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The Cauchy problem of Hartree and pure power type nonlinear Schrödinger equations

By

Shinya Kinoshita*

Abstract

This paper is concerned with the Cauchy problem of Hartree (HNLS) and pure power nonlinear Schrödinger equations (PNLS) with L^2-subcritical regularity. It is known that the global well-posedness in the scale invariant homogeneous Sobolev space with radial symmetry or some angular regularity was established provided that the initial data have small norm. We generalize these results by new weighted Strichartz estimates.

§ 1. Introduction

We consider the Cauchy problem of Hartree type nonlinear Schrödinger equations (HNLS):

$$\begin{cases} iu_t(t,x) + \Delta u(t,x) = F(u(t,x)), & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $u_t = \partial u/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$. $F(u)$ is a nonlinear functional of Hartree type:

$$F(u) = (\lambda|x|^{-\gamma} * |u|^2)u, \quad \lambda \in \mathbb{C}\setminus\{0\},\ 0<\gamma<n,$$

where $*$ denotes the convolution in $\mathbb{R}^n$. From Duhamel's formula, the solution u of (1.1) can be written as

$$u(t,x) = U(t)(\varphi + \Phi_t)(x), \tag{1.2}$$

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*Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.
e-mail: `m12018b@math.nagoya-u.ac.jp`

where

$$U(t) = e^{it\Delta}, \quad \Phi_t = \Phi_t(u) = -i\int_0^t U(-t')F(u)(t')dt'.$$

By the following scaling transformation:

$$u_\eta(t,x) = \eta^{\frac{n+2-\gamma}{2}} u(\eta^2 t, \eta x), \quad \eta > 0,$$

we see that (HNLS) has the scaling invariance in $\dot{H}^{s_c}$ with the critical index $s_c = \frac{\gamma-2}{2}$.

There are lots of works on the Cauchy problem of (HNLS). Almost all of them discussed the problem for $\varphi \in H^s$, $s \geq \max(0, s_c)$. As a fundamental result, Miao, Xu and Zhao [16] proved the local well-posedness in H^s where $s > s_c$, $s \geq 0$. Furthermore for $s \geq 1$, by the energy conservation law, they proved the global well-posedness for $0 < \gamma \leq 2$, $\gamma < n$, $\lambda \geq 0$ and for $0 < \gamma < \min(2, n)$, $\lambda < 0$, and in particular, for $s = 1$, the global well-posedness was established for $2 < \gamma < 4$, $\gamma < n$ and $\lambda \geq 0$. In addition, the smallness condition of $\|\varphi\|_{\dot{H}^{s_c}_x}$ can ensure the global existence in H^s, $s > s_c$ for $2 \leq \gamma < n$, $n \geq 3$. In [10], Hayashi and Ozawa proved the global well-posedness in L^2 for $0 < \gamma < \min(2, n)$ (see [2] for general nonlinearities). For the critical case, $s = s_c \geq 0$, (HNLS) is locally well-posed in H^{s_c} for $2 \leq \gamma < n$, and globally well-posed and the solutions behave like linear ones in H^{s_c} for $2 \leq \gamma < n$, $n \geq 3$ under the smallness condition of $\|\varphi\|_{\dot{H}^{s_c}_x}$ (see [16, 3, 2]). If initial data φ has finite energy, it is known that (HNLS) is globally well-posed in $\dot{H}^1$ for $\gamma = 4$, $\lambda \geq 0$, $n \geq 5$ (see [17], and see also [15] for radially symmetric initial data).

As opposed to the case $s \geq \max(0, s_c)$, we have few results for $s_c \leq s < 0$. Miao, Xu and Zhao [16] proved some ill-posedness results for $s < \max(0, s_c)$, while Cho, Hwang and Ozawa [4] proved the global well-posedness for radially symmetric small data $\varphi \in \dot{H}^{s_c}$, $\frac{8n-2}{6n-3} \leq \gamma < 2$:

Theorem A ([4] Theorem 5). *Let $n \geq 2$, $\frac{8n-2}{6n-3} \leq \gamma < 2$. Then there exists a positive constant $\varepsilon = \varepsilon(n, \gamma)$ such that if $\varphi \in \dot{H}^{s_c}(\mathbb{R}^n)$ is radially symmetric and satisfies $\||\nabla|^{s_c}\varphi\|_{L^2_x} < \varepsilon$, then (1.2) has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^n)) \cap L^3(\mathbb{R}; L^r(\mathbb{R}^n)).$$

Here r satisfies $\frac{1}{r} = \frac{1}{2} - \frac{2}{3n} - \frac{s}{n}$. In addition, u scatters in $\dot{H}^{s_c}(\mathbb{R}^n)$.

They also discussed the problem of global well-posedness without assuming radial symmetry:

Theorem B ([4] Theorem 2). *Let $n \geq 3$, $2 - \frac{3}{2n+2} < \gamma < 2$, $s_1 = \frac{n-1}{n+1} - \frac{\gamma-1}{2}$ and*

$$\max\left(\gamma - \frac{5n-3}{2n+2}, \frac{1}{2}\right) < s_2 < \min\left(\gamma - \frac{3n}{2n+2}, \frac{3(n-1)}{2n+2}\right).$$

Then there exist a positive constant $\varepsilon = \varepsilon(n,\gamma)$ *and* α_1, $\alpha_2 \in [2,\infty]$, β_1, $\beta_2 \in \mathbb{R}$, γ_1, $\gamma_2 \in (0,\infty)$ *such that if* $\varphi \in \dot H^{s_c} H^{s_1+s_2}_\omega(\mathbb{R}^n)$ *satisfies* $\||\nabla|^{s_c} D^{s_1+s_2}_\omega \varphi\|_{L^2_x} < \varepsilon$, *then* (1.2) *has a unique solution*

$$u \in C_b(\mathbb{R}; \dot H^{s_c} H^{s_1+s_2}_\omega(\mathbb{R}^n)) \cap L^{\alpha_1}(\mathbb{R}; |x|^{\beta_1} L^2_r H^{\gamma_1}_\omega) \cap L^{\alpha_2}(\mathbb{R}; |x|^{\beta_2} L^2_r H^{\gamma_2}_\omega).$$

In addition, u scatters in $\dot H^{s_c} H^{s_1+s_2}_\omega(\mathbb{R}^n)$.

The main goal of this paper is to widen the range of γ in Theorems A and B in the case $n \geq 3$. That is, we improve the conditions $\frac{8n-2}{6n-3} \leq \gamma < 2$ in Theorem A and $2 - \frac{3}{2n+2} < \gamma < 2$ in Theorem B to $\frac{4}{3} < \gamma < 2$. To describe it precisely, we should introduce some function spaces. We denote by H^s and $\dot H^s$, $s \in \mathbb{R}$, the usual inhomogeneous Sobolev spaces and homogeneous Sobolev spaces, respectively, and we define the norm

$$\|f\|_{L^p_r L^q_\omega} = \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r\omega)|^q d\omega \right)^{\frac{p}{q}} r^{n-1} dr \right)^{\frac{1}{p}}, \quad 1 \leq p, q < \infty.$$

We also define the modified Sobolev space $\dot H^s H^\alpha_\omega$ and its norm by

$$\dot H^s H^{\alpha,q}_\omega = \{ f \in \mathcal{S}' \setminus \mathcal{P} : \|f\|_{\dot H^s H^{\alpha,q}_\omega} < \infty \}, \quad s, \alpha \in \mathbb{R},$$
$$\|f\|_{\dot H^s H^{\alpha,q}_\omega} = \||\nabla|^s D^\alpha_\omega f\|_{L^2_r L^q_\omega}.$$

Here $\mathcal{S}$ is the Schwartz space, $\mathcal{P}$ denotes the totality of polynomials. $|\nabla| = \sqrt{-\Delta}$, and $D_\omega = \sqrt{1-\Delta_\omega}$ for the Laplace-Beltrami operator Δ_ω. We refer to [13], Appendix, [12] and [22] for the details of D_ω. We denote $\dot H^0 H^{\alpha,q}_\omega$ and $\dot H^s H^{\alpha,2}_\omega$ by $L^2_r H^{\alpha,q}_\omega$ and $\dot H^s H^\alpha_\omega$, respectively.
We denote the space $L^q(\mathbb{R}; X)$ by $L^q_t X$ and its norm by $\|\cdot\|_{L^q_t X}$ for some Banach space X, and also $L^q([0,T]; X)$ by $L^q_{I_T} X$ and its norm by $\|\cdot\|_{L^q_{I_T} X}$. We use the notation $C_b(\mathbb{R}; X) = C(\mathbb{R}; X) \cap L^\infty(\mathbb{R}; X)$.

