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Local well-posedness and parabolic smoothing effect of fifth order dispersive equations on the torus

By

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Abstract

The paper is announcement of the result obtained in [23]. We consider the Cauchy problem of fifth order dispersive equations with polynomial type nonlinearities depending on $u, \partial_x u, \partial_x^2 u$ and $\partial_x^3 u$ under the periodic boundary condition. We show the following results. When the nonlinear term is non-parabolic resonance type, we have the local well-posedness on $(-T, T)$. On the other hand, when the nonlinear term is parabolic resonance type, the local well-posedness holds with a smoothing effect only on either $[0, T)$ or $(-T, 0]$ and nonexistence result holds on the other time interval.

§ 1. Introduction and Main theorems

The paper is announcement of the result obtained in [23]. In Section 1, we will present main theorems obtained in [23]. In Section 2, we mention the modified energy, the energy inequality and the proof of it, which include main idea in this paper. In Section 3, we mention an estimate for the difference of two solutions. In Section 4, we prove the main theorems by the estimates in Sections 2 and 3. The argument in Sections 3 and 4 is an application of the energy method with Bona-Smith's approximation (see [2], [10]) , which do not include new idea. In the present paper, we treat only simplified case. For full results, see [23].

We consider the Cauchy problem of fifth order dispersive equations on $\mathbb{T}(:= \mathbb{R}/2\pi)$:

$$(\partial_t + \partial_x^5)u(t,x) = N(\partial_x^3 u, \partial_x^2 u, \partial_x u, u),\ (t,x) \in (-T,T) \times \mathbb{T}, \tag{1.1}$$

$$u(0,\cdot) = \varphi(\cdot), \tag{1.2}$$

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where the initial data φ, the unknown function u are real valued. We assume that the nonlinear term N is as follows:

$$N(\partial_x^3 u, \partial_x^2 u, \partial_x u, u) = \sum_{j=1}^{j_0} N_j(u), \quad N_j(u) = \lambda_j (\partial_x^3 u)^{a_j} (\partial_x^2 u)^{b_j} (\partial_x u)^{c_j} u^{d_j} \tag{1.3}$$

where $\lambda_j \in \mathbb{R}, j_0 \in \mathbb{N}, a_j, b_j, c_j, d_j \in \mathbb{N} \cup \{0\}$ and $p_j := a_j + b_j + c_j + d_j \geq 2$. Put $p_{max} := \max_{1 \leq j \leq j_0} p_j$. In the present paper, we are interested in the case of initial data being sufficiently smooth. Therefore, we assume s_0 is a sufficiently large constant depending only on p_{max} and consider only the case $s \in \mathbb{N}$, $s \geq s_0$ and $\varphi \in H^s(\mathbb{T})$. The case of initial data having low regularity is studied in the forthcoming paper ([14]) by Kato and the author. Here, we define a functional $P_N(f)$ to categorize the nonlinear terms.

Definition 1.1. Put

$$P_N(f) := \sum_{j=1}^{j_0} P_{N_j}(f), \quad P_{N_j}(f) := \frac{\lambda_j b_j}{2\pi} \int_{\mathbb{T}} (\partial_x^3 f)^{a_j} (\partial_x^2 f)^{b_j - 1} (\partial_x f)^{c_j} f^{d_j} \, dx.$$

We say that N is non-parabolic resonance type if $P_N \equiv 0$, namely, $P_N(f) = 0$ for any $f \in C^\infty(\mathbb{T})$. Otherwise, we say N is parabolic resonance type.

Remark that

$$P_N(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\partial}{\partial \omega_2} N(\omega_3, \omega_2, \omega_1, \omega_0) \Big|_{(\omega_3, \omega_2, \omega_1, \omega_0) = (\partial_x^3 f, \partial_x^2 f, \partial_x f, f)} dx.$$

Now, we state our main results.

Theorem 1.2 (L.W.P. for non-parabolic resonance type)**.** *Let $P_N \equiv 0$, $s \in \mathbb{N}$ and $s \geq s_0$. Then, we have the followings.*
(existence) Let $\varphi \in H^s(\mathbb{T})$. Then, there exist a time $T = T(\|\varphi\|_{H^{s_0}}) > 0$ and a solution to (1.1)–(1.2) *on $(-T, T)$ satisfying $u \in C((-T, T); H^s(\mathbb{T}))$.*
(uniqueness) Let $T > 0$, $u_1, u_2 \in L^\infty((-T, T); H^{s_0}(\mathbb{T}))$ be solutions to (1.1)–(1.2) *on $(-T, T)$. Then, $u_1(t) = u_2(t)$ on $t \in (-T, T)$.*
(continuous dependence on initial data) Assume that $\{\varphi^j\}_{j \in \mathbb{N}} \subset H^s(\mathbb{T}), \varphi \in H^s(\mathbb{T})$ satisfy $\|\varphi^j - \varphi\|_{H^s} \to 0$ as $j \to \infty$. Let u^j (resp. u) be the solution obtained above with initial data φ^j (resp. φ) and $T = T(\|\varphi\|_{H^{s_0}})$. Then $\|u^j - u\|_{L^\infty((-T,T); H^s)} \to 0$ as $j \to \infty$.

Theorem 1.3 (L.W.P. for parabolic resonance type)**.** *Let $P_N \not\equiv 0$, $s \in \mathbb{N}$ and $s \geq s_0$. Then, we have the followings.*
(existence) Let $\varphi \in H^s(\mathbb{T})$ and $P_N(\varphi) > 0$ (resp. $P_N(\varphi) < 0$). Then, there exist a

time $T = T(P_N(\varphi), \|\varphi\|_{H^{s_0}}) > 0$ *and a solution to* (1.1)–(1.2) *on* $[0,T)$ *(resp.* $(-T,0]$*) satisfying* $u \in C([0,T); H^s(\mathbb{T})) \cap C^\infty((0,T) \times \mathbb{T})$ *and* $P_N(u(t)) > P_N(\varphi)/2$ *on* $[0,T)$ *(resp.* $u \in C((-T,0]; H^s(\mathbb{T})) \cap C^\infty((-T,0) \times \mathbb{T})$ *and* $P_N(u(t)) < P_N(\varphi)/2$ *on* $(-T,0]$*).*
(uniqueness) Let $T > 0$, $u_1, u_2 \in L^\infty([0,T); H^{s_0}(\mathbb{T}))$ *(resp.* $u_1, u_2 \in L^\infty((-T,0]; H^{s_0}(\mathbb{T}))$*) be solutions to* (1.1)–(1.2) *and* $P_N(u_1(t)) > 0$ *on* $[0,T)$ *(resp.* $P_N(u_1(t)) < 0$ *on* $(-T,0]$*). Then,* $u_1(t) = u_2(t)$ *on* $t \in [0,T)$ *(resp.* $t \in (-T,0]$*).*
(continuous dependence on initial data) Assume that $\{\varphi^j\}_{j\in\mathbb{N}} \subset H^s(\mathbb{T}), \varphi \in H^s(\mathbb{T})$ *satisfy* $P_N(\varphi) > 0$ *(resp.* $P_N(\varphi) < 0$*)) and* $\|\varphi^j - \varphi\|_{H^s} \to 0$ *as* $j \to \infty$*. Let* u^j *(resp.* u*) be the solution obtained above with initial data* φ^j *(resp.* φ*) and* $T = T(P_N(\varphi), \|\varphi\|_{H^{s_0}})$*. Then* $\|u^j - u\|_{L^\infty([0,T);H^s(\mathbb{T}))} \to 0$ *(resp.* $\|u^j - u\|_{L^\infty((-T,0];H^s(\mathbb{T}))} \to 0$*) as* $j \to \infty$*.*

