xi.$$

Then the problem is reduced to proving

$$\|T\phi\|_{L^q L^r} \leq C\|\phi\|_{L^2} \tag{1.11}$$

for the wave admissible (q, r).

The formal adjoint of T is $F(t,x) \to T^*F(x)$ determined by

$$(Tf, F)_{L^2(\mathbb{R}^{1+d})} = (f, T^*F)_{L^2(\mathbb{R}^d)}, f \in \mathcal{S}(\mathbb{R}^d),\ F \in \mathcal{S}(\mathbb{R}^{1+d}).$$

So,

$$T^*F(x) = \int e^{i(x\cdot\xi - t|\xi|)}\overline{\beta(\xi)}\hat{F}(t,\xi)d\xi dt = \int e^{ix\cdot\xi}\overline{\beta(\xi)}\tilde{F}(|\xi|,\xi)d\xi,$$

where $\tilde{F}$ is the spacetime Fourier transform. The above gives the connection with the Fourier restriction problem for the forward light cone $\partial\Gamma_+ = \{(\tau,\xi) : \tau = |\xi| > 0\}$ in $\mathbb{R}^{1+d}$. In fact, from this expression, we see that

$$\widehat{T^*F}(\xi) \simeq \overline{\beta(\xi)}\tilde{F}(|\xi|,\xi) \equiv RF(\xi)$$

is just the restriction of the spacetime Fourier transform of F to $\partial\Gamma_+$, multiplied by a smooth cutoff. The cone is the graph of $\xi \to (|\xi|,\xi)$, and with respect to this parametrization, surface measure $d\sigma$ on the cone is just $d\xi$ up to a constant. Therefore, from the Plancherel's theorem T^*F is equivalent to RF in L^2, that is, the estimate (1.11) is equivalent to the following restriction theorem

Theorem 1.5. *$R : L_t^{q'}L_x^{r'} \to L^2(\partial\Gamma_+, d\sigma)$ is bounded if (q,r) is a wave admissible.*

From

$$\widehat{TT^*F}(t,\xi) \simeq e^{it|\xi|}\beta(\xi)\widehat{T^*F}(\xi) \simeq \int e^{i(t-s)|\xi|}|\beta(\xi)|^2\hat{F}(s,\xi)ds,$$

we see TT^* is a convolution operator $TT^*F = K * F$ with

$$K(t,x) := \int e^{i(x\cdot\xi + t|\xi|)}|\beta|^2 d\xi.$$

It is essentially the same with the operator T. The only difference is that in the latter we have β and not $|\beta|^2$.

Let $K_t(x) := K(t,x)$. The Riesz-Thorin interpolation between the energy inequality $\|K_t * \phi\|_{L^2} \le C\|\phi\|_{L^2}$ and dispersive inequality

$$\|K_t * \phi\|_{L^\infty} \le \frac{C}{(1+|t|)^{(d-1)/2}}\|\phi\|_{L^1}$$

gives

$$\|K_t * \phi\|_{L^r} \le \frac{C}{(1+|t|)^{\gamma(r)}}\|\phi\|_{L^{r'}}$$

for all $2 \le r \le \infty$, where $\gamma(r) = \frac{d-1}{2}\left(1 - \frac{2}{r}\right)$.

Now we know that: to prove (1.11), we only need to prove boundedness of TT^*

$$\|K * F\|_{L^qL^r} \le C\|F\|_{L^{q'}L^{r'}}$$

for (q,r) wave admissible. If we ignore the endpoint case, that is $2 \le q \le \infty$, $2 \le r < \infty$, $2/q \le \gamma(r)$ and $(2/q, \gamma(r)) \ne (1,1)$, by Minkowski's inequality and above $L^{r'}$ to L^r estimate we have

$$\|K_t * F(t)\|_{L^r} \le C \int \frac{\|F(s)\|_{L^{r'}}}{(1+|t-s|)^{\gamma(r)}} ds.$$

Dealing with the cases $2/q < \gamma(r)$ and $2/q = \gamma(r)$ separately, we have the result of frequency localized case.

- $2/q < \gamma(r)$. Because $(1+|t|)^{-\gamma(r)} \in L^{q/2}$, by using Young's inequality we get the result.
- $2/q = \gamma(r)$. We exclude the endpoint case, so $2/q = \gamma(r) < 1$. But in this case $(1+|t|)^{-\gamma(r)} \notin L^{q/2}$, we have to use the Hardy-Littlewood inequality to obtain the result in the case where $0 < 2/q = \gamma(r) < 1$.
- $q = \infty$ and $r = 2$. This is just the energy inequality.

Let $W(t)\phi = e^{itD}\phi$. Choose a radial $\beta \in C_0^\infty(\mathbb{R}^d)$ supported away from zero such that $\sum_{j\in\mathbb{Z}} \beta(\xi/2^j) = 1$ for all $\xi \ne 0$, and $\widehat{\Delta_j f}(\xi) = \beta(\xi/2^j)\hat{\phi}(\xi)$. Then $\phi = \sum \Delta_j \phi$ and $W(t)\phi = \sum W(t)\Delta_j\phi$. It is easy to know

$$W(t)\Delta_j\phi \simeq T[\phi(\cdot/2^j)](2^j t, 2^j x),$$

and

$$\begin{aligned}\|W(t)\Delta_j\phi\|_{L_t^q L_x^r} &\simeq 2^{j(-\frac{d}{r}-\frac{1}{q})}\|T(\phi(\cdot/2^j))\|_{L^q L^r}\\ &\le 2^{j(-\frac{d}{r}+\frac{d}{2}-\frac{1}{q})}\|\phi\|_{L^2}.\end{aligned}$$

From $\Delta_j\phi = \Delta_j(\sum \Delta_k\phi) = \sum_{|k-j|\le 3} \Delta_j\Delta_k\phi$, and (1.11), we have

$$\begin{aligned}\|W(t)\Delta_j\phi\|_{L_t^q L_x^r} &\lesssim \sum_{|k-j|\le 3} \|W(t)\Delta_j\Delta_k\phi\|_{L^q L^r}\\ &\lesssim \sum_{|k-j|\le 3} 2^{js}\|\Delta_k\phi\|_{L^2} \lesssim \sum_{|k-j|\le 3} \|\Delta_k\phi\|_{\dot{H}^s},\end{aligned}$$

where $s = d/2 - d/r - 1/q$.

Note that for $2 \le p < \infty$, we have $\|\phi\|_{L^p} \le \sqrt{\sum_{j\in\mathbb{Z}} \|\Delta_j\phi\|_{L^p}^2}$. Hence, we have

$$\|W(t)\phi\|_{L^q L^r} \lesssim \Big\| \sqrt{\sum \|W(t)\Delta_j\phi\|_{L_x^r}^2} \Big\|_{L_t^q} \lesssim \sqrt{\sum \|W(t)\Delta_j\phi\|_{L^q L^r}^2}.$$

Combining with the inequality in the above, and $\|\phi\|_{\dot{H}^s} \sim \sqrt{\sum \|\Delta_j\phi\|_{\dot{H}^s}^2}$, we have

$$\|e^{itD}\phi\|_{L_t^q L_x^r} \le C\|\phi\|_{\dot{H}^s}$$

at last.

Remark. For the inhomogeneous data, we have

$$\|u\|_{L^qL^r(S_T)} \le C_T(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}}),$$

where $C_T = 1+T^2$. To this end, we only need to assume $\phi = 0$, and supp$\hat{\psi} \subset \{|\xi| \le 1\}$. From

$$|\hat{u}(t,\xi)| = \frac{|\sin(t|\xi|)|}{|\xi|}|\hat{\psi}(\xi)| \le |t||\hat{\psi}(\xi)|$$

by Hölder inequality and Sobolev embedding, we have

$$\|u\|_{L^qL^r(S_T)} \le T^{1/q} \sup_{0\le t\le T} \|u(t)\|_{L^r} \le CT^{1+1/q}\|\psi\|_{\dot{H}^{s-1}}.$$

For the inhomogeneous estimates, most of the cases follows from the following lemma by Christ and Kiselev.

Lemma 1.6. *Let Y and Z be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(Y,Z)$, the space of bounded linear mappings from Y to Z. Suppose that $-\infty \le a < b \le \infty$, and set $Tf(t) = \int_a^b K(t,s)f(s)ds$. Assume that $\|Tf\|_{L^q((a,b),Z)} \le C\|f\|_{L^p((a,b),Y)}$. Set $Wf(t) = \int_a^t K(t,s)f(s)ds$. Then, if $1 \le p < q \le \infty$,*

$$\|Wf\|_{L^q((a,b),Z)} \le \frac{2^{-2(1/p-1/q)}2C}{1-2^{-(1/p-1/q)}}\|f\|_{L^p((a,b),Y)}.$$

Remark. One must take $p < q$. Because the lemma does not hold in the case $p = q \in (1,\infty)$ if $K(t,s) = 1/(t-s)$.

In 1993, Klainerman-Machedon proved that the radial version of the endpoint estimate with $(2,\infty,3)$ is true. Then a question arises naturally: can (1.10) be valid with $r=\infty$? or if it fails, how can it be improved to restricted cases such as spherically symmetry or angular regularity?

In fact, the following results obtained by Wang and myself in [3], are a supplement of the known results.

Theorem 1.7. *Let $b = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$. Then we have* (1.10) *for all admissible* (q,r,d) *except that* $(q,r) = (\max(2, \frac{4}{d-1}), \infty)$ *and* $(q,r) = (\infty,\infty)$*. On the other hand, in order for* (1.10) *to be valid for all $\phi \in \mathcal{S}$, we need (q,r,d) admissible, $(q,r) \ne (\infty,\infty)$ and $q > \frac{4}{d-1}$.*

Remark. As stated in Theorem 1.7, the only remained open problem for homogeneous estimate now is the endpoint $(2,\infty,d)$ with $d \ge 4$. Alternatively, we have

$$\|\exp(it\mathrm{D})\phi(x)\|_{L_t^2L_x^\infty} \lesssim \|\phi\|^{\theta}_{\dot{H}^{b-\epsilon}}\|\phi\|^{1-\theta}_{\dot{H}^{b+\delta}}\ .$$

For the valid region of $(1/q, 1/r)$ for (1.10), see Figure 1.

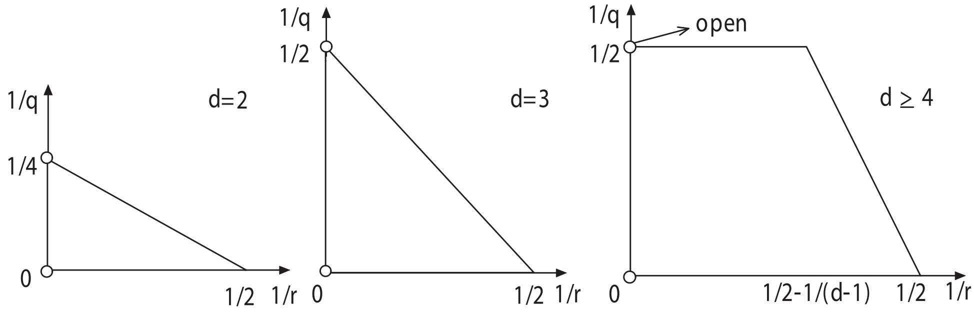

Figure 1. Classical Strichartz Estimate (1.10)

Remark. Note that the embeddings $\dot{H}^{d/2} \subset L^\infty$ and $H^{d/2} \subset L^\infty$ both fail to be valid even for radial function, then (1.10) fails for $(q, r) = (\infty, \infty)$ for any d.

Remark. As a complement to the failure of some $r = \infty$ estimate in (1.10), we have a simple but somewhat interesting result, that is, for $2 \leq q < \infty$, and $b = \frac{d}{2} - \frac{1}{q}$, we have

$$\|e^{it\mathrm{D}}\phi(x)\|_{L^\infty_x L^q_t} \lesssim \|\phi\|_{\dot{H}^b} .$$

For radial functions, the region of "admissible" triple can be vastly improved, that is an angular momentum improvement will be given,

Theorem 1.8. *Let $d \geq 3$ be the number of spatial dimensions, $\sigma_\Omega = d - 1$, $\sigma = \frac{d-1}{2}$. Then for every $\epsilon > 0$, there is a C_ϵ depending only on ϵ such that the following set of estimates hold for any $\phi \in \mathcal{S}$:*

$$\|e^{it\mathrm{D}}\phi(x)\|_{L^q_t L^r_x} \lesssim C_\epsilon \left(\|\langle\Omega\rangle^s\phi(x)\|_{\dot{H}^b}\right) , \tag{1.12}$$

where we have that $r < \infty$, $s = (1 + \epsilon)(\frac{d-1}{r} + \frac{2}{q} - \frac{d-1}{2})$, $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - b$, $\frac{1}{q} + \frac{\sigma}{r} \geqslant \frac{\sigma}{2}$, and $\frac{1}{q} + \frac{\sigma_\Omega}{r} < \frac{\sigma_\Omega}{2}$. All of the implicit constants in the above inequality depend on d, q, and r. Here $\Omega_{i,j} := x_i\partial_j - x_j\partial_i$, $\Delta_{sph} := \sum_{i<j}\Omega^2_{ij}$, $|\Omega|^s = (-\Delta_{sph})^{\frac{s}{2}}$, and

$$\|\langle\Omega\rangle^s\phi\|_{\dot{H}^b} = \|\phi\|_{\dot{H}^b} + \||\Omega|^s\phi\|_{\dot{H}^b}.$$

For its figure, see Figure 2.

Theorem 1.9. *Let (q, r, d) be radial-admissible and $(q, r) \neq (\infty, \infty)$. Then* (1.10) *valid for all radial ϕ.*

For the completeness of exposition, we also state here the corresponding estimate in the Sobolev space H^s. Here $b+$ denotes the $b + \epsilon$ with $\epsilon > 0$ arbitrarity small.

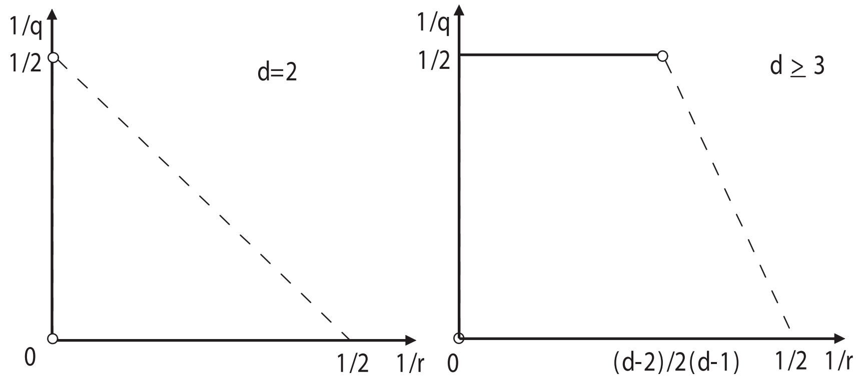

Figure 2. Radial improvement of Strichartz Estimate (1.10)

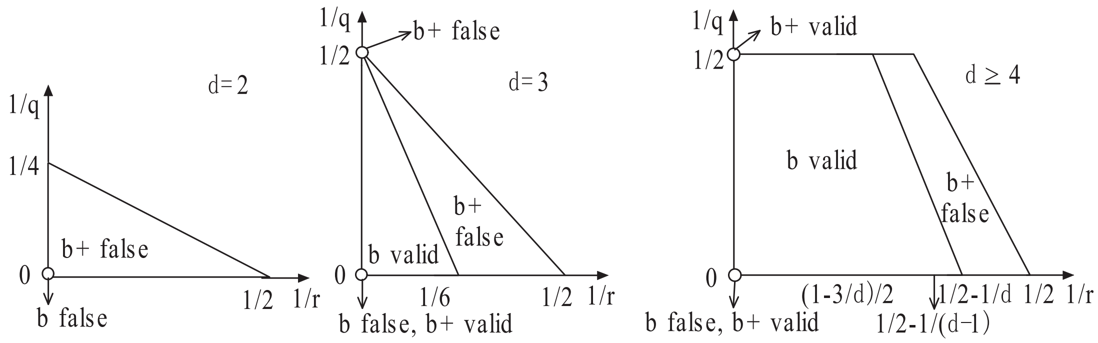

Figure 3. Theorem 1.10: Strichartz Estimate in H^s

Theorem 1.10. *Let $d \geq 2$, and let $u(t,x)$ be the solution to* (1.1) *with data (ϕ, ψ). Then the estimate*

$$\|u\|_{L_t^q L_x^r} \lesssim \|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} \tag{1.13}$$

is valid with $s = b+$ if $d \geq 3$, $b \geq 1$, (q,r,d) is admissible and $(q,r,d) \neq (2,\infty,3)$, on the other hand, (1.13) *is valid with $s = b+$ only if $b \geq 1$, (q,r,d) is admissible and $(q,r,d) \neq (2,\infty,3)$. Moreover, if $d \geq 3$, $b \geq 1$, (q,r,d) is admissible and $(q,r) \neq (2,\infty)$, (∞,∞), then* (1.13) *is valid with $s = b$. And if* (1.13) *is valid with $s = b$, we need $n \geq 3$, $b \geq 1$, (q,r,d) is admissible, $(q,r,d) \neq (2,\infty,3)$ and $(q,r) \neq (\infty,\infty)$.*

Note that (1.13) being valid for s implies its validness for $s+$, and the $s+$ failure implies the s failure.

From the figure 3 of Theorem 1.10, we see that there is a new limitation for H^s estimate, due to the fact that $b-1$ may be less than 0. However, if one substitutes ∂u for u in (1.13) with $\partial u = (\partial_t u, \nabla u)$, one can eliminate such limitation. In fact,

$$\|e^{it\mathrm{D}}\phi\|_{L_t^q L_x^r} \lesssim \|\phi\|_{H^s}$$

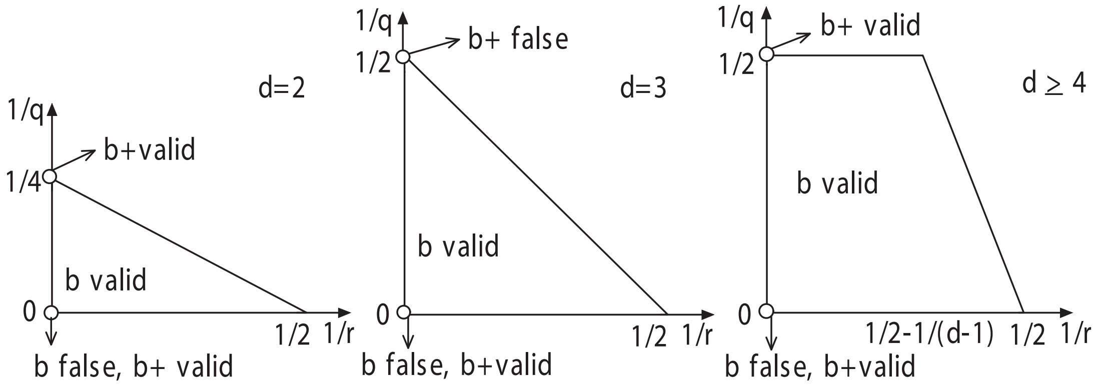

Figure 4. Theorem 1.11, Strichartz Estimate in H^s

implies

$$\|\partial u\|_{L_t^q L_x^r} \lesssim \|\phi\|_{H^{s+1}} + \|\psi\|_{H^s},$$

and we have

$$\|\cos(t\mathrm{D})\phi\|_{L_t^q L_x^r} \lesssim \|\phi\|_{H^s}.$$

Hence, we have (for it's figure, see Figure 4)

Theorem 1.11. *Let $d \geq 2$, and let $u(t,x)$ be the solution to* (1.1) *with data (ϕ, ψ). Then we have*

$$\|\partial u\|_{L_t^q L_x^r} \lesssim \|\phi\|_{H^{s+1}} + \|\psi\|_{H^s} \tag{1.14}$$

with $s = b+$ if and only if (q,r,d) is admissible and $(q,r,d) \neq (2,\infty,3)$. Moreover, for admissible (q,r,d) except that $(q,r) = (\max(2, \frac{4}{d-1}), \infty)$ and $(q,r) = (\infty,\infty)$, we have (1.14) *valid with $s = b$. On the other hand, if* (1.14) *is valid with $s = b$, then (q,r,d) is admissible and $(q,r,d) \neq (2,\infty,3), (\infty,\infty,d)$.*

From Hidano-Kurokawa's work in 2011: we can say the radial Strichartz holds true iff

$$\frac{1}{q} < (d-1)(\frac{1}{2} - \frac{1}{r}) \text{ or } (q,r) = (\infty, 2), q \geq 2, (q,r) \neq (\infty,\infty).$$

1.4.4. Generalized Strichartz estimate Radial Strichartz can be regarded as a special case of the estimates with angular regularity. The following weighted Strichartz type estimates are obtained by Wang and myself in [4]. It is related to the Keel-Smith-Sogge's estimate. One also can think of it as a generalization of general Strichartz estimate involving "non-admissible" Lebesgue exponents r, which are compensated by the weights involving powers of $|x|$. A slightly weaker version of it was also obtained by Hidano-Matcalfe-Smith-Sogge-Zhou in [2].

Let $x = r\omega \in \mathbb{R}^d$ with $r = |x|$ and $\omega \in S^{d-1}$, $\Omega_{ij} = x_i\partial_j - x_j\partial_i$,

$$\Delta = \sum_{i=1}^{d} \partial_i^2 = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_\omega,$$

$\Lambda_\omega = \sqrt{1-\Delta_\omega}$, and $\Delta_\omega = \sum_{1\le i<j\le d} \Omega_{ij}^2$.

Theorem 1.12 (Fang-Wang 2008). *If $b \in (1,d)$, $a > 0$ and $r \in [2,\infty]$, we have*

$$\| |x|^{\frac{d}{2}-\frac{d}{r}-\frac{b}{2}} e^{it\mathrm{D}^a} f(x)\|_{L^r_{t,|x|^{d-1}d|x|} L^2_\omega} \lesssim \|\mathrm{D}^{\frac{b}{2}-\frac{a}{r}} \Lambda_\omega^{\frac{1-b}{2}} f\|_{L^2_x}. \tag{1.15}$$

We also have the following generalized Strichartz estimate

Theorem 1.13 (Fang-Wang 2008). *Let $s = d(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}$, $s_{kd} = \frac{2}{q}-(d-1)(\frac{1}{2}-\frac{1}{r})$, and $\frac{d-1}{2}(\frac{1}{2}-\frac{1}{r}) < \frac{1}{q} < (d-1)(\frac{1}{2}-\frac{1}{r})$, $q \ge 2$, then we have for any $s_1 > s_{kd}$*

$$\|e^{it\mathrm{D}} f\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^{s,s_1}_\omega}.$$

The related special results can be referred to [18, 13, 11].

