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The lifespan of solutions to semilinear wave equations with scale invariant damping in one space dimension

By

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Abstract

This note is a summary of the result in Wakasa [10]. We consider the semilinear wave equations with special scale invariant damping in one space dimension. The equation can be reduced to some nonlinear wave equations under some transformation. Making use of the iteration argument of John [5], we get the sharp estimates of the lifespan. Also, we show that if the initial data are odd functions, then the critical exponent changes. This fact follows from the special property of the solution of the wave equation in one space dimension.

S 1. Introduction

We consider the initial value problem for semilinear damped wave equations:

$$\begin{cases} v_{tt} - \Delta v + \dfrac{2}{1+t} v_t = F(v), \quad \text{in } \mathbf{R}^n \times [0, \infty), \\ v(x,0) = \varepsilon f(x),\ v_t(x,0) = \varepsilon g(x), \quad x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $F(v) = |v|^p$ or $F(v) = |v|^{p-1}v$ with $p > 1$, $f, g \in C_0^\infty(\mathbf{R}^n)$, and $\varepsilon > 0$ is a "small" parameter.

First of all, we introduce a background of (1.1). We consider the following damped wave equation:

$$\begin{cases} v_{tt} - \Delta v + \dfrac{\mu}{(1+t)^\beta} v_t = 0, \quad \text{in } \mathbf{R}^n \times [0, \infty), \\ v(x,0) = f(x),\ v_t(x,0) = g(x), \quad x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

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where $\mu > 0$, $\beta > -1$ and $n \geq 1$.

When $\beta > 1$, it is known that the solution of (1.2) behaves like that of the wave equation. Actually, Wirth [14] have obtained L^p-L^q decay estimates for the solution to (1.2), and showed that the decay rate is almost the same as that of the wave equation. In this case, the damping term is "non-effective". On the other hand, the solution behaves like that of the heat equation in the case of $\beta < 1$. In fact, Wirth [15] obtained L^p-L^q decay estimates for the solution to (1.2) which is almost the same as that of the heat equation. In this case, the damping term is "effective".

In the case of $\beta = 1$, we note that this equation is invariant under a scaling $\widetilde{v}(x,t) = v(\sigma x, \sigma(1+t)-1)$ with $\sigma > 0$. For this reason, (1.2) with $\beta = 1$ is called scale invariant damped wave equation. In this case, the behavior of solutions of (1.2) is determined by μ. Wirth [13] showed that a borderline of the behavior of solutions of (1.2) is $\mu = 1$. For example, the solution of (1.2) satisfies the following L^2-L^2 estimates:

$$\|v(\cdot,t)\|_{L^2} \leq \begin{cases} C(1+t)^{1-\mu}(\|f\|_{L^2} + \|g\|_{L^2}) & \text{if } \mu \in (0,1), \\ C\log(e+t)(\|f\|_{L^2} + \|g\|_{L^2}) & \text{if } \mu = 1, \\ C(\|f\|_{L^2} + \|g\|_{L^2}) & \text{if } \mu > 1. \end{cases}$$

Moreover, the damping term is "effective" if $\mu > 1$, and "non-effective" if $0 < \mu \leq 1$ in view of the decay rate of L^p-L^q estimate.

Next, we consider the semilinear damped wave equation:

$$(1.3) \qquad \begin{cases} v_{tt} - \Delta v + \dfrac{\mu}{(1+t)^\beta} v_t = |v|^p, & \text{in } \mathbf{R}^n \times [0,\infty), \\ v(x,0) = f(x),\ v_t(x,0) = g(x), & x \in \mathbf{R}^n. \end{cases}$$

When $-1 < \beta < 1$, Lin & Nishihara & Zhai [8] showed that the solution of (1.3) blows up in finite time in the case of $1 < p \leq p_F(n) := 1 + 2/n$, where $p_F(n)$ (the Fujita exponent) is the critical exponent for the semilinear heat equation. Also they showed that (1.3) has a global in-time solution for "small" initial data if $p > p_F(n)$. Later, D'Abbicco & Lucente & Reissig [3] obtained the global existence result for a general damped wave equation which includes (1.3).

