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Two-weight Morrey norm inequality and the sequential testing

By

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Abstract

Two-weight Morrey norm inequalities for the Hardy-Littlewood maximal operators are characterized in terms of the sequential testing.

§ 1. Introduction

The purpose of this paper is to develop a theory of weights for the Hardy-Littlewood maximal operator on Morrey spaces. Our key tool is the sequential testing, which was introduced by Hänninen, Hytönen and Li in [3].

Morrey spaces, which were introduced by C. Morrey in order to study regularity questions which appear in the Calculus of Variations, describe local regularity more precisely than Lebesgue spaces and widely are used not only in harmonic analysis but also in partial differential equations (cf. [1]).

We shall consider all cubes in $\mathbb{R}^n$ which have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all such cubes. Let $0 < p < \infty$ and $0 < \lambda < n$ be two real parameters. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|^{\lambda/n}} \int_Q |f(x)|^p \, dx \right)^{1/p},$$

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where $|Q|$ denotes the volume of the cube Q. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the subset of all L^p locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$ is finite. It is easy see that $\|\cdot\|_{L^{p,\lambda}(\mathbb{R}^n)}$ is the norm if $p \geq 1$ and is the quasi norm if $p \in (0,1)$. The completeness of Morrey spaces follows easily by that of Lebesgue spaces.

Let f be a locally integrable function on $\mathbb{R}^n$. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{Q \in \mathcal{Q}} \fint_Q |f(y)|\, dy\, \mathbf{1}_Q(x),$$

where $\fint_Q f(x)\, dx$ stands for the usual integral average of f over the cube Q and $\mathbf{1}_Q$ denotes the characteristic function of the cube Q.

By weights we will always mean non-negative, locally integrable functions which are positive on a set of positive measure. Given a measurable set E and a weight w, $w(E) := \int_E w(x)\, dx$.

Let $0 < p < \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p}.$$

Let $0 < p < \infty$, $0 < \lambda < n$ and w be a weight. We define the weighted Morrey space $L^{p,\lambda}(w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^{p,\lambda}(w)} := \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|^{\lambda/n}} \int_Q |f(x)|^p w(x)\, dx \right)^{1/p}.$$

For the Hardy-Littlewood maximal operator M and $p > 1$, B. Muckenhoupt [9] showed that the weighted inequality

$$\|Mf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

holds if and only if

$$[w]_{A_p} := \sup_{Q \in \mathcal{Q}} \frac{w(Q)}{|Q|} \left(\fint_Q w(x)^{-1/(p-1)}\, dx \right)^{p-1} < \infty.$$

For $p > 1$ one says that a weight w on $\mathbb{R}^n$ belongs to the Muckenhoupt class A_p whenever $[w]_{A_p} < \infty$.

A description of all the admissible weights similar to the Muckenhoupt class A_p is an open problem for the weighted Morrey space $L^{p,\lambda}(w)$ (see [11]). In [2], we gave the following partial answer to the problem.

Proposition 1.1 ([2, Theorem 2.1]). *Let $1 < p < \infty$, $0 < \lambda < n$ and w be a weight. Then, for every cube $Q \in \mathcal{Q}$, the weighted inequality*

$$\left(\frac{1}{|Q|^{\lambda/n}} \int_Q Mf(x)^p w(x)\,dx \right)^{1/p} \le C \sup_{\substack{Q' \in \mathcal{Q} \\ Q' \supset Q}} \left(\frac{1}{|Q'|^{\lambda/n}} \int_{Q'} |f(x)|^p w(x)\,dx \right)^{1/p}$$

holds if and only if

$$\sup_{\substack{Q, Q' \in \mathcal{Q} \\ Q \subset Q'}} \frac{w(Q)}{|Q|^{\lambda/n}} \frac{|Q'|^{\lambda/n}}{|Q'|} \left(\fint_{Q'} w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty.$$

This proposition says that the weighted inequality

$$\|Mf\|_{L^{p,\lambda}(w)} \le C \|f\|_{L^{p,\lambda}(w)} \tag{1.1}$$

holds if

$$\sup_{Q \in \mathcal{Q}} \|w \mathbf{1}_Q\|_{L^{1,\lambda}(\mathbb{R}^n)} \frac{|Q|^{\lambda/n}}{|Q|} \left(\fint_Q w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty. \tag{1.2}$$

One sees that the power weight $w = |\cdot|^\alpha$ belongs to the Muckenhoupt class A_p if and only if $-n < \alpha < (p-1)n$. Meanwhile, the power weight $w = |\cdot|^\alpha$ satisfies (1.2) if and only if $\lambda - n \le \alpha < (p-1)n$.

Let H be the Hilbert transform defined by

$$Hf(x) := \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathbf{1}_{(\varepsilon,\infty)}(|x-y|)}{x-y} f(y)\,dy.$$

For $1 < p < \infty$ and $0 < \lambda < 1$, N. Samko [10] showed that the weighted inequality

$$\|Hf\|_{L^{p,\lambda}(w)} \le C \|f\|_{L^{p,\lambda}(w)}, \quad w = |\cdot|^\alpha,$$

holds if and only if $\lambda - 1 \le \alpha < \lambda + (p-1)$. Thus, our sufficient condition (1.2) seems to be quite strong. In [14], the author introduced another sufficient condition and another necessary condition for which (1.1) to hold. The conditions justify the power weight $w = |\cdot|^\alpha$ fulfills (1.1) if and only if $\lambda - n \le \alpha < \lambda + (p-1)n$.

Let ω and σ be locally finite Borel measures. Let $K\colon \mathcal{Q} \to (0,\infty)$ be an appropriate map. The maximal operator M^* is defined by

$$M^* f(x) := \sup_{Q \in \mathcal{Q}} K(Q) \int_Q |f|\,d\sigma\, \mathbf{1}_Q(x).$$

For $1 < p \le q < \infty$, Eric T. Sawyer [12] essentially showed that the weighted inequality

$$\|M^* f\|_{L^q(\omega)} \le C \|f\|_{L^p(\sigma)} \tag{1.3}$$

holds if and only if

$$\left(\int_Q (M^*\mathbf{1}_Q)^q \, d\omega\right)^{1/q} \le C\sigma(Q)^{1/p} < \infty,$$

holds for every cube $Q \in \mathcal{Q}$. This type checking condition is called "the Sawyer testing condition" which appears quite basically many branchs of harmonic analysis.

We notice that, if $\omega = u dx$, $\sigma = v^{-1/(p-1)} dx$, $K(Q) = |Q|^{-1}$ and $g = |f| v^{-1/(p-1)}$, then (1.3) implies two-weight inequality

$$\|Mg\|_{L^q(u)} \le C\|g\|_{L^p(v)}.$$

In [3], introducing a new sequential testing characterization, Hänninen, Hytönen and Li extend Sawyer's [12, Theorem A] in the case $q \ge p$ to the case $1 \le q < p$.

In this paper we extend further Sawyer's two-weight theory to Morrey spaces and give a characterization of two-weight Morrey norm inequalities for the (general) Hardy-Littlewood maximal operators in terms of the sequential testing characterization.