Next, we give a proof procedure of the Weighted Strichartz estimates with angular regularity.

Besov spaces with angular regularity ($m \ge 0$) are defined as follows

$$B^{s,m}_{p,q,\omega} = \Lambda_\omega^{-m} B^s_{p,q} = \{u \in B^s_{p,q} : \|\Lambda^m_\omega u\|_{B^s_{p,q}} < \infty\}.$$

$$\dot{B}^{s,m}_{p,q,\omega} = \Lambda_\omega^{-m} \dot{B}^s_{p,q}, H^{s,m}_\omega = B^{s,m}_{2,2,\omega}, \dot{H}^{s,m}_\omega = \dot{B}^{s,m}_{2,2,\omega}.$$

One version of the trace lemma can be stated as follows

$$r^{\frac{d-1}{2}} \|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}}.$$

Moreover, for $s \in (1/2, d/2)$, we have

$$r^{\frac{d}{2}-s} \|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{H}^s_x}.$$

For more delicate version and proof, see Fang and Wang [4], which states that

$$r^{\frac{d}{2}-s} \|f(r\omega)\|_{H^{s-1/2}_\omega} \lesssim \|f\|_{\dot{H}^s_x}.$$

Here, we want to give an elementary proof of these two estimates.

Lemma 1.14. *When $d \ge 3$, for $s \in [1, d/2)$, we have*

$$r^{\frac{d}{2}-s} \|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{H}^s_x}.$$

When $d \geq 1$, for $s \in [1/2, (d+2)/4)$, we have

$$r^{\frac{d}{2}-s}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|^{1/2}_{\dot{H}^{2s-1}}\|f\|^{1/2}_{\dot{H}^1}\ . \tag{1.16}$$

In particular, for any $d \geq 1$,

$$r^{\frac{d-1}{2}}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|^{1/2}_{L^2}\|f\|^{1/2}_{\dot{H}^1}\ . \tag{1.17}$$

Proof. For the last inequality,

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} |f(r\omega)|^2 d\omega &= -\int_{\mathbb{S}^{d-1}}\int_r^\infty \partial_\lambda |f(\lambda\omega)|^2 d\lambda d\omega \\
&\leq 2\ r^{1-d}\int_{\mathbb{S}^{d-1}}\int_r^\infty |f||\partial_\lambda f|\lambda^{d-1} d\lambda d\omega \\
&\leq 2\ r^{1-d}\|f\|_{L^2}\|\partial_\lambda f\|_{L^2}.
\end{aligned}$$

For the first inequality, since $s \geq 1$,

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} |f(r\omega)|^2 d\omega &= -\int_{\mathbb{S}^{d-1}}\int_r^\infty \partial_\lambda |f(\lambda\omega)|^2 d\lambda d\omega \\
&\leq 2\ r^{2s-d}\int_{\mathbb{S}^{n-1}}\int_r^\infty |f||\partial_\lambda f|\lambda^{d-2s} d\lambda d\omega \\
&\leq 2\ r^{2s-d}\int_{\mathbb{S}^{d-1}}\int_r^\infty |\lambda^{-s} f||\lambda^{1-s}\partial_\lambda f|\lambda^{d-1} d\lambda d\omega \\
&\leq 2\ r^{2s-d}\|\lambda^{-s} f\|_{L^2}\|\lambda^{1-s}\partial_\lambda f\|_{L^2} \\
&\leq C r^{2s-d}\|f\|^2_{\dot{H}^s}\ ,
\end{aligned}$$

where in the last inequality we have used the Hardy's inequality: if $0 \leq s < d/2$, then

$$\||x|^{-s} f\|_{L^2} \leq C_{s,d}\|f\|_{\dot{H}^s}\ .$$

Similarly, for the second inequality, since $s \in [1/2, (d+2)/4)$, we have $0 \leq 2s-1 < d/2$,

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} |f(r\omega)|^2 d\omega &= -\int_{\mathbb{S}^{n-1}}\int_r^\infty \partial_\lambda |f(\lambda\omega)|^2 d\lambda d\omega \\
&\leq 2\ r^{2s-d}\int_{\mathbb{S}^{d-1}}\int_r^\infty |f||\partial_\lambda f|\lambda^{d-2s} d\lambda d\omega \\
&\leq 2\ r^{2s-d}\int_{\mathbb{S}^{d-1}}\int_r^\infty \lambda^{-(2s-1)}|f||\partial_\lambda f|\lambda^{d-1} d\lambda d\omega \\
&\leq C\ r^{2s-d}\|f\|_{\dot{H}^{2s-1}}\|\partial_\lambda f\|_{L^2}.
\end{aligned}$$

■

As we will see, Lemma 1.14 is enough for the proof of the general trace estimates.

Theorem 1.15 (Trace lemma)**.** *For $s \in (1/2, d/2)$, we have*

$$r^{\frac{d}{2}-s}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{H}^s_x} \ . \tag{1.18}$$

Moreover, for any $d \geq 1$,

$$r^{\frac{d-1}{2}}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}} \ . \tag{1.19}$$

Proof. It suffices to give the proof for $f \in \mathcal{S}$. Applying (1.17) to the Littlewood-Paley projection $P_\lambda f$ with frequency of size λ, we see that

$$r^{\frac{d-1}{2}}\|P_\lambda f(r\omega)\|_{L^2_\omega} \lesssim \lambda^{1/2}\|P_\lambda f\|_{L^2} \ .$$

By Littlewood-Paley decomposition, we arrived

$$r^{\frac{d-1}{2}}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}} \ .$$

We are remained to prove the first inequality for $1/2 < s < 1$. Take $s_1 \in (s, 1)$, applying (1.16) to $P_\lambda f$ for s_1, we get

$$r^{\frac{d}{2}-s_1}\|P_\lambda f(r\omega)\|_{L^2_\omega} \lesssim \lambda^{s_1}\|P_\lambda f\|_{L^2},$$

and so

$$r^{\frac{d}{2}-s_1}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{s_1}_{2,1}} \ .$$

If we take $s_0 \in (1/2, s)$, we can show the same inequalities as s_1. Recall that for $s_0 \neq s_1$, we have the following fact of the real interpolation

$$[\dot{B}^{s_0}_{2,1}, \dot{B}^{s_1}_{2,1}]_{\theta,2} = \dot{H}^{(1-\theta)s_0+\theta s_1} \ ,$$

which tells us that we can actually have

$$r^{\frac{d}{2}-s}\|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{H}^s}$$

for $1/2 < s < 1$. ■

Lemma 1.16. *For any $q \in (2, \infty)$, $s = s_d = 1/2 - 1/q$,*

$$\|r^{(d-1)s}u\|_{L^q_r L^2_\omega(r^{d-1}drd\omega)} \lesssim \|u\|_{\dot{H}^s}.$$

With the help of the trace estimates, we can prove the Morawetz type estimates.

Theorem 1.17 (Morawetz type estimates)**.** *For $s \in (1/2, d/2)$, we have*

$$\| \, |x|^{-s}e^{it|D|}\varphi\|_{L^2(\mathbb{R}_+\times\mathbb{R}^d)} \lesssim \|\varphi\|_{\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^d)}. \tag{1.20}$$

Moreover, for any $d \geq 1$, we have

$$\| \, e^{it|D|}\varphi\|_{L^2(\mathbb{R}_+\times\{|x|\sim 1\})} \lesssim \| \, \varphi\|_{L^2(\mathbb{R}^d)} \ . \tag{1.21}$$

As a corollary, we have (by scaling)

$$\sup_R R^{-1/2} \| e^{it|D|}\varphi \|_{L^2(\mathbb{R}_+ \times \{|x| \lesssim R\})} \lesssim \| \varphi \|_{L^2(\mathbb{R}^d)} \,.$$

The inequality (1.21) is also known as local energy estimate, which dates back to the works of Morawetz, Strauss and others in 1970's. A more general version is due to Smith and Sogge, which states that for $\psi \in C_0^\infty$,

$$\|\psi e^{itD} f\|_{L_t^2 H^s} \lesssim_\psi \|f\|_{\dot{H}^s}, 0 \le s \le (d-1)/2 \,. \tag{1.22}$$

The inequality (1.20) is sometimes called generalized Morawetz estimate.

Remark. As a consequence of (1.21), we have

$$\|\langle x\rangle^{-b} e^{it|D|} f\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_x} \tag{1.23}$$

for any $b > 1/2$, and

$$\||x|^{-b} e^{it|D|} f\|_{L^2_{t,|x|\le 1}} \lesssim \|f\|_{L^2_x} \tag{1.24}$$

for any $b < 1/2$.

Note that by applying (1.18) to the Fourier transform of v, we see that it is equivalent to the uniform bounds

$$\Big(\int_{S^{d-1}} |\hat{v}(\lambda\omega)|^2 \, d\omega \Big)^{1/2} \lesssim \lambda^{-\frac{d}{2}+s} \| \, |x|^s v \|_{L^2(\mathbb{R}^d)} \,, \quad \lambda > 0 \,, \quad \frac{1}{2} < s < \frac{d}{2} \,,$$

which by duality is equivalent to

$$\Big\| \, |x|^{-s} \widehat{(h d\omega)}(\lambda x) \, \Big\|_{L^2_x(\mathbb{R}^d)} \lesssim \lambda^{s-\frac{d}{2}} \|h\|_{L^2_\omega(S^{d-1})} \,, \tag{1.25}$$

for $\lambda > 0$. Using this estimate we can obtain (1.20).

In fact, recall that

$$\mathcal{F}_{tx} e^{it|D|}\varphi = \int_{\mathbb{R}} e^{-it(\tau-|\xi|)} \hat{\varphi}(\xi) dt \sim \delta(\tau - |\xi|)\hat{\varphi}(\xi) \ ,$$

$$\mathcal{F}_t e^{it|D|}\varphi \sim \int_{\mathbb{R}^d} e^{ix\cdot\xi} \delta(\tau - |\xi|)\hat{\varphi}(\xi) d\xi = \int_{S^{d-1}} e^{i\tau x\cdot\omega} \tau^{d-1} \hat{\varphi}(\tau\omega) d\omega \ .$$

Thus, by Plancherel's theorem with respect to the t-variable, we find that the square of the left side of (1.20) equals

$$\begin{aligned}(2\pi)^{-1} \int_0^\infty \int_{\mathbb{R}^d} \Big| \, |x|^{-s} \int_{S^{d-1}} e^{ix\cdot\rho\omega} \rho^{d-1} \hat{\varphi}(\rho\omega) \, d\omega \, \Big|^2 \, dx \, d\rho \\ \lesssim \int_0^\infty \int_{S^{d-1}} \rho^{2(d-1)} |\hat{\varphi}(\rho\omega)|^2 \, \rho^{2s-d} \, d\omega d\rho = \| \, |D|^{s-\frac{1}{2}} \varphi \|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

by using (1.25) in the first step.

If we apply (1.19) instead, we see that, for $s = 1/2$ and $\lambda > 0$, we have

$$\left(\int_{S^{d-1}} |\hat{v}(\lambda\omega)|^2 \, d\omega\right)^{1/2} \lesssim \lambda^{\frac{1-d}{2}} \| \, |x|^{1/2}\phi(x2^{-j})v\|_{l^1_j L^2(\mathbb{R}^d)},$$

which by duality is equivalent to

$$\left\| \, |x|^{-1/2}\phi(x2^{-j}) \int_{S^{d-1}} h(\omega)e^{i\lambda x\cdot\omega} \, d\omega \, \right\|_{l^\infty_j L^2_x(\mathbb{R}^d)} \lesssim \lambda^{\frac{1-d}{2}} \|h\|_{L^2_\omega(S^{d-1})} \, , \tag{1.26}$$

for $\lambda > 0$. Using this estimate we can obtain (1.21).

In fact, recall that

$$\mathcal{F}_t e^{it|D|}\varphi \sim \int_{\mathbb{R}^d} e^{ix\cdot\xi}\delta(\tau - |\xi|)\hat{\varphi}(\xi)d\xi = \int_{S^{d-1}} e^{i\tau x\cdot\omega}\tau^{d-1}\hat{\varphi}(\tau\omega)d\omega \ .$$

Thus, by Plancherel's theorem with respect to the t-variable, we find that the square of the left side of (1.21) equals

$$\begin{aligned}(2\pi)^{-1}\int_0^\infty \int_{|x|\sim 1} \left| \, |x|^{-1/2} \int_{S^{d-1}} e^{ix\cdot\rho\omega}\rho^{d-1}\hat{\varphi}(\rho\omega)\, d\omega \right|^2 dx\, d\rho \\ \lesssim \int_0^\infty \int_{S^{d-1}} \rho^{2(d-1)}|\hat{\varphi}(\rho\omega)|^2 \, \rho^{1-d} \, d\omega d\rho = \| \, \varphi\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

by using (1.26) in the first step.

Theorem 1.18 (Fang-Wang 2008). *Let $b \in (1, d)$ and $d \geq 2$. Then for any $g \in L^2_\omega$,*

$$\|\Lambda_\omega^{\frac{b-1}{2}} |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x)\|_{L^2_x} \simeq \|g\|_{L^2_\omega}, \tag{1.27}$$

where $d\sigma$ denotes the spherical measure on S^{d-1}.

This version of trace lemma usually takes the form

$$\||x|^{-\frac{b}{2}}\widehat{gd\sigma}(x)\|_{L^2_x} \lesssim \|g\|_{H^s_\omega}. \tag{1.28}$$

The condition on b is also necessary for the estimate (1.27) or (1.28).

From the trace Lemma, we can get

Theorem 1.19 (Generalized Morawetz/Local energy estimates). *Let $b \in (1, d)$ and $a > 0$. We have*

$$\||x|^{-\frac{b}{2}}e^{it\mathrm{D}^a}f\|_{L^2_{t,x}} \simeq \|\mathrm{D}^{\frac{b-a}{2}}\Lambda_\omega^{\frac{1-b}{2}}f\|_{L^2_x}. \tag{1.29}$$

Moreover, for $b = 1$, we have the following local estimate

$$\sup_{x_0,R} R^{-\frac{1}{2}}\|e^{it\mathrm{D}^a}f\|_{L^2_{t,B(x_0,R)}} \lesssim \|\mathrm{D}^{\frac{1-a}{2}}f\|_{L^2_x}. \tag{1.30}$$

The generalized Morawetz estimates usually take the form

$$\||x|^{-\frac{b}{2}} e^{it\mathrm{D}^a} f\|_{L^2_{t,x}} \lesssim \|\mathrm{D}^{\frac{b-a}{2}} \Lambda^s_\omega f\|_{L^2_x}. \tag{1.31}$$

By using duality , scaling and Sobolev embedding on sphere, we have

Corollary 1.20 (Fang-Wang 2011). *Let $b \in (1, d)$ and $d \geq 2$. We have*

$$r^{\frac{d-b}{2}} \|f(r\omega)\|_{L^2_\omega} \lesssim \|\mathrm{D}^{\frac{b}{2}} \Lambda_\omega^{\frac{1-b}{2}} f\|_{L^2_x} \tag{1.32}$$

for any $f \in \mathcal{S}$. Moreover, if $s > \frac{d-b}{2}$, then

$$r^{\frac{d-b}{2}} \|f(r\omega)\|_{L^\infty_\omega} \lesssim \|\mathrm{D}^{\frac{b}{2}} \Lambda^s_\omega f\|_{L^2_x}. \tag{1.33}$$

Remark. (1.32) for radial functions is proved by Li-Zhou (1995) . Cho-Ozawa in 2007 proved (1.33) with $s > d-1-\frac{b}{2}$. For the case $b = 1$ it is due to Agmon-Hörmander (1976):

$$\sup_{x_0, R} R^{-\frac{1}{2}} \|\widehat{gd\sigma}(x)\|_{L^2_{B(x_0,R)}} \lesssim \|g\|_{L^2_\omega}. \tag{1.34}$$

Moreover, by duality and rescaling,

$$r^{\frac{d-1}{2}} \|f(r\omega)\|_{L^2_\omega} \lesssim \|f\|_{\dot{B}^{\frac{1}{2}}_{2,1}}. \tag{1.35}$$

Combining with (1.33) and (1.35), we have

Proposition 1.21 (Compact Embedding). *The embedding $H_\omega^{\frac{b}{2}, m} \subset L^p$ is compact for $b \in (1, d)$, $m > \frac{d-b}{2}$ and $2 < p < \frac{2d}{d-b}$. Moreover, the embedding $B^{\frac{1}{2}, m}_{2,1,\omega} \subset L^p$ is compact for $m > \frac{d-1}{2}$ and $2 < p < \frac{2d}{d-1}$.*

By using Morawetz estimates (1.29) and Sobolev estimates (1.32), we have Theorem 1.12.

The estimates stated in Theorem 1.12 is the homogeneous estimates. In practice, it is often important to give the inhomogeneous estimates. By the Christ-Kiselev lemma, we can get the inhomogeneous estimates. In conclusion, we have

Theorem 1.22. *Let $q, \tilde{q} \in [2, \infty]$, $\frac{d}{q} - \alpha, \frac{d}{\tilde{q}} - \tilde{\alpha} \in (0, \frac{d-1}{2})$, $s = \frac{d+a}{q} - \frac{d}{2} - \alpha$, $s_1 = \frac{d-1}{2} + \alpha - \frac{d}{q}$ (note that $s + s_1 = \frac{a}{q} - \frac{1}{2}$), and $\tilde{s}, \tilde{s_1}$ similarly defined. Then we have*

$$\||x|^{-\alpha} \mathrm{D}^s \Lambda^{s_1}_\omega e^{it\mathrm{D}^a} f(x)\|_{L^q_{t,|x|^{d-1}d|x|} L^2_\omega} \lesssim \|f\|_{L^2_x}, \tag{1.36}$$

and by duality, one can get

$$\|\int \mathrm{D}^s \Lambda^{s_1}_\omega e^{-is\mathrm{D}^a} F(s, x) ds\|_{L^2_x} \lesssim \||x|^\alpha F\|_{L^{q'}_{t,|x|^{d-1}d|x|} L^2_\omega}. \tag{1.37}$$

Moreover, we have the following inhomogeneous estimates

$$\| \int_0^t \mathrm{D}^s \Lambda_\omega^{s_1} e^{i(t-s)\mathrm{D}^a} F(s,x)ds \|_{L_t^\infty L_x^2} \lesssim \| |x|^\alpha F \|_{L^{q'}_{t,|x|^{d-1}d|x|} L^2_\omega}, \tag{1.38}$$

and

$$\| \int_0^t |x|^{-\alpha} \mathrm{D}^{s+\tilde{s}} \Lambda_\omega^{s_1+\tilde{s_1}} e^{i(t-s)\mathrm{D}^a} F(s,x)ds \|_{L^q_{t,|x|^{d-1}d|x|} L^2_\omega} \lesssim \| |x|^{\tilde{\alpha}} F \|_{L^{\tilde{q}'}_{t,|x|^{d-1}d|x|} L^2_\omega} \tag{1.39}$$

with $q > \tilde{q}'$.

In particular, if we choose $b \in (1, d)$ such that $\frac{d}{2} - \frac{d}{r} - \frac{b}{2} = 0$ in Theorem 1.12, we can get the following generalized Strichartz estimates with $q = r$ in presence of angular regularity.

Corollary 1.23. *Let $a > 0$, $r \in (\frac{2d}{d-1}, \infty)$ and $p \in [2, \infty)$. We have*

$$\| e^{it\mathrm{D}^a} f(x) \|_{L^r_{t,|x|^{d-1}d|x|} L^p_\omega} \lesssim \| \mathrm{D}^{\frac{d}{2} - \frac{d+a}{r}} \Lambda_\omega^{\frac{d}{r} - \frac{d-1}{p}} f \|_{L^2_x}, \tag{1.40}$$

for any $f \in \mathcal{S}$.

1.4.5. Generalized type Strichartz estimates

We shall require certain Strichartz estimates, which involve the angular mixed-norm spaces

$$\|u\|_{L_t^q L^\infty_{|x|} L^2_\theta(\mathbb{R}^2)} = \Big(\int_{\mathbb{R}} \mathrm{esssup}_{\rho > 0} \Big(\int_0^{2\pi} |f(\rho(\cos\theta, \sin\theta))|^2 \, d\theta \Big)^{q/2} dt \Big)^{1/q}.$$

Theorem 1.24 (Fang-Wang 2013). *For any $\gamma > \frac{1}{2}$, there exists a constant C_γ such that*

$$\left\| e^{-itD} f \right\|_{L_t^2 L^\infty_{|x|} L^2_\theta([0,T] \times \mathbb{R}^2)} \le C_\gamma \left(\ln(2+T) \right)^{\frac{1}{2}} \|f\|_{H^\gamma(\mathbb{R}^2)}. \tag{1.41}$$

Moreover, if $2 < q < \infty$, then

$$\left\| e^{-itD} f \right\|_{L_t^q L^\infty_{|x|} L^2_\theta(\mathbb{R} \times \mathbb{R}^2)} \le C_q \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)}, \quad \gamma = 1 - 1/q. \tag{1.42}$$

Remark. When $4 < q < \infty$, the Strichartz estimates (1.42) is weaker than the standard Strichartz estimates

$$\left\| e^{-itD} f \right\|_{L_t^q L_x^\infty(\mathbb{R} \times \mathbb{R}^2)} \le C_q \|f\|_{\dot{H}^\gamma(\mathbb{R}^2)}, \quad \gamma = 1 - 1/q, \ 4 < q < \infty. \tag{1.43}$$

By interpolating (1.42) with (1.43), we can also improve L^2_θ to L^p_θ in (1.42).