For the scale invariant damping case $\beta = 1$, Wakasugi [12] showed that the solution of (1.3) blows up in finite time if $1 < p \leq p_F(n)$ and $\mu > 1$, or $1 < p \leq 1 + 2/(n+\mu-1)$ and $0 < \mu \leq 1$. Also, Wakasugi [11] showed that an upper bound of the lifespan which is the maximal existence time of solutions of (1.3) is $C\varepsilon^{-(p-1)/\{2-n(p-1)\}}$ if $1 < p < p_F(n)$ and $\mu > 1$, and $C\varepsilon^{-(p-1)/\{2-(n+\mu-1)(p-1)\}}$ if $1 < p < 1 + 2/(n+\mu-1)$ and $0 < \mu \leq 1$, where C is a positive constant independent of ε. On the other hand, D'Abbicco [2] showed that (1.3) has a global in-time solution for "small" initial data if $p > p_F(n)$ and $n = 1$, $\mu > 5/3$ or $n = 2$, $\mu \geq 3$ or $n \geq 3$, $\mu \geq n+2$. Here we note that the small data global existence for (1.1) with $n = 1$ holds in the case of $p > p_F(1) = 3$.

For the problem (1.1), we see that the damping term is "effective". Recently, D'Abbicco & Lucente & Reissig [4] have obtained the following results to the problem (1.1). Let

$$p_c(n) = \max\{p_F(n),\ p_0(n+2)\}, \tag{1.4}$$

where $p_0(n)$ is the Strauss exponent, that is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$. Since

$$\begin{cases} p_F(n) < p_0(n+2) & \text{for } n \geq 3, \\ p_F(2) = p_0(4) = 2, & p_0(3) = 1 + \sqrt{2} < p_F(1) = 3, \end{cases}$$

we see that $p_c(n) = p_0(n+2)$ for $n \geq 2$, and $p_c(1) = p_F(1) = 3$. They showed that the problem (1.1) has a global in-time solution for "small" ε, and (f,g) has compact support if $p > p_c(n)$ in the case of $n = 2, 3$. On the other hand, the solution of (1.1) blows up in finite time if $1 < p \leq p_c(n)$ in the case of $n \geq 1$.

In the proof of [4], they reduced the problem to the following semilinear wave equations:

$$\begin{cases} u_{tt} - \Delta u = \dfrac{F(u)}{(1+t)^{p-1}}, & \text{in } \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon\{f(x) + g(x)\}, & x \in \mathbf{R}^n, \end{cases} \tag{1.5}$$

by setting $u(x,t) = (1+t)v(x,t)$. From now on, we consider (1.5) in one space dimension:

$$\begin{cases} u_{tt} - u_{xx} = \dfrac{F(u)}{(1+t)^{p-1}}, & \text{in } \mathbf{R} \times [0,\infty), \\ u(x,0) = \varepsilon f(x),\ u_t(x,0) = \varepsilon\{f(x) + g(x)\}, & x \in \mathbf{R}. \end{cases} \tag{1.6}$$

Our purpose in the present note is to show the followings for the problem (1.6). The first one is to derive an estimate of the upper bound of the lifespan in the case of $p = 3$, and show the optimality of the upper bound. Namely we give an estimate of the lifespan from below which has the same order with respect to ε as the upper bound. The second one is to show the critical exponent changes to $1 + \sqrt{2}$ from 3 when the initial data are odd functions. This fact follows from the special property of the solution of the wave equation in one space dimension. Our proof is based on the iteration argument which was introduced by John [5]. To state our result, we define the lifespan T_ε of the C^2-solution of (1.6) by

$$\begin{aligned} T_\varepsilon \equiv T_\varepsilon(f,g) := \sup\{T \in [0,\infty)\ :&\ \text{There exists a unique solution} \\ & u \in C^2(\mathbf{R} \times [0,T)) \text{ of (1.6).}\} \end{aligned}$$

for arbitrarily fixed $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$.