The remainder of this paper is organized as follows: Main results can be found in Section 3 (Theorems 3.1 and 3.2). In Section 2 we introduce a description of the Köthe dual of Morrey spaces (Proposition 2.2) and discuss the basic facts of dyadic systems. Finally, in Section 4 we apply the description of the Köthe dual of Morrey spaces to our sequential testing characterization (Theorem 4.2).

Throughout this paper all the notations are standard or will be defined as needed. The letter C will be used for constants that may change from one occurrence to another.

§ 2. Preliminaries

In what follows we introduce some basic facts.

§ 2.1. Köthe duals of Morrey spaces

In this section we shall verify the dual equations of Morrey spaces (see [8] for details).

Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L^0(\mu)$ denote the space of all equivalence classes of real-valued measurable functions on Ω with the topology of convergence in measure on μ-finite sets. A quasi-Banach (function) lattice X on (Ω, Σ, μ) is a subspace of $L^0(\mu)$, which is complete with respect to a quasi-norm $\|\cdot\|_X$ and which has the property: whenever $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ μ-a.e., $f \in X$ and $\|f\|_X \le \|g\|_X$. Moreover, we will assume that there exists $u \in X$ with $u > 0$ μ-a.e.

A quasi-Banach lattice X is said to have the *Fatou property* whenever $0 \le f_n \uparrow f$ μ-a.e., $f_n \in X$, and $\sup_{n \ge 1} \|f_n\|_X < \infty$ imply that $f \in X$ and $\|f_n\|_X \to \|f\|_X$.

The Köthe dual space X' of a quasi-Banach lattice X on (Ω, Σ, μ) is defined as the space of all $f \in L^0(\mu)$ such that $\int_\Omega |fg|\, d\mu < \infty$ for every $g \in X$. It is a Banach lattice on (Ω, Σ, μ) when equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_\Omega |fg|\, d\mu.$$

Notice that a Banach lattice X has the Fatou property if and only if $X = X'' := (X')'$ with equality of norms (see, e.g., [6, p. 30]).

Let $L^0_+(\mu)$ be a cone of all non-negative μ-measurable functions on Ω. Fix a countable subset $\mathcal{B} = \{b_j\}$, $j \in \mathbb{N}$, of $L^0_+(\mu)$. For $1 \leq p < \infty$, we denote by $L^{p,\mathcal{B}}(\mu)$ the Morrey type space of all $f \in L^0(\mu)$ supported in $\bigcup_j \operatorname{supp} b_j$ and equipped with the norm given by

$$\|f\|_{L^{p,\mathcal{B}}(\mu)} := \sup_{j\in\mathbb{N}} \left(\int_\Omega |f|^p b_j\, d\mu \right)^{1/p}.$$

To describe the Köthe dual space of the Morrey type space $L^{p,\mathcal{B}}(\mu)$, we need the following definition.

Definition 2.1. We define the class $\overline{\mathcal{B}} \subset L^0_+(\mu)$ associated with $\mathcal{B} = \{b_j\}$ by the minimal set (with respect to inclusion) that satisfies the following conditions:

(i) $\{b_j\} \subset \overline{\mathcal{B}} \subset L^0_+(\mu)$;

(ii) If $\{w_j\} \subset \overline{\mathcal{B}}$, then, for any non-negative sequence $\{c_j\}$ with $\|\{c_j\}\|_{\ell^1(\mathbb{N})} \leq 1$, one has $\sum_j c_j w_j \in \overline{\mathcal{B}}$;

(iii) For all $w \in \overline{\mathcal{B}}$,
$$\sup_{\|f\|_{L^{1,\mathcal{B}}(\mu)} \leq 1} \int_\Omega |fw|\, d\mu \leq 1;$$

(iv) (the Komlós property) If $\{w_j\} \subset \overline{\mathcal{B}}$, then there exists a subsequence $\{v_j\}$ of $\{w_j\}$ such that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} v_j = w \quad \mu\text{-a.e. and} \quad w \in \overline{\mathcal{B}}.$$

Remark. The Komlós theorem (see [7, Theorem 1a]) states: *If (Ω, Σ, μ) is a measure space, then for every bounded sequence $\{f_n\}$ in $L^1(\mu)$ there are $f \in L^1(\mu)$ and a subsequence $\{g_n\}$ of $\{f_n\}$ such that the sequence of arithmetic means $\left\{ \frac{1}{n} \sum_{k=1}^n g_k \right\}_n$ converges to f μ-a.e. Moreover, the conclusion remains true for every subsequence of $\{g_n\}$.*

Since for every Banach lattice X on (Ω, Σ, μ) the inclusion $X \hookrightarrow X''$ has a norm less than or equal to one, it follows that $\int_\Omega |x| w\, d\mu \leq \|x\|_X$, for any $x \in X$ and $w \in X'$

with $\|w\|_{X'} \le 1$. So, $X \hookrightarrow L^1(\nu)$ with $d\nu = w\,d\mu$. Here, we a priori assume that there exists $w > 0$ μ-a.e. such that $w \in X'$ and $\|w\|_{X'} \le 1$, which is a consequence of the fact that there exists $u > 0$ μ-a.e. such that $u \in X$.

This simple observation allows us to apply the Komlós theorem for any bounded sequence in X.

To discuss the Köthe duality for Morrey type spaces, we define $H^{p,\mathcal{B}}(\mu)$ by the space of all $f \in L^0(\mu)$ such that

$$\|f\|_{H^{p,\mathcal{B}}(\mu)} := \inf_{w \in \overline{\mathcal{B}}} \left(\int_\Omega |f|^p w^{1-p}\,d\mu \right)^{1/p} < \infty, \quad \text{for} \quad 1 < p < \infty,$$

and

$$\|f\|_{H^{\infty,\mathcal{B}}(\mu)} := \inf_{w \in \overline{\mathcal{B}}} \|f w^{-1}\|_{L^\infty(\mu)} < \infty, \quad \text{for} \quad p = \infty.$$

Given $1 \le p < \infty$, p' such that $1/p + 1/p' = 1$ will denote the conjugate exponent number of p. The following proposition gives a description of the Köthe dual of Morrey type spaces.

Proposition 2.2. *Let* $1 < p \le \infty$.