Remark. Note that we have also the trivial energy estimate

$$\| e^{-itD} f \|_{L_t^\infty L_{|x|}^2 L_\theta^2} \le C\|f\|_{L^2}, \tag{1.44}$$

since e^{-itD} is a unitary operator on L^2. By interpolation, we can also get more general Strichartz type estimates involving $L_t^q L_{|x|}^r L_\theta^2$ norm, where

$$\|u\|_{L_t^q L_{|x|}^r L_\theta^2(\mathbb{R}\times\mathbb{R}^2)} = \left(\int_{\mathbb{R}} \left(\int_0^\infty \left(\int_0^{2\pi} |f(\rho(\cos\theta, \sin\theta))|^2\, d\theta\right)^{r/2} \rho d\rho\right)^{q/r} dt\right)^{1/q}.$$

We prove Theorem 1.24, including the critical $L_t^2 L_{|x|}^\infty L_\theta^2$ Strichartz estimates for the wave equation when $n = 2$. The following proof essentially is the same to that in Smith,Sogge and Wang's related work [15].

Frequency Localization At first, we want to reduce the inequalities to the frequency localized counterparts.

It is easy to see that the frequency localized estimates for Theorem 1.24 are as follows

$$\|e^{-itD} f\|_{L_t^q L_{|x|}^\infty L_\theta^2(\mathbb{R}\times\mathbb{R}^2)} \le C_q \|f\|_{L^2(\mathbb{R}^2)}, \text{ if } q > 2, \text{ and } \hat{f}(\xi) = 0,\ |\xi| \notin [1/2, 1] \tag{1.45}$$

and

$$\|e^{-itD} f\|_{L_t^2 L_{|x|}^\infty L_\theta^2([0,T]\times\mathbb{R}^2)} \le C(\ln(2+T))^{1/2} \|f\|_{L^2(\mathbb{R}^2)}, \text{ if } \hat{f}(\xi) = 0,\ |\xi| \notin [1/2, 1]. \tag{1.46}$$

By scaling and Littlewood-Paley theory, we see that (1.42) and (1.45) are equivalent. To deduce (1.41) from (1.46), we will need to verify the following estimate for any $\delta > 0$

$$\sum_{j\in\mathbb{Z}} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} \le C_\delta(\ln(2+T))^{1/2}. \tag{1.47}$$

In fact, if $T \ge e$, we deal with the following two different cases.

i) $2^j \ge 1$;

$$\begin{aligned}
\sum_{j\ge 0} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\le \sum_{j\ge 0} 2^{-j\delta}(\ln(2+2^jT))^{1/2} \\
&\le C\sum_{j\ge 0} 2^{-j\delta}(j\ln 2 + \ln T)^{1/2} \\
&\le C\sum_{j\ge 0} 2^{-j\delta}(j\ln 2 + 1)^{1/2}(\ln T)^{1/2} \\
&\le C_\delta(\ln T)^{1/2}.
\end{aligned}$$

ii) $2^j \leq 1$;

$$\begin{aligned}\sum_{j<0} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\leq \sum_{j<0} 2^{j/2}(\ln(2+2^jT))^{1/2} \\ &\leq \sum_{j<0} 2^{j/2}(\ln(2+T))^{1/2} \\ &\leq C(\ln(2+T))^{1/2}.\end{aligned}$$

Else, if $T \leq e$, we also deal with two different cases.
i) $2^j \geq T^{-1}$;

$$\begin{aligned}\sum_{2^jT\geq 1} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} &\leq \sum_{2^jT\geq 1} 2^{-j\delta}(\ln(2+2^jT))^{1/2} \\ &\leq T^{\delta} \sum_{2^jT\geq 1} (2^jT)^{-\delta}(\ln(2+2^jT))^{1/2} \\ &\leq C_\delta T^\delta \leq \tilde{C}_\delta.\end{aligned}$$

ii) $1 \leq \lambda = 2^j \leq T^{-1}$;

$$\sum_{2^jT<1} 2^{j/2}(1+2^j)^{-1/2-\delta}(\ln(2+2^jT))^{1/2} \leq C\sum_{j} 2^{j/2}(1+2^j)^{-1/2-\delta} \leq C_\delta.$$

Further reduction Let us turn to the proof of (1.45) and (1.46). Due to the support assumptions for $\hat{f}$ we have that

$$\|f\|^2_{L^2(\mathbb{R}^2)} \approx \int_0^\infty \int_0^{2\pi} |\hat{f}(\rho(\cos\omega, \sin\omega))|^2 d\omega d\rho. \tag{1.48}$$

If we expand the angular part of $\hat{f}$ using Fourier series we find that if $\xi = \rho(\cos\omega, \sin\omega)$ then there are Fourier coefficients $c_k(\rho)$ which vanish when $\rho \notin [1/2, 1]$ so that

$$\hat{f}(\xi) = \sum_k c_k(\rho)e^{ik\omega},$$

and so, by (1.48) and Plancherel's theorem for $\mathbb{S}^1$ and $\mathbb{R}$ we have

$$\|f\|^2_{L^2(\mathbb{R}^2)} \approx \sum_k \int_{\mathbb{R}} |c_k(\rho)|^2\, d\rho \approx \sum_k \int_{\mathbb{R}} |\hat{c}_k(s)|^2\, ds, \tag{1.49}$$

if $\hat{c}_k$ denotes the one-dimensional Fourier transform of $c_k(\rho)$. Recall that (see Stein and Weiss [19] p. 137)

$$f(r(\cos\theta, \sin\theta)) = \frac{1}{2\pi}\sum_k \Big(i^k \int_0^\infty J_k(r\rho)c_k(\rho)\rho\, d\rho \Big) e^{ik\theta}, \tag{1.50}$$

if J_k is the k-th Bessel function, i.e.,

$$J_k(y) = \frac{(-i)^k}{2\pi} \int_0^{2\pi} e^{iy\cos\theta - ik\theta}\, d\theta. \tag{1.51}$$

Because of (1.50) and the support properties of the c_k, we find that if we fix $\beta \in C_0^\infty(\mathbb{R})$ satisfying $\beta(\tau) = 1$ for $1/2 \le \tau \le 1$ but $\beta(\tau) = 0$ if $\tau \notin [1/4, 2]$ then if we set $\alpha = \rho\beta(\rho) \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
&(e^{-itD} f)(r(\cos\theta, \sin\theta)) \\
&\quad = \frac{1}{2\pi} \sum_k \Big(i^k \int_0^\infty J_k(r\rho) e^{-it\rho} c_k(\rho)\beta(\rho)\rho\, d\rho \Big) e^{ik\theta} \\
&\quad = \frac{1}{(2\pi)^2} \sum_k \Big(i^k \int_0^\infty \int_{-\infty}^\infty J_k(r\rho) e^{i\rho(s-t)} \hat{c}_k(s)\alpha(\rho)\, ds d\rho \Big) e^{ik\theta} \\
&\quad = \frac{1}{(2\pi)^3} \sum_k \Big(\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} e^{i\rho r\cos\vartheta} e^{-ik\vartheta} e^{i\rho(s-t)} \hat{c}_k(s)\alpha(\rho)\, d\vartheta ds d\rho \Big) e^{ik\theta} \\
&\quad = \frac{1}{(2\pi)^3} \sum_k \Big(\int_{-\infty}^\infty \int_0^{2\pi} e^{-ik\vartheta} \hat{\alpha}\big((t-s) - r\cos\vartheta \big)\, \hat{c}_k(s)\, d\vartheta ds \Big) e^{ik\theta} \\
&\quad = \frac{1}{(2\pi)^3} \sum_k \Big(\int_{-\infty}^\infty \hat{c}_k(s) \psi_k(t-s, r) ds \Big) e^{ik\theta},
\end{aligned}$$

where we set

$$\psi_k(m, r) = \int_0^{2\pi} e^{-ik\theta} \hat{\alpha}\big(m - r\cos\theta \big)\, d\theta. \tag{1.52}$$

As a result, we have that for any $r \ge 0$

$$\int_0^{2\pi} \Big| (e^{-itD} f)(r(\cos\theta, \sin\theta)) \Big|^2 d\theta = \frac{1}{(2\pi)^5} \sum_k \Big| \int_{-\infty}^\infty \hat{c}_k(s)\psi_k(t-s, r)\, ds \Big|^2. \tag{1.53}$$

Now we claim that we have the estimate

$$\| \psi_k(m, r) \langle m \rangle^{\frac{1}{2}} \|_{L^2_m} \le C\,, \tag{1.54}$$

where $\langle m \rangle = \sqrt{1 + m^2}$ and C is independent of $k \in \mathbb{Z}$ and $r \ge 0$. If this is true, then

$$\begin{aligned}
\|(e^{-itD} f)(r, \theta)\|_{L^2_\theta} &\le C \|\hat{c}_k(s)\psi_k(t-s, r)\|_{l^2_k L^1_s} \\
&\le C \|\hat{c}_k(s) \langle t-s \rangle^{-1/2} \|_{l^2_k L^2_s} \| \langle t-s \rangle^{\frac{1}{2}} \psi_k(t-s, r)\|_{l^\infty_k L^2_s} \\
&\le C \|\hat{c}_k(s) \langle t-s \rangle^{-1/2} \|_{l^2_k L^2_s},
\end{aligned}$$

and we can immediately get the required estimates (1.45) and (1.46), if we note that

$$\langle t-s \rangle^{-1/2} \in L^q \text{ if } q > 2, \text{ and } \| \langle t-s \rangle^{-1/2} \|_{L^2_{t\in[0,T]}} \le C(\ln(2+T))^{1/2}.$$

The estimate for $\psi_k(m,r)$ Now we present the proof of the key estimate (1.54) for $\psi_k(m,r)$, to conclude the proof of the Strichartz estimate in Theorem 1.24.

We begin with the proof of the following pointwise estimates (which is precisely Lemma 2.1 of Smith, Sogge and Wang [15]).

Lemma 1.25. *Let $\alpha \in \mathcal{S}(\mathbb{R})$ and $N \in \mathbb{N}$ be fixed. Then there is a uniform constant C_N, which is independent of $m \in \mathbb{R}$ and $r \geq 0$ so that the following inequalities hold. First,*

$$\int_0^{2\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C_N \langle m \rangle^{-N}, \quad \text{if } \ 0 \leq r \leq 1, \ \text{or } \ |m| \geq 2r. \tag{1.55}$$

If $r > 1$ and $|m| \leq 2r$ then

$$\int_0^{2\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C\Big(r^{-1} + r^{-1/2} \langle\, r - |m|\, \rangle^{-1/2} \Big). \tag{1.56}$$

Consequently, for any $\delta > 0$, we have the weaker estimate for (1.54)

$$\| \psi_k(m,r)\, \langle m \rangle^{\frac{1}{2} - \delta} \|_{L^2_m} \leq C_\delta\,, \tag{1.57}$$

with the constant C_δ independent of $r > 0$.

Proof. We first realize that (1.55) is trivial since $\hat{\alpha} \in \mathcal{S}$. To prove (1.56), it suffices to show that

$$\int_0^{\pi/4} |\hat{\alpha}(m - r\cos\theta)|\, d\theta + \int_{\pi - \pi/4}^{\pi} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C r^{-1/2} \langle\, r - |m|\, \rangle^{-1/2}, \tag{1.58}$$

and also

$$\int_{\pi/4}^{\pi - \pi/4} |\hat{\alpha}(m - r\cos\theta)|\, d\theta \leq C r^{-1}. \tag{1.59}$$

In order to prove (1.58), it suffices to prove that the first integral is controlled by the right side. For if we apply this estimate to the function $\hat{\alpha}(-s)$, we then see that the second integral satisfies the same bounds. We can estimate the first integral if we make the substitution $u = 1 - \cos\theta$, in which case, we see that it equals

$$\begin{aligned}
\int_0^{1-1/\sqrt{2}} |\hat{\alpha}((m-r) + ru)|\, \frac{du}{\sqrt{2u - u^2}} &\leq \int_0^{1-1/\sqrt{2}} |\hat{\alpha}((m-r) + ru)|\, \frac{du}{\sqrt{u}} \\
&\leq C r^{-1/2} \int_0^{\infty} |\hat{\alpha}((m-r) + u)|\, \frac{du}{\sqrt{u}} \\
&\leq C' r^{-1/2} \langle\, r - m\, \rangle^{-1/2} \\
&\leq C' r^{-1/2} \langle\, r - |m|\, \rangle^{-1/2},
\end{aligned}$$

as desired, which completes the proof of (1.58).

To prove (1.59) we just make the change of variables $u = r\cos\theta$ and note that $|du/d\theta| \approx r$ on the region of integration, which leads to the inequality as $\hat{\alpha} \in \mathcal{S}$.

Finally, we check that inequalities (1.55) and (1.56) imply (1.57). If $r \le 1$, it is trivial. Else, if $r \ge 1$, we can prove (1.57) as follows

$$\begin{aligned}
\|\psi_k(m,r)\,\langle m\rangle^{\frac{1}{2}-\delta}\|^2_{L^2_m} &\le C + C\int_{|m|\le 2r} r^{-2}\,\langle m\rangle^{1-2\delta}\,dm \\
&\quad + C\int_{|m|\le 2r} r^{-1}\,\langle m\rangle^{1-2\delta}\,\langle r-|m|\rangle^{-1}\,dm \\
&\le C + Cr^{-2}\,\langle r\rangle^{2-2\delta} \\
&\quad + C\int_{r/2\le m\le 2r} r^{-1}\,\langle m\rangle^{1-2\delta}\,\langle r-m\rangle^{-1}\,dm \\
&\quad + C\int_{0\le m\le r/2} r^{-1}\,\langle m\rangle^{1-2\delta}\,\langle r-m\rangle^{-1}\,dm \\
&\le C + C\int_{r/2\le m\le 2r} r^{-2\delta}\,\langle r-m\rangle^{-1}\,dm \\
&\quad + C\int_{0\le m\le r/2} r^{-2}\,\langle m\rangle^{1-2\delta}\,dm \\
&\le C + Cr^{-2\delta}\ln(2+r) \le C_\delta \ (\text{if } \delta > 0).
\end{aligned}$$

■

Here, we remark that the reason we need to introduce a parameter $\delta > 0$ is due to the estimate (1.58) (the bound r^{-1} will be enough for us to get the estimate with $\delta = 0$).

To prove the stronger estimate (1.54), we need to consider the effect of oscillated factor $e^{-ik\theta}$ in the definition of $\psi_k(m,r)$, and the support property of the function α.

To begin, we give some more reductions. At first, without loss of generality, we can assume $m \ge 0$. In this case, we need only to give the estimate for $\theta \in [0, \frac{3\pi}{4}]$ and $\theta \in [\frac{3\pi}{4}, \pi]$. For the case $\theta \in [\frac{3\pi}{4}, \pi]$, since $m - r\cos\theta \simeq m + r$ and $\hat{\alpha} \in \mathcal{S}$, the estimate is admissible for our purpose. So we need only to give a refined estimate for the integral of the type

$$I_k(m,r) = \int_0^{3\pi/4} e^{-ik\theta}\hat{\alpha}\big(m - r\cos\theta\big)\,d\theta \tag{1.60}$$

when $m \le 2r$, $r > 1$. Moreover, we observe from (1.58) and (1.59) that

$$|\psi_k(m,r)|,\ |I_k(m,r)| \le Cr^{-1},\ \text{if } |m| \le r/2,$$

which are also admissible estimates. This means that we need only to consider the case

$r/2 < m < 2r$ with $r > 1$. Now we are ready to give the second estimate about $\psi_k(m,r)$ (which is resemble to Proposition 4.1 of [18]).

Lemma 1.26. *Let $\alpha \in \mathcal{S}$ with support in $[1/4, 2]$, $r > 1$ and $r/2 < m < 2r$. If $r < m+1$, then for any $N \geq 0$, we have*

$$|I_k(m,r)| \leq C_N r^{-1/2} \langle r - m \rangle^{-N} . \tag{1.61}$$

Else if $r > m+1$ and if $d = \sqrt{r^2 - m^2}$, we have

$$|I_k(m,r)| \leq C r^{-1/2} \langle r - m \rangle^{-1/2} \Big(\langle r - m \rangle^{-1} + \min(k/d, d/k) \Big) . \tag{1.62}$$

Here, when $k = 0$, the estimate is understood to be $|I_0(m,r)| \leq C r^{-1/2} \langle r - m \rangle^{-3/2}$.

Before giving the proof of Lemma 1.26, we give the proof of (1.54). By Lemma 1.25, Lemma 1.26 and the discussion before Lemma 1.26, we know that

$$|\psi_k(m,r)| \leq C \begin{cases} \langle m \rangle^{-N} , & |m| \geq 2r \text{ or } r \leq 1, \\ r^{-1}, & |m| \leq r/2 \text{ and } r > 1, \\ \langle r + |m| \rangle^{-N} + r^{-1/2} \langle |m| - r \rangle^{-N} , & \begin{array}{l} r < |m|+1, r/2 \leq |m| \leq 2r, \\ \text{and } r > 1, \end{array} \\ \langle r + |m| \rangle^{-N} + r^{-1/2} \langle |m| - r \rangle^{-3/2} & \\ \quad + r^{-1/2} \langle |m| - r \rangle^{-1/2} \min(k/d, d/k), & \begin{array}{l} r \geq |m|+1, r/2 \leq |m| \leq 2r \\ \text{and } r > 1. \end{array} \end{cases}$$

Then a simple calculation will give us the key estimate (1.54).

In fact, the case where $r \leq 1$ is trivial. So we need only to consider the case with $r > 1$, in which case, we write the integral into the sums as follows

$$\begin{aligned} & \int_{\mathbb{R}} |\psi_k(m,r)|^2 \langle m \rangle \, dm \\ = & \left(\int_{|m| \leq r/2} + \int_{|m| \geq 2r} + \int_{\max(r-1, r/2) < |m| < 2r} + \int_{r/2 < |m| < r-1} \right) |\psi_k(m,r)|^2 \langle m \rangle \, dm \\ = & I + II + III + IV \ . \end{aligned}$$

The first two terms I and II can be estimated as before. For III,

$$\begin{aligned} III \leq & C + \int_{\max(r-1, r/2) < |m| < 2r} r^{-1} \langle |m| - r \rangle^{-2N} \langle m \rangle \, dm \\ \leq & C + C \int_{\max(r-1, r/2) < |m| < 2r} \langle |m| - r \rangle^{-2N} \, dm \leq C. \end{aligned}$$

Now we turn to the estimate for IV,

$$\begin{aligned}
IV &\leq C + \int_{r/2<|m|<r-1} (r^{-1}\left\langle |m|-r\right\rangle^{-3} + r^{-1}\left\langle |m|-r\right\rangle^{-1}\min(k^2/d^2, d^2/k^2))\left\langle m\right\rangle dm \\
&\leq C + C\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-3} dm \\
&\quad + C\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \\
&\leq C,
\end{aligned}$$

where in the last inequality, we used the fact that

$$\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \lesssim 1.$$

In fact, if $k^2 \leq r$, then

$$\begin{aligned}
&\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \\
&\lesssim \int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-1} k^2 d^{-2} dm \\
&\leq C\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-2} k^2 r^{-1} dm \\
&\leq Ck^2/r \leq C.
\end{aligned}$$

Else, if $k^2 > r$, we have

$$\begin{aligned}
&\int_{r/2<|m|<r-1} \left\langle |m|-r\right\rangle^{-1}\min(k^2/d^2, d^2/k^2) dm \\
&\leq \int_{|m|<r-k^2/r} \left\langle |m|-r\right\rangle^{-1} k^2 d^{-2} dm \\
&\quad + \int_{\max(r/2, r-k^2/r)<|m|<r-1} \left\langle |m|-r\right\rangle^{-1} d^2 k^{-2} dm \\
&\leq C\int_{|m|<r-k^2/r} \left\langle |m|-r\right\rangle^{-2} k^2 r^{-1} dm \\
&\quad + C\int_{\max(r/2, r-k^2/r)<|m|<r-1} r k^{-2} dm \\
&\leq C\left\langle k^2/r\right\rangle^{-1} k^2/r + Crk^{-2}\min(k^2/r-1, r/2-1) \leq \tilde{C}.
\end{aligned}$$

This proves our key estimate (1.54).

Finally, we give the proof of Lemma 1.26, which will conclude the proof of (1.54).

Proof. If $r < m+1$, we have

$$m - r\cos\theta = r(1-\cos\theta) + m - r \geq m - r \geq -1.$$

Let $u = 1 - \cos\theta$. So we get

$$\langle m - r\cos\theta\rangle \simeq 1 + r(1-\cos\theta) + (2+m-r) \simeq \langle m-r\rangle + \langle ru\rangle\,.$$

Since $\alpha \in \mathcal{S}$,

$$\begin{aligned}
|I_k(m,r)| &\le \int_0^{\frac{3\pi}{4}} |\hat{\alpha}\big(m - r\cos\theta\big)|\; d\theta \\
&\le C\int_0^{\frac{3\pi}{4}} \langle m - r\cos\theta\rangle^{-2N}\; d\theta \\
&\le C\int_0^{\frac{3\pi}{4}} \langle m - r\rangle^{-N}\langle ru\rangle^{-N}\; d\theta \\
&\le C\int_0^{1+1/\sqrt{2}} \langle m - r\rangle^{-N}\langle ru\rangle^{-N}\; \frac{du}{\sqrt{2u-u^2}} \\
&\le C\int_0^{1+1/\sqrt{2}} \langle m - r\rangle^{-N}\langle ru\rangle^{-N}\; \frac{du}{\sqrt{u}} \\
&\le Cr^{-1/2}\langle m - r\rangle^{-N},
\end{aligned}$$

which gives us (1.61).