Our results are the following. The first one is radially symmetric case, and the second is general case:

Theorem 1.1. *Let* $n \geq 3$, $\frac{4}{3} < \gamma < 2$ *and* $\delta = \delta(n,\gamma)$ *be sufficiently small. Then there exist a positive constant* $\varepsilon = \varepsilon(n,\gamma)$ *and exponents* $q_1, q_2, \ell \in [2,\infty]$ *such that if* $\varphi \in \dot H^{s_c}(\mathbb{R}^n)$ *is radially symmetric and satisfies* $\||\nabla|^{s_c}\varphi\|_{L^2_x} < \varepsilon$, *then* (1.2) *has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot H^{s_c}(\mathbb{R}^n)) \cap L^{q_1}(\mathbb{R}; |x|^{s_c-\delta} L^2(\mathbb{R}^n)) \cap L^{q_2}(\mathbb{R}; |x|^{s_c} L^\ell(\mathbb{R}^n)).$$

Theorem 1.2. *Let* $n \geq 3$, $\frac{4}{3} < \gamma < 2$ *and* $\delta = \delta(n,\gamma)$ *be sufficiently small. Then there exist a positive constant* $\varepsilon = \varepsilon(n,\gamma)$ *and exponents* $q_1, q_2, \ell, \sigma \in [2,\infty]$ *such that if*

$\varphi \in \dot{H}^{s_c} H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$ *satisfies* $\||\nabla|^{s_c} D_\omega^{\frac{3}{4}(2-\gamma)+\delta}\varphi\|_{L_x^2} < \varepsilon$, *then* (1.2) *has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c} H_\omega^{\frac{3}{4}(2-\gamma)+\delta}) \cap L^{q_1}(\mathbb{R}; |x|^{s_c-\delta} L_r^2 H_\omega^{\frac{3}{4}(2-\gamma)+\frac{3}{2}\delta}) \cap L^{q_2}(\mathbb{R}; |x|^{s_c} L_r^\ell H_\omega^{\frac{3}{4}(2-\gamma)+(\frac{3}{2}-\frac{1}{n})\delta,\sigma}).$$

Remark 1. Actually, the solutions of Theorems 1.1 and 1.2 scatter in $\dot{H}^{s_c}(\mathbb{R}^n)$ and $\dot{H}^{s_c} H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$, respectively. See [4] for the details.

Next, we consider the subcritical case, $s_c < s < 0$. The following two theorems show the local well-posedness in time for large initial data. The important difference from the critical case is that they include the case $n = 2$ and $0 < \gamma \le \frac{4}{3}$ under the restriction $-\frac{\gamma}{4} < s$.

Theorem 1.3. *Let* $n \ge 2$, $0 < \gamma < 2$,

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

and suppose that $\delta = \delta(n, s, \gamma)$ *is sufficiently small. Then there exist a positive time* T *and exponents* $\alpha \in \mathbb{R}$, $q_1, q_2, \ell \in [2, \infty]$ *such that if* $\varphi \in \dot{H}^s(\mathbb{R}^n)$ *is radially symmetric then* (1.2) *has a unique radial solution*

$$u \in C([0, T]; \dot{H}^s(\mathbb{R}^n)) \cap L^{q_1}([0, T]; |x|^{s-\delta} L^2(\mathbb{R}^n)) \cap L^{q_2}([0, T]; |x|^\alpha L^\ell(\mathbb{R}^n)).$$

Theorem 1.4. *Let* $n \ge 2$, $0 < \gamma < 2$,

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

and suppose that $\delta = \delta(n, s, \gamma)$ *is sufficiently small. Then there exist a positive time* T *and exponents* $\alpha, \beta \in \mathbb{R}$, $q_1, q_2, \ell, \sigma \in [2, \infty]$ *such that if* $\varphi \in \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^n)$ *then* (1.2) *has a unique solution*

$$u \in C([0, T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}) \cap L^{q_1}([0, T]; |x|^{s-\delta} L_r^2 H_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}) \cap L^{q_2}([0, T]; |x|^\alpha L_r^\ell H_\omega^{\beta,\sigma}).$$

Remark 2. If $-s$ is sufficiently close to 0 then the necessary angular regularity for φ is sufficiently small. This seems to be natural since we do not need angular regularity assumption if $s \ge 0$.

Next, we study the Cauchy problem of pure power type nonlinear Schrödinger equations (PNLS):

$$\begin{cases} iu_t + \Delta u = G(u), & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Here $G(u)$ is a nonlinear functional of pure power type:

$$G(u) = \lambda |u|^{p-1} u, \qquad \lambda \in \mathbb{C} \setminus \{0\}, \quad 1 < p.$$

Similarly to (HNLS) case, the following scaling transformation

$$\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

shows that (PNLS) has the scaling invariance in $\dot{H}^{s_{c,p}}$ with the scale critical index $s_{c,p} = \frac{n}{2} - \frac{2}{p-1}$. There exist a lot of works on the Cauchy problem of (PNLS). See [23, 3, 8, 20, 18, 19].

In [11], Hidano proved the global existence for radially symmetric small initial data $\varphi \in \dot{H}^{s_{c,p}}$ if $n \geq 3$ and $1 + \frac{4}{n+1} < p < 1 + \frac{4}{n}$. After that, Fang and Wang [9] proved the global existence for small initial data $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{\frac{1}{p-1}}$ if $3 \leq n \leq 6$ and $1 + \sqrt{\frac{2}{n-1}} < p < 1 + \frac{4}{n}$. We relax the conditions of n and p in the general case. Our result is the following:

Theorem 1.5. *Let $3 \leq n \leq 14$, $p_0 < p < 1 + 4/n$ where p_0 is a unique solution of*

$$\begin{cases} 1 + \frac{4}{n+1} \leq p_0 < 1 + \frac{4}{n}, \\ 2p_0^3 + 6(n-2)p_0^2 + (n^2 - 13n + 10)p_0 - n(n-3) = 0, \end{cases}$$

and suppose that $\delta = \delta(n,p)$ is sufficiently small. Then there exist a positive constant $\varepsilon = \varepsilon(n,p)$ and exponents $\alpha \in \mathbb{R}$, $q, \ell, \sigma \in [2, \infty]$ such that if $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^n)$ satisfies $\| |\nabla|^{s_{c,p}} D_\omega^{s_0} \varphi \|_{L_x^2} < \varepsilon$ where

$$s_0 = \begin{cases} \frac{1}{p-1}(7 - 3p) + \delta & (\text{if } n = 3), \\ \frac{1}{2(p-1)^2}\left(-(n+1)p^2 + (n+7)p - 2\right) + \delta & (\text{if } n \geq 4), \end{cases}$$

then the integral equation

$$u(t,x) = U(t)(\varphi + \Phi_{t,p})(x), \tag{1.3}$$

where

$$\Phi_{t,p} = \Phi_{t,p}(u) = -i \int_0^t U(-t') G(u)(t') dt',$$

has a unique solution

$$u \in C_b(\mathbb{R}; \dot{H}^{s_{c,p}} H_\omega^{s_0}) \cap L^q(\mathbb{R}; |x|^\alpha L^\ell H_\omega^{s_0,\sigma}).$$

Remark 3. Similarly to (HNLS) case, the solution of Theorem 1.5 scatters in $\dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^n)$, and if $n = 3, 4$ the necessary angular regularity for φ gets close to 0 as $-s_{c,p}$ approaches 0.

The paper is organized as follows. In Section 2, we introduce some estimates as preliminaries. In Section 3, we consider the Cauchy problem of (HNLS). To avoid redundancy, we only establish Theorem 1.4. In Section 4, we establish Theorem 1.5. Lastly as appendix, we consider the Cauchy problem of inhomogeneous power type nonlinear Schrödinger equations.

§ 2. Preliminaries

In this section, we introduce some estimates which will be used for the proof of the main results. Throughout the paper, we use the notation $a \lesssim b$ to denote the estimate $a \leq Cb$ for some positive constant C.

First, we introduce weighted Strichartz estimates for $U(t)$.

Lemma 2.1 ([9] Theorem 1.15, [4] Lemma 2)**.** *Let $n \geq 2$, $2 \leq q \leq \infty$.*
(i) *If c, δ_1 satisfy*

$$-\frac{n}{q} < c < -\frac{n}{q} + \frac{n-1}{2}, \quad \delta_1 \leq -\frac{n}{q} + \frac{n-1}{2} - c,$$

then we have

$$\||x|^c |\nabla|^{c+\frac{n+2}{q}-\frac{n}{2}} D_\omega^{\delta_1}[U(t)\varphi]\|_{L_t^q L_r^q L_\omega^2} \lesssim \|\varphi\|_{L_x^2}. \tag{2.1}$$

(ii) *If c, δ_2 satisfy*

$$-\frac{n}{q} < c < -\frac{1}{q}, \quad \delta_2 \leq -c - \frac{1}{q},$$

then we have

$$\||x|^c |\nabla|^{c+\frac{2}{q}} D_\omega^{\delta_2}[U(t)\varphi]\|_{L_t^q L_x^2} \lesssim \|\varphi\|_{L_x^2}.$$

By interpolating between the inequality (2.1) and the classical Strichartz estimates, we immediately get the following weighted Strichartz estimates.