Theorem 1.4 (non existence for parabolic resonance type). *Let* $P_N \not\equiv 0$, $\varphi \in H^{s_0}(\mathbb{T}) \setminus C^\infty(\mathbb{T})$ *and* $P_N(\varphi) < 0$ *(resp.* $P_N(\varphi) > 0$*). Then, for any small* $T > 0$*, there does not exist any solution to* (1.1)–(1.2) *on* $[0,T)$ *(resp.* $(-T,0]$*) satisfying* $u \in C([0,T); H^{s_0}(\mathbb{T}))$ *(resp.* $u \in C((-T,0]; H^{s_0}(\mathbb{T}))$*).*

Remark. Theorem 1.2 is a typical result for dispersive equations in the following sense: they can be solved on both positive and negative time intervals and the regularity of the solution is same as that of initial data. Theorems 1.3 and 1.4 are typical results for parabolic equations in the following sense: they can be solved on either positive or negative time interval with strong smoothing effect and they are ill-posed on the other time interval. Since (1.1) are semilinear dispersive equations, Theorem 1.2 is a natural result. On the other hand, Theorems 1.3 and 1.4 are somewhat surprising. These theorems mean that when the nonlinear term is parabolic resonance type, the nonlinear term cannot be treated as a perturbation of the linear part and the effect by the second derivative in the nonlinear part is dominant.

Finally, we mention known results for related problems. For the case of $x \in \mathbb{R}$, there are many results related to fifth order dispersive equations ([4], [5], [11], [12], [13], [15], [16], [17], [22]). In [16], Kenig, Ponce and Vega consider the following $(2j+1)$ st order dispersive equations:

$$(\partial_t + \partial_x^{2j+1})u = N(\partial_x^{2j}u, \dots, \partial_x u, u).$$

Employing the gauge transformation introduced by Hayashi [7], Hayashi and Ozawa [8], [9] and the smoothing effect for the linear part:

$$\|\partial_x^j e^{t\partial_x^{2j+1}}\varphi\|_{L_x^\infty L_t^2} \lesssim \|\varphi\|_{L^2},$$

they proved the local well-posedness on $(-T,T)$ in $H^{s_1}(\mathbb{R}) \cap H^{s_2}(\mathbb{R}; x^2\,dx)$ for sufficiently large integers s_1, s_2. The result means that the local solution is controlled by the linear part of the equation in the case $x \in \mathbb{R}$ unlike the parabolic resonance type in the case $x \in \mathbb{T}$. In [12], Kwon proved the local well-posedness of

$$(\partial_t + \partial_x^5)u = c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u \tag{1.4}$$

in $H^s(\mathbb{R})$ for $s > 5/2$. The standard energy estimate gives only the following:

$$\frac{d}{dt}\|\partial_x^s u(t)\|_{L^2}^2 \lesssim \|\partial_x^3 u\|_{L^\infty}\|\partial_x^s u(t)\|_{L^2} + \Big|\int_{\mathbb{R}} \partial_x u \partial_x^{s+1} u \partial_x^{s+1} u \, dx\Big| \tag{1.5}$$

for $s \in \mathbb{N}$. It is the main difficulty in this problem that the last term can not be estimated by $\|u(t)\|_{H^s}$. To overcome the difficulty, Kwon introduced the following modified energy:

$$E_s^*(u(t)) := \|D^s u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + C_s \int_{\mathbb{R}} u(t) D^{s-2}\partial_x u(t) D^{s-2}\partial_x u(t), \tag{1.6}$$

where $D := \mathcal{F}_\xi^{-1}|\xi|\mathcal{F}_x$. The last term is the correction term and used to cancel out the last term in (1.5).

For the case of $x \in \mathbb{T}$, the linear part does not have the smoothing effect and only a few results are known. In [18], Saut proved the existence of solutions to nonlinear $(2j+1)$ order dispersive equations which have Hamiltonian structure. In [19], Schwarz Jr. proved the existence and the uniqueness to the equations in the KdV hierarchy. Since the equations in the KdV hierarchy have Hamilton structures, the class of equations treated in [18] is larger than that in [19]. Both results require some special structure to the nonlinear terms and nonlinearities of parabolic resonance type are excluded, that is to say only the case of $P_N \equiv 0$ is treated in [18] and [19]. As far as the author knows, no result exist for nonlinearities having no structure when $x \in \mathbb{T}$.

§ 2. Main idea and outline of the proof

We use the energy method with Bona-Smith's approximation to show main theorems. In this method, so called energy inequality and an estimate of the difference of two solutions play an important role. Therefore, we focus on them. In this section, we present the energy inequality and the outline of the proof of it, which include the main idea in the paper.

[10] is a good book to study the energy method with Bona-Smith's approximation. In this book, the energy method is applied to the KdV equation. The nonlinear term of it depends only on u and $\partial_x u$. In our problem, the nonlinear term depends not only on $u, \partial_x u$, but also on $\partial_x^2 u$ and $\partial_x^3 u$. Therefore, the standard energy method does not work. To overcome this difficulty, we use the modified energy introduced by Kwon in [12], which is mentioned in Section 1 (see also Segata [20]). The right-hand side of (1.4) includes only two terms. So, the correction term of (1.6) is not so complicated. However,

it seems difficult to construct the correction term corresponding to the nonlinear term of (1.1) because it is so complicated. In our proof, we first use the Fourier transform. Next, we extract resonance parts which have loss of derivatives. Finally, we construct the correction term of the modified energy to cancel out the resonance parts by using the normal form reduction. See [1], [3], [6] and [21] for the normal form reduction. In this way, we have the following modified energy for (1.1):

$$\begin{aligned} E_s(u) :=& \frac{1}{2}\|\partial_x^s u\|_{L^2}^2 + \|u\|_{L^2}^2 + \sum_{j=1}^{j_0} C_s \|u\|_{L^2}^{2s(p_j-1)+2} \\ &+ \sum_{j=1}^{j_0} \sum_{\vec{k}^{(p_j)} \in \mathbb{Z}_0^{(p_j)}} \frac{(ik_{p_j})^{s+1}(ik_{p_j+1})^{s+1} M_{NR,j}}{\Phi^{(p_j)}(\vec{k}^{(p_j)})} \prod_{l=1}^{p_j+1} \widehat{u}(k_l). \end{aligned} \tag{2.1}$$

Here, we give some notations used in (2.1). Put $\mathbb{Z}_0^{(p)} := \{(k_1, \cdots, k_p, k_{p+1}) \in \mathbb{Z}^{p+1} \mid k_1 + k_2 + \cdots + k_{p+1} = 0\}$. Put

$$\Phi^{(p)}(\vec{k}^{(p)}) := -i \sum_{l=1}^{p+1} k_l^5 \quad \text{for} \quad \vec{k}^{(p)} := (k_1, \ldots, k_p, k_{p+1}) \in \mathbb{Z}^{p+1}. \tag{2.2}$$