(I) *Suppose that* $\overline{\mathcal{B}}$ *fulfills the condition* (ii). *Then* $H^{p,\mathcal{B}}(\mu)$ *is a Banach space;*

(II) *Suppose that* $\overline{\mathcal{B}}$ *fulfills the condition* (iv). *Then* $H^{p,\mathcal{B}}(\mu)$ *has the Fatou property;*

(III) *Suppose that* $\overline{\mathcal{B}}$ *fulfills the conditions* (i) *and* (iii). *Then the following Köthe duality formulas hold with equality of norms*:

$$H^{p,\mathcal{B}}(\mu)' = L^{p',\mathcal{B}}(\mu);$$

(IV) *Suppose that* $\overline{\mathcal{B}}$ *fulfills the conditions* (i)–(iv). *Then the following Köthe duality formulas hold with equality of norms*:

$$H^{p,\mathcal{B}}(\mu) = L^{p',\mathcal{B}}(\mu)'.$$

§ 2.2. Dyadic systems

Let $\mathcal{D}$ be a countable collection of measurable subsets of $\mathbb{R}^n$ with the following property:

$$\forall P, R \in \mathcal{D};\; P \cap R \in \{P, R, \emptyset\}. \tag{2.1}$$

We will refer to the elements of $\mathcal{D}$ as the dyadic cubes. The main example to keep in mind is the standard dyadic cubes given by

$$\mathcal{D}^0 := \{2^{-k}([0,1)^n + m);\ k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

The dyadic maximal operator $M_{\mathcal{D}}^{\sigma}$ adapted to a dyadic grid $\mathcal{D}$ and a locally finite Borel measure σ is defined by

$$M_{\mathcal{D}}^{\sigma} f(x) := \sup_{Q \in \mathcal{D}} \fint_Q |f| \, d\sigma \, \mathbf{1}_Q(x).$$

Lemma 2.3 ([4]). *Let $1 < p < \infty$. Let σ be a locally finite Borel measure. Then*

$$\|M_{\mathcal{D}}^{\sigma} f\|_{L^p(\sigma)} \le p' \|f\|_{L^p(\sigma)}.$$

For $\alpha \in \{0, \frac{1}{3}\}^n$, we define the 2^n dyadic systems by

$$\mathcal{D}^{\alpha} := \{2^{-k}([0,1)^n + m + \alpha);\ k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

It is not difficult to verify that each of these satisfies the dyadic property (2.1).

Proposition 2.4 ([4]). *If $Q \in \mathcal{Q}$, then there exist $\alpha \in \{0, \frac{1}{3}\}^n$ and $Q' \in \mathcal{D}^{\alpha}$ such that $Q \subset Q'$ and $\ell(Q') \le 6\ell(Q)$. Here, $\ell(Q)$ denotes the sides length of the cube Q.*

Let

$$M^{\alpha} f(x) := \sup_{Q \in \mathcal{D}^{\alpha}} \fint_Q |f| \, dy \, \mathbf{1}_Q(x), \quad \alpha \in \{0, \frac{1}{3}\}^n,$$

be the dyadic maximal function related to $\mathcal{D}^{\alpha}$. The following corollary links our analysis (real world) to dyadic analysis (dyadic world) for the Hardy-Littlewood maximal operator.

Corollary 2.5 ([4]). *We have the pointwise estimate for the Hardy-Littlewood maximal operator*:

$$\max_{\alpha \in \{0, \frac{1}{3}\}^n} M^{\alpha} f(x) \le M f(x) \le 6^n \max_{\alpha \in \{0, \frac{1}{3}\}^n} M^{\alpha} f(x).$$

Fix a collection $\mathcal{D}$ of dyadic cubes. For a family $\mathcal{F} \subset \mathcal{D}$ and $F \in \mathcal{F}$, we denote

$$ch_{\mathcal{F}}(F) := \{ \text{maximal}\, F' \subsetneq F;\ F' \in \mathcal{F}\}, \qquad E_{\mathcal{F}}(F) := F \setminus \bigcup_{F' \in ch_{\mathcal{F}}(F)} F'.$$

The sets $E_{\mathcal{F}}(F)$ are pairwise disjoint.

Let σ be a locally finite Borel measure. We say that $\mathcal{F}$ is σ-sparse if

$$\sigma(E_{\mathcal{F}}(F)) \geq \frac{\sigma(F)}{2},$$

or equivalently

$$\sum_{F' \in ch_{\mathcal{F}}(F)} \sigma(F') \leq \frac{\sigma(F)}{2}.$$

For $Q \in \mathcal{D}$, we denote

$$\pi_{\mathcal{F}}(Q) := \min\{F \supset Q;\ F \in \mathcal{F}\}.$$

§ 3. Two-weight estimates for linearized maximal operators

In this section we shall investigate two-weight estimates of maximal operators in (general) Morrey spaces.

§ 3.1. Linear case

Let σ be a locally finite Borel measure and fix a collection $\mathcal{D}$ of dyadic cubes. Let $\mathcal{F} \subset \mathcal{D}$ be a σ-sparse family. Let $0 < p < \infty$ and $\lambda\colon \mathcal{D} \to (0, \infty)$ be a map. For σ-measurable function f, define

$$\|f\|_{L^{p,\lambda}(\sigma)} := \sup_{Q \in \mathcal{D}} \left(\frac{1}{\lambda(Q)} \int_Q |f|^p \, d\sigma \right)^{1/p}.$$

The (generalized) Morrey space $L^{p,\lambda}(\sigma)$ is defined to be the subset of all L^p locally σ-integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{L^{p,\lambda}(\sigma)}$ is finite. We always assume that there exists $u \in L^{p,\lambda}(\sigma)$ with $u > 0$ σ-a.e. for the restriction of a map λ.

For the sequence $\{\alpha_F\}_{F \in \mathcal{F}}$, define

$$\|\{\alpha_F\}\|_{\ell^{p,\lambda}(\mathcal{F})} := \sup_{S \in \mathcal{D}} \left(\frac{1}{\lambda(S)} \sum_{\substack{F \in \mathcal{F} \\ F \subset S}} |\alpha_F|^p \right)^{1/p}.$$

The (sequence) Morrey space $\ell^{p,\lambda}(\mathcal{F})$ is defined to be the subset of all sequences $\{\alpha_F\}_{F \in \mathcal{F}}$ for which $\|\{\alpha_F\}\|_{\ell^{p,\lambda}(\mathcal{F})}$ is finite.

In what follows we assume that all functions are non-negative.

Let $K\colon \mathcal{D} \to (0, \infty)$ be another map. The maximal operator M^* is defined by

$$M^* f(x) := \sup_{Q \in \mathcal{D}} K(Q) \int_Q f \, d\sigma \, \mathbf{1}_Q(x).$$

Let $\mathcal{E} := \{E(Q) \subset Q;\ Q \in \mathcal{D}\}$ be a collection of pairwise disjoint sets. The linearized maximal operator $M_{\mathcal{E}}$ and its localized version $M_{\mathcal{E},R}$, $R \in \mathcal{D}$, are defined by

$$M_{\mathcal{E}} f(x) := \sum_{Q \in \mathcal{D}} K(Q) \int_Q f \, d\sigma \, \mathbf{1}_{E(Q)}(x)$$

and

$$M_{\mathcal{E},R} f(x) := \sum_{\substack{Q \in \mathcal{D} \\ Q \subset R}} K(Q) \int_Q f \, d\sigma \, \mathbf{1}_{E(Q)}(x).$$

We notice that, if the collection $\mathcal{D}$ is finite, for each function f there exists a collection $\mathcal{E}$ of pairwise disjoint sets $E(Q) \subset Q$ such that $M^* f(x) = M_{\mathcal{E}} f(x)$. For example, we can choose

$$E(Q) = \left\{ x \in Q;\ M^* f(x) = K(Q) \int_Q f \, d\sigma \right\} \setminus \bigcup_{\substack{Q' \in \mathcal{D} \\ Q' \supsetneq Q}} \left\{ x \in Q';\ M^* f(x) = K(Q') \int_{Q'} f \, d\sigma \right\}.$$

This implies

$$\|M^*\|_{L^{p,\lambda}(\sigma) \to L^q(\omega)} = \sup_{\mathcal{E}} \|M_{\mathcal{E}}\|_{L^{p,\lambda}(\sigma) \to L^q(\omega)},$$

where the supremum is taken over all collections $\mathcal{E}$ of pairwise disjoint sets $E(Q) \subset Q$.