Lemma 2.2. *Let $n \geq 2$, $2 \leq \sigma \leq \ell \leq \infty$ and*

$$\begin{cases} \frac{1}{2} - \frac{1}{\sigma} \leq \frac{1}{q} < \frac{1}{2} + \frac{1}{\ell} - \frac{1}{\sigma} & \text{(if } n = 2), \\ \frac{n}{2}\left(\frac{1}{2} - \frac{1}{\sigma}\right) \leq \frac{1}{q} \leq \frac{1}{2} + \frac{1}{\ell} - \frac{1}{\sigma} & \text{(if } n \geq 3). \end{cases}$$

If d, δ satisfy

$$\frac{n^2}{4} - \frac{n}{q} - \frac{n^2}{2\sigma} < d < \frac{n}{4} - \frac{1}{q} - \frac{n-1}{\ell} + \frac{n-2}{2\sigma},$$
$$\delta \leq -d + \frac{n}{4} - \frac{1}{q} - \frac{n-1}{\ell} + \frac{n-2}{2\sigma},$$

then we have

$$\||x|^d |\nabla|^{d-\frac{n}{2}+\frac{2}{q}+\frac{n}{\ell}} D_\omega^{\delta}[U(t)\varphi]\|_{L_t^q L_r^\ell L_\omega^\sigma} \lesssim \|\varphi\|_{L_x^2}. \tag{2.2}$$

Proof. Suppose that

$$\theta = \frac{2}{q} - \frac{2}{\ell} + \frac{2}{\sigma} - n\left(\frac{1}{2} - \frac{1}{\sigma}\right),$$
$$\frac{1}{q_0} = \frac{n}{2(1-\theta)}\left(\frac{1}{2} - \frac{1}{\sigma}\right),$$
$$\frac{2}{q_0} = n\left(\frac{1}{2} - \frac{1}{r_0}\right),$$
$$\frac{1}{q_1} = \frac{1}{\theta}\left(\frac{1}{q} - \frac{n}{2}\left(\frac{1}{2} - \frac{1}{\sigma}\right)\right).$$

It follows from the classical Strichartz estimates and Lemma 2.1 that

$$\|U(t)\varphi\|_{L_t^{q_0}L_x^{r_0}} \lesssim \|\varphi\|_{L_x^2}, \tag{2.3}$$
$$\||x|^c|\nabla|^{c+\frac{n+2}{q_1}-\frac{n}{2}}D_\omega^{\delta_1}[U(t)\varphi]\|_{L_t^{q_1}L_r^{q_1}L_\omega^2} \lesssim \|\varphi\|_{L_x^2}, \tag{2.4}$$

if

$$-\frac{n}{q_1} < c < -\frac{n}{q_1} + \frac{n-1}{2}, \quad \delta_1 \le -c + \frac{n-1}{2} - \frac{n}{q_1}.$$

By the complex interpolation between (2.3) and (2.4), we get (2.2). □

The following lemma is necessary to handle the nonlinear term.

Lemma 2.3 ([5] Lemma 4.3)**.** *Let $p, q, q_1 \in [1,\infty]$, $0 \le \delta < \gamma < (n-1)/p'$,*

$$\frac{1}{q_1} \ge \frac{1}{q} - \frac{1}{p'} + \frac{\gamma}{n-1}, \quad \frac{\gamma}{n-1} \ne \frac{1}{q_1} - \frac{1}{p}.$$

Then we have

$$\||x|^\delta(|x|^{-\frac{n}{p}-\gamma} * f)\|_{L_r^pL_\omega^{q_1}} \lesssim \||x|^{-(\gamma-\delta)}f\|_{L_r^1L_\omega^{q,1}},$$

where $L_\omega^{q,1}$ is the Lorentz space on the unit sphere.

The following lemma will be utilized for the time restriction $t' < t$. The general case was proved in [6], and see also [21].

Lemma 2.4 ([6] Theorem 1.1)**.** *Let $1 \le r < q \le \infty$, and X, Y be Banach spaces. If*

$$\|U(t)\varphi\|_{L_t^q(Y)} \lesssim \|\varphi\|_{L_x^2} \quad \text{and} \quad \|\int_{-\infty}^{\infty} U(-t')g(t')dt'\|_{L_x^2} \lesssim \|g\|_{L_t^r(X)},$$

then we have

$$\|\int_{-\infty}^{t} U(t-t')g(t')dt'\|_{L_t^q(Y)} \lesssim \|g\|_{L_t^r(X)}.$$

§ 3. (HNLS)

In this section, we consider the Cauchy problem of (HNLS). For convenience, we restate Theorems 1.1-1.4 with the explicit exponents.

Theorem 3.1. *Let $n \geq 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(n,\gamma)$ be sufficiently small. Then there exists a positive constant $\varepsilon = \varepsilon(n,\gamma)$ such that if $\varphi \in \dot{H}^{s_c}(\mathbb{R}^n)$ is radially symmetric and satisfies $\||\nabla|^{s_c}\varphi\|_{L^2_x} < \varepsilon$, then (1.2) has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^n)) \cap L^{2q_{1,s_c}}(\mathbb{R}; |x|^{s_c-\delta}L^2(\mathbb{R}^n)) \cap L^{q_{2,s_c}}(\mathbb{R}; |x|^{s_c}L^{\ell_1}(\mathbb{R}^n))$$

where

$$\frac{1}{q_{1,s_c}} = -2s_c + \delta, \qquad \frac{1}{q_{2,s_c}} = \frac{\gamma}{4} - \frac{\delta}{2},$$
$$\frac{1}{\ell_1} = \frac{1}{2} + \frac{2}{n} - \frac{3}{2n}\gamma + \frac{\delta}{n}.$$

Theorem 3.2. *Let $n \geq 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(n,\gamma)$ be sufficiently small. Then there exists a positive constant $\varepsilon = \varepsilon(n,\gamma)$ such that if $\varphi \in \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$ satisfies $\||\nabla|^{s_c}D_\omega^{\frac{3}{4}(2-\gamma)+\delta}\varphi\|_{L^2_x} < \varepsilon$, then (1.2) has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}) \cap L^{2q_{1,s_c}}(\mathbb{R}; |x|^{s_c-\delta}L^2_r H_\omega^{\frac{3}{4}(2-\gamma)+\frac{3}{2}\delta})$$
$$\cap L^{q_{2,s_c}}(\mathbb{R}; |x|^{s_c}L^{\ell_1}_r H_\omega^{\frac{3}{4}(2-\gamma)+(\frac{3}{2}-\frac{1}{n})\delta,\sigma_1})$$

where

$$\frac{1}{q_{1,s_c}} = -2s_c + \delta, \qquad \frac{1}{q_{2,s_c}} = \frac{\gamma}{4} - \frac{\delta}{2},$$
$$\frac{1}{\ell_1} = \frac{1}{2} + \frac{2}{n} - \frac{3}{2n}\gamma + \frac{\delta}{n}, \quad \frac{1}{\sigma_1} = \frac{1}{2} + \frac{2}{n} - \frac{3}{2n}\gamma + \frac{2}{n}\delta.$$

Theorem 3.3. *Let $n \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

and suppose that $\delta = \delta(n,s,\gamma)$ is sufficiently small. Then there exists a positive time T such that if $\varphi \in \dot{H}^s(\mathbb{R}^n)$ is radially symmetric then (1.2) has a unique radial solution

$$u \in C([0,T]; \dot{H}^s(\mathbb{R}^n)) \cap L^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}([0,T]; |x|^{s-\delta}L^2(\mathbb{R}^n)) \cap L^{q_2}([0,T]; |x|^{\alpha}L^{\ell_2}(\mathbb{R}^n)),$$

where

$$\frac{1}{q_1} = 1 - \frac{\gamma}{2} - s + \delta, \qquad \frac{1}{q_2} = \frac{\gamma}{4} - \frac{\delta}{2},$$
$$\alpha = \begin{cases} s-\delta & (\text{if } n = 2), \\ s & (\text{if } n \geq 3), \end{cases} \quad \frac{1}{\ell_2} = \begin{cases} \frac{1}{2} - \frac{\gamma}{4} - s + \delta & (\text{if } n = 2), \\ \frac{1}{2} - \frac{\gamma}{2n} - \frac{2}{n}s + \frac{\delta}{n} & (\text{if } n \geq 3). \end{cases}$$

Theorem 3.4. *Let $n \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < \ 0,$$

and suppose that $\delta = \delta(n, s, \gamma)$ is sufficiently small. Then there exists a positive time T such that if $\varphi \in \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^n)$ then (1.2) has a unique solution

$$u \in C([0,T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}) \cap L^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}([0,T]; |x|^{s-\delta} L_r^2 H_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}) \\ \cap L^{q_2}([0,T]; |x|^{\alpha} L_r^{\ell_2} H_\omega^{\beta,\sigma_0}),$$

where

$$\frac{1}{q_1} = 1 - \frac{\gamma}{2} - s + \delta, \qquad \frac{1}{q_2} = \frac{\gamma}{4} - \frac{\delta}{2},$$

$$\alpha = \begin{cases} s - \delta & (\text{if } n = 2), \\ s & (\text{if } n \geq 3), \end{cases} \quad \beta = \begin{cases} -\frac{3}{2}s + \frac{3}{2}\delta & (\text{if } n = 2), \\ -\frac{3}{2}s + \frac{3}{2}\delta - \frac{\delta}{n} & (\text{if } n \geq 3), \end{cases}$$

$$\frac{1}{\ell_2} = \begin{cases} \frac{1}{2} - \frac{\gamma}{4} - s + \delta & (\text{if } n = 2), \\ \frac{1}{2} - \frac{\gamma}{2n} - \frac{2}{n}s + \frac{\delta}{n} & (\text{if } n \geq 3), \end{cases} \quad \frac{1}{\sigma_0} = \frac{1}{2} - \frac{\gamma}{2n} - \frac{2}{n}s + \frac{2}{n}\delta.$$

Since the proofs of Theorems 3.1, 3.2 and 3.3 are analogous to that of Theorem 3.4, here we establish only Theorem 3.4. We should mention that if $n = 2$, as Theorems 3.1 and 3.2, we cannot prove the small data global existence for $\varphi \in \dot{H}^{s_c}(\mathbb{R}^n)$. See Remark 4 below for the details.