Note that $\Phi^{(p)}(\vec{k}^{(p)}) = i(k_1 + \cdots + k_p)^5 - i\sum_{l=1}^{p} k_l^5$ on $\mathbb{Z}_0^{(p)}$. Put

$$D_{a,b,c} := \prod_{l=1}^{a}(ik_l)^3 \prod_{l=a+1}^{a+b}(ik_l)^2 \prod_{l=a+b+1}^{a+b+c}(ik_l). \tag{2.3}$$

For an integer $p \geq 2$ and a real number $h > 0$, we put

$$M_{H,h}^{(p)}(\vec{k}^{(p)}) := \begin{cases} 1, \ \text{when} \ \min\{|k_p|^{1/h}, |k_{p+1}|^{1/h}\} \geq C \max\{|k_1|, \ldots, |k_{p-1}|\}, \\ 0, \ \text{otherwise}, \end{cases} \tag{2.4}$$

$$M_{NZ}^{(p)}(\vec{k}^{(p)}) := \begin{cases} 1, \ \text{when} \ k_1 + \cdots + k_{p-1} \neq 0, \\ 0, \ \text{when} \ k_1 + \cdots + k_{p-1} = 0, \end{cases} \tag{2.5}$$

where $C > 0$ is a sufficiently large constant. Put

$$\begin{aligned} &h_j := \max_{1 \leq j \leq j_0} \left\{ \frac{p_j}{2} + 3a_j + 2b_j + c_j - 2, 2 \right\}, \\ &M_{NR,j} := M_{H,h_j}^{(p_j)} M_{NZ}^{(p_j)} \lambda_j \Big((s - 3/2)a_j i(k_1 + \cdots + k_{p_j-1}) D_{a_j-1,b_j,c_j} + b_j D_{a_j,b_j-1,c_j} \Big). \end{aligned}$$

Remark. Note that (2.1) is real valued. Since u is real valued, it follows that $\overline{\widehat{u}}(k_l) = \widehat{\overline{u}}(-k_l) = \widehat{u}(-k_l)$. Obviously, $\overline{M_{NR,j}(\vec{k}^{(p_j)})} = M_{NR,j}(-\vec{k}^{(p_j)})$ and $\overline{\Phi^{(p_j)}(\vec{k}^{(p_j)})} = \Phi^{(p_j)}(-\vec{k}^{(p_j)})$. Thus, computing the complex conjugate of the correction term of (2.1), we can easily check that.

Remark. By the presence of $\Phi^{(p)}(\vec{k}^{(p)})$ and Lemma 2.3 below, the last term of (2.1) can be controlled by the sum of the other terms. Therefore, we have

$$E_s(u) \sim \|u\|_{H^s}^2 + \|u\|_{L^2}^{2s(p_{max}-1)+2}. \tag{2.6}$$

The following estimate is the energy estimate, which is the main estimate in the present paper.

Theorem 2.1. *Assume that* $r := (2s+1)(p_{max}-1)/2, s \in \mathbb{N}$ *and* $s \geq s_0$. *Let* u *be a sufficiently smooth solution to* (1.1)–(1.2) *on* $[0,T)$. *Then, for any* $t \in [0,T)$, *it follows that*

$$\frac{d}{dt}E_s(u(t)) + P_N(u(t))\|\partial_x^{s+1}u(t)\|_{L^2}^2 \lesssim E_s(u(t))(1+E_{s_0}(u(t)))^r. \tag{2.7}$$

Before we prove Theorem 2.1, we give some lemmas. The following lemma is the Gagliardo-Nirenberg inequality for periodic functions. For the proof, see Section 2 in [19].

Lemma 2.2. *Assume that integers* l *and* m *satisfy* $0 \leq l \leq m-1$ *and a real number* p *satisfies* $2 \leq p \leq \infty$. *Put* $\alpha = (l+1/2-1/p)/m$. *Then, we have*

$$\|\partial_x^l f\|_{L_x^p} \lesssim \begin{cases} \|f\|_{L^2}^{1-\alpha}\|\partial_x^m f\|_{L^2}^{\alpha}, & (\text{when } 1 \leq l \leq m-1), \\ \|f\|_{L^2}^{1-\alpha}\|\partial_x^m f\|_{L^2}^{\alpha} + \|f\|_{L^2}, & (\text{when } l = 0), \end{cases}$$

for any $f \in H^m$. *Especially,* $\|\partial_x^l f\|_{L^p} \lesssim \|f\|_{L^2}^{1-\alpha}\|f\|_{H^m}^{\alpha}$.

The following lemma plays an important role to recover the derivative loss by using an effect of oscillation. The proof follows from a direct calculation.

Lemma 2.3. *Let* $p \geq 2$, $h \geq 5/4$ *and* $|k_p|^{1/h} \geq C\max_{1\leq l\leq p-1}\{|k_l|\}$ *for sufficiently large* $C = C(p) > 0$. *Then,*

$$|\Phi^{(p)}(\vec{k}^{(p)})| \gtrsim |k_p|^4|k_1+\cdots+k_{p-1}| \sim |k_{p+1}|^4|k_1+\cdots+k_{p-1}|$$

on $\vec{k}^{(p)} \in \mathbb{Z}_0^{(p)}$.

Main idea of the proof of Theorem 2.1. One of the key points is how we get the second term in the left-hand side of (2.7), which plays an important role to get the smoothing effect. The other key point is why we need the correction term of the modified energy (2.1). From the both points of view, $\partial_x^2 u$ is important. Therefore, we explain the idea of the proof only for the following simple example instead of (1.1):

$$(\partial_t + \partial_x^5)u(t,x) = u^{p-1}\partial_x^2 u. \tag{2.8}$$

In this case,

$$N = u^{p-1}\partial_x^2 u, \qquad P_N(u) = \frac{1}{2\pi}\int_{\mathbb{T}} u^{p-1}\,dx,$$

and

$$\begin{aligned} E_s(u) =& \frac{1}{2}\|\partial_x^s u\|_{L^2}^2 + \|u\|_{L^2}^2 + C_s\|u\|_{L^2}^{2s(p-1)+2} \\ &+ \sum_{(k_1,\dots,k_{p+1})\in\mathbb{Z}_0^{(p)}} \frac{(ik_p)^{s+1}(ik_{p+1})^{s+1}M_{H,h}^{(p)}M_{NZ}^{(p)}}{\Phi^{(p)}(\vec{k}^{(p)})}\prod_{l=1}^{p+1}\widehat{u}(k_l) \end{aligned} \tag{2.9}$$

by the definition (2.1), where we omitted the index j since $1 \le j \le j_0$ and $j_0 = 1$. Namely, $p := p_1$ and $M_{H,h}^{(p)} := M_{H,h_1}^{(p_1)}$. Calculating the L^2 product of the linear part of (2.8) and $\partial_x^{2s}u$, by integration by parts, we have

$$\langle(\partial_t + \partial_x^5)u, \partial_x^{2s}u\rangle = \frac{(-1)^s}{2}\frac{d}{dt}\|\partial_x^s u(t)\|_{L^2}^2.$$