Let $\mathcal{F} \subset \mathcal{D}$ be a σ-sparse family. The (small) linearized maximal operator $M_{\mathcal{E},\mathcal{F};R}$, $R \in \mathcal{F}$, is defined by

$$M_{\mathcal{E},\mathcal{F};R} f(x) := \sum_{\substack{Q \in \mathcal{D} \\ \pi_{\mathcal{F}}(Q) = R}} K(Q) \int_Q f \, d\sigma \, \mathbf{1}_{E(Q)}(x).$$

The following is our first theorem.

Theorem 3.1. *Let ω and σ be the locally finite Borel measures. Let $0 < q < \infty$, $1 < p < \infty$ and $\lambda \colon \mathcal{D} \to (0, \infty)$ be a map. Consider the following two statements*:

(a) *There exists a constant $c_1 > 0$ such that*

$$\|M_{\mathcal{E}} f\|_{L^q(\omega)} \le c_1 \|f\|_{L^{p,\lambda}(\sigma)}$$

holds for every function $f \in L^{p,\lambda}(\sigma)$;

(b) *There exists a constant $c_2 > 0$ such that*

$$\left(\sum_{F \in \mathcal{F}} \left(\alpha_F \frac{\|M_{\mathcal{E},\mathcal{F};F} \mathbf{1}_F\|_{L^q(\omega)}}{\sigma(F)^{1/p}} \right)^q \right)^{1/q} \le c_2$$

holds for any σ-sparse family $\mathcal{F} \subset \mathcal{D}$, where the nonnegative sequence $\{\alpha_F\}_{F\in\mathcal{F}}$ satisfies

$$\|\{\alpha_F\}\|_{\ell^{p,\lambda}(\mathcal{F})} \le 1. \tag{3.1}$$

Then, for the least possible constants c_1 and c_2,

(I) (b) *implies* (a) *with $c_1 \le Cc_2$;*

(II) (a) *implies* (b) *with $c_2 \le Cc_1$, provided that*

$$\sum_{\substack{F\in\mathcal{F}\\ F\supsetneq S}} \left(\frac{\lambda(S)^{-1}\sigma(S)}{\lambda(F)^{-1}\sigma(F)}\right)^{1/p} \le C \quad \textit{for all} \quad S \in \mathcal{D}. \tag{3.2}$$

Proof. First, we prove (II). It follows that, by letting $\beta_F = \alpha_F \sigma(F)^{-1/p}$,

$$\begin{aligned}
\sum_{F\in\mathcal{F}} \|M_{\mathcal{E},\mathcal{F};F}[\beta_F \mathbf{1}_F]\|_{L^q(\omega)}^q &= \sum_{F\in\mathcal{F}} \sum_{\substack{Q\in\mathcal{D}\\ \pi_{\mathcal{F}}(Q)=F}} (K(Q)\sigma(Q)\beta_F)^q\, \omega(E(Q)) \\
&= \sum_{Q\in\mathcal{D}} \left(K(Q)\sigma(Q)\beta_{\pi_{\mathcal{F}}(Q)}\right)^q \omega(E(Q)) \\
&\le \sum_{Q\in\mathcal{D}} \left(K(Q)\int_Q \left(\sum_{F\in\mathcal{F}} \beta_F \mathbf{1}_F\right) d\sigma\right)^q \omega(E(Q)) \\
&= \int_{\mathbb{R}^n} \left(M_{\mathcal{E}}\left(\sum_{F\in\mathcal{F}} \beta_F \mathbf{1}_F\right)\right)^q d\omega,
\end{aligned}$$

where we have used the uniqueness of the parent. By the statement (a),

$$\sum_{F\in\mathcal{F}} \|M_{\mathcal{E},\mathcal{F};F}[\beta_F \mathbf{1}_F]\|_{L^q(\omega)}^q \le c_1^q \left(\sup_{S\in\mathcal{D}} \frac{1}{\lambda(S)} \int_S \left(\sum_{F\in\mathcal{F}} \beta_F \mathbf{1}_F\right)^p d\sigma\right)^{q/p}.$$

For $S \in \mathcal{D}$ and the function g with $\operatorname{supp}(g) \subset S$ and $\|g\|_{L^{p'}(\sigma)} \le 1$,

$$\begin{aligned}
\sum_{F\in\mathcal{F}} \beta_F \int_{F\cap S} g\, d\sigma &= \sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \beta_F \int_F g\, d\sigma + \sum_{\substack{F\in\mathcal{F}\\ F\supsetneq S}} \beta_F \int_S g\, d\sigma \\
&=: (i_1) + (i_2).
\end{aligned}$$

For (i_1),

$$\begin{aligned}(i_1) &= \sum_{F\subset S}\beta_F \fint_F g\,d\sigma\,\sigma(F) \le 2\sum_{F\subset S}\beta_F \fint_F g\,d\sigma\,\sigma(E_{\mathcal{F}}(F)) \\ &\le 2\left(\int_S \sum_{F\subset S}\frac{\alpha_F^p \mathbf{1}_{E_{\mathcal{F}}(F)}}{\sigma(F)}\,d\sigma\right)^{1/p}\cdot\left(\int_S (M_{\mathcal{D}}^\sigma g)^{p'}\,d\sigma\right)^{1/p'} \\ &\le 2p\left(\sum_{F\subset S}\alpha_F^p\right)^{1/p},\end{aligned}$$

where we have used Lemma 2.3.

For (i_2), thanks to the condition (3.2),

$$\begin{aligned}(i_2) &\le \left(\frac{\sigma(S)}{\lambda(S)}\right)^{1/p}\sum_{F\supsetneq S}\beta_F \\ &= \sum_{F\supsetneq S}\left(\frac{\lambda(S)^{-1}\sigma(S)}{\lambda(F)^{-1}\sigma(F)}\right)^{1/p}\left(\frac{\sigma(F)}{\lambda(F)}\right)^{1/p}\beta_F \\ &= \sum_{F\supsetneq S}\left(\frac{\lambda(S)^{-1}\sigma(S)}{\lambda(F)^{-1}\sigma(F)}\right)^{1/p}\left(\frac{\alpha_F^p}{\lambda(F)}\right)^{1/p} \\ &\le C\sup_{F\supsetneq S}\left(\frac{\alpha_F^p}{\lambda(F)}\right)^{1/p}.\end{aligned}$$

Thus, by duality and (3.1) we obtain

$$\sup_{S\in\mathcal{D}}\frac{1}{\lambda(S)}\int_S\left(\sum_{F\in\mathcal{F}}\beta_F\mathbf{1}_F\right)^p d\sigma \le C,$$

which proves (II).

Next, we prove (I). Fix (large) $Q_0\in\mathcal{D}$. We shall estimate

$$(ii) := \|M_{\mathcal{E},Q_0}f\|_{L^q(\omega)}.$$

It follows that

$$(ii)^q = \sum_{Q\subset Q_0}\left(K(Q)\int_Q f\,d\sigma\right)^q\omega(E(Q)) = \sum_{Q\subset Q_0}\left(\fint_Q f\,d\sigma\right)^q (K(Q)\sigma(Q))^q\omega(E(Q)).$$

We now employ the argument of the principal cubes (cf. [5, 13]).