The following theorem shows that the optimality of the upper bound of [11] in the subcritical case $1 < p < 3$.

Theorem 1.1. *Let $F(u)=|u|^p$ or $F(u)=|u|^{p-1}u$ with $1<p\le 3$ in (1.6). Assume that both $f\in C^2(\mathbf{R})$ and $g\in C^1(\mathbf{R})$ have compact support contained in $\{x\in\mathbf{R}\ :\ |x|\le 1\}$. Then, there exists a positive constant $c=c(f,g,p)$ such that*

$$T_\varepsilon \ge \begin{cases} c\varepsilon^{-(p-1)/(3-p)} & \text{if } 1<p<3,\\ \exp(c\varepsilon^{-2}) & \text{if } p=3, \end{cases} \tag{1.7}$$

holds for $\varepsilon>0$.

To derive a blow-up result, we require the following assumptions on the data:

$$\begin{aligned}&\text{Let } f\equiv 0 \text{ and } g\in C^1(\mathbf{R}) \text{ does not vanish identically.}\\ &\text{Assume } g(x)\ge 0 \text{ for all } x\in\mathbf{R} \text{ and } \int_{-1}^{1} g(y)dy>0.\end{aligned} \tag{1.8}$$

Then, we have the following.

Theorem 1.2. *Let $F(u)=|u|^p$ or $F(u)=|u|^{p-1}u$ with $1<p\le 3$ in (1.6). Assume (1.8). Then, there exist positive constants $\varepsilon_0=\varepsilon_0(g,p)$ and $C=C(g,p)$ such that*

$$T_\varepsilon \le \begin{cases} C\varepsilon^{-(p-1)/(3-p)} & \text{if } 1<p<3,\\ \exp(C\varepsilon^{-2}) & \text{if } p=3, \end{cases} \tag{1.9}$$

holds for any ε with $0<\varepsilon\le\varepsilon_0$.

Remark. Due to (1.9), the lower bound in the case of $p=3$ in (1.7) is optimal. However, if the initial data are odd functions, we obtain different estimates of the lifespan and different critical exponent.

The following theorems show that the global existence and blow-up results when the initial data are odd functions. We define

$$Y_\kappa=\{(f,g)\in C^2(\mathbf{R})\times C^1(\mathbf{R}):\ \|(f,g)\|_{Y_\kappa}<\infty\},$$

$$\|(f,g)\|_{Y_\kappa}=\sup_{x\ge 0}\left\{(1+|x|)^{1+\kappa}\left(\sum_{j=0}^{1}|f^{(j)}(x)|+|g(x)|\right)\right\}.$$

Then we have the following.

Theorem 1.3. *Let $F(u)=|u|^{p-1}u$ with $p>2$ in (1.6). Suppose $p>2$, $(f,g)\in Y_\kappa$ with $\kappa>\max\{1/p,(3-p)/(p-1)\}$ and f, g are odd functions. Then, there exist positive constants $\varepsilon_0=\varepsilon_0(f,g,p,\kappa)$ and $c=c(f,g,p,\kappa)$ such that*

$$T_\varepsilon=\infty \quad \text{if} \quad p>1+\sqrt{2}, \tag{1.10}$$

$$
T_\varepsilon \geq \begin{cases} c\varepsilon^{-p(p-1)/(1+2p-p^2)} & \text{if} \quad 2 < p < 1+\sqrt{2}, \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if} \quad p = 1+\sqrt{2}, \end{cases} \tag{1.11}
$$

hold for any ε *with* $0 < \varepsilon \leq \varepsilon_0$.

Remark. One can obtain a similar estimate to (1.11) also in the case of $1 < p \leq 2$. See Remark 5.

Remark. We note that the estimates of (1.11) in the case of $2 < p < 1+\sqrt{2}$ holds when f, g are odd functions, and satisfy the same assumptions in Theorem 1.1. Hence, the estimates of (1.11) is an improvement of (1.7) for small ε, because

$$
\frac{p-1}{3-p} < \frac{p(p-1)}{1+2p-p^2} \tag{1.12}
$$

is equivalent to $p > 1$.