Throughout the section, we assume $n \geq 2$ and use the explicit exponents

$$\left(\frac{1}{q}, \frac{1}{q_1}, \frac{1}{q_2}\right) = \left(\frac{\gamma}{4} + s - \frac{\delta}{2},\ 1 - \frac{\gamma}{2} - s + \delta,\ \frac{\gamma}{4} - \frac{\delta}{2}\right),$$

$$\left(\frac{1}{\ell}, \frac{1}{\ell_1}, \frac{1}{\ell_2}\right) = \begin{cases} \left(\frac{1}{2} - \frac{\gamma}{4} + \delta, \frac{\gamma}{2} + s - 2\delta, \frac{1}{2} - \frac{\gamma}{4} - s + \delta\right) & (\text{if } n = 2), \\ \left(\frac{1}{2} - \frac{\gamma}{2n} + \frac{\delta}{n}, \frac{\gamma}{n} + \frac{2}{n}s - \frac{2}{n}\delta, \frac{1}{2} - \frac{\gamma}{2n} - \frac{2}{n}s + \frac{\delta}{n}\right) & (\text{if } n \geq 3), \end{cases}$$

$$\frac{1}{\sigma_0} = \frac{1}{2} - \frac{\gamma}{2n} - \frac{2}{n}s + \frac{2}{n}\delta,$$

$$\frac{n-1}{\sigma} = \begin{cases} \frac{\gamma}{2} + \frac{s}{2} - \frac{\delta}{2} & (\text{if } n = 2), \\ \frac{n-1}{n}\gamma + \frac{5}{2}s - \frac{4}{n}s - \frac{5}{2}\delta + \frac{3}{n}\delta & (\text{if } n \geq 3), \end{cases}$$

with sufficiently small $\delta = \delta(n, s, \gamma)$. Here q' and ℓ' are given by $1/q + 1/q' = 1$ and $1/\ell + 1/\ell' = 1$, respectively. Note that

$$\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2}, \qquad \frac{1}{\ell'} = \frac{1}{\ell_1} + \frac{1}{\ell_2}.$$

Lemma 3.5. *Let* $\max(s_c, -\gamma/4) < s < 0$*. Then we have*

$$\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta} U(t)\Phi_t\|_{L^\infty_{I_T} L^2_x} + W_1(U(t)\Phi_t) + W_2(U(t)\Phi_t) \lesssim T^\theta [W_1(u)]^2 W_2(u)$$

where

$$\theta = \frac{2+2s-\gamma}{2},$$
$$W_1(u) = \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L^2_x},$$
$$W_2(u) = \begin{cases} \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\sigma_0}_\omega} & (\text{if } n=2), \\ \||x|^{s} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta-\frac{\delta}{n}} u\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\sigma_0}_\omega} & (\text{if } n\geq 3). \end{cases}$$

Proof. (I) $(n \geq 3)$
First, we assume $n \geq 3$ and prove

$$\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta} U(t)\Phi_t\|_{L^\infty_{I_T} L^2_x} \lesssim T^\theta [W_1(u)]^2 W_2(u). \tag{3.1}$$

Let us set

$$s_1 = -\frac{n-2}{n}s + \frac{n-2}{2n}\delta, \qquad s_2 = -\frac{n+4}{2n}s + \frac{n+2}{2n}\delta.$$

Note that $s_1 + s_2 = -\frac{3}{2}s + \delta$. Since $2 \leq \sigma_0 \leq \ell \leq \infty$,

$$\frac{n}{2}\left(\frac{1}{2} - \frac{1}{\sigma_0}\right) \leq \frac{1}{q} \leq \frac{1}{2} + \frac{1}{\ell} - \frac{1}{\sigma_0},$$
$$\frac{n^2}{4} - \frac{n}{q} - \frac{n^2}{2\sigma_0} < -s < \frac{n}{4} - \frac{1}{q} - \frac{n-1}{\ell} + \frac{n-2}{2\sigma_0},$$

it follows from Lemma 2.2 that

$$\||x|^{-s}|\nabla|^s D_\omega^{s_1}[U(t)\varphi]\|_{L^q_t L^\ell_r L^{\sigma_0}_\omega} \lesssim \|\varphi\|_{L^2_x}.$$

By the dual estimate, we have

$$\|\int_{-\infty}^{\infty} U(-t')F(u)(t')dt'\|_{L^2_x} \lesssim \||x|^s|\nabla|^{-s} D_\omega^{-s_1} F(u)\|_{L^{q'}_t L^{\ell'}_r L^{\sigma_0'}_\omega}, \tag{3.2}$$

where $1/\sigma_0' = 1 - 1/\sigma_0$. By applying Lemma 2.4 to $\|U(t)\varphi\|_{L^\infty_t L^2_x} = \|\varphi\|_{L^2_x}$ and (3.2), we have

$$\||\nabla|^s D_\omega^{s_1+s_2} U(t)\Phi_t\|_{L^\infty_{I_T} L^2_x} \lesssim \||x|^s D_\omega^{s_2} F(u)\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega}.$$

By Leibniz rule and Sobolev embedding on the unit sphere (see Appendix in [13]), we have

$$\begin{aligned}
&\||x|^s D_\omega^{s_2} F(u)\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega} \\
&\lesssim \|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L^{q_1}_{I_T} L^{\ell_1}_r L^{\sigma}_\omega} \||x|^s u\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\alpha}_\omega} \\
&\qquad + \||x|^{-\gamma} * |u|^2\|_{L^{q_1}_{I_T} L^{\ell_1}_r L_\omega^{\frac{\sigma(n-1)}{n-1-\sigma s_2}}} \||x|^s D_\omega^{s_2} u\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\beta}_\omega} \\
&\quad \lesssim \|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L^{q_1}_{I_T} L^{\ell_1}_r L^{\sigma}_\omega} \||x|^s D_\omega^{(n-1)(-1+\frac{2}{\sigma_0}+\frac{1}{\sigma})} u\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\sigma_0}_\omega}.
\end{aligned}$$

Here the exponents α, β satisfy

$$\frac{1}{\alpha} = 1 - \frac{1}{\sigma_0} - \frac{1}{\sigma}, \qquad \frac{1}{\beta} = 1 - \frac{1}{\sigma_0} - \frac{1}{\sigma} + \frac{s_2}{n-1}.$$

We deduce from Lemma 2.3 that

$$\|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L^{\ell_1}_r L^{\sigma}_\omega} \lesssim \||x|^{-\gamma+\frac{n}{\ell_1}} D_\omega^{s_2}(|u|^2)\|_{L^1_r L_\omega^{\frac{n-1}{n-1-(\gamma-\frac{n-1}{\sigma}-\frac{1}{\ell_1})},1}}. \tag{3.3}$$

To estimate the right hand side of (3.3), we utilize Leibniz rule and Sobolev embedding in the Lorentz spaces on the unit sphere:

$$\|D_\omega^s(u\bar{u})\|_{L^{p,1}_\omega} \lesssim \|D_\omega^s u\|_{L^{p_0,2}_\omega} \|u\|_{L^{p_1,2}_\omega}, \tag{3.4}$$

for $s \in (0,1)$, p, p_0, $p_1 \in (1,\infty)$ and $1/p = 1/p_0 + 1/p_1$.

$$\|u\|_{L^{p,2}_\omega} \lesssim \|D_\omega^s u\|_{L^2_\omega}, \tag{3.5}$$

for $-\frac{n-1}{p} = s - \frac{n-1}{2}$, $s > 0$. The above two estimates are verified as follows. From the arguments in Appendix [13] and the general Marcinkiewicz interpolation theorem (Theorem 5.3.2 in [1]), (3.4) and (3.5) are easily transferred from the Euclidean case. Thus it suffices to prove the followings:

$$\||\nabla|^s(u\bar{u})\|_{L^{p,1}_x} \lesssim \||\nabla|^s u\|_{L^{p_0,2}_x} \|u\|_{L^{p_1,2}_x}, \tag{3.6}$$

for $s \in (0,1)$, p, p_0, $p_1 \in (1,\infty)$ and $1/p = 1/p_0 + 1/p_1$, and

$$\|u\|_{L^{q,2}_x} \lesssim \||\nabla|^s u\|_{L^2_x}, \tag{3.7}$$

for $-\frac{n}{q} = s - \frac{n}{2}$, $s > 0$. (3.6) is immediately verified by the proof of Leibniz rule in the Lebesgue spaces (see Proposition 3.3 in [7]), the simple inequality

$$\|u\bar{u}\|_{L^{p,1}_x} \lesssim \|u\|_{L^{p_0,2}_x} \|u\|_{L^{p_1,2}_x},$$

and the general Marcinkiewicz interpolation theorem. Similarly, (3.7) is proved by (real) interpolating Sobolev embedding in the Lebesgue spaces. By using (3.4) and (3.5), we get

$$\begin{aligned}
&\||x|^{2(s-\delta)} D_\omega^{s_2}(|u|^2)\|_{L_r^1 L_\omega^{\frac{n-1}{n-1-(\gamma-\frac{n-1}{\sigma}-\frac{1}{\ell_1})},1}} \\
&\lesssim \||x|^{s-\delta} D_\omega^{s_2} u\|_{L_r^2 L_\omega^{\frac{2(n-1)}{n-1-(\gamma-\frac{n-1}{\sigma}-\frac{1}{\ell_1}-s_2)},2}} \||x|^{s-\delta} u\|_{L_r^2 L_\omega^{\frac{2(n-1)}{n-1-(\gamma-\frac{n-1}{\sigma}-\frac{1}{\ell_1}+s_2)},2}} \\
&\qquad \lesssim \||x|^{s-\delta} D_\omega^{(s_2+\gamma-\frac{n-1}{\sigma}-\frac{1}{\ell_1})/2} u\|_{L_x^2}^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L_{I_T}^{q_1} L_r^{\ell_1} L_\omega^\sigma} &\lesssim \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|_{L_{I_T}^{2q_1} L_x^2}^2 \\
&\lesssim T^\theta \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L_x^2}^2.
\end{aligned}$$