Calculating the L^2 product of the nonlinear term and $\partial_x^{2s}u$, by integration by parts and the Leibniz rule, we have

$$\begin{aligned} \langle u^{p-1}\partial_x^2 u, \partial_x^{2s}u\rangle =& (-1)^{s-1}\langle\partial_x^{s-1}(u^{p-1}\partial_x^2 u), \partial_x^{s+1}u\rangle \\ =& (-1)^{s-1}\langle u^{p-1}\partial_x^{s+1}u, \partial_x^{s+1}u\rangle + C_s\langle\partial_x(u^{p-1})\partial_x^s u, \partial_x^{s+1}u\rangle \\ &+ C_s\langle\partial_x^2(u^{p-1})\partial_x^{s-1}u, \partial_x^{s+1}u\rangle + (\text{the latter terms}) \end{aligned}$$

By integration by parts,

$$\begin{aligned} (\text{the 2nd term}) &= C_s\int_{\mathbb{T}}\partial_x(u^{p-1})\partial_x^s u\,\partial_x^{s+1}u\,dx \\ &= \frac{C_s}{2}\int_{\mathbb{T}}\partial_x(u^{p-1})\partial_x\{(\partial_x^s u)^2\}\,dx \\ &= -\frac{C_s}{2}\int_{\mathbb{T}}\partial_x^2(u^{p-1})(\partial_x^s u)^2\,dx, \end{aligned}$$

which include at most s-th derivative of u. Thus, we have $|(\text{the 2nd term})| \lesssim \|u\|_{H^s}^2\|u\|_{H^{s_0}}^{p-1}$.

$$\begin{aligned} (\text{the 3rd term}) &= C_s\int_{\mathbb{T}}\partial_x^2(u^{p-1})\partial_x^{s-1}u\,\partial_x^{s+1}u\,dx \\ &= -C_s\int_{\mathbb{T}}\partial_x\big(\partial_x^2(u^{p-1})\partial_x^{s-1}u\big)\partial_x^s u\,dx, \end{aligned}$$

which include at most s-th derivative of u. Thus, we have $|(\text{the 3rd term})| \lesssim \|u\|_{H^s}^2\|u\|_{H^{s_0}}^{p-1}$. By the Gagliardo-Nirenberg inequality, the latter terms also satisfy the same estimate.

For the 1st term, we have

$$
\begin{aligned}
\text{(the 1st term)} =&(-1)^{s-1} P_N(u) \|\partial_x^{s+1} u\|_{L^2}^2 \\
&+ (-1)^{s-1} \langle \big(u^{p-1} - \frac{1}{2\pi}\int_{\mathbb{T}} u^{p-1}\, dx\big)\partial_x^{s+1} u, \partial_x^{s+1} u\rangle.
\end{aligned}
\tag{2.10}
$$

Here, we get the second term of (2.7) from the first term of (2.10). Below, we will get the correction term of the modified energy (2.9) from the second term of (2.10). By Plancherel's theorem,

$$
\begin{aligned}
&\langle \big(u^{p-1} - \frac{1}{2\pi}\int_{\mathbb{T}} u^{p-1}\, dx\big)\partial_x^{s+1} u, \partial_x^{s+1} u\rangle \\
=&\sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1} \prod_{l=1}^{p+1} \widehat{u}(t,k_l) \\
=&\sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} M_H M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1} \prod_{l=1}^{p+1} \widehat{u}(t,k_l) \\
&+ \sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} (1-M_H) M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1} \prod_{l=1}^{p+1} \widehat{u}(t,k_l) \\
=:& I + II.
\end{aligned}
$$

When $(k_1,\dots,k_{p+1}) \in \operatorname{supp}\{(1-M_H)\}$, there exist $j \le p-1$ such that $|k_j| \gtrsim |k_p| + |k_{p+1}|$. Therefore, we can easily show that $|II| \lesssim \|u\|_{H^s}^2 \|u\|_{H^{s_0}}^{p-1}$. Now, we apply the normal form reduction to I. Put $\widehat{v}(t,k_l) := e^{itk_l^5}\widehat{u}(t,k_l)$. Then, by differentiation by parts

$$
\begin{aligned}
I =& \sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} e^{t\Phi^{(p)}} M_H M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1} \prod_{l=1}^{p+1} \widehat{v}(t,k_l) \\
=&\frac{d}{dt}\Big(\sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} \frac{e^{t\Phi^{(p)}} M_H M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1}}{\Phi^{(p)}(\vec{k}^{(p)})} \prod_{l=1}^{p+1} \widehat{v}(t,k_l)\Big) \\
&- \sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} \frac{e^{t\Phi^{(p)}} M_H M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1}}{\Phi^{(p)}(\vec{k}^{(p)})} \frac{d}{dt}\prod_{l=1}^{p+1} \widehat{v}(t,k_l) \\
=:& I_a + I_b.
\end{aligned}
$$

We use $\widehat{v}(t,k_l) = e^{itk_l^5}\widehat{u}(t,k_l)$ again. Then,

$$
I_a = \frac{d}{dt}\Big(\sum_{(k_1,\dots,k_{p+1})\in \mathbb{Z}_0^{(p)}} \frac{M_H M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1}}{\Phi^{(p)}(\vec{k}^{(p)})} \prod_{l=1}^{p+1} \widehat{u}(t,k_l)\Big).
$$

Here we got the correction term of the modified energy (2.9). By (2.8), we have

$$\begin{aligned}\frac{d}{dt}\widehat{v}(t,k_l) &= e^{itk_l^5}\Big(\frac{d}{dt}\widehat{u}(t,k_l)+(ik_l)^5\widehat{u}(t,k_l)\Big)\\ &= e^{itk_l^5}\widehat{(\partial_t+\partial_x^5)u}(t,k_l),\\ &= e^{itk_l^5}\widehat{u^{p-1}\partial_x^2 u}(t,k_l),\end{aligned}$$

which include the second derivative. We substitute it for I_b. The symbol $(ik_p)^{s+1}(ik_{p+1})^{s+1}$ includes two derivatives loss. Therefore, I_b includes $2+2=4$ derivatives loss. However, by Lemma 2.3, we have $|\Phi^{(p)}| \gtrsim |k_p|^4 \sim |k_{p+1}|^4$. Therefore, the 4 derivatives loss can be recovered by $\Phi^{(p)}$ in I_b. Consequently, we get $|I_b| \lesssim \|u\|_{H^s}^2\|u\|_{H^{s_0}}^{2p-2}$.

Collecting the obtained results, we conclude

$$\begin{aligned}&\Big|\frac{1}{2}\frac{d}{dt}\|\partial_x^s u(t)\|_{L^2}^2+\frac{d}{dt}\Big(\sum_{(k_1,\dots,k_{p+1})\in\mathbb{Z}_0^{(p)}}\frac{M_{H,h}M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1}}{\Phi^{(p)}(\vec{k}^{(p)})}\prod_{l=1}^{p+1}\widehat{u}(t,k_l)\Big)\\ &\quad+P_N(u)\|\partial_x^{s+1}u\|_{L^2}^2\Big|\\ &\lesssim \|u\|_{H^s}^2(\|u\|_{H^{s_0}}^{p-1}+\|u\|_{H^{s_0}}^{2p-2}) \lesssim \|u\|_{H^s}^2(1+\|u\|_{H^{s_0}}^{2r}) \lesssim E_s(u)(1+E_{s_0}(u))^r.\end{aligned}$$