Now we turn to the proof for the case $r \ge m+1$. We can imagine that the behavior is worst in the region that $m - r\cos\theta \sim 0$. To illustrate this, we introduce $\theta_0 \in (0, \frac{\pi}{2}]$ such that

$$r\cos\theta_0 = m,\ \sin\theta_0 = \frac{\sqrt{r^2-m^2}}{r} \equiv \frac{d}{r}. \tag{1.63}$$

Then the local behavior of the function $r\cos\theta - m$ near $\theta = \theta_0$ looks like

$$r\cos\theta - m \sim -d(\theta-\theta_0) + \mathcal{O}((\theta-\theta_0)^2),$$

since $r\cos\theta_0 - m = 0$ and $\frac{d}{d\theta}(r\cos\theta - m)|_{\theta=\theta_0} = -r\sin\theta_0 = -d$. Based on this information, we make the change of variable

$$\beta = d(\theta-\theta_0),\ \phi(\beta) = m - r\cos(\theta_0 + \beta/d)\,. \tag{1.64}$$

For the function ϕ, we can find that $\phi(0) = 0$, $\phi'(\beta) = \frac{r}{d}\sin\theta$. Moreover, we have the following

Lemma 1.27. *Let $\phi(\beta)$ be the function defined by* (1.64)*, and $\theta \in [\theta_1, \frac{3\pi}{4}]$ with $\theta_1 \in (0, \frac{\pi}{4})$ such that $\frac{r}{d}\sin\theta_1 = \frac{1}{2}$. Then we have*

$$\frac{1}{2} \le \phi'(\beta) \le 1 + |\phi(\beta)| \simeq \langle\phi(\beta)\rangle\,. \tag{1.65}$$

In addition,

$$|\phi(\beta)| = |\phi(\beta) - \phi(0)| \ge \frac{1}{2}|\beta|.$$

Proof. We need only to give the proof of the inequality

$$\phi'(\beta) \leq 1 + |\phi(\beta)|.$$

In fact, if $\beta = 0$ (i.e., $\theta = \theta_0$), we know the inequality is true with identity (see (1.63)). For $\beta \leq 0$, we have $\phi(\beta) \leq 0$ and the inequality amounts to $\phi'(\beta) \leq 1 - \phi(\beta)$, which is equivalent to

$$\frac{r}{d}\sin\theta \leq 1 + r\cos\theta - m,\ \ \theta \in [\theta_1, \theta_0].$$

Now we can see that this inequality is trivial by the monotonicity of the trigonometric functions

$$\frac{r}{d}\sin\theta \leq \frac{r}{d}\sin\theta_0 = 1 = 1 + r\cos\theta_0 - m \leq 1 + r\cos\theta - m.$$

If we consider instead the case $\beta \geq 0$, we know that it is equivalent to

$$\frac{r}{d}\sin\theta \leq 1 + m - r\cos\theta,\ \ \theta \in [\theta_0, \frac{3\pi}{4}]. \tag{1.66}$$

Once again, by the monotonicity of the trigonometric functions, we need only to prove the inequality for $\theta \in [\theta_0, \frac{\pi}{2}]$. In the latter case, consider $F(\theta) = 1 + m - r\cos\theta - \frac{r}{d}\sin\theta$. We observe that (recall $r \geq m+1$)

$$F'(\theta) = r\sin\theta - \frac{r}{d}\cos\theta \geq r\sin\theta_0 - \frac{r}{d}\cos\theta_0 = d - \frac{m}{d} \geq \frac{(m+1)^2 - m^2 - m}{d} \geq 0\ .$$

Recall $F(\theta_0) = 0$, we know that $F(\theta) \geq 0$ for $\theta \in [\theta_0, \frac{\pi}{2}]$ and so is (1.66). This completes the proof of the inequality (1.65). ∎

Now let us continue the proof of the estimate for I_k. We write

$$I_k(m,r) = J_k(m,r) + K_k(m,r) \tag{1.67}$$

with

$$\begin{aligned} J_k(m,r) &= \int_{\theta_1}^{3\pi/4} e^{-ik\theta}\hat{\alpha}\big(m - r\cos\theta\big)\ d\theta \\ &= \frac{e^{-ik\theta_0}}{d}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)} e^{-i\frac{k}{d}\beta}\hat{\alpha}\big(\phi(\beta)\big)\ d\beta \equiv \frac{e^{-ik\theta_0}}{d}L_k. \end{aligned} \tag{1.68}$$

We first give the easier estimate for K_k. In fact, if $\theta \in [0, \theta_1]$,

$$r\cos\theta - m \geq r\cos\theta_1 - m = \frac{\sqrt{3r^2+m^2}}{2} - m \geq \frac{3(r^2-m^2)}{4\sqrt{3r^2+m^2}} \simeq r - m.$$

Note that $\theta_1 \sim \sin\theta_1 = \frac{d}{2r}$, this means that

$$|K_k(m,r)| \leq C\int_0^{\theta_1}(r-m)^{-N-1}d\theta \leq C\frac{d}{r}(r-m)^{-N-1} \leq Cr^{-1/2}(r-m)^{-N}. \tag{1.69}$$

Now we turn to the estimate for J_k in terms of L_k. We want to exploit the effect of the oscillated factor $e^{-i\frac{k}{d}\beta}$, together with the support property of the function α. Recall that $i\frac{d}{k}\partial_\beta e^{-i\frac{k}{d}\beta}=e^{-i\frac{k}{d}\beta}$, we use integration by parts in β to get

$$\begin{aligned}
|L_k(m,r)| =& \left|\frac{d}{k}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\partial_\beta(e^{-i\frac{k}{d}\beta})\hat\alpha\big(\phi(\beta)\big)\ d\beta\right|\\
=& \frac{d}{k}\left|e^{-i\frac{k}{d}\beta}\hat\alpha\big(\phi(\beta)\big)|_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}-\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}e^{-i\frac{k}{d}\beta}\phi'(\beta)(\hat\alpha)'\big(\phi(\beta)\big)\ d\beta\right|\\
\le& C\frac{d}{k}\left(|\hat\alpha\big(m-r\cos\frac{3\pi}{4}\big)|+|\hat\alpha\big(m-r\cos\theta_1\big)|+\int_{\mathbb{R}}\langle\phi(\beta)\rangle^{1-N}\ d\beta\right)\\
\le& C\frac{d}{k}\left(1+\int_{\mathbb{R}}\langle\beta\rangle^{1-N}\ d\beta\right)\\
\le& C\frac{d}{k}\ \text{, if } k\neq 0,
\end{aligned}$$

where we have used Lemma 1.27 in the first and second inequality.

To prove another inequality for $|L_k|$, we need only to exploit the support property of α. Since $\mathrm{supp}\alpha\subset[\frac14,2]$, we can introduce $\tilde\alpha(\rho)=i\alpha(\rho)/\rho\in\mathcal{S}$ so that $\hat\alpha=(\hat{\tilde\alpha})'$ and

$$\hat\alpha(\phi(\beta))=(\hat{\tilde\alpha})'(\phi(\beta))=\frac{1}{\phi'(\beta)}\partial_\beta(\hat{\tilde\alpha}(\phi(\beta))).$$

Thus we have

$$\begin{aligned}
|L_k(m,r)| =& \left|\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\partial_\beta(\hat{\tilde\alpha}(\phi(\beta)))\ d\beta\right|\\
\le& \left|\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\hat{\tilde\alpha}(\phi(\beta))\right)\right|_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\\
&-\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\partial_\beta\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\right)\hat{\tilde\alpha}(\phi(\beta))\ d\beta\Bigg|\\
\le& C\left(|\hat{\tilde\alpha}\big(m-r\cos\frac{3\pi}{4}\big)|+|\hat{\tilde\alpha}\big(m-r\cos\theta_1\big)|\right)\\
&+\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\partial_\beta\left(e^{-i\frac{k}{d}\beta}\frac{1}{\phi'(\beta)}\right)\hat{\tilde\alpha}(\phi(\beta))\ \right|d\beta\\
\le& C\langle r-m\rangle^{-N}+C\frac{k}{d}\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\frac{1}{\phi'(\beta)}\hat{\tilde\alpha}(\phi(\beta))\right|d\beta\\
&+C\int_{d(\theta_1-\theta_0)}^{d(3\pi/4-\theta_0)}\left|\frac{\phi''(\beta)}{(\phi'(\beta))^2}\hat{\tilde\alpha}(\phi(\beta))\right|d\beta\\
\le& C\langle r-m\rangle^{-1}+C\frac{k}{d}\ ,
\end{aligned}$$

where we have used the fact that $r\cos\theta_1 - m \gtrsim r-m$, $\phi'(\beta) \geq \frac{1}{2}$ (for $\theta \in [\theta_1, \frac{3\pi}{4}]$) and $\phi''(\beta) = \frac{r}{d^2}\cos\theta = \mathcal{O}((r-m)^{-1})$. Combining with the previous inequality, we have proved

$$|L_k(m,r)| \leq C\langle r-m\rangle^{-1} + C\min\left(\frac{k}{d}, \frac{d}{k}\right)$$

and so is the inequality (1.62) by (1.67), (1.68) and (1.69). This completes the proof.■

§ 2. Apply to nonlinear wave equations

§ 2.1. A brief recall of the linear theory

Consider the Cauchy problem on $\mathbb{R}^{1+d}$ for the inhomogeneous wave equation

$$\Box u = F(t,x), \quad u|_{t=0} = \phi, \quad \partial_t u|_{t=0} = \psi. \tag{2.1}$$

Let us recall the concept of the solutions at first:

Classical solution: the solution of (2.1) which is at least C^2 so that the derivatives involved make sense pointwise.

Weak solution: the solution formulas make sense for data ϕ, ψ and F with very little regularity. For example, let $\phi, \psi \in L^1_{loc}(\mathbb{R}^d)$ and $F \in L^1_{loc}(S_T)$ is a weak solution of (2.1) in S_T if

$$\int_{S_T} u\Box\varphi dtdx = \int_{S_T} F\varphi dx - \int_{\mathbb{R}^d}\phi(x)\partial_t\varphi(0,x)dx + \int_{\mathbb{R}^d}\psi(x)\varphi(0,x)dx$$

for all $\varphi \in C_0^\infty$ with support in $(-\infty, T)\times\mathbb{R}^d$.

This concept of weak solution for the Cauchy problem is defined for locally integrable initial data. One can check that a weak solution belonging to $C^2(S_T)$ is a classical solution.

Distribution solution: if for all $\varphi \in C_0^\infty(\mathbb{R}^{1+d})$ we have $(u, \Box\varphi) = 0$, we say u is a distribution solution of $\Box u = 0$. In this case no initial condition makes sense. If u is a time-dependent distribution of class, for instance, $u \in C^1(\mathbb{R}, \mathcal{D}'(\mathbb{R}^d))$, then we can merely assume the initial data to be distribution on $\mathbb{R}^d$.

Typically, we consider the initial data in $H^s_x \times H^{s-1}_x$, and call a **strong solution** of (2.1) to be a distributional solution which also lies in $C([0,T];H^s)\cap C^1([0,T];H^{s-1})$, while a **weak solution** lies in $L^\infty(0,T;H^s)$ with one weak time derivative in $L^\infty(0,T;H^{s-1})$. Usually, the category of strong solutions is the broadest category of solution in which we can hope to have a good existence and uniqueness theory. For the weak solutions one can hope to have existence, but not uniqueness. When we consider a low regularity problem, to show the the wellposedness one often needs to strengthen the notation of a strong solution by adding some additional properties of the solution map from data to the solution. In fact, wellposed solutions are highly compatible with classical solutions, which can be viewed as the strong limit of the classical solutions.

Theorem 2.1. *For all $\phi, \psi \in \mathcal{D}'(\mathbb{R}^d)$, there exists unique time-dependent distribution $u \in C^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^d))$ which solves Cauchy problem* (2.1) *with* $F = 0$.

Of course, we can understand the equation $\Box u = F$ in the sense of distributions on S_T also. The following is the basic theorem for the L^2 theory of the linear wave equations.

Theorem 2.2. *Let $(\phi, \psi) \in H^s \times H^{s-1}$, $s \in \mathbb{R}$, and $F \in L^1([0,T], H^{s-1})$. Then, for every $T > 0$, there is a unique solution u of the Cauchy problem* (2.1) *on $S_T = (0,T) \times \mathbb{R}^d$, which belongs to $C([0,T], H^s) \cap C^1([0,T], H^{s-1})$. Moreover, u satisfies the energy inequality*

$$\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}} \leq C_T \left(\|\phi\|_{H^s} + \|\psi\|_{H^{s-1}} + \int_0^t \|F(\tau)\|_{H^{s-1}} d\tau \right)$$

for all $0 \leq t \leq T$, where $C_T = C_s(1+T)$.

§ 2.2. General introduction

In this part we turn to the nonlinear wave equations. We shall face the problem of local and global solutions in time to the Cauchy problem in classical and low regularity sense. In general, for the equation with general nonlinearities, the result will be quite weak, and usually requires highly regular data and strong conditions for continuing a given local solution to larger times. One can find that better nonlinearities will obtain better solution properties, but nonlinear phenomena usually is complicated. So the properties often depend on the structure of the linear equation, and the sign, structures and the growth of the nonlinearity. For example, for the nonlinear wave equation in 3 dimensions

$$\partial_{tt} u - \Delta u + f(u) = 0,$$

if $f(u) = \pm u^2$, the solution of the equation with small data will blow up in the finite time, and so is for $f(u) = -u^3$. But if $f(u) = u^3$, the solution will be global in time.

In this note we do not aim at giving a collection of the results which have been obtained, but aim at introducing the methods, well adaptable to various problems, for deriving local or global existence results.

When we deal with the nonlinear problems, we always consider the case that can be viewed as a perturbation of the corresponding linear system at first. The basic idea is to find the solution of the nonlinear problem as a fixed point of the solution operator of the linearized inhomogeneous Cauchy problem. The existence of that fixed point will follow from Banach's contraction principle in a Banach space adapted to the special problem. The main difficulty is to find the appropriate Banach space. Symmetries can give us some important inspiration.

The **perturbation method** is the major method, but the perturbation methods only work when the solution is very close to its approximation. This requires the initial datum to be small (or a small perturbation of a special datum), or the time interval to be small (or perhaps some spacetime integral of the solution to be well controlled on this time interval). To realize such method, the main technique is to construct an approximation sequence by contracting map to construct solutions , which is called **iterative methods**. Such methods tend to yield a fairly strong type of wellposedness. However, when the regularity of the data is extremely low, or equation behavior is in an extremely nonlinear fashion, such methods can fail. We need to use a completely different approach, that is called **viscosity methods** (penalization, weak compactness, or regularization method) to find a weak solution to the original equation. The difficulty of the viscosity method is to upgrade this solution to the strong one or to establish wellposedness properties, or to show persistence of regularity. Furthermore, the conservation laws are often not preserved by weak limits.

What we really need in establishing contraction are growth estimates for some norms of the solution under certain conditions, usually uniform bounds. They are not only crucial to local theory, but also to continuing the solution to be global in time. From the first part we know that the structure or symmetry of the linear equation will provide various estimates, including the decay estimates of the linear solution, which depend on the tools of the analysis. This is the information worth exploiting in the study of nonlinear problem. Classical solutions enjoy the conservation laws and other formal identities, which lead to an a priori bound for suitable norm. **An a priori estimate** is an important common trick to obtain a control on classical solutions. Such method can exploit various delicate cancellations arising from the structure of the equations. In the study of nonlinear problem, we mainly deal with the solutions which are highly compatible with classical ones by taking appropriate limits. In some cases one needs to regularize the nonlinearity in addition to the initial data. In such situations the continuity of the solution map is not quite sufficient, and one needs to supplement it with some stability properties of the solution.

When dealing with large solutions over long times, perturbative techniques no longer work by themselves, and one must combine them with **non-perturbative methods**, which include conservation law, monotonicity formulae, and algebraic transformations of the equation. The perturbative theory guarantees a well-behaved solution provided that certain integrals of the solution stay bounded and the non-perturbative theory guarantees control of these integrals provided that the solution remains well-behaved. Local theory plays a mostly **qualitative** role in the global argument, justifying the local existence of the solution as well as the conservation law, but does not provide the key **quantitative** bounds.

§ 2.3. Energy Method

2.3.1. Local theory

We consider the following Cauchy problem

$$\begin{cases} \Box u = F(u, \partial u), \\ u|_{t=0} = \phi, \partial_t u|_{t=0} = \psi, \end{cases} \tag{2.2}$$

where F is a given real valued smooth function with $F(0) = 0$, and $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$.

Theorem 2.3. *Let $s > \frac{d}{2}+1$. Then for all $(\phi, \psi) \in H^s \times H^{s-1}$, there exist $T > 0$,depending continuously on $E_s := \|\phi\|_{H^s} + \|\psi\|_{H^{s-1}}$, and a unique $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ solving (2.2) on S_T. The solution continuously depends on data and can be extended to T^*, which is the supremum of all $T > 0$ such that either $T^* = \infty$ or $\partial u \notin L^\infty(S_{T^*})$.*

Moreover, if $(\phi, \psi) \in C_c^\infty(\mathbb{R}^d)$, then $u \in C^\infty([0,T] \times \mathbb{R}^d)$.

To understand the idea of the proof better, we only give a brief structure of the proof for the model equation $\Box u = |\partial_t u|^2$.

Existence To this end, we denote $X_T = C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ for $T > 0$. It is a Banach space with norm

$$\|u\|_{X_T} = \sup_{0 \le t \le T} (\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}).$$

We introduce an iteration scheme, let $u_{-1} = 0$, and define u_j inductively by $\Box u_j = |\partial_t u_{j-1}|^2$ satisfying the initial data. Then, by Sobolev's inequality and multiply inequality

$$\|uv\|_{H^{s-1}} \le C_s \|u\|_{H^{s-1}} \|v\|_{H^{s-1}},$$

we have $u_{j-1} \in X_T$ implies $(\partial_t u_j)^2 \in C([0,T]; H^{s-1})$ and $u_j \in X_T$ for $s - 1 > \frac{d}{2}$. That is, the sequence of iterates is well-defined in X_T for any $T > 0$.

Now, if we can prove $\{u_n\}$ is Cauchy in X_T for $T > 0$ small enough, we have the limit $u \in X_T$ which will be the solution of the Cauchy problem on S_T. This is because we can show $\Box u_j \to \Box u$ and $(\partial_t u_j)^2 \to (\partial_t u)^2$ in the sense of distribution respectively from $u_j \to u$ in X_T.

To prove $\{u_n\}$ is Cauchy in X_T for $T > 0$ small enough, we have to show the sequence is bounded by induction: $\|u_j\|_{X_T} \le 2CE_s$ for all $j = 0, 1, \cdots$ if $0 < T \le 1/(8C^2 E_s)$; and to show

$$\|u_{j+1} - u_j\|_{X_T} \le \frac{1}{2} \|u_j - u_{j-1}\|_{X_T}$$

by using energy inequality, multiply inequality and above uniform bound, and therefore Cauchy.

In fact, for the former, the case $\|u_{-1}\|_{X_T} \leq 2CE_s$ is trivial. If $\|u_{j-1}\|_{X_T} \leq 2CE_s$ holds, then from the energy inequality and multiply inequality, we have

$$\|u_j\|_{X_T} \leq CE_s + CT\|u_{j-1}\|_{X_T}^2 \leq CE_s + CT(2CE_s)^2,$$

where C depends on s and d if $T \leq 1$. If $T \leq 1/(8C^2E_s)$, we have $\|u_j\|_{X_T} \leq 2CE_s$.

For the latter, note that

$$\Box(u_{j+1} - u_j) = \partial_t(u_j - u_{j-1})\partial_t(u_j + u_{j-1})$$

with vanishing initial data at $t = 0$. From the energy inequality, and the uniform bound of u_j, we have

$$\begin{aligned}\|u_{j+1} - u_j\|_{X_T} &\leq CT\left(\|u_j\|_{X_T} + \|u_{j-1}\|_{X_T}\right)\|u_j - u_{j-1}\|_{X_T} \\ &\leq 4TC^2E_s\|u_j - u_{j-1}\|_{X_T}.\end{aligned}$$

Thus we also get the desired bound.

Uniqueness To prove the uniqueness, we assume that there are two solutions of the same Cauchy problem we considered. Then

$$\Box(u - v) = \partial_t(u + v)\partial_t(u - v)$$

and setting

$$A(t) = \|(u - v)(t)\|_{H^s} + \|\partial_t(u - v)(t)\|_{H^{s-1}},$$

by the energy inequality and calculus inequality, we have

$$A(t) \leq C\int_0^t A(\tau)d\tau$$

for $0 \leq t \leq T$ for some constant C independent of t. Then by Gronwall's inequality, $A(t) = 0$ for $0 \leq t \leq T$. Therefore, we have $u = v$ in S_T.

Continuous dependence of data Similarly, we can get the following inequality to obtain the continuous dependence: if the solution $u(\phi, \psi)$ with data (ϕ, ψ) exists up to some $T > 0$, then there are constants $C, \delta > 0$ such that if

$$\|\phi - \phi'\|_{H^s} + \|\psi - \psi'\|_{H^{s-1}} \leq \delta,$$

the solution $u(\phi', \psi')$ with data (ϕ', ψ') exists up to time T also, and

$$\|u(\phi, \psi) - v(\phi', \psi')\|_{X_T} \leq C\left(\|\phi - \phi'\|_{H^s} + \|\psi - \psi'\|_{H^{s-1}}\right).$$

Continuation We assume the solution u on S_T satisfies $\partial u \in L^\infty(S_T)$, then we can show

$$\sup_{0\leq t<T}\left(\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}\right) < \infty,$$

that is, we can extend the C^∞ solution to the closure of S_T(that the extended u is in C^∞ on $t = T$ follows from the fact the equation of (2.2) permits to express all derivatives of u on $t = T$ uniquely in terms of the values of u and $\partial_t u$ on $t = T$), and would have compact support in x. By the local existence theorem u could be extended as a solution of (2.2) to a larger strip $[0, T + \epsilon] \times \mathbb{R}^d$, for some ϵ.

To prove it, by the energy inequality and calculus inequality, we have

$$\begin{aligned} A(t) &\le C_{s,T}\left(E_s + \int_0^t \|\partial_t u(\tau)\|_{L^\infty}\|u(\tau)\|_{H^{s-1}}d\tau\right) \\ &\le C_{s,T}\left(E_s + \|\partial u\|_{L^\infty(S_T)}\int_0^t A(\tau)d\tau\right) \end{aligned}$$

with $A(t) = \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}$, and Gronwall's inequality gives the result.

Remark. From the continuation of the smooth solution, we know that the existence problem (2.2) can be reduced purely to a priori estimates, once one has a local existence theorem. How a priori estimates imply lower bounds for T is stated in the following: Let the positive number τ be such that whenever a solution u of (2.2) exists in a strip S_s, with $0 < s \le \tau$, then all $\partial^\alpha u$ are bounded in S_s. It follows that $T > \tau$.

We have to point out, for $d > 1$, we cannot only use the a priori estimate on equations between derivatives of u holding along special curves, say "charactristic", because there are always unwanted higher derivatives that make their appearance. So the main tool in existence proofs will be energy inequalities and Sobolev type inequalities, which estimate point values of lower derivatives in terms of L^2 type norms of higher ones.