We define the collection of principal cubes

$$\mathcal{F} = \bigcup_{k=0}^{\infty}\mathcal{F}_k,$$

where $\mathcal{F}_0 = \{Q_0\}$,

$$\mathcal{F}_{k+1} = \bigcup_{F \in \mathcal{F}_k} ch_{\mathcal{F}}(F)$$

and $ch_{\mathcal{F}}(F)$ is defined by the set of all maximal dyadic cubes $Q \subset F$ such that

$$\fint_Q f\,d\sigma > 2 \fint_F f\,d\sigma.$$

Observe that

$$\sum_{F' \in ch_{\mathcal{F}}(F)} \sigma(F') \le \left(2 \fint_F f\,d\sigma\right)^{-1} \sum_{F' \in ch_{\mathcal{F}}(F)} \int_{F'} f\,d\sigma \le \frac{\sigma(F)}{2},$$

and, hence,

$$\sigma(E_{\mathcal{F}}(F)) = \sigma\left(F \setminus \bigcup_{F' \in ch_{\mathcal{F}}(F)} F'\right) \ge \frac{\sigma(F)}{2},$$

where the sets $E_{\mathcal{F}}(F)$ are pairwise disjoint. Thus, $\mathcal{F}$ is σ-sparse. For $Q \in \mathcal{D}$, we further define the stopping parents

$$\pi_{\mathcal{F}}(Q) = \min\{F \supset Q;\ F \in \mathcal{F}\}.$$

We notice that, when $\pi_{\mathcal{F}}(Q) = F$,

$$\fint_Q f\,d\sigma \le 2 \fint_F f\,d\sigma.$$

This fact and the uniqueness of the parent yield

$$\begin{aligned}
(ii)^q &= \sum_{F \in \mathcal{F}} \sum_{\substack{Q \in \mathcal{D} \\ \pi_{\mathcal{F}}(Q) = F}} \left(\fint_Q f\,d\sigma\right)^q (K(Q)\sigma(Q))^q \omega(E(Q)) \\
&\le 2^q \sum_{F \in \mathcal{F}} \left(\fint_F f\,d\sigma\right)^q \sum_{\substack{Q \in \mathcal{D} \\ \pi_{\mathcal{F}}(Q) = F}} (K(Q)\sigma(Q))^q \omega(E(Q)) \\
&= 2^q \sum_{F \in \mathcal{F}} \left(\fint_F f\,d\sigma\right)^q \|M_{\mathcal{E},\mathcal{F};F} \mathbf{1}_F\|_{L^q(\omega)}^q \\
&= 2^q \sum_{F \in \mathcal{F}} \left(\sigma(F)^{1/p} \fint_F f\,d\sigma \cdot \frac{\|M_{\mathcal{E},\mathcal{F};F} \mathbf{1}_F\|_{L^q(\omega)}}{\sigma(F)^{1/p}}\right)^q.
\end{aligned}$$

Thus, we have

$$(ii) \le C \left(\sum_{F \in \mathcal{F}} \left(\alpha_F \frac{\|M_{\mathcal{E},\mathcal{F};F} \mathbf{1}_F\|_{L^q(\omega)}}{\sigma(F)^{1/p}}\right)^q\right)^{1/q} \|f\|_{L^{p,\lambda}(\sigma)},$$

where

$$\alpha_F := \frac{\sigma(F)^{1/p}}{C\|f\|_{L^{p,\lambda}(\sigma)}} \fint_F f\,d\sigma.$$

To prove (I), we need only verify that $\{\alpha_F\}_{F\in\mathcal{F}}$ satisfies (3.1). For $S\in\mathcal{D}$, we have

$$\begin{aligned}
\frac{1}{\lambda(S)} \sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \left(\sigma(F)^{1/p} \fint_F f\,d\sigma\right)^p &= \frac{1}{\lambda(S)} \sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \sigma(F) \left(\fint_F f\,d\sigma\right)^p \\
&\le \frac{2}{\lambda(S)} \sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \left(\fint_F f\,d\sigma\right)^p \sigma(E_{\mathcal{F}}(F)) \\
&\le \frac{2}{\lambda(S)} \int_S (M_{\mathcal{D}}^{\sigma}[f\mathbf{1}_S])^p\,d\sigma \\
&\le \frac{C}{\lambda(S)} \int_S f^p\,d\sigma \\
&\le C\|f\|_{L^{p,\lambda}(\sigma)}^p,
\end{aligned}$$

which means that α_F satisfies (3.1) and completes the proof. □

Remark. In the statement (b) of Theorem 3.1, for the case $q\ge 1$, the linearized maximal operator $M_{\mathcal{E},\mathcal{F};F}$ can be replaced by the (big) linearized maximal operator $M_{\mathcal{E},F}$. Indeed, we need only verify then (II) holds. There holds

$$\begin{aligned}
\sum_{F\in\mathcal{F}} \|M_{\mathcal{E},F}\beta_F\|_{L^q(\omega)}^q &= \sum_{F\in\mathcal{F}} \sum_{Q\subset F} (K(Q)\sigma(Q)\beta_F)^q\,\omega(E(Q)) \\
&\le \sum_{Q\in\mathcal{D}} (K(Q)\sigma(Q))^q\,\omega(E(Q)) \left(\sum_{F\in\mathcal{F}} \beta_F^q\right).
\end{aligned}$$

By the use of $\|\cdot\|_{\ell^1}\ge\|\cdot\|_{\ell^q}$,

$$\begin{aligned}
\sum_{F\in\mathcal{F}} \|M_{\mathcal{E},F}\beta_F\|_{L^q(\omega)}^q &\le \sum_{Q\in\mathcal{D}} (K(Q)\sigma(Q))^q\,\omega(E(Q)) \left(\sum_{F\in\mathcal{F}} \beta_F\right)^q \\
&= \sum_{Q\in\mathcal{D}} \left(K(Q)\int_Q \left(\sum_{F\in\mathcal{F}} \beta_F\mathbf{1}_F\right) d\sigma\right)^q \omega(E(Q)) \\
&= \int_{\mathbb{R}^n} M_{\mathcal{E}} \left(\sum_{F\in\mathcal{F}} \beta_F\mathbf{1}_F\right)^q d\omega.
\end{aligned}$$

Remark. The analysis of the sequential testing, like as Theorem 3.1, is first due to Hänninen, Hytönen and Li in [3]. They extend Sawyer's [12, Theorem A] in the case $q\ge p$ to the case $1\le q<p$.

§ 3.2. m linear case

In what follows we assume that all functions are non-negative. Let σ_i, $i=1,\dots,m$, be the locally finite Borel measures and fix a collection $\mathcal{D}$ of dyadic cubes. Let $K\colon \mathcal{D}\to(0,\infty)$ be a map. The m linear maximal operator M^* is defined by

$$M^*[(f_i)](x) := \sup_{Q\in\mathcal{D}} K(Q)\left(\prod_{i=1}^m \int_Q f_i\,d\sigma_i\right)\mathbf{1}_Q(x).$$

Analogous to the linear case, we can also define the collection $\mathcal{E}$ and the corresponding operator $\mathcal{M}_{\mathcal{E}}$ and its localized version $\mathcal{M}_{\mathcal{E},R}$, $R\in\mathcal{D}$.