Since the correction term is real valued by Remark 2, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|\partial_x^s u(t)\|_{L^2}^2+\sum_{(k_1,\dots,k_{p+1})\in\mathbb{Z}_0^{(p)}}\frac{M_{H,h}M_{NZ}(ik_p)^{s+1}(ik_{p+1})^{s+1}}{\Phi^{(p)}(\vec{k}^{(p)})}\prod_{l=1}^{p+1}\widehat{u}(t,k_l)\Big)\\ &\quad+P_N(u)\|\partial_x^{s+1}u\|_{L^2}^2\\ &\lesssim E_s(u)(1+E_{s_0}(u))^r.\end{aligned}\tag{2.11}$$

Since s_0 is sufficiently large and $s \ge s_0$, we can easily show

$$\frac{d}{dt}(\|u\|_{L^2}^2+C_s\|u\|_{L^2}^{2s(p-1)+2}) \lesssim E_s(u)(1+E_{s_0}(u))^r. \tag{2.12}$$

by the Sobolev embedding and (2.6). Here, we used the assumption $r=(2s+1)(p_{max}-1)/2$. From (2.11) and (2.12), we conclude (2.7). □

§ 3. Estimate of the difference of two solutions

For the proof of the uniqueness and the continuous dependence, we need an estimate of the difference of two solutions. In this section we present it. For $s \ge 0$, we put

$$\begin{aligned}F_s(u_1,u_2) &:= \frac{1}{2}\|\partial_x^s u_0\|_{L^2}^2\\ &+\sum_{j=1}^{j_0}\sum_{\vec{k}^{(p_j)}\in\mathbb{Z}_0^{(p_j)}}\frac{(ik_{p_j})^{s+1}(ik_{p_j+1})^{s+1}M_{NR,j}}{\Phi^{(p_j)}(\vec{k}^{(p_j)})}\widehat{u}_0(k_{p_j})\widehat{u}_0(k_{p_j+1})\prod_{l=1}^{p_j-1}\widehat{u}_1(k_l),\end{aligned}\tag{3.1}$$

where $u_0 := u_1 - u_2$. See (2.2)–(2.5), for the definitions of the Fourier multipliers $\Phi^{(p)}(\vec{k}^{(p)}), D_{a,b,c}, M^{(p)}_{H,h}, M^{(p)}_{NZ}$.

Remark. Assume that $\|u_1\|_{H^{s_0}} + \|u_2\|_{H^{s_0}} \lesssim 1$. Then, by the presence of $\Phi^{(p_j)}$ and Lemma 2.3, the second term of (3.1) can be controlled by the first term and we obtain

$$F_s(u_1, u_2) \sim \|\partial_x^s u_0\|^2_{L^2}. \tag{3.2}$$

Proposition 3.1. *Let $s \geq 0$. Assume that $u_1, u_2 \in L^\infty([0,T) : H^{s_0+s}(\mathbb{T}))$ satisfy* (1.1) *on $t \in [0,T)$. Then, it follows that*

$$\begin{aligned}
&\frac{d}{dt}F_s(u_1(t), u_2(t)) + P_N(u_1(t))\|\partial_x^{s+1}(u_1(t) - u_2(t))\|^2_{L^2}\\
&\lesssim F_s(u_1(t), u_2(t))(1 + \|u_1(t)\|^2_{H^{\max\{s_0,s\}}} + \|u_2(t)\|^2_{H^{\max\{s_0,s\}}})^{(p_{\max}-1)/2}\\
&+F_0(u_1(t), u_2(t))\|u_2(t)\|^2_{H^{s_0+s-1}}\\
&\qquad \times (1 + \|u_1(t)\|^2_{H^{\max\{s_0,s\}}} + \|u_2(t)\|^2_{H^{\max\{s_0,s\}}})^{(p_{\max}-3)/2}.
\end{aligned} \tag{3.3}$$

on $t \in [0,T)$.

Outline of the proof. In the same manner as the proof of Theorem 2.1, we explain the idea of the proof only for the simple case (2.8). Since $j_0 = 1, a_1 = 0, b_1 = 1, \lambda_1 = 1$, by the definition (3.1), we have

$$\begin{aligned}
F_s(u_1, u_2) &= \frac{1}{2}\|\partial_x^s u_0\|^2_{L^2} + \\
&\sum_{(k_1,\dots,k_{p+1})\in\mathbb{Z}_0^{(p)}} \frac{(ik_p)^{s+1}(ik_{p+1})^{s+1}M^{(p)}_{H,h}M^{(p)}_{NZ}}{\Phi^{(p)}(\vec{k}^{(p)})}\widehat{u}_1(k_1)\cdots\widehat{u}_1(k_{p-1})\widehat{u}_0(k_p)\widehat{u}_0(k_{p+1})
\end{aligned} \tag{3.4}$$

where $u_0 := u_1 - u_2$ and we omitted the index j. Calculating the L^2 product of $\partial_x^{2s}(u_1 - u_2)$ and the difference of the linear parts of (2.8) for u_1 and u_2, by integration by parts, we have

$$\langle(\partial_t + \partial_x^5)(u_1 - u_2), \partial_x^{2s}(u_1 - u_2)\rangle = \frac{(-1)^s}{2}\frac{d}{dt}\|\partial_x^s u_0(t)\|^2_{L^2}.$$

Calculating the L^2 product of $\partial_x^{2s}(u_1 - u_2)$ and the difference of the nonlinear terms for u_1 and u_2, we have

$$\begin{aligned}
&\langle u_1^{p-1}\partial_x^2 u_1 - u_2^{p-1}\partial_x^2 u_2, \partial_x^{2s}(u_1 - u_2)\rangle\\
=&(-1)^{s-1}\langle \partial_x^{s-1}(u_1^{p-1}\partial_x^2 u_1 - u_2^{p-1}\partial_x^2 u_2), \partial_x^{s+1}u_0\rangle\\
=&(-1)^{s-1}\langle u_1^{p-1}\partial_x^{s+1}u_1 - u_2^{p-1}\partial_x^{s+1}u_2, \partial_x^{s+1}u_0\rangle\\
&+ C_s\langle \partial_x(u_1^{p-1})\partial_x^s u_1 - \partial_x(u_2^{p-1})\partial_x^s u_2, \partial_x^{s+1}u_0\rangle + \cdots
\end{aligned} \tag{3.5}$$

Here, we used integration by parts and the Leibniz rule. As we see in the proof of Theorem 2.1, the most difficult term to estimate is the first term. Therefore, we omit the proof of the estimate for the other terms and consider only the first term here.

$$\begin{aligned}\text{(the first term)} =&(-1)^{s-1}\langle u_1^{p-1}\partial_x^{s+1}(u_1-u_2),\partial_x^{s+1}u_0\rangle\\ &+(-1)^{s-1}\langle (u_1^{p-1}-u_2^{p-1})\partial_x^{s+1}u_2,\partial_x^{s+1}u_0\rangle\\ =&:I+II.\end{aligned}$$