To derive a blow-up result when the initial data are odd functions, we require the following assumptions on the data:

(1.13) Let $f \in C^1(\mathbf{R})$, $g \in C^2(\mathbf{R})$ are odd functions. Assume $f(x) > 0$, $g(x) > 0$ for all $x \in (0, \infty)$ and $f'(0) > 0$.

Then, we have the following.

Theorem 1.4. *Let* $F(u) = |u|^{p-1}u$ *with* $1 < p \leq 1+\sqrt{2}$ *in (1.6). Assume (1.13). Then, there exist positive constants* $\varepsilon_0 = \varepsilon_0(f, g, p)$ *and* $C = C(f, g, p)$ *such that*

$$
T_\varepsilon \leq \begin{cases} C\varepsilon^{-p(p-1)/(1+2p-p^2)} & \text{if} \quad 1 < p < 1+\sqrt{2}, \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if} \quad p = 1+\sqrt{2}, \end{cases} \tag{1.14}
$$

holds for any ε *with* $0 < \varepsilon \leq \varepsilon_0$.

Remark. There exists some initial data which satisfies the assumptions in Theorem 1.1 and Theorem 1.4. However, the estimates (1.7) does not contradict to (1.14) for small ε, because of (1.12).

This note is organized as follows. In the next section, we prepare some definitions and lemmas. The proofs of Theorem 1.1 and Theorem 1.2 shall be discussed in Section 3 and Section 4, respectively. The proofs of Theorem 1.3 and Theorem 1.4 are obtained in Section 5 and Section 6, respectively.

S 2. Preliminaries

In this section, we give some definitions and useful lemmas.

We define

$$u^0(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\} + \frac{1}{2}\int_{x-t}^{x+t} \{f(y) + g(y)\}dy \tag{2.1}$$

for $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$, and

$$L(\Phi)(x,t) = \frac{1}{2}\iint_{D(x,t)} \frac{\Phi(y,s)}{(1+s)^{p-1}} dyds \tag{2.2}$$

for $\Phi \in C(\mathbf{R} \times [0,\infty))$, where

$$D(x,t) = \{(y,s) \in \mathbf{R} \times [0,\infty) : 0 \le s \le t, x-t+s \le y \le x+t-s\}.$$

For $(f,g) \in C^2(\mathbf{R}) \times C^1(\mathbf{R})$, if $u \in C(\mathbf{R} \times [0,\infty))$ is a solution of

$$u(x,t) = \varepsilon u^0(x,t) + L(F(u))(x,t), \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{2.3}$$

then $u \in C^2(\mathbf{R} \times [0,\infty))$ is the solution to the initial value problem (1.6).

When $F(u) = |u|^{p-1}u$ and (f,g) are odd functions, if u is the C^2-solution of (1.6) then $-u(-x,t)$ is the solution to the problem (1.6). Making use of the uniqueness of the C^2-solution to (1.6), we see that $u(x,t)$ is odd function with respect to x. Therefore, in that case, we see that it is sufficient to consider the following integral equation:

$$u(x,t) = \varepsilon u^0(x,t) + \widetilde{L}(|u|^{p-1}u)(x,t), \quad (x,t) \in [0,\infty)^2, \tag{2.4}$$

where we set

$$\widetilde{L}(\Phi)(x,t) = \frac{1}{2}\iint_{\widetilde{D}(x,t)} \frac{\Phi(y,s)}{(1+s)^{p-1}} dyds \tag{2.5}$$

for $\Phi \in C([0,\infty)^2)$. Here $\widetilde{D}(x,t)$ is defined by

$$\widetilde{D}(x,t) = \{(y,s) \in [0,\infty)^2 : 0 \le s \le t, |x-t+s| \le y \le x+t-s\}.$$

Next, we prepare some useful lemmas for proving Theorem 1.3.