Let the symmetric group S_m be the set of all permutations of the set $\{1,\dots,m\}$, that is, the set of all bijections from the set $\{1,\dots,m\}$ to itself.

For $i=1,\dots,m$, let $\mathcal{F}_i\subset\mathcal{D}$ be a σ_i-sparse family. For $Q\in\mathcal{D}$, define

$$\begin{cases} \pi_{\mathcal{F}_i}(Q) := \min\{F\supset Q;\ F\in\mathcal{F}_i\}, \\ \pi(Q) := \left(\pi_{\mathcal{F}_1}(Q),\dots,\pi_{\mathcal{F}_m}(Q)\right). \end{cases}$$

We recall the elementary fact that, if $P,R\in\mathcal{D}$, then $P\cap R\in\{P,R,\emptyset\}$. This fact implies, if $\pi(Q)=(F_1,\dots,F_m)$, then

$$Q\subset F_{\phi(1)}\subset\cdots\subset F_{\phi(m)} \quad\text{for some}\quad \phi\in S_m. \tag{3.3}$$

Moreover,

$$\pi_{\mathcal{F}_{\phi(j)}}(F_{\phi(i)}) = F_{\phi(j)} \quad\text{for all}\quad 1\le i<j\le m. \tag{3.4}$$

From these observations, we define $\langle\mathcal{F}_i\rangle$ to be the set $(F_i)\in\prod_{i=1}^m\mathcal{F}_i$ that satisfies (3.3) and (3.4).

For $(F_i)\in\langle\mathcal{F}_i\rangle$, the m linear operator $M_{\mathcal{E},\langle\mathcal{F}_i\rangle;(F_i)}$ is defined by

$$M_{\mathcal{E},\langle\mathcal{F}_i\rangle;(F_i)}[(f_i)](x) := \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F_i)}} K(Q)\left(\prod_{i=1}^m\int_Q f_i\,d\sigma_i\right)\mathbf{1}_{E(Q)}(x).$$

The following is an m linear version of Theorem 3.1.

Theorem 3.2. *Let ω and σ_i, $i=1,\dots,m$, be the locally finite Borel measures. Let $0<q<\infty$ and $1<p_i<\infty$. Let $\lambda_i\colon\mathcal{D}\to(0,\infty)$ be a map. Consider the following two statements*:

(a) *There exists a constant $c_1>0$ such that*

$$\|M_{\mathcal{E}}[(f_i)]\|_{L^q(\omega)} \le c_1\prod_{i=1}^m \|f_i\|_{L^{p_i,\lambda_i}(\sigma_i)}$$

holds for every function $f_i\in L^{p_i,\lambda_i}(\sigma_i)$;

(b) *There exists a constant $c_2 > 0$ such that*

$$\left(\sum_{(F_i) \in \langle \mathcal{F}_i \rangle} \left(\left(\prod_{i=1}^m \alpha^i_{F_i} \right) \frac{\| M_{\mathcal{E}, \langle \mathcal{F}_i \rangle ; (F_i)} [(\mathbf{1}_{F_i})] \|_{L^q(\omega)}}{\prod_{i=1}^m \sigma_i(F_i)^{1/p_i}} \right)^q \right)^{1/q} \le c_2$$

holds for any σ_i-sparse family $\mathcal{F}_i \subset \mathcal{D}$, where the nonnegative sequence $\{\alpha^i_F\}_{F \in \mathcal{F}_i}$ satisfies

$$\| \{\alpha^i_F\} \|_{\ell^{p_i, \lambda_i}(\mathcal{F}_i)} \le 1, \quad i = 1, \ldots, m. \tag{3.5}$$

Then, for the least possible constants c_1 and c_2,

(I) (b) *implies* (a) *with $c_1 \le C c_2$;*

(II) (a) *implies* (b) *with $c_2 \le C c_1$, provided that*

$$\sum_{\substack{F \in \mathcal{F}_i \\ F \supsetneq S}} \left(\frac{\lambda_i(S)^{-1} \sigma_i(S)}{\lambda_i(F)^{-1} \sigma_i(F)} \right)^{1/p_i} \le C \quad \text{for all} \quad S \in \mathcal{D}. \tag{3.6}$$

Proof. The proof of Theorem 3.2 follows in the same manner as that of Theorem 3.1. We show only the necessary modifications. First, we prove (II). Let $\beta^i_F := \alpha^i_F \sigma_i(F)^{-1/p_i}$. There holds by the uniqueness of the parent

$$\begin{aligned}
\sum_{(F_i) \in \langle \mathcal{F}_i \rangle} \| M_{\mathcal{E}, \langle \mathcal{F}_i \rangle ; (F_i)} [(\beta^i_{F_i}] \|^q_{L^q(\omega)} &= \sum_{(F_i) \in \langle \mathcal{F}_i \rangle} \sum_{\substack{Q \in \mathcal{D} \\ \pi(Q) = (F_i)}} K(Q)^q \left(\prod_i \sigma_i(Q) \beta^i_{F_i} \right)^q \omega(E(Q)) \\
&= \sum_{Q \in \mathcal{D}} K(Q)^q \left(\prod_i \sigma_i(Q) \beta^i_{\pi_{\mathcal{F}_i}(Q)} \right)^q \omega(E(Q)) \\
&= \sum_{Q \in \mathcal{D}} K(Q)^q \left(\prod_i \int_Q \left(\sum_{F \in \mathcal{F}_i} \beta^i_F \mathbf{1}_F \right) d\sigma_i \right)^q \omega(E(Q)) \\
&= \int_{\mathbb{R}^n} \left(M_{\mathcal{E}} \left[\left(\sum_{F \in \mathcal{F}_i} \beta^i_F \mathbf{1}_F \right) \right] \right)^q d\omega.
\end{aligned}$$

By the statement (a),

$$\sum_{(F_i) \in \langle \mathcal{F}_i \rangle} \| M_{\mathcal{E}, \langle \mathcal{F}_i \rangle ; (F_i)} [(\beta^i_{F_i}] \|^q_{L^q(\omega)} \le c_1^q \prod_{i=1}^m \left(\sup_{S \in \mathcal{D}} \frac{1}{\lambda_i(S)} \int_S \left(\sum_{F \in \mathcal{F}_i} \beta^i_F \mathbf{1}_F \right)^{p_i} d\sigma_i \right)^{q/p_i}.$$

The remainder estimates of this part are the same as that of the proof of Theorem 3.1 and we omit them here.