This completes (3.1) if $n \geq 3$.

(II) $(n = 2)$

Next, we assume $n = 2$ and establish (3.1). The strategy is almost the same as in the case of $n \geq 3$ above. We set

$$s_3 = \frac{\delta}{2}, \qquad s_4 = -\frac{3}{2}s + \frac{\delta}{2}.$$

We deduce from Lemma 2.2 that

$$\||x|^{-s-\delta} |\nabla|^s D_\omega^{s_3}[U(t)\varphi]\|_{L_t^q L_r^\ell L_\omega^{\sigma_0}} \lesssim \|\varphi\|_{L_x^2}.$$

By the similar argument as above, we get

$$\begin{aligned}
&\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta} U(t)\Phi_t\|_{L_{I_T}^\infty L_x^2} \lesssim \||x|^{s+\delta} D_\omega^{s_4} F(u)\|_{L_{I_T}^{q'} L_r^{\ell'} L_\omega^{\sigma_0'}} \\
&\quad \lesssim \||x|^{2\delta} D_\omega^{s_4}(|x|^{-\gamma} * |u|^2)\|_{L_{I_T}^{q_1} L_r^{\ell_1} L_\omega^\sigma} \||x|^{s-\delta} D_\omega^{-1+\frac{2}{\sigma_0}+\frac{1}{\sigma}} u\|_{L_{I_T}^{q_2} L_r^{\ell_2} L_\omega^{\sigma_0}}.
\end{aligned}$$

It follows from Lemma 2.3 that

$$\||x|^{2\delta} D_\omega^{s_4}(|x|^{-\gamma} * |u|^2)\|_{L_r^{\ell_1} L_\omega^\sigma} \lesssim \||x|^{-\gamma+\frac{2}{\ell_1}+2\delta} D_\omega^{s_4}(|u|^2)\|_{L_r^1 L_\omega^{\frac{1}{1-(\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})},1}}. \tag{3.8}$$

By Leibniz rule and Sobolev embedding in the Lorentz spaces on the unit sphere, we have

$$\||x|^{2(s-\delta)} D_\omega^{s_4}(|u|^2)\|_{L_r^1 L_\omega^{\frac{1}{1-(\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})},1}} \lesssim \||x|^{s-\delta} D_\omega^{(s_4+\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})/2} u\|_{L_x^2}^2.$$

Then we have

$$\begin{aligned}\||x|^{2\delta} D_\omega^{s_4}(|x|^{-\gamma} * |u|^2)\|_{L^{q_1}_{I_T} L^{\ell_1}_r L^{\sigma}_\omega} &\lesssim \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|^2_{L^{2q_1}_{I_T} L^2_x} \\ &\lesssim T^\theta \||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} u\|^2_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L^2_x}.\end{aligned}$$

This completes (3.1).

Remark 4. It should be noted that to get the estimate (3.8) above we need the condition $1/\ell_1 > \gamma - 1$. This causes the exception of $n = 2$ in the scaling critical ($s = s_c$) results, that is Theorems 3.1 and 3.2.

(III) Lastly, we prove

$$W_1(U(t)\Phi_t) + W_2(U(t)\Phi_t) \lesssim T^\theta [W_1(u)]^2 W_2(u), \tag{3.9}$$

which completes the lemma. Here we only consider the case for $n \geq 3$. The same method can be utilized for $n = 2$. Since

$$-\frac{n}{2q_1} + \frac{2+2s-\gamma}{4} n < s - \delta < -\frac{1}{2q_1} + \frac{2+2s-\gamma}{4},$$

we deduce from Lemma 2.1 (ii) that

$$\||x|^{s-\delta} |\nabla|^{-s} D_\omega^{\frac{\delta}{2}} [U(t)\varphi]\|_{L_t^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L^2_x} \lesssim \|\varphi\|_{L^2_x}. \tag{3.10}$$

Applying Lemma 2.4 to (3.2) and (3.10), we have

$$\||x|^{s-\delta} |\nabla|^{-s} D_\omega^{\frac{\delta}{2}} U(t)\Phi_t\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L^2_x} \lesssim \||x|^s |\nabla|^{-s} D_\omega^{-s_1} F(u)\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega},$$

which implies

$$\||x|^{s-\delta} D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta} U(t)\Phi_t\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}} L^2_x} \lesssim \||x|^s D_\omega^{s_2} F(u)\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega}.$$

As above, this estimate implies

$$W_1(U(t)\Phi_t) \lesssim T^\theta [W_1(u)]^2 W_2(u).$$

For W_2, since $2 \leq \sigma_0 \leq \ell_2 \leq \infty$,

$$\begin{gathered}\frac{n}{2}\left(\frac{1}{2} - \frac{1}{\sigma_0}\right) \leq \frac{1}{q_2} \leq \frac{1}{2} + \frac{1}{\ell_2} - \frac{1}{\sigma_0}, \\ \frac{n^2}{4} - \frac{n}{q_2} - \frac{n^2}{2\sigma_0} < s < \frac{n}{4} - \frac{1}{q_2} - \frac{n-1}{\ell_2} + \frac{n-2}{2\sigma_0},\end{gathered}$$

we deduce from Lemma 2.2 that

$$\| |x|^s |\nabla|^{-s} D_\omega^{\frac{\delta}{2}-\frac{\delta}{n}} [U(t)\varphi] \|_{L_t^{q_2} L_r^{\ell_2} L_\omega^{\sigma_0}} \lesssim \|\varphi\|_{L_x^2}. \tag{3.11}$$

Applying Lemma 2.4 to (3.2) and (3.11), we have

$$\| |x|^s D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta-\frac{\delta}{n}} U(t)\Phi_t \|_{L_{I_T}^{q_2} L_r^{\ell_2} L_\omega^{\sigma_0}} \lesssim \| |x|^s D_\omega^{s_2} F(u) \|_{L_{I_T}^{q'} L_r^{\ell'} L_\omega^{\sigma_0'}},$$

which gives

$$W_2(U(t)\Phi_t) \lesssim T^\theta [W_1(u)]^2 W_2(u).$$

This completes (3.10). □

Proof of Theorem 3.4. We prove the existence by Banach's fixed-point theorem. Fix a positive constant ρ and a positive time T, to be chosen later, and we define a complete metric space $(X_{\rho,T}, d_X)$ by

$$\begin{aligned} X_{\rho,T} =& \{u \in C([0,T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^n)); \|u\|_{L_{I_T}^\infty \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(u) + W_2(u) \le \rho\}, \\ d_X(u,v) =& \|u-v\|_{L_{I_T}^\infty \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(u-v) + W_2(u-v), \end{aligned}$$

and the mapping

$$\mathcal{N}_X(u) = U(t)(\varphi + \Phi_t) \qquad \text{on } X_{\rho,T}.$$

Our strategy is to prove that $\mathcal{N}_X$ is a contraction mapping on $X_{\rho,T}$ for sufficiently small T.