Persistence of higher regularity For the regularity, if $\phi, \psi \in C_0^\infty$, then $\phi, \psi \in H^s$ for every $s \in \mathbb{R}$. Fix $s_0 > \frac{d}{2} + 1$. By the first part of the theorem, we have

$$u \in C([0, T], H^{s_0}) \cap C^1([0, T], H^{s_0-1}) \tag{2.3}$$

on S_T. Again, for every $s > s_0$ there exists $T_s > 0$ such that $u \in C([0, T_s], H^s) \cap C^1([0, T_s], H^{s-1})$. Of course, an H^s solution is in particular an H^{s_0} solution. By uniqueness of H^{s_0} solution, we deduce that both solutions are the same where the two solutions are defined. We prove $T_s = T$. In fact, it follows from the extension part, since $\partial u \in L^\infty(S_T)$ by Sobolev's Lemma and (2.3). Again, by Sobolev's Lemma, we have

$$\partial_t^j \partial_x^\alpha u \in C([0, T] \times \mathbb{R}^d) \tag{2.4}$$

for $j = 0, 1$ and all α. From the equation, we see $\partial_t^2 \partial_x^\alpha u \in C([0, T] \times \mathbb{R}^d)$. Applying ∂_t to both sides of the equation, it is easy to know $j = 3$ holds true. Again and again, by taking successive time derivatives of the equation, we obtain (2.4) for all j by induction, and we have $u \in C^\infty([0, T] \times \mathbb{R}^d)$.

2.3.2. Global solutions Consider the nonlinear Cauchy problem on $\mathbb{R}^{1+d}$,

$$\Box u = F(\partial u), \quad u|_{t=0} = \varepsilon\phi, \quad \partial_t u|_{t=0} = \varepsilon\psi, \tag{2.5}$$

where F is a given C^∞ real function which vanishes to second order at the origin.

Theorem 2.4. *Let $\phi, \psi \in C_0^\infty(\mathbb{R}^d)$. For $d \geq 4$, there exists $\varepsilon_0 > 0$ such that above Cauchy problem has a solution $u \in C^\infty([0,\infty) \times \mathbb{R}^d)$ if $\varepsilon \leq \varepsilon_0$.*

For the lower dimensions, there are smooth solutions in $0 \leq t \leq T$, with $T = e^{c/\varepsilon}$ if $d = 3$, $T = c/\varepsilon^2$ if $d = 2$, and c/ε if $d = 1$.

If the nonlinearities depend on u as well the situation can be changed dramatically. For instance, if $F = u^2$, then there is global existence for $d \geq 5$ (Klainerman 1980), no longer global existence for small data when $d = 4$ (Sideris 1984, or Zhou 1995), and the life-span for $d = 3$ is no longer $O(e^{c/\varepsilon^2})$ but only $O(\varepsilon^{-2})$.

The following are several basic facts:

- From F's assumption we have $|F(z)| \leq G_2(|z|)|z|^2$, $|\partial F(z)| \leq G_2(|z|)|z|$, and $|\partial^\alpha F(z)| \leq G_2(|z|), m = |\alpha| \geq 2$ for all $z \in \mathbb{R}^{1+d}$, where $G_2, \cdots$ are continuous, increasing functions.
- $\Box\Gamma^\alpha = \sum_{|\beta|\leq|\alpha|} c_{\alpha\beta}\Gamma^\beta\Box$ with contants $c_{\alpha\beta}$.
- For $\alpha \neq 0$, $\Gamma^\alpha[F(\partial u)]$ is linear combination of terms
$$[\partial^\alpha F](\partial u)\Gamma^{\beta_1}\partial u \cdots \Gamma^{\beta_m}\partial u,$$
where $1 \leq m \leq |\alpha|$, $\sum_1^m |\beta_i| = |\alpha|$. At most one, say β_m can have order $|\beta_m| > \frac{|\alpha|}{2}$.
- Let $N = d + 4$. If $|\alpha| \leq N$ and $|\beta_j| \leq \frac{|\alpha|}{2}$, then $|\beta_j| + 1 + \frac{d+2}{2} \leq N$.

Outline of the proof for Theorem 2.4 Let $N = d + 4$, and let $u \in C^\infty([0,T) \times \mathbb{R}^d)$ solve (2.5) for some $T > 0$. Denote $A(t) = \sum_{|\alpha|\leq N} \|\Gamma^\alpha \partial u(t,\cdot)\|_{L^2}, 0 \leq t < T$ on $[0,T) \times \mathbb{R}^d)$. From the initial data, we have $A(0) \leq \frac{A\varepsilon}{2}$, where A depends only on ϕ and ψ.

From Sobolev inequality

$$\|\partial u\|_{L^\infty([0,T)\times\mathbb{R}^d)} \leq C \sup_{0\leq t<T} A(t)$$

and blow up criteria, it is easy to know, to prove the theorem, we only need to show

there exists $\varepsilon_0 > 0$ such that if $T > 0$ and $u \in C^\infty([0,T) \times \mathbb{R}^d)$ solves the problem on $[0,T) \times \mathbb{R}^d$ with $\varepsilon \leq \varepsilon_0$, then $A(t) \leq A\varepsilon$ for all $0 \leq t < T$.

To prove the reduced problem, we set $E = \{t \in [0,T) : A(s) \leq A\varepsilon \text{ for all } 0 \leq s \leq t\}$. E is nonempty. Since $A(t)$ is continuous in t, E is relatively closed in $[0,T)$. If we can prove E is relatively open in $[0,T)$, we have $E = [0,T)$.

Fix $t_0 \in E$ with $t_0 < T$, there exists $t_1 > t_0$ such that $A(t) \leq 2A\varepsilon$ for $0 \leq t \leq t_1$ by continuity. We will prove that this implies $A(t) \leq A\varepsilon$ for $0 \leq t \leq t_1$ if ε is small enough.

It suffices to prove that

$$A(t) \leq A\varepsilon/2 + C_A\varepsilon \int_0^t \frac{A(s)}{(1+s)^{(d-1)/2}} ds, \tag{2.6}$$

this is because it follows

$$A(t) \leq \frac{A\varepsilon}{2} e^{C_A\varepsilon \int_0^t \frac{ds}{(1+s)^{(d-1)/2}}}. \tag{2.7}$$

For $d \geq 4$, we can choose $\varepsilon > 0$ so that $e^{C_A\varepsilon \int_0^t \frac{ds}{(1+s)^{(d-1)/2}}} \leq 2$ and finish the proof. (The ε_0 can be chosen from $= 2$.)

To prove the inequality (2.6), we use the bound of initial data and energy inequality, to get

$$\begin{aligned} A(t) &\leq A(0) + C_N \int_0^t \sum_{|\alpha| \leq N} \|\Box \Gamma^\alpha u(s,\cdot)\|_{L^2} ds \\ &\leq \frac{A\varepsilon}{2} + C_N \int_0^t \sum_{|\alpha| \leq N} \|\Gamma^\alpha [F(\partial u)](s,\cdot)\|_{L^2} ds. \end{aligned}$$

We now estimate

$$\|\Gamma^\alpha [F(\partial u)](t,\cdot)\|_{L^2}, |\alpha| \leq N.$$

If $\alpha = 0$, we have

$$\|F(\partial u)(t,\cdot)\|_{L^2} \leq G_2(\|\partial u\|_{L^\infty}) \|\partial u\|_{L^\infty} \|\partial u\|_{L^2}.$$

The first factor on the right hand side is bounded by a continuous functions of A, since $|\partial u(t,\cdot)|_{L^\infty} \leq CA(t) \leq 2CA\varepsilon$. the second is bounded by $CA(t)/((1+t)^{(d-1)/2})$ from K-S inequality, and the third is bounded by $2A\varepsilon$ from the assumption $A(t) \leq 2A\varepsilon$.

If $\alpha \neq 0$, we use the basic points 3 to write $\Gamma^\alpha u(t,\cdot)$ as a sum of terms of the

$$[\partial^\alpha F](\partial u) \Gamma^{\beta_1} \partial u \cdots \Gamma^{\beta_m} \partial u,$$

whose L^2 norms in space can be bounded by

$$\|[\partial^\alpha F](\partial u)\|_{L^\infty} \Pi_{i=1}^{m-1} \|\Gamma^{\beta_i} \partial u(t,\cdot)\|_{L^\infty} \|\Gamma^{\beta_m} \partial u(t,\cdot)\|_{L^2}.$$

For $m \geq 2$, note that the first factor is bounded by a continuous function of A, and the last factor is bounded by $2A\varepsilon$. From the Klainerman- Sobolev inequality, and the basic facts, we can show, it is bounded by $C_A\varepsilon A(t)(1+t)^{-(d-1)/2}$. For $m = 1$, we can get the bound if we use $|\partial F(z)| \leq G_2(|z|)|z|$ instead of $|\partial^\alpha F(z)| \leq G_2(|z|), m = |\alpha| \geq 2$. This completes the proof of the first part.

When $d = 1, 2, 3$ the function $(1+s)^{-(d-1)/2}$ is no longer integrable at infinity, but we still get the bound $A(t) \leq A\varepsilon$, provided ε is sufficiently small and T satisfies

$$T \leq \begin{cases} e^{c/\varepsilon}, & n = 3, \\ c/\varepsilon^2, & n = 2, \\ c/\varepsilon, & n = 1. \end{cases}$$

In fact, for $d = 3$, $\int_0^t \frac{ds}{(1+s)} = \log(1+t)$, so (2.7) becomes

$$A(t) \leq \frac{A\varepsilon}{2}(1+t)^{C_A\varepsilon},$$

and if $(1+T)^{C_A\varepsilon} \leq 2$, we have $A(t) \leq A\varepsilon$ for $0 \leq t < T$. Now we show $(1+T)^{C_A\varepsilon} \leq 2$. In fact, if $C_A\varepsilon \leq \frac{1}{2}$, then $(1+T)^{C_A\varepsilon} \leq \sqrt{2}T^{C_A\varepsilon} \leq 2$ if we take $T \leq 2^{1/(2C_A\varepsilon)}$.

For $d = 2$, $\int_0^t \frac{ds}{(1+s)^{1/2}} = 2\sqrt{1+t} - 2 \leq C\sqrt{t}$, (2.7) gives

$$A(t) \leq \frac{A\varepsilon}{2} e^{C_A\varepsilon\sqrt{t}},$$

and $A(t) \leq A\varepsilon$ for $0 \leq t < T$ follows from $\sqrt{T} \leq \frac{\log 2}{C_A\varepsilon}$.

For $d = 1$, $\int_0^t ds = t$, (2.7) gives

$$A(t) \leq \frac{A\varepsilon}{2} e^{C_A\varepsilon t},$$

and $A(t) \leq A\varepsilon$ for $0 \leq t < T$ follows from $T \leq \frac{\log 2}{C_A\varepsilon}$.

Remark. The proof of the global existence used the continuity method or bootstrap principle. In fact, it is a continuous analogue of the principle of mathematical induction. In the proof, we know that the Klainerman-Sobolev inequality is essential. One also can find that local theory plays a qualitative role in the global argument, justifying the local existence of the solution and the conservation law, but not providing the key quantitative bounds.

From the proof of the theorem we know that, if F vanishes to third order at 0, then we have $|F(z)| \leq G_3(|z|)|z|^3$, $|\partial F(z)| \leq G_3(|z|)|z|^2$, $|\partial^2 F(z)| \leq G_3(|z|)|z|$, and $|\partial^m F(z)| \leq G_3(|z|), m \geq 3$. In this way we can get one extra power of $\|\partial u(t,\cdot)\|_{L^\infty}$, and by K-S inequality we can use $(1+t)^{-(d-1)}$ instead of $(1+t)^{-(d-1)/2}$, so we get global existence when $d = 3$.

In general, for $d = 3$ the existence of global smooth solutions for small data fails for the equations (2.5). For example, every non-trivial C^3 solution of $\Box u = (\partial_t u)^2$ with compactly supported Cauchy data blows up in finite time (due to John).

$\Box u = (\partial_t u)^2 - \sum_{j=1}^3 (\partial_j u)^2$ always has a global C^∞ solution if the data (ϕ, ψ) in C_0^∞ is small (due to Nirenberg). The key observation is if we let $v(t,x) = 1 - e^{-u(t,x)}$, then v solves the linear problem

$$\Box v = 0,\ v|_{t=0} = 1 - e^{-\varepsilon\phi},\ \partial_t v|_{t=0} = \varepsilon\psi e^{-\varepsilon\psi},$$

which has a global smooth solution. The inverse of the map is $u(t,x) = -\log(1-v(t,x))$ if $|v| < 1$, but from the decay estimate $\|v(t,\cdot)\|_{L^\infty} \le \frac{A}{1+t}$ for all $t \ge 0$, where A is a constant which depends linearly on the L^∞ norm of $v(0,x), \partial v(0,x)$, we can ensure that v is globally small $\|v(t,\cdot)\|_{L^\infty} < 1$ for all $t \ge 0$ only if we take $\varepsilon > 0$ small enough, depending on (ϕ, ψ).

Note that for a system of the form $\Box u = F(\partial u)$ with F vanishing to second order at the origin, the quadratic part of F determines the global existence, and from both examples, we know the global property depends strongly on the algebra structure in the bilinear form of $F(\partial u)$. Comparing with them, we can know what the difference is. The solutions of free wave equation with C_0^∞ data have gradients that decay like $1/t^2$ away from the associated light cone where $t = |x|$. Even on the light cone, most components of the gradient enjoy this fast decay rate. The only bad directional derivative that just has the $1/t$ decay in general is the one which is normal to the light cone $L_- u = (\partial_t - \partial_r)u$, so we can ask the nonlinearity to ensure that in the quadratic terms at least one of the factors involves a derivative that is orthogonal to L_-, that is the following so-called " **Null condition"**.

We consider a system of N equations

$$\Box u = F(u, \partial u), \quad (t,x) \in \mathbb{R}^{1+3}, \tag{2.8}$$

where $u = (u^1, \cdots, u^N)$, $F = (F^1, \cdots, F^N)$. A vector $\xi = (\xi_0, \cdots, \xi_3) \in \mathbb{R}^{1+3}$ is null, if $\xi \neq 0$ and $\xi_0^2 = \xi_1^2 + \cdots + \xi_3^2$, that is, ξ lies on the light cone in Minkowski space $\mathbb{R}^{1+3}$.

F satisfies the null condition if F vanishes to second order at the origin, $F(z) = F_{(2)}(z) + R(z)$, where R is C^∞ and vanishes to third order at 0,

$$F^I_{(2)}(u, \partial u) = \sum_{J,K=1}^N \sum_{\mu,\nu=0}^3 a^{I\mu\nu}_{JK} \partial_\mu u^I \partial_\nu u^K,$$

where the a's are real constants satisfying, for all $I, J, K = 1, \cdots, N$,

$$\sum_{\mu,\nu=0}^3 a^{I\mu\nu}_{JK} \xi_\mu \xi_\nu = 0$$

for all null vectors ξ.

The following theorem says that for $d = 3$, if the nonlinearity has a null condition, one still can prove the global existence for all sufficiently small data. But it seems still open if there is always blow-up for nontrivial sufficiently small data, when the null condition does not hold.

Theorem 2.5. *Assume that F in (2.8) satisfies the null condition. Then there exists $\varepsilon_0 = \varepsilon_0(\phi, \psi) > 0$ such that the Cauchy problem has a smooth global solution provided $\varepsilon < \varepsilon_0$.*

If Q is a real bilinear form on $\mathbb{R}^4 \times \mathbb{R}^4$ such that $Q(\xi, \xi) = 0$ for all null vectors ξ, then Q is a linear combination, with real coefficients, of the null forms

$$Q_0(\xi, \eta) = \xi_0 \eta_0 - \sum_{i=1}^{3} \xi_i \eta_i,$$

$$Q_{\mu\nu}(\xi, \eta) = \xi_\mu \eta_\nu - \xi_\nu \eta_\mu, 0 \leq \mu < \nu \leq 3.$$

It is not difficult to know from above, that F in (2.8) satisfies null condition iff each component $F^I(u, \partial u)$ is of the form

$$\sum_{J,K} a^I_{JK} Q_0(\partial u^J, \partial u^K) + \sum_{J,K} \sum_{0 \leq \mu < \nu \leq 3} b^{I\mu\nu}_{JK} Q_{\mu\nu}(\partial u^J, \partial u^K) + R^I(u, \partial u),$$

where the a's and b's are real constants and R^I is C^∞ and vanishes to third order at 0.

From this observation, we only need to prove Theorem 2.5 for the following problem

$$\Box u = \sum_{J,K} a^I_{JK} Q_0(\partial u^J, \partial u^K) + \sum_{J,K} \sum_{0 \leq \mu < \nu \leq 3} b^{I\mu\nu}_{JK} Q_{\mu\nu}(\partial u^J, \partial u^K) \tag{2.9}$$

with initial data $(\varepsilon\phi, \varepsilon\psi)$.

The following lemma is of key importance. Null forms have better decay properties, due to cancellations, than generic bilinear forms. For the invariant vector fields $\Gamma_0, \cdots, \Gamma_m$, let $\Gamma_j(t, x; \xi)$ be the symbol of Γ_j, obtained by replacing ∂ by the vector $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$, and denoting by Γ the vector $(\Gamma_0, \cdots, \Gamma_m)$, we have $|\Gamma|^2 = \sum |\Gamma_j|^2$.

Lemma 2.6. *Let Q be a bilinear form on $\mathbb{R}^4$. Then there exists a constant C such that*

$$|Q(\xi, \eta)| \leq \frac{C}{1 + |t| + |x|} |\Gamma(t, x; \xi)||\Gamma(t, x; \eta)| \tag{2.10}$$

for all $(t, x), \xi, \eta \in \mathbb{R}^{1+3}$, if and only if Q satisfies $Q(\xi, \xi) = 0$ for all null vectors ξ.

We also have

$$(1 + |t| + |x|) \sum_{|\alpha| \leq M} |\Gamma^\alpha Q(\partial v, \partial w)|$$

$$\leq C_M \left(\sum_{1 \leq |\alpha| \leq M+1} |\Gamma^\alpha v(t, x)| \right) \left(\sum_{1 \leq |\alpha| \leq \frac{M}{2}+1} |\Gamma^\alpha w(t, x)| \right)$$

$$+ \left(\sum_{1 \leq |\alpha| \leq \frac{M}{2}+1} |\Gamma^\alpha v(t, x)| \right) \left(\sum_{1 \leq |\alpha| \leq M+1} |\Gamma^\alpha w(t, x)| \right)$$

Outline of the proof of Theorem 2.5 Suppose $u \in C^\infty(S_{T_0})$ solves (2.9) . We shall prove that there is a $\varepsilon_0 > 0$, independent of T_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, $u, \partial u \in L^\infty(S_{T_0})$. From the local existence theory, we have $T_0 > 0$. It also can be reduced to proving the a priori estimate

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot)\|_{L^\infty} \le \frac{A\varepsilon}{1+t} \tag{2.11}$$

for $0 \le t < T_0$, provided $\varepsilon < \varepsilon_0$, where A depends on (ϕ, ψ), independent of T_0 and ε, and k is an integer large enough.

The proof of (2.11) uses the continuity method. Because of the critical dimension $d = 3$, the energy has growth in t, so in the proof we have to use the conform energy inequality (1.3) and Hörmander's inequality (1.6).

Let E be the set of $T \in [0, T_0)$ such that (2.11) holds for all $t \in [0, T]$. Obviously, $0 \in E$ if we take A large enough, and E is a closed set. So we only need to prove E is open in $[0, T_0)$. By the continuity and Huygen's principle for the solution, there exists $T' > T$ such that

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot)\|_{L^\infty} \le \frac{2A\varepsilon}{1+t}$$

for $0 \le t \le T'$. If we can show $T' \in E$, then E is open.

To this end , we use conform energy estimate and null forms. Let

$$A(t) = \sum_{|\alpha|\le k+3} \|\Gamma^\alpha u(t,\cdot)\|_{L^2},$$

and we can get

$$\begin{aligned} A(t) &\le CA(0) + C \sum_{|\alpha|\le k+2} \int_0^t \|(1+s+|\cdot|)\Gamma^\alpha \Box u(s,\cdot)\|_{L^2} ds \\ &\le CA(0) + C\int_0^t A(s) \left(\sum_{|\alpha|\le \frac{k+2}{2}+1} \|\Gamma^\alpha u(s,\cdot)\|_{L^\infty} \right) ds \\ &\le CA(0) + C'\varepsilon \int_0^t \frac{A(s)}{1+s} ds, \end{aligned}$$

for $0 \le t \le T'$ if we take $k \ge 4$. Then, by using the Gronwall's inequality, we get the energy estimate

$$A(t) \le CA(0)(1+t)^{C'\varepsilon}.$$

Next, we use Hömander's inequality to get (2.11). To this end, let w_α solve

$$\Box w_\alpha = 0,\ w_\alpha|_{t=0} = (\Gamma^\alpha u)|_{t=0}, \partial_t w_\alpha = (\partial_t \Gamma^\alpha u)|_{t=0}.$$

It is obviously we have

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha w_\alpha(t,\cdot)\|_{L^\infty} \le \frac{A\varepsilon}{2(1+t)}.$$

Therefore, we only need to show

$$\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} \le \frac{A\varepsilon}{2(1+t)}. \tag{2.12}$$

From Hörmander's inequality, the commutation relation between $\Box$ and the invariant vector fields, and the estimate of null forms Q in Lemma 2.6, we have

$$\begin{aligned}(1+t)\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} &\le C \sum_{|\beta|\le 2}\int_0^t\int_{\mathbb{R}^3}\sum_{|\alpha|\le k}|\Gamma^\beta\Box\Gamma^\alpha u(s,y)|\frac{dyds}{1+s}\\ &\le C\sum_{|\alpha|\le k+3}\int_0^t\int_{\mathbb{R}^3}|\Gamma u(s,y)|^2\frac{dyds}{(1+s)^2}\\ &\le CA(0)^2\int_0^t (1+s)^{2C'\varepsilon-2}ds.\end{aligned}$$

Note that $A(0)=O(\varepsilon)$. If we take $2C'\varepsilon<1$, the integral is uniformly bounded in t, so we can get the bound

$$(1+t)\sum_{|\alpha|\le k} \|\Gamma^\alpha u(t,\cdot) - w_\alpha(t,\cdot)\|_{L^\infty} \le C\varepsilon^2$$

for $0\le t\le T'$ at last. Let $C\varepsilon<\frac{A}{2}$. The proof is complete.