Next, we prove (I). Fix (large) $Q_0 \in \mathcal{D}$. We shall estimate

$$(i) := \|M_{\mathcal{E},Q_0}[(f_i)]\|_{L^q(\omega)}.$$

It follows that

$$\begin{aligned}(i)^q &= \sum_{Q\subset Q_0} K(Q)^q \left(\prod_i \int_Q f_i\,d\sigma_i\right)^q \omega(E(Q))\\ &= \sum_{Q\subset Q_0} \left(\prod_i ⨍_Q f_i\,d\sigma_i\right)^q K(Q)^q \left(\prod_i \sigma_i(Q)\right)^q \omega(E(Q)).\end{aligned}$$

In the same manner as the above, we define the collection of principal cubes $\mathcal{F}_i$ for the pair (f_i,σ_i), $i=1,\ldots,m$. Then $\mathcal{F}_i$ is σ_i-sparse. We notice that, when $\pi_{\mathcal{F}_i}(Q)=F$,

$$⨍_Q f_i\,d\sigma_i \le 2 ⨍_F f_i\,d\sigma_i.$$

This fact and the uniqueness of the parent yield

$$\begin{aligned}(i)^q &= \sum_{(F_i)\in\langle\mathcal{F}_i\rangle} \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F_i)}} \left(\prod_i ⨍_Q f_i\,d\sigma_i\right)^q K(Q)^q \left(\prod_i \sigma_i(Q)\right)^q \omega(E(Q))\\ &\le 2^{mq} \sum_{(F_i)\in\langle\mathcal{F}_i\rangle} \left(\prod_i ⨍_{F_i} f_i\,d\sigma_i\right)^q \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F_i)}} K(Q)^q \left(\prod_i \sigma_i(Q)\right)^q \omega(E(Q))\\ &= 2^{mq} \sum_{(F_i)\in\langle\mathcal{F}_i\rangle} \left(\prod_i ⨍_{F_i} f_i\,d\sigma_i\right)^q \|M_{\mathcal{E},\langle\mathcal{F}_i\rangle;(F_i)}[(\mathbf{1}_{F_i})]\|_{L^q(\omega)}^q\\ &= 2^{mq} \sum_{(F_i)\in\langle\mathcal{F}_i\rangle} \left(\left(\prod_i \sigma_i(F_i)^{1/p_i} ⨍_{F_i} f_i\,d\sigma_i\right)\cdot \frac{\|M_{\mathcal{E},\langle\mathcal{F}_i\rangle;(F_i)}[(\mathbf{1}_{F_i})]\|_{L^q(\omega)}}{\prod_i \sigma_i(F_i)^{1/p_i}}\right)^q.\end{aligned}$$

Thus, we have

$$(i) \le C \left(\sum_{(F_i)\in\langle\mathcal{F}_i\rangle} \left(\left(\prod_i \alpha_{F_i}^i\right) \frac{\|M_{\mathcal{E},\langle\mathcal{F}_i\rangle;(F_i)}[(\mathbf{1}_{F_i})]\|_{L^q(\omega)}}{\prod_i \sigma_i(F_i)^{1/p_i}}\right)^q\right)^{1/q} \cdot \prod_i \|f_i\|_{L^{p_i,\lambda_i}(\sigma_i)},$$

where

$$\alpha_{F_i}^i = \frac{\sigma_i(F_i)^{1/p_i}}{C\|f_i\|_{L^{p_i,\lambda_i}(\sigma_i)}} ⨍_{F_i} f_i\,d\sigma_i.$$

To prove (I), we need only verify that $\{\alpha_F^i\}_{F\in\mathcal{F}_i}$ satisfies (3.5). But, this is done in the same way as that of the proof of Theorem 3.1. So, we finish the proof. $\square$

§ 4. Application of Köthe duals

The statement (b) of Theorem 3.1 is expressed by the dual form. In this section, for some cases, we shall investigate another expression of the quantities in terms of the description of the Köthe dual of Morrey type spaces introduced in Subsection 2.1.

Let σ be a locally finite Borel measure. Fix a collection $\mathcal{D}$ of dyadic cubes and a σ-sparse family $\mathcal{F} \subset \mathcal{D}$. Let $0 < p < \infty$ and $\lambda \colon \mathcal{D} \to (0, \infty)$ be a map. For $S \in \mathcal{D}$, define the sequence $\mathbf{1}(S)_F$ with the index set $\mathcal{F}$ by

$$\mathbf{1}(S)_F := \begin{cases} 1, & \text{when} \quad F \subset S, \\ 0, & \text{otherwise.} \end{cases}$$

Let $b(S)_F := \mathbf{1}(S)_F / \lambda(S)$ and define $\mathcal{B}_\lambda := \{b(S)_F;\ S \in \mathcal{D}\}$. Then, for the sequence $\{\alpha_F\}_{F \in \mathcal{F}}$, we have

$$\begin{aligned} \|\{\alpha_F\}\|_{\ell^{p,\lambda}(\mathcal{F})} &= \sup_{S \in \mathcal{D}} \left(\frac{1}{\lambda(S)} \sum_{\substack{F \in \mathcal{F} \\ F \subset S}} |\alpha_F|^p \right)^{1/p} = \sup_{S \in \mathcal{D}} \|\{|\alpha_F|^p b(S)_F\}\|_{\ell^1(\mathcal{F})}^{1/p} \\ &= \sup_{\{b(S)_F\} \in \mathcal{B}_\lambda} \|\{|\alpha_F|^p b(S)_F\}\|_{\ell^1(\mathcal{F})}^{1/p}. \end{aligned}$$

The Köthe dual space $\ell^{p,\lambda}(\mathcal{F})'$ of the space $\ell^{p,\lambda}(\mathcal{F})$ is the space of all sequences $\{\alpha_F\}_{F \in \mathcal{F}}$ equipped with the norm

$$\|\{\alpha_F\}\|_{\ell^{p,\lambda}(\mathcal{F})'} := \sup_{\|\{\beta_F\}\|_{\ell^{p,\lambda}(\mathcal{F})} \le 1} \|\{\alpha_F \beta_F\}\|_{\ell^1(\mathcal{F})}.$$

We wish to give a description of $\ell^{p,\lambda}(\mathcal{F})'$ by applying Proposition 2.2. To this end, we first let $\widehat{\mathcal{B}_\lambda}$ denote the set of all sequences $\{a_F\}_{F \in \mathcal{F}}$ such that

$$a_F = \sum_{S \in \mathcal{D}} c(S) b(S)_F,$$

where $c \colon \mathcal{D} \to [0, \infty)$ is any map that satisfies $\sum_{S \in \mathcal{D}} c(S) \le 1$. We notice that, for any $\{a_F\} \in \widehat{\mathcal{B}_\lambda}$,

$$\sup_{\|\{\beta_F\}\|_{\ell^{1,\lambda}(\mathcal{F})} \le 1} \|\{a_F \beta_F\}\|_{\ell^1(\mathcal{F})} \le 1. \tag{4.1}$$

The class $\overline{\mathcal{B}_\lambda}$ is defined to be the subset of all sequences $\{a_F\}_{F \in \mathcal{F}}$ for which there exists $\{a_F^{(j)}\} \in \widehat{\mathcal{B}_\lambda}$, $j \in \mathbb{N}$, such that

$$a_F = \lim_{j \to \infty} a_F^{(j)} \quad \text{for all} \quad F \in \mathcal{F}.$$

We can easily check the following conditions:

(i) $\mathcal{B}_\lambda \subset \overline{\mathcal{B}_\lambda}$;

(ii) If

$$\{a_F^{(j)}\} \subset \overline{\mathcal{B}_\lambda}, \quad j \in \mathbb{N},$$

then, for any non-negative sequence $\{c_j\}_{j\in\mathbb{N}}$ with $\|\{c_j\}\|_{\ell^1(\mathbb{N})} \le 1$, one has

$$\{\sum_j c_j a_F^{(j)}\} \in \overline{\mathcal{B}_\lambda};$$

(iii) For all $\{a_F\} \in \overline{\mathcal{B}_\lambda}$,

$$\sup_{\|\{\beta_F\}\|_{\ell^{1,\lambda}(\mathcal{F})}\le 1} \|\{a_F\beta_F\}\|_{\ell^1(\mathcal{F})} \le 1.$$

(This fact can be verified by the use of (4.1) and the Fatou theorem.)