It follows from $\|U(t)\varphi\|_{L_t^\infty L_x^2} = \|\varphi\|_{L_x^2}$, (3.10) and (3.11) (if $n=2$, (3.10) and $\| |x|^{s-\delta} |\nabla|^{-s} D_\omega^{\frac{\delta}{2}} [U(t)\varphi] \|_{L_t^{q_2} L_r^{\ell_2} L_\omega^{\sigma_0}} \lesssim \|\varphi\|_{L_x^2}$) that there exists a positive constant C_1 such that

$$\|U(t)\varphi\|_{L_{I_T}^\infty \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(U(t)\varphi) + W_2(U(t)\varphi) \le C_1 \|\varphi\|_{\dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}}. \tag{3.12}$$

For $u \in X_{\rho,T}$, we deduce from Lemma 3.5 that there exists a positive constant C_2 such that

$$\begin{aligned} \|U(t)\Phi_t\|_{L_{I_T}^\infty \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(U(t)\Phi_t) + W_2(U(t)\Phi_t) &\le C_2 T^\theta [W_1(u)]^2 W_2(u) \\ &\le C_2 T^\theta \rho^3. \end{aligned} \tag{3.13}$$

For u, $v \in X_{\rho,T}$, we have

$$\begin{aligned} d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) =& \|U(t)(\Phi_t(u) - \Phi_t(v))\|_{L_{I_T}^\infty \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} \\ &+ W_1(U(t)(\Phi_t(u) - \Phi_t(v))) + W_2(U(t)(\Phi_t(u) - \Phi_t(v))). \end{aligned}$$

By the arguments similar to the proof of Lemma 3.5, we have

$$d_X(\mathcal{N}_X(u),\mathcal{N}_X(v)) \lesssim \||x|^s D_\omega^{s_2}(F(u)-F(v))\|_{L_{I_T}^{q'} L_r^{\ell'} L_\omega^{\sigma_0'}}.$$

It follows from the following equality

$$\begin{aligned} F(u)-F(v) =&\lambda(|x|^{-\gamma} * |u|^2)u - \lambda(|x|^{-\gamma} * |v|^2)v \\ =&\lambda(|x|^{-\gamma} * (u(\bar{u}-\bar{v}) + (u-v)\bar{v}))u + \lambda(|x|^{-\gamma} * |v|^2)(u-v), \end{aligned}$$

and the same estimates as in Lemma 3.5 that

$$\begin{aligned} &d_X(\mathcal{N}_X(u),\mathcal{N}_X(v)) \\ &\quad\lesssim T^\theta (W_1(u)+W_2(u)+W_1(v)+W_2(v))^2 (W_1(u-v)+W_2(u-v)). \end{aligned}$$

Then there exists a positive constant C_3 such that

$$d_X(\mathcal{N}_X(u),\mathcal{N}_X(v)) \le C_3 T^\theta \rho^2 d_X(u,v). \tag{3.14}$$

Now we define $C = \max(C_1, C_2, C_3)$ and choose ρ, T such that

$$C\|\varphi\|_{\dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} \le \frac{\rho}{2}, \quad CT^\theta \rho^2 \le \frac{1}{2}.$$

Then, from (3.12)-(3.14), $\mathcal{N}_X$ is a contraction mapping on $X_{\rho,T}$. □

§ 4. (PNLS)

In this section, we establish Theorem 1.5. We then consider the problem in the scaling critical homogeneous Sobolev space $\dot{H}^{s_{c,p}}(\mathbb{R}^n)$. Let us recall that $s_{c,p} = \frac{n}{2} - \frac{2}{p-1}$. For convenience, we restate Theorem 1.5 with the explicit exponents.

Theorem 4.1. *Let $3 \le n \le 14$, $p_0 < p < 1 + 4/n$ where p_0 is a unique solution of*

$$\begin{cases} 1 + \frac{4}{n+1} \le p_0 < 1 + \frac{4}{n}, \\ 2p_0^3 + 6(n-2)p_0^2 + (n^2-13n+10)p_0 - n(n-3) = 0, \end{cases}$$

and suppose that $\delta = \delta(n,p)$ is sufficiently small. Then there exists a positive constant $\varepsilon = \varepsilon(n,p)$ such that if $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^n)$ satisfies $\||\nabla|^{s_{c,p}} D_\omega^{s_0} \varphi\|_{L_x^2} < \varepsilon$ where

$$s_0 = \begin{cases} \frac{1}{p-1}(7-3p) + \delta & (\text{if } n = 3), \\ \frac{1}{2(p-1)^2}\big(-(n+1)p^2 + (n+7)p - 2\big) + \delta & (\text{if } n \ge 4), \end{cases}$$

then (1.3) *has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_{c,p}} H_\omega^{s_0}) \cap L^{pq'}(\mathbb{R}; |x|^\alpha L^{p\ell'} H_\omega^{s_0,\sigma})$$

where

$$\alpha = \begin{cases} \frac{7}{p} - \frac{4}{p-1} + \frac{2(p-1)}{7p}\delta & (\text{if } n = 3), \\ \frac{n}{2} - \frac{2}{p-1} + \frac{(n-4)(p-1)}{5p}\delta & (\text{if } n \geq 4), \end{cases} \quad \frac{1}{q} = \begin{cases} \frac{p-1}{7}\delta & (\text{if } n = 3), \\ 1 - \frac{p}{2} + \frac{2(p-1)}{5}\delta & (\text{if } n \geq 4), \end{cases}$$

$$\frac{1}{\ell} = \begin{cases} 2 - \frac{2}{p-1} & (\text{if } n = 3), \\ 1 - \frac{4}{n} + \frac{p}{2} + \frac{p}{n} - \frac{4}{n(p-1)} + \frac{p-1}{5}\delta - \frac{8(p-1)}{5n}\delta & (\text{if } n \geq 4), \end{cases}$$

$$\frac{1}{\sigma} = \begin{cases} -\frac{3}{2} + \frac{4}{p} + \frac{4(p-1)}{7p}\delta & (\text{if } n = 3), \\ \frac{1}{2n(p-1)}(-np - 2p + n + 10) + \frac{8(p-1)}{5np}\delta & (\text{if } n \geq 4). \end{cases}$$

Similarly to (HNLS) case, by using weighted Strichartz estimates, we establish the following crucial estimate.

Lemma 4.2. *Let* $3 \leq n \leq 14$, $p_\delta < p < 1 + 4/n$ *where* p_δ *satisfies the following:*

(if $n = 3$) $\quad p_\delta = 2 + (p_\delta - 1)\sqrt{\frac{3}{14}\delta},$

(if $n \geq 4$)

$$1 + \frac{4}{n+1} < p_\delta < 1 + \frac{4}{n},$$

$$2p_\delta^3 + 6(n-2)p_\delta^2 + (n^2 - 13n + 10)p_\delta - n(n-3) = \frac{2(p-1)^2}{5}(-3np_\delta + 8p_\delta + 8n - 8)\delta.$$

Then we have

$$\||\nabla|^{s_{c,p}} D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} + \||x|^\alpha D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \lesssim \||x|^\alpha D_\omega^{s_0} u\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p,$$

where the exponents q, ℓ, σ, s_0, α *are same as in Theorem* 4.1.

Remark 5. Since δ is sufficiently small, it is easy to see that the above p_δ exists and is unique.

Proof. (I) ($n = 3$)
First, we assume $n = 3$ and prove

$$\||\nabla|^{s_{c,p}} D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} \lesssim \||x|^\alpha D_\omega^{s_0} u\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p. \tag{4.1}$$

Let us set that

$$c_0 = -3 + \frac{4}{p-1} - \frac{2}{7}(p-1)\delta,$$
$$s_1 = \frac{p-1}{7}\delta, \quad s_2 = \frac{1}{p-1}(7-3p) + \frac{8-p}{7}\delta.$$

Note that $s_1 + s_2 = s_0$. Since $2 \le \ell \le q \le \infty$ and

$$-\frac{3}{q} < c_0 < 1 - \frac{1}{q} - \frac{2}{\ell},$$

we deduce from Lemma 2.2 that

$$\||x|^{c_0}|\nabla|^{s_{c,p}} D_\omega^{s_1}[U(t)\varphi]\|_{L_t^q L_r^\ell L_\omega^2} \lesssim \|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\|\int_{-\infty}^{\infty} U(-t')G(u)(t')dt'\|_{L_x^2} \lesssim \||x|^{-c_0}|\nabla|^{-s_{c,p}} D_\omega^{-s_1} G(u)\|_{L_t^{q'} L_r^{\ell'} L_\omega^2}. \tag{4.2}$$

By applying Lemma 2.4 to $\|U(t)\varphi\|_{L_t^\infty L_x^2} = \|\varphi\|_{L_x^2}$ and (4.2), we have

$$\||\nabla|^{s_{c,p}} D_\omega^{s_1+s_2} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} \lesssim \||x|^{-c_0} D_\omega^{s_2}(|u|^{p-1}u)\|_{L_t^{q'} L_r^{\ell'} L_\omega^2}.$$

Since $0 \le s_2 \le \min([p](=2), \frac{1}{p-1}(\frac{2p}{\sigma} - 1))$, where $[p]$ denotes the integral part of p, it follows from Moser type estimates and Sobolev embedding on the unit sphere that

$$\|D_\omega^{s_2}(|u|^{p-1}u)\|_{L_\omega^2} \lesssim \|D_\omega^{s_2} u\|_{L_\omega^{\sigma_0}}^p$$
$$\lesssim \|D_\omega^{s_0} u\|_{L_\omega^\sigma}^p.$$

Here we have used the exponent

$$\frac{1}{\sigma_0} = \frac{1}{2p}(1 + (p-1)s_2).$$

This gives

$$\||\nabla|^{s_{c,p}} D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} \lesssim \||x|^{-\frac{c_0}{p}} D_\omega^{s_0} u\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p,$$

which completes the proof of (4.1) if $n = 3$.