2.3.3. Low regularity problem From above we know that the null condition can improve the global property of the Cauchy problem. Now we show that the null condition also can improve the regularity assumptions in the local existence theorem.

Usually the local existence result holds true for semilinear wave equations with data in $H^\gamma(\mathbb{R}^d)\times H^{\gamma-1}(\mathbb{R}^d)$ if $\gamma>(d+2)/2$. For the Nuremberg's example, one might expect that the result should hold if $\gamma>3/2$. On the other hand, Lindblad showed that for $\Box u=(\partial_t u)^2$ one cannot have $\gamma\le 2$. But for the special case of the equation which satisfies the null condition, there are some improvement. For example, for $d=3$, one has an improvement of $1/2$ of a derivative in the regularity assumptions. More precisely,

Theorem 2.7 (Sogge). *Let $d=3$ and fix $(\phi,\psi)\in H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Then there is a $T>0$ so that* (2.9) *has a unique solution verifying $u\in C([0,T];H^2)\cap C^1([0,T];H^1)$, and $Q(u^J,u^K)\in H^1([0,T]\times\mathbb{R}^3), \forall J,K,$ whenever Q is a null form.*

To prove it, one only needs to note that the condition of null form $Q \in H^1$ implies that the right hand side of (2.9) belongs to $L^1([0,T];H^1)$ for $T < \infty$. The reason is that the null forms, rather than arbitrary bilinear forms acting on the gradients, will create cancellation which will allow us to avoid using the Hardy-Littlewood inequality in the proof. From the existence of linear equations one can obtain $u \in C([0,T];H^2) \cap C^1([0,T];H^1)$, so the key point is to prove $Q \in H^1$.

As an example, we prove the following inequality

$$\|Q(\partial_t u_1, u_2)\|_{L^2(\mathbb{R}^{1+3})} \le C\|\phi_1\|_{H^2}\|\phi_2\|_{H^2},$$

where u_i is the solution of $\Box u_i = 0$ with initial data $(\phi_i, 0), i = 1, 2$.

Note that we can write $u_i^{\pm}(t,x) = (u_i^+(t,x) + u_i^-(t,x))/2$, where

$$u_i^{\pm}(t,x) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{ix\cdot\xi \pm it|\xi|}\hat{\phi}_i(\xi)d\xi,$$

$Q(\partial_t u_1, u_2)$ is the sum of four terms $Q(\partial_t u_1^{\pm}, u_2^{\pm})/4$. Therefore, we only need to prove

$$\|Q(\partial_t u_1^+, u_2^+)\|_{L^2(\mathbb{R}^{1+3})} + \|Q(\partial_t u_1^+, u_2^-)\|_{L^2(\mathbb{R}^{1+3})} \le C\|\phi_1\|_{H^2}\|\phi_2\|_{H^2}.$$

To do it, we still denote Q to be one of the null form Q_0 and $Q_{ij}, 1 \le i,j \le 3$, and we have

$$Q(\partial_t u_1^+, u_2^+) = (2\pi)^{-6}\int\int e^{ix\cdot(\xi+\eta)+it(|\xi|+|\eta|)} i|\xi| q(\xi,\eta)\hat{\phi}_1\hat{\phi}_2 d\xi d\eta,$$

where q is $q_0(\xi,\eta) = \xi\cdot\eta - |\xi||\eta|$, or

$$q_{ij} = \begin{cases} -(\xi_i\eta_j - \xi_j\eta_i), & 1 \le i < j \le 3, \\ -(|\xi|\eta_j - |\eta|\xi_j), & 0 = i < j \le 3. \end{cases}$$

If we use polar coordinates $\eta = \rho\omega, \rho > 0, \omega \in S^2$, and for fixed ω, make the change of variables $(\tau,\zeta) = (|\xi| + \rho, \xi + \rho\omega)$, then

$$\begin{aligned} \|Q(\partial_t u_1^+, u_2^+)\|_{L^2}^2 &\le C\int\int\int \left| |\xi|^2\hat{\phi}_1(\xi)\rho^3\hat{\phi}_2(\rho\omega)q(\frac{\xi}{|\xi|},\omega)\left|\frac{d(\tau,\zeta)}{d(\rho,\xi)}\right|^{-1}\right|^2 d\tau d\zeta d\sigma(\omega) \\ &\le C\int\int\int \left|\rho^2\hat{\phi}_2(\rho\omega)|\xi|^2\hat{\phi}_1(\xi)\right|^2 \rho^2 d\rho d\sigma(\omega) d\xi \\ &= C(2\pi)^{-6}\|\phi_1\|_{H^2}^2\|\phi_2\|_{H^2}^2, \end{aligned}$$

where we have used the fact

$$|q(\xi,\eta)| \le C|\xi||\eta|\cdot\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right|,$$

$$\left|\frac{d(\tau,\zeta)}{d(\rho,\xi)}\right| = \left|1-\omega\cdot\frac{\xi}{|\xi|}\right| \geq c\left|\omega-\frac{\xi}{|\xi|}\right|^2,$$

and Plancherel's theorem.

For the case $Q(\partial_t u_1^+, u_2^-)$, we can argument in a similar way. We only need to note in this time $|q(\xi,\eta)| \leq C|\xi||\eta|\left|\frac{\xi}{|\xi|}-\frac{\eta}{|\eta|}\right|$, and let $(\tau,\zeta)=(|\xi|-\rho,\xi+\rho\omega)$.

The following example tells you Strichartz estimate also can make an improvement of the regularity assumption in the local existence theorem.

Consider the cubic semilinear wave equations in $\mathbb{R}^3$

$$\Box u = u^3$$

with the data $(\phi,\psi)\in \dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}$ to be small enough. We know from the energy methods, we have local wellposedness in $C([0,T];\dot{H}^1_x)$, but from scaling heuristics, the local wellposedness can be in $C([0,T];\dot{H}^s_x)$ for $s>1/2$. We can see it by using Strichartz estimates.

To this end, denote by $X(u)$ the supremum of

$$\|u\|_{L^4(\mathbb{R}^{1+3}_+)} + \|u(t)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t u\|_{\dot{H}^{-\frac{1}{2}}}$$

over $t\geq 0$. (we prove in global case)

As usual, the iterates are defined inductively by $u_{-1}=0$ and $\Box u_j = u_{j-1}^3$, with data (ϕ,ψ), for $j\geq 0$. Then by using

$$\|uvw\|_{L^{4/3}} \leq \|u\|_{L^4}\|v\|_{L^4}\|w\|_{L^4}$$

and Strichartz's estimate, we have

$$X(u_j)\leq CE_0 + CX(u_{j-1})^3,$$

where $E_0 = \|\phi\|_{\dot{H}^{\frac{1}{2}}} + \|\psi\|_{\dot{H}^{-\frac{1}{2}}}$. So we can get $X(u_j)\leq 2CE_0$, from $X(u_{j-1})$ and $C(2CE_0)^2\leq\frac{1}{2}$.

From

$$\Box(u_{j+1}-u_j) = (u_j-u_{j-1})u_j^2 - u_{j-1}(u_j+u_{j-1})(u_j-u_{j-1})$$

with vanishing data, we have

$$X(u_{j+1}-u_j)\leq C'[X(u_j)+X(u_{j-1})]^2X(u_j-u_{j-1})\leq C'(4CE_0)^2X(u_j-u_{j-1}),$$

and $\{u_j\}$ is Cauchy if $C'16C^2E_0^2\leq\frac{1}{2}$.

Recall that the proof of the classical local theory relies on energy inequality for the wave equation, Sobolev's lemma $\|u\|_{L^\infty}\leq\|u\|_{H^s}$ for $s>\frac{d}{2}$ and Morser's inequality

$$\|F(u,\partial u)(t)\|_{H^{s-1}} \leq f(\|(u,\partial u)\|_{L^\infty})\|(u,\partial u)\|_{H^s}$$

provided $s > \frac{d}{2} + 1$.

In general the lower bound on s is sharp. For instance, $\Box u = (\partial_t u)^k$ with $H^s \times H^{s-1}$, $s < \frac{d}{2} + 1 - \frac{1}{k-1}$, data is not well-posed, and this approaches $\frac{d}{2} + 1$ as $k \to \infty$.

Question: For given F in the local theorem, what is the minimal s for which the conclusion of Theorem holds for data in $\dot{H}^s \times \dot{H}^{s-1}$?

The critical wellposedness exponent s_c. This is the unique $s \in \mathbb{R}$ such that the homogenous data space $\dot{H}^s \times \dot{H}^{s-1}$ is invariant under scaling of the equation.

It is easy to see

$$\|f(\lambda x)\|_{\dot{H}^s} = \lambda^{s-n/2} \|f\|_{\dot{H}^s}.$$

From the scaling heuristics, usually one can hope the Cauchy problem is local wellposed for the data in $H^s \times H^{s-1}$, $s > s_c$; for the smooth data with small $H^{s_c} \times H^{s_c-1}$ norm, there exists a global smooth solution; and the Cauchy problem is ill posed for data in $\dot{H}^s \times \dot{H}^{s-1}, s < s_c$.

This is because one has the following principle according to the Tao's book:

In the subcritical case $s > s_c$, we expect the high frequencies of the solutions to evolve linearly for all time. The low frequencies of the solution will evolve linearly for short times, but nonlinearly for long times.

In the critical case $s = s_c$, we expect high frequencies to evolve linearly for all time if their H^{s_c} norm is small, but to quickly develop nonlinear behavior when the norm is large. The low frequencies of the solution will evolve linearly for all time if their H^{s_c} norm is small, but will eventually develop nonlinear behavior when the norm is large.

In the supercritical case $s < s_c$, the high frequencies are very unstable and will develop nonlinear behavior very quickly. The low frequencies are in principle more stable and linear, though in practice they can be quickly disrupted by the unstable behavior in the high frequencies.

Lindblad shows the uniqueness fails of the nonlinear equation $\Box u = u^3$ with data $(0, 0)$ in the space $C([0, \infty); \dot{H}^s) \cap C^1([0, \infty); \dot{H}^{s-1})$ when the regularity is supercritical, $s < \frac{1}{2}$. Obviously, $u = 0$ is a solution. One can show

$$u(t, x) = \frac{\sqrt{2} H(t - |x|)}{t}, \quad (t, x) \in \mathbb{R}^{1+3}_+$$

is a nonzero solution, where H is the Heaviside function.

The scale-invariant regularity which coincides with one of the other special regularity often is important. For instance, the regularity of $\dot{H}^1_x$ (energy, or Hamiltonian), $\dot{H}^{1/2}_x$ (symplectic structure, Lorentz invariance, and conformal invariance) and L^2_x (the limiting regularity to make sense of the equation $\Box u = |u|^{\kappa-1} u$ distributionaly) usually are important.

§ 2.4. Strauss conjecture

We consider the following model semilinear wave equations ($p > 1$)

$$\begin{cases} \Box u = F_\kappa(u) := |u|^\kappa, & x \in \mathbb{R}^3, t > 0, \\ u(x,0) = \varepsilon\phi(x), u_t(x,0) = \varepsilon\psi(x), (\phi,\psi) \in C_0^\infty(B_R). \end{cases} \tag{2.13}$$

Fundamental problem: for what range of κ, does the problem admit global solutions with small enough $\varepsilon > 0$? Then there should be a threshold value, denoted by $\kappa_c(3)$.

The problem enjoys scaling invariance:

$$u(t,x) \to u_\lambda(t,x) = \lambda^{-2/(\kappa-1)} u(t/\lambda, x/\lambda) \ .$$

Invariance in the homogeneous Sobolev space $\|f\|_{\dot{H}^s} = \|(-\Delta)^{s/2} f\|_{L^2}$ suggests a lower bound of the regularity index $s_c = \frac{3}{2} - \frac{2}{\kappa-1}$.

Similarly, invariance under Lorentz transform suggests another lower bound of the regularity index $s_l = 1 - \frac{1}{\kappa-1}$. Observe that $s_c \geq s_l \Leftrightarrow \kappa \geq 3 \Leftrightarrow s_c \geq 1/2$. There is another index of regularity (even for radial solutions) $s_d = \frac{1}{2} - \frac{1}{\kappa}$.

Observe that $\kappa > 1, s_c > s_d \Leftrightarrow \kappa > 1 + \sqrt{2}$. As is natural, we expect these powers play certain roles for existence results.

2.4.1. Nonexistence of global solutions

The nonexistence of global solutions for $1 < \kappa \leq 1 + \frac{2}{d-1}$ is due to Kato [8] for generic initial data. In particular, $1 < \kappa < \infty$ for $d = 1$. $\kappa < 1 + \sqrt{2}$ for $d = 3$ is due to John [7].

2.4.2. John's result and Strauss conjecture

Theorem 2.8. *Suppose $\phi \in C^3(\mathbb{R}^3)$ and $\psi \in C^2(\mathbb{R}^3)$ have compact support. Then if $\kappa > 1 + \sqrt{2}$, the equation (2.13) has a global solution $u \in C^2(\mathbb{R}_+^{1+3})$ with small enough data.*

The original proof of John involved an iteration argument in the space with norm $\sup_{t>0}(1+t)\|(1+(t-|x|)^{\kappa-2}u^*(t,\cdot))\|_{L_x^\infty}$, where $u^*(t,x) = \sup_{y\in\mathbb{R}^3,|y|=|x|}|u(t,y)|$ denotes the radial majorant of u.

Based on similarity of the wave equation and Schrödinger equation, Strauss 1981 made the conjecture that for $d \geq 2$, the critical power, $\kappa_c(d)$, is given by the positive root of

$$(d-1)\kappa^2 - (d+1)\kappa - 2 = 0.$$

In particular,

$$\kappa_c(2) = \frac{3+\sqrt{17}}{2}, \ \kappa_c(3) = 1 + \sqrt{2}, \ \kappa_c(4) = 2, \ \kappa_c(d) < 2, d \geq 5 \ .$$

Many people made contributions to Struauss conjecture, say, John, Glassey, Sideris, Schaeffer, Lindblad, Sogge, Li, Zhou, Kubo, Takamura-Wakasa, Yordanov-Zhang, Georgiev. Finally, it was solved by Tataru.

We should point out the proof of Tataru for all dimensions requires the data to have compact support, but if $d \leq 8$, Lindblad-Sogge proved the existence without assuming compact support. Can we remove it?

What is the optimal regularity assumptions on the data so that the Strauss conjecture holds true?

Suppose that w solves inhomogeneous wave equation $\Box u = F$ with zero data. F is supported in the forward light cone $\{(t,x) \in \mathbb{R}_+^{1+3} : |x| < t\}$. Then (proved by Lindblad and Sogge)

$$\|(t^2 - |x|^2)^a w^*\|_{L^p(\mathbb{R}_+^{1+3})} \leq C_{p,b}\|(t^2 - |x|^2)^b F^*\|_{L^{p'}(\mathbb{R}_+^{1+3})},$$

provided $2 \leq p \leq 4$, $b < 1/p$ and $b = a + 4/p - 1$.

The following is the outline of the proof. As for the detail one can refer to [16]. At first, suppose for $\kappa > 3$ the existence result is true, to obtain global existence for a given $\kappa < 3$. For $\kappa > 1 + \sqrt{2}$, one can choose $b = b/\kappa + 4/\kappa + 1 - 1$, and $b < 1/(\kappa + 1)$ to use the weighted Strichartz estimate. Then, let $b_\kappa = b/\kappa$. From $\kappa + 1 - b_\kappa > 3$ when $\kappa > 1 + \sqrt{2}$, for $\Box u_0 = 0$ with the data in theorem, we can show the following inequality is true

$$\sum_{j=0}^{2} \|(t^2 - |x|^2)^{b_\kappa/\kappa}(\nabla^j u_0)^*\|_{L^{\kappa+1}([R,+\infty)\times\mathbb{R}^3)} \leq \frac{A_0\epsilon}{4}.$$

And then, one can use continuity argument to show

$$\sum_{j=0}^{2} \|(t^2 - |x|^2)^{b_\kappa/\kappa}(\nabla^j u)^*\|_{L^{\kappa+1}([R,T_*)\times\mathbb{R}^3)}$$

is bounded.

Suppose that w solves the inhomogeneous wave equation with zero data. Suppose further that F (and hence w) is supported in the forward light cone $\{(t,x) \in \mathbb{R}^{1+d} : |x| < t\}$, $d \geq 2$. Tataru proved the weighted Strichartz estimate

$$\|(t^2 - |x|^2)^b w\|_{L^{\frac{2(d+1)}{d-1}}(\mathbb{R}_+^{1+d})} \leq C\|(t^2 - |x|^2)^b F\|_{L^{\frac{2(d+1)}{d+3}}(\mathbb{R}_+^{1+d})}, b < \frac{d-1}{2(d+1)}.$$

2.4.3. Low regularity Strauss conjecture An interesting problem is: " Under what kind of the low regularity assumptions on the data, the Strauss conjecture still holds true?"

The following Strichartz estimates are due to Sogge (see [16]). Suppose u is a weak solution of the inhomogeneous linear wave equation with data $\dot{H}^\gamma \times \dot{H}^{\gamma-1}$. Then

$$
\begin{aligned}
&\|u\|_{L^{\frac{2\kappa}{1+\gamma}} L^{\frac{2\kappa}{2-\gamma}}(S_T)} + \|u(T,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} \\
&\quad \le C \left(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L^{\frac{2}{1+\gamma}} L^{\frac{2}{2-\gamma}}(S_T)} \right),
\end{aligned} \tag{2.14}
$$

provided that $3 \le \kappa \le 5/(1-\gamma)$ and $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$.

If $2 < \kappa \le 3$, and $\gamma = 1 - \frac{1}{\kappa-1}$,

$$
\begin{aligned}
&\|u\|_{L^{\frac{2}{\gamma}} L^{\frac{2\kappa}{2-\gamma}}(S_T)} + \|u(T,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} \\
&\quad \le C_\gamma \left(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L^{\frac{2}{1+\gamma}} L^{\frac{2}{2-\gamma}}(S_T)} \right).
\end{aligned} \tag{2.15}
$$

If $4 \le q < \infty$, and $\gamma = \gamma(q) = 3/2 - 4/q$,

$$
\begin{aligned}
&\|u\|_{L^q(S_T)} + \|(\sqrt{-\Delta_x})^{\gamma-1/2} u\|_{L^4(S_T)} + \|u(T,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\partial_t u(T,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} \\
&\quad \le C_q \left(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} + \|(\sqrt{-\Delta_x})^{\gamma-1/2} F\|_{L^{4/3}(S_T)} \right).
\end{aligned} \tag{2.16}
$$

Remark. If $F = 0$, this inequality is essentially equivalent to

$$
\|u\|_{L^q(\mathbb{R}_+^{1+3})} \le C_q \left(\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)} \right).
$$

But no such an inequality holds for any $q < 4$. For the spherical symmetry case, it still holds for $3 < q < \infty$.

For the homogeneous linear wave equation with spherically symmetric data, we have the following form Strichartz estimates

$$
\|u\|_{L^{\frac{\kappa(\kappa-1)}{3-\kappa}} L^\kappa(\mathbb{R}_+^{1+3})} \le C_\kappa (\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)}), \tag{2.17}
$$

with $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$, $1 + \sqrt{2} < \kappa < 3$, and

$$
\|u\|_{L^{\kappa^2} L^\kappa([0,T]\times\mathbb{R}^3)} \le C_\kappa T^{\frac{1+2\kappa-\kappa^2}{\kappa^2}} (\|\phi\|_{\dot{H}^\gamma(\mathbb{R}^3)} + \|\psi\|_{\dot{H}^{\gamma-1}(\mathbb{R}^3)}), \tag{2.18}
$$

with $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$, $2 < \kappa < 1 + \sqrt{2}$.

From these Strichartz estimates, we wish to find the minimal γ, depending on κ, such that the conditions on the data (ϕ, ψ) are strong enough to ensure that for some $0 < T \le \infty$ there is a weak solution

$$
(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1}).
$$

For $\kappa > 2$, and $\gamma = \gamma(\kappa) = \max\{3/2 - 2/(\kappa-1), 1 - 1/(\kappa-1)\}$, we have

Theorem 2.9 (Sogge 95).

For $\kappa > 2$, there is a $T > 0$ and a unique (weak) solution $(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$ and $u \in L_t^s L_x^{2(\kappa-1)}([0,T] \times \mathbb{R}^3)$, $s = \max\{2(\kappa-1), 2(\kappa-1)/(\kappa-2)\}$.

If T_* denotes the supremum of all $T > 0$ of the above solution, then either $T_* = \infty$ or $u \notin L^{2(\kappa-1)}([0,T_*) \times \mathbb{R}^3)$.

If $\kappa \geq 3$, there exists a unique global weak solution for small data.

Because of the better Strichartz estimates for the spherical symmetry case, there are better existence results for the equations of the form $\Box u = F_\kappa(u)$ with radial data. In this case, Strauss conjecture holds true under minimal regularity assumptions.

Theorem 2.10 (Sogge 95). *Assume ϕ and ψ are spherically symmetric functions. Then for $1+\sqrt{2} < \kappa < 3$ there is a unique global weak solution $u \in L_t^{\frac{\kappa(\kappa-1)}{3-\kappa}} L_x^\kappa(\mathbb{R}_+^{1+3})$; for $2 < \kappa < 1+\sqrt{2}$ there is a unique solution $u \in L_t^{\kappa^2} L_x^\kappa([0,T_\epsilon] \times \mathbb{R}^3)$ with $T_\epsilon = \epsilon^{\frac{\kappa(\kappa-1)}{\kappa^2-2\kappa-1}}$.*

Remark. From the scaling argument we know $\gamma \geq 3/2 - 2/(\kappa-1)$ is necessary for wellposedness.

We can show that $u_\epsilon(t,x) = \epsilon^{-2/(\kappa-1)} u(t/\epsilon, x/\epsilon)$ solves the same equation with data $(0, \psi_\epsilon)$, with $\psi_\epsilon = \epsilon^{-2/(\kappa-1)-1}\psi(x/\epsilon)$ for $\psi \in C_0^\infty(\mathbb{R}^3)$. The lifespan $T_\epsilon = \epsilon T_*$. From $\|\psi_\epsilon\|_{\dot{H}^{\gamma-1}} / \|\psi\|_{\dot{H}^{\gamma-1}} = \epsilon^{3/2-2/(\kappa-1)-\gamma}$, we know that if $\gamma < 3/2 - 2/(\kappa-1)$ both the lifespan and the norm of the data go to zero with ϵ. There is no local existence in a strip.