Since

$$\begin{aligned}I =&(-1)^{s-1}P_N(u_1(t))\|\partial_x^{s+1}u_0(t)\|_{L^2}^2\\ &+(-1)^{s-1}\langle (u_1^{p-1}-P_N(u_1(t)))\partial_x^{s+1}u_0,\partial_x^{s+1}u_0\rangle,\end{aligned}$$

in the same manner as (2.10), we obtain the second term of the left-hand side of (3.3), the second term of (3.4) and error terms which are bounded by the first term of the right-hand side of (3.3). Since

$$u_1^{p-1}-u_2^{p-1}=u_0\sum_{m=0}^{p-2}(-1)^m u_1^{p-2-m}u_2^m,$$

by integration by parts,

$$\begin{aligned}|II|\lesssim&\sum_{m=0}^{p-2}|\langle(\partial_x u_0)u_1^{p-2-m}u_2^m\partial_x^{s+1}u_2,\partial_x^s u_0\rangle|\\ &+|\langle u_0\partial_x(u_1^{p-2-m}u_2^m\partial_x^{s+1}u_2),\partial_x^s u_0\rangle|\\ =&:\sum_{m=0}^{p-2}II_{a,m}+II_{b,m}.\end{aligned}\tag{3.6}$$

When $s=0$ or 1, it is easy to check that $II_{a,m}$ is bounded by the first term of the right-hand side of (3.3). When $s\geq 2$, by the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}II_{a,m}&\lesssim\|\partial_x^s u_0\|_{L^2}\|\partial_x u_0\|_{L^2}\|\partial_x^{s+1}u_2\|_{L^\infty}(\|u_1\|_{L^\infty}+\|u_2\|_{L^\infty})^{p-2}\\ &\lesssim\|\partial_x^s u_0\|_{L^2}^{1+1/s}\|u_0\|_{L^2}^{1-1/s}\|\partial_x^{s+3/2+3/2(s-1)}u_2\|_{L^2}^{1-1/s}\|\partial_x^s u_2\|_{L^2}^{1/s}(\|u_1\|_{H^1}+\|u_2\|_{H^1})^{p-2}.\end{aligned}$$

Since $|x|^{1+1/s}|y|^{1-1/s}\lesssim|x|^2+|y|^2$, we have

$$\begin{aligned}&\|\partial_x^s u_0\|_{L^2}^{1+1/s}\|\partial_x^s u_2\|_{L^2}^{1/s}(\|u_1\|_{H^1}+\|u_2\|_{H^1})\big(\|u_0\|_{L^2}\|\partial_x^{s+3/2+3/2(s-1)}u_2\|_{L^2}\big)^{1-1/s}\\ \lesssim&\|\partial_x^s u_0\|_{L^2}^2(\|u_1\|_{H^{\max\{s_0,s\}}}+\|u_2\|_{H^{\max\{s_0,s\}}})^2+\|u_0\|_{L^2}^2\|u_2\|_{H^{s+s_0-1}}^2.\end{aligned}$$

Therefore, $II_{a,m}$ is bounded by the right-hand side of (3.3) from (3.2). In the same manner,

$$II_{b,m}\lesssim\|u_0\|_{L^2}\|\partial_x^s u_0\|_{L^2}\|u_2\|_{H^{s+3}}(\|u_1\|_{H^2}+\|u_2\|_{H^2})^{p-2},$$

which is bounded by the right-hand side of (3.3). □

§ 4. Proof of the main theorems

In this section, we give the outline of the proof of Theorems 1.2, 1.3 and 1.4. Here, we introduce the following smoothing operator.

Definition 4.1. For $\eta \in (0,1], s \geq 0,\ f \in H^s$, we put

$$\widehat{J_{\eta,s} f}(k) := \exp(-\eta(1+|k|^2)^{s/2})\widehat{f}(k).$$

For the proof of the following lemma, see Lemma 6.4 in [10].

Lemma 4.2. *Let $0 \leq j \leq s,\ 0 \leq l$ and $f \in H^s(\mathbb{T})$. Then, $J_{\eta,s} f \in H^\infty(\mathbb{T})$ satisfies*

$$\begin{aligned} &\|J_{\eta,s} f - f\|_{H^s} \to 0 \quad (\eta \to 0), \\ &\|J_{\eta,s} f - f\|_{H^{s-j}} \lesssim \eta^{j/s} \|f\|_{H^s}, \quad \|J_{\eta,s} f\|_{H^{s-j}} \lesssim \|f\|_{H^{s-j}}, \\ &\|J_{\eta,s} f\|_{H^{s+l}} \lesssim \eta^{-l/s} \|f\|_{H^s}. \end{aligned}$$

The approximation argument used in (Step 3) and (Step 4) below was introduced by Bona-Smith in [2]. First, we give the proof of Theorem 1.2. By time reversibility, we only need to consider positive time interval $[0,T)$.

(The outline of the proof of Theorem 1.2).

(Step 1) We consider the following regularized problem:

$$(\partial_t - \varepsilon\partial_x^6 + \partial_x^5) u_\varepsilon(t,x) = N(\partial_x^3 u_\varepsilon, \partial_x^2 u_\varepsilon, \partial_x u_\varepsilon, u_\varepsilon),\ (t,x) \in [0,T) \times \mathbb{T}, \tag{4.1}$$

$$u_\varepsilon(0,\cdot) = \varphi(\cdot) \in H^s(\mathbb{T}), \tag{4.2}$$

where $\varepsilon \in (0,1]$ and $s \geq s_0$. By the parabolic smoothing effect of $-\varepsilon\partial_x^6$ and the standard fixed point argument, we have the local well-posedness of (4.1)–(4.2) and the solution is in $C^\infty((0,T_\varepsilon) \times \mathbb{T})$ where T_ε satisfies $T_\varepsilon = +\infty$ or $\liminf_{t \to T_\varepsilon} \|u_\varepsilon(t)\|_{H^s} = \infty$ (See Section 6.1 in [10]). In the same manner as the proof of Theorem 2.1, we have

$$\frac{d}{dt} E_s(u_\varepsilon(t)) + P_N(u_\varepsilon(t)) \|\partial_x^{s+1} u_\varepsilon(t)\|_{L^2}^2 \lesssim E_s(u(t))(1 + E_{s_0}(u_\varepsilon(t)))^r, \tag{4.3}$$

where $r := (2s+1)(p-1)/2$ and the implicit constant does not depend on ε. Recall that $P_N \equiv 0$ and apply the Gronwall inequality to (4.3) with $s = s_0$. Then, we have the following a priori estimate:

$$\sup_{0 \leq t \leq T} E_{s_0}(u_\varepsilon(t)) \leq \frac{1 + E_{s_0}(\varphi)}{\big(1 - CTr_0(1 + E_{s_0}(\varphi))^{r_0}\big)^{1/r_0}} \lesssim 1 + E_{s_0}(\varphi) \tag{4.4}$$

where $T = T(E_{s_0}(\varphi)) = T(\|\varphi\|_{H^{s_0}})$ and $r_0 := (2s_0+1)(p-1)/2$. Combining (4.3) and (4.4), we have

$$\frac{d}{dt}E_s(u_\varepsilon(t)) \lesssim E_s(u(t))(1+E_{s_0}(\varphi))^r.$$

Therefore, by the Gronwall inequality, we have the following a priori estimate:

$$\sup_{0\le t\le T'} E_s(u_\varepsilon(t)) \le E_s(\varphi)\exp\{C(1+E_{s_0}(\varphi))^r T'\} \le 2E_s(\varphi)$$

for sufficiently small $T' = T'(r, E_{s_0}(\varphi)) = T'(s, E_{s_0}(\varphi))$. Iterating this argument, we obtain

$$\sup_{0\le t\le T} E_s(u_\varepsilon(t)) \le C(s, E_{s_0}(\varphi))E_s(\varphi)$$