(II) ($n \ge 4$)

Next we assume $n \ge 4$ and obtain (4.1). Similarly to the $n = 3$ case, we set

$$c_1 = -\frac{n}{2}p + 2 + \frac{1}{p-1} - \frac{(n-4)(p-1)}{5}\delta,$$
$$s_1 = \frac{p}{n} - \frac{n}{2} + \frac{3}{2} - \frac{4}{n} + \frac{2}{p-1} - \frac{4}{n(p-1)} + \frac{3(p-1)}{5}\delta - \frac{8(p-1)}{5n}\delta,$$
$$s_2 = \frac{1}{p-1}\left(-\frac{p^2}{n} - 2p + \frac{5}{n}p - \frac{n}{2} + \frac{5}{2} + \frac{2}{p-1}\right) + \frac{8-3p}{5}\delta + \frac{8(p-1)}{5n}\delta.$$

Note that $s_1+s_2=s_0$. Since $2\le \ell\le q\le\infty$ and

$$-\frac{n}{q}<c_1<\frac{n-1}{2}-\frac{1}{q}-\frac{n-1}{\ell},$$

we deduce from Lemma 2.2 that

$$\||x|^{c_1}|\nabla|^{s_{c,p}}D_\omega^{s_1}[U(t)\varphi]\|_{L_t^qL_r^\ell L_\omega^2}\lesssim\|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\|\int_{-\infty}^{\infty}U(-t')G(u)(t')dt'\|_{L_x^2}\lesssim\||x|^{-c_1}|\nabla|^{-s_{c,p}}D_\omega^{-s_1}G(u)\|_{L_t^{q'}L_r^{\ell'}L_\omega^2},\tag{4.3}$$

which gives

$$\||\nabla|^{s_{c,p}}D_\omega^{s_1+s_2}U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2}\lesssim\||x|^{-c_1}D_\omega^{s_2}(|u|^{p-1}u)\|_{L_t^{q'}L_r^{\ell'}L_\omega^2}.$$

Since $0\le s_2\le\min\big([p](=1),\frac{n-1}{p-1}\big(\frac{p}{\sigma}-\frac{1}{2}\big)\big)$, it follows from Moser type estimates and Sobolev embedding on the unit sphere that

$$\|D_\omega^{s_2}(|u|^{p-1}u)\|_{L_\omega^2}\lesssim\|D_\omega^{s_0}u\|_{L_\omega^\sigma}^p,$$

which completes (4.1).

(III)

Lastly, we establish

$$\||x|^\alpha D_\omega^{s_0}U(t)\Phi_{t,p}\|_{L_t^{pq'}L_r^{p\ell'}L_\omega^\sigma}\lesssim\||x|^\alpha D_\omega^{s_0}u\|_{L_t^{pq'}L_r^{p\ell'}L_\omega^\sigma}^p.\tag{4.4}$$

To avoid redundancy, here we assume $n\ge4$. We can prove (4.4) in case of $n=3$ by the same way as below. Since $2\le\sigma\le p\ell'\le\infty$,

$$\begin{aligned}\frac{n}{2}\left(\frac{1}{2}-\frac{1}{\sigma}\right)&\le\frac{1}{pq'}\le\frac{1}{2}+\frac{1}{p\ell'}-\frac{1}{\sigma},\\ \frac{n^2}{4}-\frac{n}{pq'}-\frac{n^2}{2\sigma}&<\alpha<\frac{n}{4}-\frac{1}{pq'}-\frac{n-1}{p\ell'}+\frac{n-2}{2\sigma},\end{aligned}$$

we deduce from Lemma 2.2 that

$$\||x|^\alpha|\nabla|^{-s_{c,p}}U(t)\varphi\|_{L_t^{pq'}L_r^{p\ell'}L_\omega^\sigma}\lesssim\|\varphi\|_{L_x^2}.\tag{4.5}$$

By applying Lemma 2.4 to (4.3) and (4.5), we have

$$\||x|^\alpha D_\omega^{s_0}U(t)\Phi_{t,p}\|_{L_t^{pq'}L_r^{p\ell'}L_\omega^\sigma}\lesssim\||x|^{-c_1}D_\omega^{s_2}(|u|^{p-1}u)\|_{L_t^{q'}L_r^{\ell'}L_\omega^2}.$$

By the same argument as above, this completes (4.4). □

Proof of Theorem 4.1. Obviously, p_0 in Theorem 4.1 is less than p_δ in Lemma 4.2, and if $\delta = \delta(n,p)$ is sufficiently small then p_δ is sufficiently close to p_0. Thus it suffices to prove Theorem 4.1 for any p such that $p_\delta < p < 1 + 4/n$. Similarly to the (HNLS) case, we prove Theorem 4.1 by the contraction mapping theorem. Let the exponents s_0, s_1, α, c_0, c_1 be the same as in Lemma 4.2. Fix a positive constant ε, to be chosen later, and we define a complete metric space (X_ε, d_X) by

$$\begin{aligned} X_\varepsilon =& \{u \in C(\mathbb{R}; \dot H^{s_{c,p}} H^{s_0}_\omega); \|u\|_{L^\infty_t \dot H^{s_{c,p}} H^{s_0}_\omega} + \||x|^\alpha D^{s_0}_\omega u\|_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega} \le \varepsilon\}, \\ & d_X(u,v) = \|u-v\|_{L^\infty_t \dot H^{s_{c,p}}} + \||x|^\alpha (u-v)\|_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega}, \end{aligned}$$

and the mapping

$$\mathcal{N}_X(u) = U(t)(\varphi + \Phi_{t,p}) \qquad \text{on } X_\varepsilon.$$

We show that $\mathcal{N}_X$ is a contraction mapping on X_ε for sufficiently small ε. It follows from (4.5) and Lemma 4.2 that there exists a positive constant C such that

$$\begin{aligned} &\|U(t)\varphi\|_{L^\infty_t \dot H^{s_{c,p}} H^{s_0}_\omega} + \||x|^\alpha D^{s_0}_\omega U(t)\varphi\|_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega} \le C\|\varphi\|_{\dot H^{s_{c,p}} H^{s_0}_\omega}, \\ &\|U(t)\Phi_{t,p}\|_{L^\infty_t \dot H^{s_{c,p}} H^{s_0}_\omega} + \||x|^\alpha D^{s_0}_\omega U(t)\Phi_{t,p}\|_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega} \le \\ &\qquad\qquad\qquad\qquad\qquad\qquad C\||x|^\alpha D^{s_0}_\omega u\|^p_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega}. \end{aligned}$$

Next, we prove

$$\begin{aligned} &d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \\ &\quad \le (\||x|^\alpha D^{s_0}_\omega u\|^{p-1}_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega} + \||x|^\alpha D^{s_0}_\omega v\|^{p-1}_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega}) d_X(u,v) \end{aligned} \tag{4.6}$$

for any u, $v \in X_\varepsilon$.
Similarly to the proof of Lemma 4.2, we have

$$\begin{aligned} &d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \\ &\lesssim \||\nabla|^{s_{c,p}} (\int_0^t U(t-t')(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t'))dt')\|_{L^\infty_t L^2_x} \\ &\quad + \||x|^\alpha (\int_0^t U(t-t')(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t'))dt')\|_{L^{pq'}_t L^{p\ell'}_r L^\sigma_\omega} \\ &\lesssim \||x|^{p\alpha} D^{-s_1}_\omega (|u|^{p-1}u - |v|^{p-1}v)\|_{L^{q'}_t L^{\ell'}_r L^2_\omega}. \end{aligned}$$

Note that $p\alpha$ satisfies

$$p\alpha = \begin{cases} -c_0 & (\text{if } n = 3), \\ -c_1 & (\text{if } n \ge 4). \end{cases}$$

By Sobolev embedding on the unit sphere, we have

$$\||x|^{p\alpha} D^{-s_1}_\omega (|u|^{p-1}u - |v|^{p-1}v)\|_{L^{q'}_t L^{\ell'}_r L^2_\omega} \lesssim \||x|^{p\alpha} (|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{q'}_t L^{\ell'}_r L^{\sigma_0}_\omega}$$

where $\frac{1}{\sigma_0} = \frac{1}{2} + \frac{s_1}{n-1}$. By Holder's inequality with

$$\frac{1}{\sigma_0} = \left(\frac{1}{2} + \frac{s_1}{n-1} - \frac{1}{\sigma}\right) + \frac{1}{\sigma},$$

we have

$$\begin{aligned}
&\||x|^{p\alpha}(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{\sigma_0}_\omega} \\
&\quad = \||x|^{(p-1)\alpha}(|u|^{p-1} + |v|^{p-1})|x|^{\alpha}(u-v)\|_{L^{\sigma_0}_\omega} \\
&\quad \lesssim \||x|^{(p-1)\alpha}(|u|^{p-1} + |v|^{p-1})\|_{L^{\sigma_1}_\omega} \||x|^{\alpha}(u-v)\|_{L^{\sigma}_\omega} \\
&\quad \lesssim (\||x|^{\alpha}u\|^{p-1}_{L^{\sigma_1(p-1)}_\omega} + \||x|^{\alpha}v\|^{p-1}_{L^{\sigma_1(p-1)}_\omega})\||x|^{\alpha}(u-v)\|_{L^{\sigma}_\omega},
\end{aligned}$$

where

$$\frac{1}{\sigma_1} = \frac{1}{2} + \frac{s_1}{n-1} - \frac{1}{\sigma}.$$

Since

$$-\frac{n-1}{\sigma_1(p-1)} = s_0 - \frac{n-1}{\sigma},$$

Sobolev embedding on the unit sphere gives

$$\||x|^{\alpha}u\|_{L^{\sigma_1(p-1)}_\omega} \lesssim \||x|^{\alpha}D^{s_0}_\omega u\|_{L^{\sigma}_\omega},$$

which completes (4.6).