We now check the following condition:

(iv) (the Komlós property) If $\{a_F^{(j)}\} \subset \overline{\mathcal{B}_\lambda}$, then there exists a subsequence $\{b_F^{(j)}\}$ of $\{a_F^{(j)}\}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n b_F^{(j)} = a_F, \qquad \text{for all } F \in \mathcal{F}, \text{ and } \{a_F\} \in \overline{\mathcal{B}_\lambda}\ .$$

By Remark 2.1, it suffices to show that there exists $u_F > 0$ such that $\{u_F\}_{F\in\mathcal{F}} \in \ell^{1,\lambda}(\mathcal{F})$.

Recall that we always assume that there exists $u \in L^{p,\lambda}(\sigma)$ with $u > 0$ σ-a.e. Let $p > 1$ and $u > 0$ such that $\|u\|_{L^{p,\lambda}(\sigma)} = 1$. Define

$$u_F := \sigma(F)\left(\fint_F u\,d\sigma\right)^p.$$

Then it follows that $u_F > 0$ and, for any $S \in \mathcal{D}$,

$$\begin{aligned}\frac{1}{\lambda(S)}\sum_{\substack{F\in\mathcal{F}\\ F\subset S}} u_F &= \frac{1}{\lambda(S)}\sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \sigma(F)\left(\fint_F u\,d\sigma\right)^p \le \frac{2}{\lambda(S)}\sum_{\substack{F\in\mathcal{F}\\ F\subset S}} \left(\fint_F u\,d\sigma\right)^p \sigma(E_{\mathcal{F}}(F))\\ &\le \frac{2}{\lambda(S)}\int_S M_{\mathcal{D}}^\sigma[u\mathbf{1}_S]^p\,d\sigma \le \frac{2p'}{\lambda(S)}\int_S u^p\,d\sigma \le 2p'\|u\|_{L^{p,\lambda}(\sigma)}^p = 2p'.\end{aligned}$$

This means that the condition (iv) is fulfilled.

Let $1 < p \le \infty$. We define $h^{p,\lambda}(\mathcal{F})$ by the space of all sequence $\{\alpha_F\}_{F\in\mathcal{F}}$ such that

$$\|\{\alpha_F\}\|_{h^{p,\lambda}(\mathcal{F})} := \inf_{\{a_F\}\in\overline{\mathcal{B}_\lambda}} \|\{|\alpha_F|^p a_F^{1-p}\}\|_{\ell^1(\mathcal{F})}^{1/p} < \infty, \quad \text{for} \quad 1 < p < \infty,$$

and

$$\|\{\alpha_F\}\|_{h^{\infty,\lambda}(\mathcal{F})} := \inf_{\{a_F\}\in\overline{\mathcal{B}_\lambda}} \|\{|\alpha_F|a_F^{-1}\}\|_{\ell^\infty(\mathcal{F})} < \infty, \quad \text{for} \quad p=\infty.$$

By Proposition 2.2, we have the following.

Proposition 4.1. *Let $1 < p \le \infty$. Then the space $h^{p,\lambda}(\mathcal{F})$ is a Banach space with the Fatou property. Moreover, the following Köthe duality formulas hold with equality of norms*:

$$h^{p,\lambda}(\mathcal{F})' = l^{p',\lambda}(\mathcal{F}) \quad \textit{and} \quad h^{p,\lambda}(\mathcal{F}) = l^{p',\lambda}(\mathcal{F})'.$$

The following is our last theorem.

Theorem 4.2.

(I) *Let $1 < q = p < \infty$. Then the statement (b) of Theorem 3.1 is equivalent to*

$$\left\|\left\{\left(\frac{\|M_{\mathcal{E},\mathcal{F};F}\mathbf{1}_F\|_{L^p(\omega)}}{\sigma(F)^{1/p}}\right)^p\right\}\right\|_{h^{\infty,\lambda}(\mathcal{F})}^{1/p} \le c_2.$$

(II) *Let $0 < q < \infty$, $1 < p < \infty$ and $q < p$. Then the statement (b) of Theorem 3.1 is equivalent to*

$$\left\|\left\{\left(\frac{\|M_{\mathcal{E},\mathcal{F};F}\mathbf{1}_F\|_{L^q(\omega)}}{\sigma(F)^{1/p}}\right)^q\right\}\right\|_{h^{r/q,\lambda}(\mathcal{F})}^{1/q} \le c_2,$$

where $q/r + q/p = 1$.

Proof. The theorem is the direct consequences of Proposition 4.1. □

Remark. For the remender case $1 < p < q < \infty$, we have had no mathematical language to express appropriately.

References

[1] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin, (1983).

[2] T. Iida, E. Sato, Y. Sawano and H. Tanaka, *Weighted norm inequalities for multilinear fractional operators on Morrey spaces*, Studia Math., **205** (2011), no. 2, 139–170.

[3] T. S. Hänninen, T. P. Hytönen and K. Li, *Two-weight L^p-L^q bounds for positive dyadic operators: unified approach to $p \le q$ and $p > q$*, arXiv:1412.2593 [math.CA].

[4] T. P. Hytönen, *Weighted norm inequalities*, Lecture notes of a course at the University of Helsinki, Winter 2011.

[5] ———, *The A_2 theorem: Remarks and complements*, Harmonic analysis and partial differential equations, 91–106. American Mathematical Society, Providence, RI, 2014. arXiv:1212.3840 [math.CA].

[6] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Second edition, Pergamon Press, Oxford-Elmsford, N.Y., 1982.

[7] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hung. **18** (1967), 217–229.

[8] M. Mastyło, Y. Sawano and H. Tanaka, *Morrey type space and its Köthe dual space*, submitted.

[9] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165** (1972), 207–226.

[10] N. Samko, *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl., **350** (2009), no. 1, 56–72.

[11] ———, *On a Muckenhoupt-type condition for Morrey spaces*, Mediterr. J. Math., **10** (2013), no. 2, 941–951.

[12] E. T. Sawyer, *A characterization of a two-weight norm inequality for maximal operators*, Studia Math., **75** (1982), no. 1, 1–11.

[13] H. Tanaka, *A characterization of two-weight trace inequalities for positive dyadic operators in the upper triangle case*, Potential Anal., **41** (2014), no. 2, 487–499.

[14] ———, *Two-weight norm inequalities on Morrey spaces*, Ann. Acad. Sci. Fenn. Math., **40** (2015), 773–791.