Therefore, by (2.6),

$$\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^s}^2 \le C(s, E_{s_0}(\varphi))(\|\varphi\|_{H^s}^2 + \|\varphi\|_{L^2}^{2s(p_{max}-1)+2}) =: K_s(\varphi). \tag{4.5}$$

Therefore, $T \le T_\varepsilon$ and the solution u_ε exists on $[0,T)$. By (4.1), (4.5) and the Sobolev inequality, we also have

$$\sup_{0\le t\le T}\|\partial_t u_\varepsilon(t)\|_{H^{s-6}}^2 \le (1+\|u_\varepsilon(t)\|_{H^s}^2)^{p_{max}} \lesssim (1+K_s(\varphi))^{p_{max}}. \tag{4.6}$$

Let $\varepsilon\to 0$. Then, by the standard limiting argument with (4.5) and (4.6), we have a solution u to (1.1)–(1.2) such that

$$\sup_{0\le t\le T}\|u(t)\|_{H^s}^2 \le K_s(\varphi). \tag{4.7}$$

(Step 2) By Proposition 3.1 with $s=0$ and the Gronwall inequality, we have

$$\begin{aligned}&\sup_{0\le t\le T} F_0(u_1(t), u_2(t))\\ &\lesssim F_0(u_1(0), u_2(0))\exp\left(CT(1+\|u_1\|_{L^\infty([0,T):H^{s_0})}^2 + \|u_2\|_{L^\infty([0,T):H^{s_0})}^2)^{(p_{max}-1)/2}\right) = 0,\end{aligned}$$

which yields the uniqueness on $[0,T)$.

(Step 3) Fix $s\ge s_0$. We will prove the persistence of regularity, that is $u\in C([0,T) : H^s(\mathbb{T}))$. Let $\varphi_\eta := J_{\eta,s}\varphi \in H^\infty(\mathbb{T})$. By (Step 1), we have the solution $u_\eta \in L^\infty([0,T); H^\infty(\mathbb{T}))$ to (1.1) with initial data φ_η and

$$\sup_{0\le t\le T}\|u_\eta(t)\|_{H^\sigma}^2 \le K_\sigma(\varphi_\eta) \tag{4.8}$$

for any σ such that $s_0\le\sigma$. Note that

$$K_{s-j}(\varphi_\eta) \lesssim K_{s-j}(\varphi),\quad K_{s+l}(\varphi_\eta) \lesssim \eta^{-2l/s}K_s(\varphi) \tag{4.9}$$

for $0 \leq j \leq s$ and $0 \leq l$ by Lemma 4.2. Moreover, we can take $T = T(\|\varphi_\eta\|_{H^{s_0}}) \sim T(\|\varphi\|_{H^{s_0}})$ since $\|\varphi_\eta\|_{H^{s_0}} \sim \|\varphi\|_{H^{s_0}}$ by Lemma 4.2. Let $0 < \eta' < \eta < 1$. By Proposition 3.1 with $u_1 := u_{\eta'}, u_2 := u_\eta$, (4.8) and (4.9), it follows that

$$\frac{d}{dt}F_0(u_\eta(t), u'_\eta(t)) \lesssim F_0(u_\eta(t), u_{\eta'}(t))(1 + K_{s_0}(\varphi))^{(p_{\max}-1)/2}.$$

Therefore, by the Gronwall inequality, (3.2) and Lemma 4.2,

$$\begin{aligned} &\sup_{0\leq t\leq T} \|u_\eta(t) - u_{\eta'}(t)\|_{L^2}^2 \sim \sup_{0\leq t\leq T} F_0(u_\eta(t), u'_\eta(t)) \\ &\lesssim F_0(\varphi_\eta, \varphi_{\eta'}) \sim \|\varphi_\eta - \varphi_{\eta'}\|_{L^2}^2 \lesssim \|\varphi_\eta - \varphi\|_{L^2}^2 + \|\varphi - \varphi_{\eta'}\|_{L^2}^2 \lesssim \eta^2\|\varphi\|_{H^s}^2 \end{aligned} \tag{4.10}$$

where $T = T(\|\varphi\|_{H^{s_0}})$. By Proposition 3.1 with $u_1 := u_{\eta'}, u_2 := u_\eta$, (4.8) and (4.9), it follows that

$$\begin{aligned} \frac{d}{dt}F_s(u_\eta(t), u_{\eta'}(t)) &\lesssim F_s(u_\eta(t), u_{\eta'}(t))(1 + K_s(\varphi))^{(p_{\max}-1)/2} \\ &\quad + F_0(u_\eta(t), u_{\eta'}(t))\eta^{2(-s_0+1)/s}K_s(\varphi)(1 + K_s(\varphi))^{(p_{\max}-3)/2}. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we obtain

$$\frac{d}{dt}F_s(u_\eta(t), u_{\eta'}(t)) \lesssim \big(F_s(u_\eta(t), u_{\eta'}(t)) + \eta^{2(s-s_0+1)/s}\big)(1 + K_s(\varphi))^{(p_{\max}-1)/2}. \tag{4.12}$$

By the Gronwall inequality and (3.2),

$$\begin{aligned} &\sup_{0\leq t\leq T''} \|\partial_x^s(u_\eta(t) - u_{\eta'}(t))\|_{L^2}^2 \sim \sup_{0\leq t\leq T''} F_s(u_\eta(t), u_{\eta'}(t)) \\ &\lesssim F_s(\varphi_\eta, \varphi_{\eta'}) + \eta^{2(s-s_0+1)/s} \sim \|\partial_x^s(\varphi_\eta - \varphi_{\eta'})\|_{L^2}^2 + \eta^{2(s-s_0+1)/s} \to 0 \quad (\eta \to 0) \end{aligned} \tag{4.13}$$

where T'' depends on $K_s(\varphi)$, that is to say $T'' = T''(\|\varphi\|_{H^s})$. From (4.10) and (4.13), we conclude u_η is a Cauchy sequence in $C([0, T'') : H^s(\mathbb{T}))$. The limit u_∞ is a solution to (1.1)–(1.2). Thus, by the uniqueness, the solution u obtained in (Step 1) is equal to u_∞ and in $C([0, T'') : H^s(\mathbb{T}))$. Iterating this argument, we can extend the time interval of the persistence result from $[0, T'')$ to $[0, T)$ since we have a priori estimate (4.7).