One needs $\gamma \geq 1 - 1/(\kappa-1)$, which is based on the fact that

$$u_{\alpha\beta}(t,x) = \frac{c_\alpha(1-\beta^2)^{\alpha/2}}{(\epsilon - (t - \beta x_1))^\alpha}, c_\alpha = \alpha(\alpha+1)^{\alpha/2}, \alpha = \frac{2}{\kappa-1}$$

satisfies $\Box u_{\alpha\beta} = |u_{\alpha\beta}|^\kappa$ and blows up when $t - \beta x_1 = \epsilon$.

Outline of the proof of the existence part of Theorem 2.9 To prove it, we need to divid it into different cases:

Case $2 < \kappa \leq 3$. As usual, let $u_{-1} = 0$. We define $u_m, m = 0, 1, \cdots,$ from $\Box u_m = F_\kappa(u_{m-1})$ with data (ϕ, ψ) inductively. Then we can show that there is an $\varepsilon_0 = \varepsilon_0(\kappa) > 0$, so that

$$A_m(T) \leq 2A_0(T), B_{m+1}(T) \leq \frac{1}{2} B_m(T) \tag{2.19}$$

if $2A_0(T)T^{\frac{1}{\kappa-1}-\frac{1}{2}} \leq \varepsilon_0$, with

$$A_m(T) = \|u_m\|_{L^{\frac{2}{\gamma}} L^{\frac{2\kappa}{2-\gamma}}(S_T)}, \ B_m(T) = \|u_m - u_{m-1}\|_{L^{\frac{2}{\gamma}} L^{\frac{2\kappa}{2-\gamma}}(S_T)},$$

and $\gamma = 1 - \frac{1}{\kappa-1}$.

Note that $B_0(T) = A_0(T)$, and

$$A_0(T) \le C_\gamma(\|\phi\|_{\dot H^\gamma} + \|\psi\|_{\dot H^{\gamma-1}}).$$

From the Strichartz estimate (2.15), we can choose T such that the condition of (2.19) is satisfied. It follows that $\{u_m\}$ converges to a limit in $L^{\frac{2}{\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)$ and hence in the sense of distributions.

From

$$\|F_\kappa(u_{m+1}) - F_\kappa(u_m)\|_{L^{\frac{2}{\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)} \le \|V_\kappa(u_{m+1}, u_m)\|_{L^2(S_T)}\|u_m - u_{m-1}\|_{L^{\frac{2}{\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)}$$

and $\|V_\kappa(u_{m+1}, u_m)\|_{L^2(S_T)} \le 1/2C_\gamma$, we can see

$$\|F_\kappa(u_{m+1}) - F_\kappa(u_m)\|_{L^{\frac{2}{\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)} \le C2^{-m},$$

and $F_\kappa(u_m) \to F_\kappa(u)$ in $L^{\frac{2}{\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)$, where $V_\kappa(u,v) = (F_\kappa(u) - F_\kappa(v))/(u-v)$. Therefore, $F_\kappa(u_m)$ converges weakly to $F_\kappa(u)$. To here, we have shown that u must be a week solution of (2.13). From the Strichartz estimate (2.15) and (2.19) we see $(u_m, \partial_t u_m)$ must be a Cauchy sequence in $C([0,T]; \dot H^\gamma \times \dot H^{\gamma-1})$ converging to $(u, \partial_t u)$ if the data is in C_0^∞, and this assumption about the data can be removed by standard approximation argument using (2.15).

Case $3 \le \kappa \le 5$. We will prove this case under the following claim:

$$A_m(T) \le 2A_0(T), B_{m+1}(T) \le \frac{1}{2}B_m(T) \tag{2.20}$$

if $2A_0(T) \le \varepsilon_0$, with

$$A_m(T) = \|u_m\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)},\ B_m(T) = \|u_m - u_{m-1}\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T)},$$

and $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$.

Note that

$$\|u_0\|_{L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}(\mathbb{R}^{1+3})} \le C_\gamma(\|\phi\|_{\dot H^\gamma} + \|\psi\|_{\dot H^{\gamma-1}}).$$

From the Strichartz estimate (2.14), we have $2A_0(T) \le \varepsilon_0$ for all T if the data has small norm, or, if not, this inequality will be satisfied for some $T > 0$ by dominated convergence theorem. Therefore, if we let $T = \infty$ in the first case and T be this finite time in the second, we can argue as before to conclude that there must be a weak solution of (2.13) and $(u, \partial_t u) \in C([0,T]; \dot H^\gamma \times \dot H^{\gamma-1})$ as well as

$$u \in L^{\frac{2\kappa}{1+\gamma}}L^{\frac{2\kappa}{2-\gamma}}(S_T) = L^{\frac{4\kappa(\kappa-1)}{5\kappa-9}}L^{\frac{4\kappa(\kappa-1)}{\kappa+3}}(S_T).$$

Note that for $0 \le t \le T$,

$$\|u(t,\cdot)\|_{L^{\frac{6}{3-2\gamma}}(\mathbb{R}^3)} \le C\|u(t,\cdot)\|_{\dot H^\gamma(\mathbb{R}^3)}.$$

We have $u \in L^\infty L^{\frac{6}{3-2\gamma}}(S_T) = L^\infty L^{\frac{3(\kappa-1)}{2}}(S_T)$. Hence, by the Hölder inequality, we have

$$\|u\|_{L^{2(\kappa-1)}(S_T)} \le \|u\|^{\theta}_{L^{\frac{4\kappa(\kappa-1)}{5\kappa-9}} L^{\frac{4\kappa(\kappa-1)}{\kappa+3}}(S_T)} \|u\|^{1-\theta}_{L^\infty L^{\frac{3(\kappa-1)}{2}}(S_T)},$$

which completes the proof of existence part when $3 \le \kappa \le 5$.

Case $\kappa > 5$. In this case, we let

$$A_m(T) = \|D^{1-2/(\kappa-1)} u\|_{L^4(S_T)} + \|u\|_{L^{2(\kappa-1)}(S_T)},$$

and $B_m = \|u_m - u_{m-1}\|_{L^4(S_T \cap \Gamma_{R,0})}$, where $\Gamma_{R,0} = \{(t,x) \in \mathbb{R}^{1+3}_+ : |x| < R - t, t \ge 0\}$ and $R < \infty$. Then one can prove that there is an $\varepsilon_0 > 0$ so that, for $m = 0, 1, 2 \cdots$

$$A_m(T) \le 2A_0(T), B_{m+1}(T) \le \frac{1}{2} B_m(T) \tag{2.21}$$

if $A_0(T) \le \varepsilon_0$.

As before we can always choose $T > 0$ so that (2.21) holds, and if the data has small enough norm we can take $T = \infty$. Note that $2(\kappa - 1) > 4$. The Hölder inequality implies that $B_0(T) \le C_R A_0(T)$. Thus we can show that u_m must tend to a limit in $L^4_{loc}(S_T)$ and hence in $\mathcal{D}'$ and almost everywhere. Also we can show $F_\kappa(u_m)$ converges to $F_\kappa(u)$ in L^1_{loc} and hence u is a weak solution of (2.13).

From Fatou's lemma, we know

$$\|u\|_{L^{2(\kappa-1)}(S_T)} \le \lim \inf_{m\to\infty} \|u_m\|_{L^{2(\kappa-1)}(S_T)} \le 2A_0(T) < \infty.$$

We have $u \in L^{2(\kappa-1)}(S_T)$. Note that $u_m \to u$ in the sense of distribution as $m \to \infty$. We have $|(u, \varphi)| \le 2A_0 \|D^{1-2/(\kappa-1)} \varphi\|_{L^{4/3}}$ and hence $D^{1-2/(\kappa-1)} u \in L^4(S_T)$.

Note that also

$$\|D^\sigma F_\kappa(u)\|_{L^q} \le C \|F'_\kappa(u)\|_{L^p} \|D^\sigma u\|_{L^r}, \frac{1}{q} = \frac{1}{p} + \frac{1}{r},$$

for $0 < \sigma \le 1$, which comes from the Liebnitz rule of fractional derivatives, and $F_\kappa(u) \in C^1$ satisfies $C_0^{-1} \le |uF'_\kappa(u)|/|F_\kappa(u)| \le C_0$ for some constant C_0. Then, we can show $D^{1-\frac{2}{\kappa-1}} F_\kappa(u) \in L^{4/3}(S_T)$, by using Strichartz estimate (2.16) to show $(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1})$.

The uniqueness for $2 \le \kappa \le 3$ follows from the following theorem due to Sogge,

Theorem 2.11. *Assume that $V \in L^2([0,T] \times \mathbb{R}^3)$. The equation*

$$\Box u = Vu, \ u(0,\cdot) = \phi, \ \partial_t u(0,\cdot) = \psi,$$

with $(\phi, \psi) \in \dot{H}^\gamma \times \dot{H}^{\gamma-1}$, has a unique solution satisfying

$$(u, \partial_t u) \in C([0,T]; \dot{H}^\gamma \times \dot{H}^{\gamma-1}) \text{ and } u \in L^{\frac{2}{\gamma}} L^{\frac{2\kappa}{2-\gamma}}(S_T).$$

Moreover, there is a universal constant C_γ *so that for* $0 \le t \le T$ *and that for* $(\phi, \psi) \in \dot{H}^\gamma \times \dot{H}^{\gamma-1}$

$$\begin{aligned}\|u(t,\cdot)\|_{\dot{H}^\gamma} + \|\partial_t u\|_{\dot{H}^{\gamma-1}} \\ \le 2\exp\left(C_\gamma \int_{S_T} |V(t,x)|^2 dtdx\right)(\|\phi\|_{\dot{H}^\gamma} + \|\psi\|_{\dot{H}^{\gamma-1}}).\end{aligned}$$

Also, if $1/2 \le \gamma < 3/2$ *suppose that* $u \in L^{\frac{8}{3-2\gamma}}$*, and* $(D^{\gamma-1/2})u \in L^4(S_T)$*. Then, if* $\Box u = F(u)$ *with* (ϕ, ψ)*,*

$$\begin{aligned}\|u(t,\cdot)\|_{\dot{H}^\gamma} + \|\partial_t u\|_{\dot{H}^{\gamma-1}} \\ \le 2\exp\left(C_\gamma \int_{S_T} |F'(u(t,x))|^2 dtdx\right)(\|\phi\|_{\dot{H}^\gamma} + \|\psi\|_{\dot{H}^{\gamma-1}}).\end{aligned}$$

For the case $\kappa > 3$, if u_1 and u_2 are two solutions of (2.13), then $w = u_1 - u_2$ satisfies $\Box w = Vw$ with zero Cauchy data and

$$V = (F_\kappa(u_1) - F_\kappa(u_2))/(u_1 - u_2) \in L^2(S_T).$$

The uniqueness follows from

$$\begin{aligned}\|u_1 - u_2\|_{L^4(S_T \cap \Lambda_{R,0})} &\le C\|V \cdot (u_1 - u_2)\|_{L^{4/3}(S_T \cap \Lambda_{R,0})} \\ &\le C\|V\|_{L^2(S_T)}\|u_1 - u_2\|_{L^4(S_T \cap \Lambda_{R,0})}.\end{aligned}$$

This is because $2(\kappa-1) > 4, u_1 - u_2 \in L^4(S_T \cap \Lambda_{R,0})$, and we have $\|u_1 - u_2\|_{L^4(S_T \cap \Lambda_{R,0})} = 0$ if T is small enough, and hence $u_1 = u_2$ in $S_T \cap \Lambda_{R,0}$. Repeating this argument a finite number of times will show that the same is true for any fixed $T > 0$, so we finished the proof of the uniqueness.

For the continuation, if one takes $V = F_\kappa(u)/u$, then it is also clear from above theorem.

Outline of the proof the existence part of Theorem 2.10 For the case $1 + \sqrt{2} < \kappa < 3$, $\gamma = \frac{3}{2} - \frac{2}{\kappa-1}$, we construct u_m as before. If we can prove

$$A_m(T) = \|u_m\|_{L^{\frac{\kappa(\kappa-1)}{3-\kappa}} L^\kappa(\mathbb{R}^{1+3}_+)} \le 2^{-m} C_\kappa \varepsilon, \tag{2.22}$$

$0 < \varepsilon < \varepsilon(\kappa)$, for $\varepsilon(\kappa)$ small enough, we have u_m converges to a weak solution of (2.13) in $L^{\frac{\kappa(\kappa-1)}{3-\kappa}} L^\kappa(\mathbb{R}^{1+3}_+)$. From (2.17), (2.22) holds if $m = 0$. Under the induction hypothesis, we can show

$$\|u_n\|_{L^{\frac{\kappa(\kappa-1)}{3-\kappa}} L^\kappa(\mathbb{R}^{1+3}_+)} \le 2C_\kappa \varepsilon, n < m,$$

from (2.22). Note that

$$F_\kappa(u_{m-1}) - F_\kappa(u_{m-2}) = O(|u_{m-1}|^{\kappa-1} + |u_{m-2}|^{\kappa-1})|u_{m-1} - u_{m-2}|.$$

By using Hölder's inequality, and the improved inhomogeneous Strichartz estimate in spherical symmetry case

$$\|u\|_{L^{\frac{q(q-1)}{3-q}}L^q(S_T)} \le C_q \|F_\kappa\|_{L^{\frac{q-1}{3-q}}L^1(S_T)},$$

we can show

$$A_m \le 2C_2(2\varepsilon)^{\kappa-1}A_{m-1}.$$

And then, one has (2.22) must hold if $2C_2(2\varepsilon)^{\kappa-1} < 1/2$, and we finish the proof in the case of $1+\sqrt{2} < \kappa < 3$.

For the case $2 \le \kappa < 1+\sqrt{2}$, let

$$B_m = \|u_m - u_{m-1}\|_{L^{\kappa^2}L^\kappa([0,T_\varepsilon]\times\mathbb{R}^3)}.$$

By using (2.18) and induction, we can show that

$$B_m \le C_\kappa \varepsilon T_\varepsilon^{\frac{1+2\kappa-\kappa^2}{\kappa^2}} 2^{-m}, 0 < \varepsilon < \varepsilon(\kappa),$$

if $\varepsilon(\kappa)$ is small enough, and have the result. In the proof, we must use the fact $\kappa^2(\kappa-2) < \kappa$ if $\kappa < 1+\sqrt{2}$, and

$$\|u\|_{L^{\kappa^2}L^\kappa}([0,T]\times\mathbb{R}^3) \le CT^{\frac{1+2\kappa-\kappa^2}{\kappa^2}}\|F_\kappa\|_{L^\kappa L^1([0,T]\times\mathbb{R}^3)}$$

when $2 \le \kappa < 1+\sqrt{2}$.

Now we see that if $\kappa \ge 3$, there exists a unique global weak solution for small data, but for radial data, the Strauss' conjecture holds true. A natural question is: can we add some regularity on the angular variable such that the Strauss' conjecture still holds?

Recall the weighted Strichartz estimates for $2 \le q \le \infty$ and $\Box u = 0$

$$\||x|^{\frac{d}{2}-\frac{d+1}{q}-\gamma}u\|_{L^q_t L^q_{|x|}L^2_\omega} \le C\|u'(0)\|_{\dot H^{\gamma-1}}\ ,\ \frac{1}{2}-\frac{1}{q} < \gamma < \frac{d}{2}-\frac{1}{q}.$$

By Duhamel's principle,

$$\||x|^{\frac{d}{2}-\frac{d+1}{\kappa}-\gamma}u\|_{L^\kappa_t L^\kappa_{|x|}L^2_\omega} \le C\|u'(0)\|_{\dot H^{\gamma-1}} + C\|\Box u\|_{L^1_t \dot H^{\gamma-1}}$$

for $2 \le \kappa \le \infty$, $\frac{1}{2}-\frac{1}{\kappa} < \gamma < \frac{d}{2}-\frac{1}{\kappa}$.

By duality of the generalized Sobolev

$$\||x|^{\frac{d}{2}-\frac{d+1}{\kappa}-\gamma}u\|_{L^\kappa_t L^\kappa_{|x|}L^2_\omega} \le C\|u'(0)\|_{\dot H^{\gamma-1}} + C\||x|^{-\frac{d}{2}+1-\gamma}\Box u\|_{L^1_t L^1_{|x|}L^2_\omega}$$

for $2 \le \kappa \le \infty$, $\frac{1}{2}-\frac{1}{\kappa} < \gamma < \frac{d}{2}-\frac{1}{\kappa}$, $\frac{1}{2} < 1-\gamma < \frac{d}{2}$.

We can apply the above estimates to (2.13) for the small data $(\phi,\psi) \in \dot H^{s_c,s_1}_\omega \times \dot H^{s_c-1,s_1}_\omega$ with some s_1.

Theorem 2.12. *Let $d \geq 2$, $\kappa_h < \kappa < \kappa_{conf}$ (i.e., $\frac{1}{2d} < s_c < \frac{1}{2}$) and $s_1 > \frac{1}{2} - s_c$, where $\kappa_{conf} = 1 + \frac{4}{d-1}$ and $\kappa_h = 1 + \frac{4d}{(d+1)(d-1)}$. Suppose that*

$$(\phi, \psi) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c - 1, s_1}$$

with small enough norm. Then there is a unique global weak solution u to (2.13) satisfying

$$u \in C_t \dot{H}_x^{s_c} \cap C_t^1 \dot{H}_x^{s_c} \cap L_{t,x}^q \text{ with } q = \frac{(d+1)(\kappa - 1)}{2}.$$

Moreover, we can prove Strauss' Conjecture with a kind of mild rough data for $d \leq 4$ in the sense of the following theorem.

Theorem 2.13. *Let $2 \leq d \leq 4$, $\kappa_c < \kappa < \kappa_{conf}$ and $s_1 = \frac{1}{\kappa - 1}$. Suppose that*

$$(\phi, \psi) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c - 1, s_1}$$

with small enough norm. Then there is a unique global weak solution $u \in C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c - 1, s_1}$ to (2.13) satisfying

$$|x|^{-\alpha} u \in L^\kappa_{t, |x|^{d-1} d|x|} H_\omega^{s_2},$$

for $\alpha = \frac{d+1}{\kappa} - \frac{2}{\kappa - 1}$ and $s_2 = s_1 + s_c - s_d$.

Next, we give the proof of Theorem 2.13 by using the weighted Strichartz estimates (1.15) in Theorem 1.12 and Sobolev inequality (2.25) in Corollary 1.20.

Since the method of the proof is just the usual contraction argument, we need only to give some of the key inequalities here.

First, for $s_1 = \frac{1}{\kappa - 1}$, $s_2 = s_1 + s_c - s_d$, $\alpha = \frac{d+1}{\kappa} - \frac{2}{\kappa - 1}$, and any solution u of the equation $(\partial_t^2 - \Delta) u = 0$ with data $(\phi, \psi) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c - 1, s_1}$, we have $u \in C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c - 1, s_1}$ and get from (1.15) that

$$|x|^{-\alpha} u \in L^\kappa_{t, |x|^{d-1} d|x|} H_\omega^{s_2},$$

if $s_c - s_d \in (0, \frac{d-1}{2})$, i.e., $\kappa > \kappa_c$.

Since $d \leq 4$ and $\kappa > \kappa_c \geq 2$, we have $\frac{d-1}{2} < 2 \leq [\kappa]$, where $[\kappa]$ stands for the integer part of κ. Note that by the Moser estimate

$$\|\Lambda_\omega^s F_\kappa(u)\|_{L_\omega^r} \lesssim \|u\|_{L_\omega^q}^{\kappa - 1} \|\Lambda_\omega^s u\|_{L_\omega^p} \tag{2.23}$$

for $s \in [0, m]$ and $p, q, r \in (1, \infty)$ with $\frac{1}{r} = \frac{\kappa - 1}{q} + \frac{1}{p}$, where $F_\kappa \in C^m$ with

$$F_\kappa(0) = 0, |\partial^\alpha F_\kappa(x)| \lesssim |x|^{\kappa - |\alpha|}, 1 \leq |\alpha| \leq m \leq \kappa,$$

and the Sobolev embedding, we get the following estimate on S^{d-1}

$$\|F_\kappa(u)\|_{H^a_\omega} \lesssim \|u\|^\kappa_{H^b_\omega} \tag{2.24}$$

with $b \geq \frac{\kappa-1}{\kappa}\frac{d-1}{2} + \frac{a}{\kappa}$ if $0 \leq a < \frac{d-1}{2}$ (and thus $a \leq [\kappa]$). Then by letting $b = s_2$ and $a = \kappa s_2 - \frac{d-1}{2}(\kappa - 1) = \frac{d-1}{2} - \frac{1}{\kappa-1} < \frac{d-1}{2}$, we have

$$|x|^{-\alpha\kappa} F_\kappa(u) \in L^1_{t,|x|^{d-1}d|x|} H^a_\omega.$$

Recall the estimate

$$\|\mathrm{D}^{-\frac{b}{2}} \Lambda_\omega^{\frac{b-1}{2}} \phi\|_{L^2_x} \lesssim \| |x|^{\frac{b-d}{2}} \phi(x)\|_{L^1_{|x|^{d-1}d|x|} L^2_\omega} \tag{2.25}$$

for any $\phi \in \mathcal{S}$, if $\frac{1}{2} - s_c \in (0, \frac{d-1}{2})$, i.e., $1 + \frac{2}{d-1} < \kappa < \kappa_{conf}$. Then

$$F_\kappa(u) \in L^1_t \dot{H}^{s_c - 1, a + \frac{1}{2} - s_c}_\omega.$$

Note that $a + \frac{1}{2} - s_c = \frac{1}{\kappa-1} = s_1$. We have $F_\kappa(u) \in L^1_t \dot{H}^{s_c-1,s_1}_\omega$.

By the classical energy estimate, we can get again $(v, \partial_t v) \in C_t(\dot{H}^{s_c,s_1}_\omega \times \dot{H}^{s_c-1,s_1}_\omega)$ and

$$|x|^{-\alpha} v \in L^\kappa_{t,|x|^{d-1}d|x|} H^{s_2}_\omega,$$

if $(\partial_t^2 - \Delta)v = F_\kappa(u)$ with initial data (ϕ, ψ).