From (4.6), there exists a positive constant C' such that

$$d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \le C'\varepsilon^{p-1} d_X(u,v).$$

Now we choose ε and an initial data φ such that

$$\max(C, C')\varepsilon^{p-1} \le \frac{1}{2}, \qquad C\|\varphi\|_{\dot{H}^{s_{c,p}}H^{s_0}_\omega} \le \frac{\varepsilon}{2},$$

then the functional $\mathcal{N}_X$ becomes a contraction mapping on X_ε. □

§ 5. Appendix

We consider the Cauchy problem of nonlinear Schrödinger equations with inhomogeneous nonlinearities:

$$\begin{cases} iu_t(t,x) + \Delta u(t,x) = w(x)|u(t,x)|^{p-1}u(t,x), & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{5.1}$$

Here we assume that $|w(x)| \lesssim |x|^{-a}$. Note that if $|w(x)| = |x|^{-a}$, the scale critical index for (5.1) is

$$s_{c,a} = \frac{n}{2} - \frac{2-a}{p-1}.$$

We prove that there exists a solution of (5.1) for $s_{c,a} < 0$ and $\varphi \in \dot{H}^{s_{c,a}}(\mathbb{R}^n)$. In [4], the small data global well-posedness was established for each a and p if an initial data φ is radially symmetric or under some angular regularity assumption. The following theorem shows that we can get the small data global well-posedness without angular conditions if the exponent a is positive.

Theorem 5.1. *Let $n \geq 3$, $0 < a < 2$ and*

$$\begin{cases} p_0 < p < 1 + \frac{4-2a}{n} & (\text{If } 0 < a < 1 + \frac{2n-1}{n^2-4}), \\ 1 + \frac{4-2a}{n+1} < p < 1 + \frac{4-2a}{n} & (\text{If } 1 + \frac{2n-1}{n^2-4} \leq a < 2), \end{cases}$$

where $p_0 \in (1, 1 + \frac{4-2a}{n})$ satisfies

$$n(n-2){p_0}^2 - 2(n - 4 - 2an + 4a)p_0 - n^2 - 4n + 4a = 0.$$

Then there exists a positive constant $\varepsilon = \varepsilon(n, p, a)$ such that if $\varphi \in \dot{H}^{s_{c,a}}$ satisfies $\||\nabla|^{s_{c,a}}\varphi\|_{L^2_x} < \varepsilon$, then the integral equation

$$u(t, x) = U(t)(\varphi + \Phi_{t,a})(x), \tag{5.2}$$

where

$$\Phi_{t,a} = \Phi_{t,a}(u) = -i \int_0^t U(-t')(w(x)|u(t')|^{p-1}u(t'))dt',$$

has a unique solution

$$u \in C(\mathbb{R}; \dot{H}^{s_{c,a}}) \cap L^{pq'}(\mathbb{R}; |x|^{-\frac{1}{p}(n+a-\frac{2-a}{p-1}-\frac{n+2}{q})}L^{pq'}).$$

Here q satisfies the condition in Lemma 5.3 below.

Remark 6.
(i) It should be noted that if a is sufficiently small then p_0 is sufficiently close to $1+\frac{4-2a}{n}$.
(ii) If we try to estimate the nonlinearity $w(x)|u|^{p-1}u$ with $a < 0$, loss of regularity on the sphere arises and we need some angular regularity condition for φ to get the well-posedness. Precisely, the estimate (5.3) in Corollary 5.2 below for positive values of d does not hold. Therefore, we assume $a > 0$ in Theorem 5.1.

Since we do not have to mind an angular condition, the proof of Theorem 5.1 is simple relatively. First we restate Lemma 2.2 with $q = \ell = \sigma$ for convenience.

Corollary 5.2. *Let $n \geq 2$ and*

$$\frac{n}{2(n+2)} \leq \frac{1}{q} \leq \frac{1}{2}, \quad \frac{n^2}{4} - \frac{n^2+2n}{2q} < d < \frac{n}{4} - \frac{n+2}{2q}.$$

Then we have

$$\||x|^d|\nabla|^{d-\frac{n}{2}+\frac{n+2}{q}}[U(t)\varphi]\|_{L^q_t L^q_x} \lesssim \|\varphi\|_{L^2_x}. \tag{5.3}$$

The following lemma can be established by simple calculation. We omit the details.

Lemma 5.3. *Let $n \geq 3$, $0 < a < 2$ and*

$$\begin{cases} p_0 < p < 1 + \frac{4-2a}{n} & (\text{If } 0 < a < 1 + \frac{2n-1}{n^2-4}), \\ 1 + \frac{4-2a}{n+1} < p < 1 + \frac{4-2a}{n} & (\text{If } 1 + \frac{2n-1}{n^2-4} \leq a < 2). \end{cases} \tag{5.4}$$

Then there exists q such that

$$\max\left(\frac{n}{2(n+2)}, 1 - \frac{p}{2}\right) \leq \frac{1}{q} \leq \min\left(\frac{1}{2}, 1 - \frac{n}{2(n+2)}p\right),$$
$$\frac{2}{n^2-4}\left(\frac{n^2}{4} - n + \frac{2-a}{p-1}\right) < \frac{1}{q} < \frac{2}{n+2}\left(\frac{n}{4}p + \frac{n-2}{2} + a - \frac{2-a}{p-1}\right).$$

Lemma 5.4. *Let $n \geq 3$, $0 < a < 2$. Suppose that p and q satisfy the condition (5.4) and the conditions in Lemma 5.3, respectively. Then we have*

$$\| |\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} + \| |x|^{-\frac{a+c}{p}} U(t)\Phi_{t,a}\|_{L_t^{pq'} L_x^{pq'}} \lesssim \| |x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}},$$

where $c = n - \frac{2-a}{p-1} - \frac{n+2}{q}$.

Proof. (I) First, we prove

$$\| |\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} \lesssim \| |x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}}. \tag{5.5}$$

If $\frac{2}{n^2-4}\left(\frac{n^2}{4} - n + \frac{2-a}{p-1}\right) < \frac{1}{q}$, the following inequality

$$\frac{n^2}{4} - \frac{n^2+2n}{2q} < c < \frac{n}{4} - \frac{n+2}{2q}$$

holds. Then we deduce from Corollary 5.2 that

$$\| |x|^c |\nabla|^{s_{c,a}} U(t)\varphi\|_{L_t^q L_x^q} \lesssim \|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\left\| \int_{-\infty}^{\infty} U(-t')(w(x)|u(t')|^{p-1}u(t'))dt' \right\|_{L_x^2} \lesssim \| |x|^{-c} |\nabla|^{-s_{c,a}} (w(x)|u|^{p-1}u)\|_{L_t^{q'} L_x^{q'}}, \tag{5.6}$$

which means

$$\begin{aligned} \| |\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} &\lesssim \| |x|^{-(a+c)} |u|^{p-1} u\|_{L_t^{q'} L_x^{q'}} \\ &= \| |x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}}. \end{aligned}$$

This completes the proof of (5.5).
(II) Next, we prove

$$\||x|^{-\frac{a+c}{p}}U(t)\Phi_{t,a}\|_{L_t^{pq'}L_x^{pq'}} \lesssim \||x|^{-\frac{a+c}{p}}u\|^p_{L_t^{pq'}L_x^{pq'}}.$$

From the inequalities $1-\frac{p}{2} \le \frac{1}{q} \le 1-\frac{n}{2(n+2)}p$ and

$$\frac{1}{q} < \frac{2}{n+2}(\frac{n}{4}p + \frac{n-2}{2} + a - \frac{2-a}{p-1}),$$

we have $\frac{n}{2(n+2)} \le \frac{1}{pq'} \le \frac{1}{2}$ and

$$\frac{n^2}{4} - \frac{n^2+2n}{2pq'} < -\frac{a+c}{p} < \frac{n}{4} - \frac{n+2}{2pq'}.$$

Then we deduce from Corollary 5.2 that

$$\||x|^{-\frac{a+c}{p}}|\nabla|^{-s_{c,a}}U(t)\varphi\|_{L_t^{pq'}L_x^{pq'}} \lesssim \|\varphi\|_{L_x^2}. \tag{5.7}$$

Applying Lemma 2.4 to (5.6) and (5.7), we have

$$\begin{aligned}\||x|^{-\frac{a+c}{p}}U(t)\Phi_{t,a}\|_{L_t^{pq'}L_x^{pq'}} &\lesssim \||x|^{-c}w(x)|u|^{p-1}u\|_{L_t^{q'}L_x^{q'}}\\ &\lesssim \||x|^{-\frac{a+c}{p}}u\|^p_{L_t^{pq'}L_x^{pq'}}.\end{aligned}$$

This completes the proof. □

From Lemmas 5.3 and 5.4, Theorem 5.1 is established with the contraction mapping argument. The way of the proof is the same as in that of Theorems 3.4 and 4.1. We leave the details to the readers.

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