(Step 4) We will show the continuous dependence on initial data. In the same manner as (Step 3), we only need to show it on $[0, T'')$. Let $\varphi_\eta^j := J_{\eta,s}\varphi^j, \varphi_\eta := J_{\eta,s}\varphi$ and u_η^j (resp. u_η) be the solution to (1.1) with initial data φ_η^j (resp. φ_η). By taking $\eta' \to 0$ in (4.10) and (4.13),

$$\sup_{0\leq t\leq T''} \|u_\eta(t) - u(t)\|_{H^s}^2 \lesssim \eta^2\|\varphi\|_{H^s}^2 + \|\varphi_\eta - \varphi\|_{H^s}^2 + \eta^{2(s-s_0+1)/s}. \tag{4.14}$$

In the same manner,

$$\sup_{0\leq t\leq T''} \|u_\eta^j(t) - u^j(t)\|_{H^s}^2 \lesssim \eta^2\|\varphi^j\|_{H^s}^2 + \|\varphi_\eta^j - \varphi^j\|_{H^s}^2 + \eta^{2(s-s_0+1)/s}.$$

Since

$$\begin{aligned}\|\varphi_\eta^j-\varphi^j\|_{H^s} &\lesssim \|J_{\eta,s}(\varphi^j-\varphi)\|_{H^s}+\|\varphi_\eta-\varphi\|_{H^s}+\|\varphi-\varphi^j\|_{H^s}\\ &\lesssim \|\varphi^j-\varphi\|_{H^s}+\|\varphi_\eta-\varphi\|_{H^s},\end{aligned}$$

we have

$$\begin{aligned}&\sup_{0\le t\le T''}\|u_\eta^j(t)-u^j(t)\|_{H^s}^2\\ &\lesssim \eta^2\|\varphi\|_{H^s}^2+\|\varphi^j-\varphi\|_{H^s}^2+\|\varphi_\eta-\varphi\|_{H^s}^2+\eta^{2(s-s_0+1)/s}.\end{aligned}\tag{4.15}$$

By Proposition 3.1 with $u_1:=u_\eta^j, u_2:=u_\eta$, in the same manner as (4.10), we have

$$\begin{aligned}&\sup_{0\le t\le T}\|u_\eta^j(t)-u_\eta(t)\|_{L^2}^2\sim\sup_{0\le t\le T}F_0(u_\eta^j(t),u_\eta(t))\\ &\lesssim F_0(\varphi_\eta^j,\varphi_\eta)\sim\|\varphi_\eta^j-\varphi_\eta\|_{L^2}^2\lesssim\|\varphi^j-\varphi\|_{L^2}^2.\end{aligned}\tag{4.16}$$

By Proposition 3.1 with $u_1:=u_\eta^j, u_2:=u_\eta$, it follows that

$$\begin{aligned}\frac{d}{dt}F_s(u_\eta^j(t),u_\eta(t)) &\lesssim F_s(u_\eta^j(t),u_\eta(t))(1+K_s(\varphi))^{(p_{\max}-1)/2}\\ &\quad+F_0(u_\eta^j(t),u_\eta(t))\eta^{2(-s_0+1)/s}K_s(\varphi)(1+K_s(\varphi))^{(p_{\max}-3)/2}.\end{aligned}$$

Inserting (4.16) into it, we have

$$\begin{aligned}&\frac{d}{dt}F_s(u_\eta^j(t),u_\eta(t))\\ &\lesssim \big(F_s(u_\eta^j(t),u_\eta(t))+\|\varphi^j-\varphi\|_{L^2}^2\eta^{2(-s_0+1)/s}\big)(1+K_s(\varphi))^{(p_{\max}-1)/2}\end{aligned}\tag{4.17}$$

on $[0,T)$. Applying the Gronwall inequaity, we obtain

$$\begin{aligned}\sup_{0\le t\le T''}\|\partial_x^s(u_\eta^j(t)-u_\eta(t))\|_{L^2}^2 &\sim \sup_{0\le t\le T''}F_s(u_\eta^j(t),u_\eta(t))\\ &\lesssim F_s(\varphi_\eta^j,\varphi_\eta)+\|\varphi^j-\varphi\|_{L^2}^2\eta^{2(-s_0+1)/s}\\ &\sim\|\partial_x^s(\varphi_\eta^j-\varphi_\eta)\|_{L^2}^2+\|\varphi^j-\varphi\|_{L^2}^2\eta^{2(-s_0+1)/s}.\end{aligned}\tag{4.18}$$

By (4.16) and (4.18), we conclude

$$\sup_{0\le t\le T''}\|u_\eta^j(t)-u_\eta(t)\|_{H^s}\lesssim(1+\eta^{(-s_0+1)/s})\|\varphi^j-\varphi\|_{H^s}.\tag{4.19}$$

Here we take $j=j(\eta)$ such that $j\to\infty$ and $\eta^{(-s_0+1)/s}\|\varphi^j-\varphi\|_{H^s}\to 0$ as $\eta\to 0$. From (4.14), (4.15) and (4.19), we conclude

$$\begin{aligned}&\sup_{0\le t\le T''}\|u^j(t)-u(t)\|_{H^s}\\ &\lesssim\sup_{0\le t\le T''}\|u^j(t)-u_\eta^j(t)\|_{H^s}+\|u_\eta^j(t)-u_\eta(t)\|_{H^s}+\|u_\eta(t)-u(t)\|_{H^s}\to 0\quad(\eta\to 0).\end{aligned}$$

□

Next, we mention the outline of the proof of Theorem 1.3 briefly. The differences between Theorem 1.3 and Theorem 1.2 on $[0,T)$ are only the condition $P_N(\varphi)>0$ and the parabolic smoothing effect $u\in C^\infty((0,T)\times\mathbb{T})$. $P_N(u(t))$ is continuous function if u is sufficiently smooth. Therefore, roughly speaking, we have $P_N(u(t))>P_N(\varphi)/2>0$ for sufficiently small interval $[0,T)$. By using the second term of (2.7), we obtain a bound not only for $\|u(t)\|_{H^s}$ but also for $\int_0^t P_N(u(t))\|\partial_x^{s+1}u(t')\|_{L^2}^2\,dt'$ in (4.7). Therefore, we obtain $u(t)\in H^{s+1}$ a.e. $t\in(0,T)$. For any $\delta\in(0,T)$, we choose $0<t_1<\delta(1-2^{-1})$ such that $u(t_1)\in H^{s+1}$. We repeat the same argument with initial time t_1. Then, we have $u(t)\in H^{s+2}$ a.e. $t\in(t_1,T)$. We choose $t_1<t_2<\cdots<t_m<\delta(1-2^{-m})$ and iterate this argument to obtain $u(t)\in H^{s+m+1}(\mathbb{T})$ on $t\in(t_m,T)$, that includes $u(t)\in C^\infty(\mathbb{T})$ on $t\in(\delta,T)$. Since we can choose any small $\delta>0$, we conclude $u(t)\in C^\infty(\mathbb{T})$ on $t\in(0,T)$. By (1.1), we also have $u(t)\in C^\infty((0,T)\times\mathbb{T})$. Note that we used the fact that T depends only on $\|\varphi\|_{H^{s_0}}$ and does not depend $\|\varphi\|_{H^s}$ in this argument.

Finally, we mention the proof of Theorem 1.4 We prove it by contradiction. Assume $\varphi\in H^{s_0}(\mathbb{T})\setminus C^\infty(\mathbb{T})$, $P_N(\varphi)<0$ and there exist a solution $u\in C([0,T);H^{s_0}(\mathbb{T}))$. We take sufficiently small $t_0>0$. Then, we have $P_N(u(t_0))<0$ since $P_N(u(t))$ is continuous. We apply Theorem 1.3. Then we obtain a backward solution on $(t_0-T''',t_0]$ with initial data $u(t_0)$. Since $T'''=T'''(\|u(t_0)\|_{H^{s_0}})$ and $\|u(t_0)\|_{H^{s_0}}\sim\|\varphi\|_{H^{s_0}}$, we can choose sufficiently small t_0 satisfying $0\in(t_0-T''',t_0]$. Therefore, by the parabolic smoothing effect in Theorem 1.3, we conclude that the solution is in $C^\infty((t_0-T''',t_0]\times\mathbb{T})$. It contradicts to the uniqueness and $u(0)=\varphi\notin C^\infty(\mathbb{T})$.

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