If we define the solution space X to be

$$X = \{u \in C_t \dot{H}^{s_c,s_1}_\omega \cap C^1_t \dot{H}^{s_c-1,s_1}_\omega : \| |x|^{-\alpha} u\|_{L^\kappa_{t,|x|^{d-1}d|x|} H^{s_2}_\omega} \leq C\epsilon\},$$

then, at last, combining all above, we can prove that the problem (2.13) with $\kappa_c < \kappa < \kappa_{conf}$ is global well-posed for small data in the space $C_t \dot{H}^{s_c,s_1}_\omega \cap C^1_t \dot{H}^{s_c-1,s_1}_\omega$ with $s_1 = \frac{1}{\kappa-1}$. Thus we get Theorem 2.13.

§ 2.5. Energy critical problem

We focus on the following three dimensional energy-critical defocusing nonlinear wave equations

$$\Box u = -|u|^4 u,\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x). \tag{2.26}$$

For critical equations, there is a delicate balance between the linear and nonlinear parts of the equation. Both high frequencies and low frequencies can exhibit nonlinear behavior, at short times and long times respectively. It forces us to work with scale-invariant norms, which severely limits the tools available. The energy of the solution could concentrate to a point in finite time, causing the lifespan of the local theory to shrink to zero as time progresses. The key theme will be of interaction between scales.

Recall that this equation has a scaling symmetry $u(t,x) = 1/\lambda^{1/2}u(t/\lambda, x/\lambda)$ and has a conserved energy

$$E[u(t)] := \int_{\mathbb{R}^3} \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\bigtriangledown u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 dx$$

which is invariant under the above scaling.

Note that endpoint Sobolev embedding allows us to control the nonlinear component of the energy by some quantity depending only on the linear component. If the energy is small, then we can expect to have linear behavior.

Local wellpssedness in $\dot{H}^1_x \times L^2_x$ can be proven by perturbative theory. In the proof one needs the key Strichartz estimate

$$\begin{aligned}\|\bigtriangledown_{t,x} u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)} + \|u\|_{L^\infty_t L^6_x(I\times\mathbb{R}^3)} + \|u\|_{L^4_t L^{12}_x(I\times\mathbb{R}^3)} \\ \leq \|u_0\|_{\dot{H}^1_x(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^1_t L^2_x(I\times\mathbb{R}^3)}.\end{aligned}$$

Theorem 2.14. *Smooth data leads to smooth global solutions of* (2.26)*.*
Finite energy initial data leads to $\dot{H}^1_x \times L^2_x$ well-posed solutions of (2.26) *on arbitarity large bounded time intervals.*

The first part was established in spherically symmetric case by Struwe in 1988, and in general by Grillakis in 1992. The second was established in 1994 by Shatah and Struwe, and Kapitanski. Solutions for small energy was proved by Rauch.

By time reversal symmetry we only need to construct solutions forwards in time. We will argue by contradiction, supposing that the $\dot{H}^1_x \times L^2_x$ solution breaks down at some maximal time of existence $0 < T_* < \infty$,

Assuming finite energy, we have $E[u] \leq E_0$, and

$$\|\bigtriangledown_{t,x} u\|_{L^\infty_t L^2_x([0,T_*)\times\mathbb{R}^3)} + \|u\|_{L^\infty_t L^6_x([0,T_*)\times\mathbb{R}^3)} + \|u\|_{L^4_t L^{12}_x([0,T_*)\times\mathbb{R}^3)} \lesssim_{E_0} 1.$$

These bounds will not immediately allow us to use the perturbative theory to extend the solution beyond T_*, as there is no smallness condition on E_0 and the L^∞_t norm does not decay upon localizing the time interval $[0, T_*)$.

One needs to use perturbation and non perturbation argument to obtain some useful decay estimates.

To quantify some sense in which the solution is becoming badly behaved as $t \to T_*$, we need a good blow up criterion. The standard persistence of regularity theory would allow one to obtain the blow up criterion

$$\|u\|_{L^\infty_t L^\infty_x([0,T_*)\times\mathbb{R}^3)} = \infty.$$

But it is not useful here,

- it is a subcritical criterion rather than a critical one, very hard to disprove in the fine-scale limit $t \to T_*$. We need to use a scale invariant perturbation theory to find a variant of it, say use Strichartz estimates.
- it is a global in space rather than a local in space. We can use finite speed of propagation.

From small energy having a global $H^1_x \times L^2_x$ solution theory, we have a weak blowup criterion, namely we must have $E[u(t)] \geq \epsilon_0$ for some absolute constant $\epsilon_0 > 0$. But it is too weak to be of any use. By finite speed of propagation, we can strengthen to a localized one.

- If u is an $H^1_x \times L^2_x$ well-posed solution with a maximal time of existence $0 < T_* < \infty$, then there exists $x \in \mathbb{R}^3$ such that

$$\limsup_{t \to T_*} E_{B(x,3(T_*-t))}[u(t)] \geq \epsilon_0$$

for some absolute constant ϵ_0.

We can show to exclude blow up and prove the result, we only need to prevent a non-zero fraction of the energy from concentrating in the interior of the backwards light cone

$$\cup_{0<t\leq T_*}\{t\} \times B(x, T_* - t) = \{(t,y) \times \mathbb{R}^3 : |y-x| < (T_* - t)\}.$$

But this is rather difficult to do directly because of the derivatives of the solution which lies with the linear components $\int(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\bigtriangledown u(t,x)|^2)dx$ of the energy. It is difficult to make it small.

One can show that a global solution will exist whenever the $L^4_t L^{12}_x$ norm is small enough. Let $\mathcal{D}_+(\Omega, t_0, t_1) = \{(t,x) \in [t_0,t_1) \times \mathbb{R}^3 : B(x, t-t_0) \subset \Omega\}$ denote the truncated forward domain of dependence, where $\Omega \subset \mathbb{R}^3$ is any open set, and $B(x,r)$ denotes the ball of radius r centred at x. We can now localize the small energy wellposedness theory and obtain

- If u is an $H^1_x \times L^2_x$ well-posed solution with a maximal time of existence $0 < T_* < \infty$, then there exists $x \in \mathbb{R}^3$ such that

$$\limsup_{t \to T_*^-} \|u_t\|_{L^4_t L^{12}_x(\mathcal{D}_+(B(x,2(T_*-t)),t,T_*))} \geq \epsilon_0(E_0)$$

for some absolute constant ϵ_0, where u_t has initial data $u_t[t] = u[t]$.

- Then one can use the perturbation theory to get a stronger criterion,

$$\limsup_{t \to T_*^-} \int_{B(x,T_*-t)} |u(t,y)|^6 dy \geq \epsilon_2$$

for some constant $\epsilon_2(E_0) > 0$. So, to prove the theorem, we only need to obtain decay for the potential energy , which has no derivatives.

If u is an $H^1_x \times L^2_x$ solution with energy at most E_0 with a maximal time of existence $0 < T_* < \infty$, then

$$\int_{B(x,T_*-t)} |u(t,x)|^6 dx = o_{t\to T_*^-}(1),$$

where we use $o_{t\to T_*^-}(1)$ to denote any quantity depending on t whose magnitude goes to zero as $t \to T_*^-$. Hence we can use Morawetz identity to complete the proof.

§ 2.6. Almost critical regularity problem

Let $\Box \equiv \partial_t^2 - \Delta$, $\partial = (\partial_t, \partial_x)$ and P_α be polynomials for $\alpha \in \mathbb{N}^3$, we consider the following Cauchy problem

$$\Box u = \sum_{|\alpha|=3} P_\alpha(u)(\partial u)^\alpha \tag{2.27}$$

on $[0,T] \times \mathbb{R}^2$, together with the initial data at time $t = 0$

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x)\ . \tag{2.28}$$

In the case of classical C_0^∞ initial data with size of order ϵ, the almost global existence with

$$T_\epsilon \geq \exp(c\epsilon^{-2}) \tag{2.29}$$

(for some small constant $c > 0$) can be proved by the standard energy methods, see e.g. Sogge [16]. Moreover, the lifespan T_ϵ is also sharp for the problem with nonlinearity $|\partial_t u|^3$.

Our object here is to prove the corresponding result with low regularity. Note that the equation (2.27) with P_α being constants C_α is invariant under the scaling transformation $u(t,x) \to \lambda^{-\frac{1}{2}} u(\lambda t, \lambda x)$. This scaling preserves the critical Sobolev space $\dot{H}^{s_c}$ with exponent

$$s_c = \frac{3}{2}\ , \tag{2.30}$$

which is then, heuristically, a lower bound for the range of admissible s such that the problem (2.27)-(2.28) is well-posed in $C_t H^s_x \cap C^1_t H^{s-1}_x$. (See e.g. Theorem 2 in [4] for the ill posed result with $s < s_c$ and nonlinearity $(\partial_t u)^3$.)

The local well posedness for the problem of this type with low regularity has been extensively studied by many mathematicians, say Ponce-Sideris, Tataru and so on. For this problem, besides scaling, there is one more mechanism due to Lorentz invariance such that the problem is not well posed in $C_t H_x^s \cap C_t^1 H_x^{s-1}$ with $s = s_c + \epsilon$ for arbitrary small $\epsilon \ll 1$. Instead, the local well posedness is true for $s > \frac{7}{4}$.

To state our main result, we need to introduce the Sobolev space with angular regularity $b > 0$,

$$f \in H_\theta^{s,b} \Leftrightarrow f \in H^s, \text{ and } (1-\partial_\theta^2)^{b/2} f \in H^s \tag{2.31}$$

where the (r, θ) is the polar coordinates. Now we are ready to state our main result.

Theorem 2.15. *Let $d = 2$, $s > s_c = 3/2$ and $b > 1/2$. Then there exist two small positive constant ϵ_0 and c, such that the problem (2.27) admits a unique almost global solution $(u, \partial_t u) \in C_{T_\epsilon}(H_\theta^{s,b} \times H_\theta^{s-1,b})$ with $\partial_{t,x} u \in L^2_{T_\epsilon} L^\infty_{|x|} H^b_\theta$ on $[0, T_\epsilon] \times \mathbb{R}^2$ with $T_\epsilon = \exp(c\epsilon^{-2})$, whenever $(u_0, u_1) \in H_\theta^{s,b} \times H_\theta^{s-1,b}$ with norm bounded by $\epsilon \le \epsilon_0$.*

Remark. Here we note that, by adding some angular regularity, the Sobolev regularity required to ensure almost global existence is only $s > s_c$, which is $1/4$ less than the usual requirement of $s > 7/4$.

To begin, let us prove the fractional Leibniz rule in the Sobolev space with angular regularity.

Lemma 2.16. *Let $d = 2$, $s \in (0, 1)$, $b > \frac{1}{2}$ and $\psi \in \mathcal{S}(\mathbb{R}^2)$ be a radial function. Then we have*

$$\|\psi * f\|_{L_r^\infty L_\theta^2} \lesssim \|\psi\|_{L^1} \|f\|_{L_r^\infty L_\theta^2} , \tag{2.32}$$

and the fractional Leibniz rule

$$\|fg\|_{H_\theta^{s,b}} \lesssim \|f\|_{L^\infty_{|x|} H^b_\theta} \|g\|_{H_\theta^{s,b}} + \|g\|_{L^\infty_{|x|} H^b_\theta} \|f\|_{H_\theta^{s,b}} . \tag{2.33}$$

Moreover, we have

$$\|fg\|_{H_\theta^{s,b}} \lesssim \|f\|_{L^\infty_{|x|} H^b_\theta \cap \dot H_\theta^{1,b}} \|g\|_{H_\theta^{s,b}} , \tag{2.34}$$

$$\|fg\|_{\dot H_\theta^{1,b} \cap L^\infty_{|x|} H^b_\theta} \lesssim \|f\|_{\dot H_\theta^{1,b} \cap L^\infty_{|x|} H^b_\theta} \|g\|_{\dot H_\theta^{1,b} \cap L^\infty_{|x|} H^b_\theta} . \tag{2.35}$$

Proof. At first, we give the proof for (2.32). Recall

$$(\psi * f)(x) = \int \psi(y) f(x-y) dy.$$

We set $x = (r\cos\omega, r\sin\omega)$, $y = (\lambda\cos\theta, \lambda\sin\theta)$. Then $x - y = (\rho\cos\alpha, \rho\sin\alpha)$, with

$$\rho = \sqrt{r^2 + \lambda^2 - 2r\lambda\cos(\omega-\theta)},\ \alpha = \omega + \arcsin\left(\frac{\lambda}{\rho}\sin(\omega-\theta)\right).$$

Introduce a new variable $a = \omega - \theta \in [0, 2\pi]$. Then $\rho = \rho(\lambda, r, a)$ and $\alpha = \alpha(\lambda, r, \omega, a) = \omega + h(\lambda, r, a)$ for some function h. Now, for fixed r,

$$\begin{aligned}
\|\psi * f\|_{L^2_\omega} &= \| \int_0^\infty \int_0^{2\pi} \psi(\lambda) f(x-y) \lambda d\lambda d\theta \|_{L^2_\omega} \\
&\le \|f(\rho\cos\alpha, \rho\sin\alpha)\|_{L^\infty_\lambda L^2_\omega L^1_\theta} \int_0^\infty |\psi(\lambda)| \lambda d\lambda \\
&\simeq \|\psi\|_{L^1} \|f(\rho\cos\alpha, \rho\sin\alpha)\|_{L^\infty_\lambda L^2_\omega L^1_a} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\alpha(\lambda, r, \omega, a), \rho(\lambda, r, a)\sin\alpha(\lambda, r, \omega, a))\|_{L^\infty_\lambda L^2_\omega L^2_a} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\omega, \rho(\lambda, r, a)\sin\omega)\|_{L^\infty_\lambda L^2_a L^2_\omega} \\
&\lesssim \|\psi\|_{L^1} \|f(\rho(\lambda, r, a)\cos\omega, \rho(\lambda, r, a)\sin\omega)\|_{L^\infty_\lambda L^\infty_a L^2_\omega} \\
&\le \|\psi\|_{L^1} \|f(\rho\cos\omega, \rho\sin\omega)\|_{L^\infty_\rho L^2_\omega} ,
\end{aligned}$$

which proves (2.32). The estimate (2.32) tells us that the space $L^\infty_r L^2_\theta$ is stable under the frequency localization.

Based on (2.32), and the fact that H^b_θ is an algebra under multiplication when $b > 1/2$, we can easily apply Littlewood-Paley decomposition to prove the fraction Leibniz rule (2.33), (2.34), and (2.35). ■

Now we are ready to give the proof of Theorem 2.15, based on the endpoint estimate (1.41) and the fraction Leibniz rule (2.33).

To understand the idea of the proof, we only prove the easier case when $P_\alpha(u)$ do not depend on u. By (1.41) and energy estimate, for fixed $s > \frac{3}{2}$ and $b > \frac{1}{2}$, we have

$$\|e^{-itD} f\|_{L^2_T L^\infty_{|x|} H^b_\theta} \le C_0 (\ln(2+T))^{1/2} \|f\|_{H^{s-1,b}_\theta} \tag{2.36}$$

and

$$\|e^{-itD} f\|_{L^\infty_T H^{s-1,b}_\theta} \le C_0 \|f\|_{H^{s-1,b}_\theta} \tag{2.37}$$

with some constant $C_0 > 1$. Recall that we have the initial data $(u_0, u_1) \in H^{s,b}_\theta \cap H^{s-1,b}_\theta$ with

$$\|u_0\|_{H^{s,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta} = \epsilon \le \epsilon_0, \tag{2.38}$$

where ϵ_0 will be fixed later (see (2.40)).

Given the metric

$$d(u, v) = (\ln(2+T))^{-1/2} \|\partial_{t,x}(u-v)\|_{L^2_T L^\infty_{|x|} H^b_\theta} + \|\partial_{t,x}(u-v)\|_{L^\infty_T H^{s-1,b}_\theta}, \tag{2.39}$$

we define the complete domain with $T = \exp(c\epsilon^{-2})$ and $c \ll 1$ to be chosen later (see (2.40)),

$$X = \{u \in C_T H^{s,b}_\theta \cap C^1_T H^{s-1,b}_\theta : u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ d(u, 0) < \infty\}.$$

Then for any $u \in X$, we use Πu to denote the solution to the linear wave equation

$$\Box \Pi u = (\partial u)^\alpha$$

with initial data (u_0, u_1). Note that for $u \in X$, we have $(\partial u)^\alpha \in L^1_T H^{s-1,b}_\theta$ by using the fraction Leibniz rule (2.33), and so $\Pi u \in C_T H^{s,b}_\theta \cap C^1_T H^{s-1,b}_\theta$ is well defined. Thus, by energy estimates (2.37) and Strichartz estimates (2.36), we have

$$d(\Pi u, 0) \le C_1(\|u_0\|_{H^{s,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta}) + C_1\|(\partial u)^\alpha\|_{L^1_T H^{s-1,b}_\theta}$$

for some $C_1 \ge C_0$. Based on this estimate, we define a complete domain $D_\epsilon \subset X$ so that the map Π will be a contraction map in D_ϵ (for ϵ_0 and c small enough),

$$D_\epsilon = \{u \in X;\ d(u, 0) \le 2C_1\epsilon\}\ .$$

By using the fraction Leibniz rule (2.33) and noting that $T = \exp(c\epsilon^{-2}) > e > 2$, we have for some $C_2 \ge C_1$,

$$\begin{aligned} d(\Pi u, 0) &\le C_1(\|u_0\|_{H^{s,b}_\theta} + \|u_1\|_{H^{s-1,b}_\theta}) + C_1\|(\partial u)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\ &\le C_1\epsilon + C_2\|\partial u\|^2_{L^2_T L^\infty_{|x|} H^b_\theta}\|\partial u\|_{L^\infty_T H^{s-1,b}_\theta} \\ &\le C_1\epsilon + C_2\ln(2+T) d(u,0)^3 \\ &\le C_1\epsilon + C_2 d(u,0)^3 + C_2 c\epsilon^{-2} d(u,0)^3\ . \end{aligned}$$

Moreover, for any $u, v \in X$, we have for some $C_3 \ge C_2$,

$$\begin{aligned} d(\Pi u, \Pi v) &\le C_1\|(\partial u)^\alpha - (\partial v)^\alpha\|_{L^1_T H^{s-1,b}_\theta} \\ &\le C_3 \ln(2+T)(d(u,0)^2 + d(v,0)^2) d(u,v) \\ &\le (C_3 + C_3 c\epsilon^{-2})(d(u,0)^2 + d(v,0)^2) d(u,v). \end{aligned}$$

Now we fix the constants ϵ_0 and c such that we have

$$c\epsilon_0^{-2} \ge 1,\ 2C_2(2C_1\epsilon_0)^3 c\epsilon_0^{-2} \le C_1\epsilon_0,\ 2C_3 c\epsilon_0^{-2}(2\times 2C_1\epsilon_0)^2 \le \frac{1}{2},$$

which can be satisfied if we set

$$c = \frac{1}{2^6 C_1^2 C_3},\ \epsilon_0 = \sqrt{c}\ . \tag{2.40}$$

Then we know that if $\epsilon \le \epsilon_0$, the map Π is a contraction map on the complete set D_ϵ, and the fixed point $u \in D_\epsilon$ of the map Π gives the required unique almost global solution.

Acknowledgement

The author thanks the anonymous referee for the carefully reading and constructive remarks.

References

[1] S. Alinhac, *Hyperbolic Differential Equations.* Springer-Verlag Berlin Heidelberg 2009.

[2] Y. Du, J. Metcalfe, C. D. Sogge, Y. Zhou, *Concerning the Strauss conjecture and almost global existence for nonlinear Dirichlet-wave equations in 4-dimensions*, Comm. Partial Differential Equations **33** (2008), no. 7-9, 1487–1506.

[3] D. Fang, C. Wang, *Some Remarks on Strichartz Estimates for Homogeneous Wave Equation*, Nonlinear Analysis TMA **65** (2006), Issue 3, 697–706.

[4] D. Fang, C. Wang, *Weighted Strichartz Estimates with Angular Regularity and their Applications*, Forum Math. **23** (2011), no. 1, 181–205.

[5] K. Hidano and K. Yokoyama, *A remark on the almost global existence theorems of Keel, Smith and Sogge*, Funkcial. Ekvac. **48** (2005), 1-34.

[6] L. Hörmander *Lectures on nonlinear hyperbolic differential equations.* Springer-Verlag Berlin Heidelberg 1997.

[7] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions* , Manuscripta Math. **28** (1979), no. 1-3, 235–268.

[8] Kato, Tosio, *Blow-up of solutions of some nonlinear hyperbolic equations.* Comm. Pure Appl. Math. **33** (1980), no. 4, 501–505.

[9] M. Keel, H. Smith, C. D. Sogge, *Almost global existence for some semilinear wave equations*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 265–279.

[10] M. Keel, H. F. Smith, C. D. Sogge, *Almost global existence for quasilinear wave equations in three space dimensions*, J. Amer. Math. Soc. **17** (2004), no. 1, 109–153 (electronic).

[11] J.-C. Jiang, C. Wang, X. Yu, *Generalized and weighted Strichartz estimates*, Communications on Pure and Applied Analysis, **11** (2012), Number 5, 1723–1752.

[12] H. Lindblad, C. D. Sogge, *On Existence and Scattering with Minimal Regularity for Semilinear Wave Equations*, J. Func. Anal. **130** (1995), 357–426.

[13] S. Machihara, M. Nakamura, K. Nakanishi, T. Ozawa, *Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation*, J. Func. Anal. **219** (2005), 1–20.

[14] S. Selberg, *Lecture notes Math 632, PDE.* Johns Hopkins University, 2001.

[15] Smith, H. F., Sogge, C. D., Wang, C. (2012) *Strichartz estimates for Dirichlet-wave equations in two dimensions with applications.* Trans. Amer. Math. Soc. **364**:3329–3347.

[16] C. D. Sogge, *Lectures on non-linear wave equations. Second edition.* International Press, Boston, MA, 2008.

[17] C. D. Sogge, C. Wang, *Concerning the wave equation on asymptotically Euclidean manifolds*, J. Anal. Math. **112** (2010), no. 1, 1–32.

[18] J. Sterbenz, *Angular regularity and Strichartz estimates for the wave equation*, With an appendix by Igor Rodnianski, Int. Math. Res. Not. **2005** No.4, 187–231.

[19] Stein, E. M., Weiss, G. (1971). *Introduction to Fourier analysis on Euclidean spaces.* Princeton: Princeton University Press.

[20] T. Tao, *Nonlinear Dispersive Equations: Local And Global Analysis*, 2006.