Lemma 2.1. ([7], Lemma 2.1) *Let $\nu > 0$. Then there exists a positive constant C_ν such that*

$$\int_{|x-t|}^{x+t} \frac{dy}{(1+y)^{1+\nu}} \le \frac{C_\nu \min\{x,t\}}{(1+x+t)(1+|x-t|)^\nu} \tag{2.6}$$

for $(x,t) \in [0,\infty)^2$.

For the proof, see e.g. Lemma 2.1 in Kubo & Osaka & Yazici [7].

Making use of this Lemma, we obtain the following basic estimate.

Lemma 2.2. ([10]) *Let $p > 2$ and $\sigma > 0$, and let $E_\sigma(\tau)$ be a function defined by*

$$E_\sigma(\tau) = \begin{cases} 1 & \textit{if} \quad \sigma > 1, \\ \log(3+\tau) & \textit{if} \quad \sigma = 1, \\ (1+\tau)^{1-\sigma} & \textit{if} \quad \sigma < 1 \end{cases} \tag{2.7}$$

for $\tau \geq 0$. Then there exists a positive constant $C_{p,\sigma}$ such that

$$\begin{aligned} \iint_{\widetilde{D}(x,t)} & \frac{(1+s)dyds}{(1+s+y)^p(1+|s-y|)^\sigma} \\ & \leq C_{p,\sigma} E_\sigma(T) \frac{1+t}{(1+x+t)(1+|x-t|)^{p-2}} \end{aligned} \tag{2.8}$$

for $(x,t) \in [0,\infty) \times [0,T]$.

S 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First of all, we introduce a Banach space

$$\begin{aligned} X = \{u \in C(\mathbf{R} \times [0,T]) \ : & \ \|u\|_{L^\infty(\mathbf{R}\times[0,T])} < \infty, \\ & \text{supp } u(x,t) \subset \{|x| \leq t+1\}\}, \end{aligned} \tag{3.1}$$

which is equipped with a norm

$$\|u\|_{L^\infty(\mathbf{R}\times[0,T])} = \sup_{(x,t)\in\mathbf{R}\times[0,T]} |u(x,t)|. \tag{3.2}$$

We shall construct a solution of the integral equation (2.3) in X under suitable assumption on T such as (3.6) below. Define a sequence of functions $\{u_n\}_{n\in\mathbf{N}} \subset X$ by

$$u_n = u_0 + L(|u_{n-1}|^p), \quad u_0 = \varepsilon u^0, \tag{3.3}$$

where L and u^0 are given by (2.2) and (2.1), respectively. It follows that

$$\|u_0\|_{L^\infty(\mathbf{R}\times[0,T])} \leq M\varepsilon,$$

where $M = \|f\|_{L^\infty(\mathbf{R})} + \|f+g\|_{L^1(\mathbf{R})}$. Since (f,g) has a compact support, M is a finite number, so that $u_0 \in X$.

The following *a priori* estimate plays a key role in the proof of Theorem 1.1.

Lemma 3.1. ([10]) *Let $V \in X$, $1 < p \leq 3$, and let $D(\tau)$ be a function defined by*

$$D(\tau) = \begin{cases} (1+\tau)^{3-p} & \textit{if} \quad 1 < p < 3, \\ \log(1+\tau) & \textit{if} \quad p = 3 \end{cases} \tag{3.4}$$

for $\tau \geq 0$. Then, there exists a positive constant C_p such that

$$\|L(V)\|_{L^\infty(\mathbf{R}\times[0,T])} \leq C_p D(T) \|V\|_{L^\infty(\mathbf{R}\times[0,T])}. \tag{3.5}$$

Now, we move on to the proof of Theorem 1.1. First of all, we take $T > 0$ such that

$$2^{p+1} p C_p D(T) M^{p-1} \varepsilon^{p-1} \leq 1, \tag{3.6}$$

where C_p is the one in Lemma 3.1.

Analogously to the proof of Theorem 1.2 in [9] (see p.16 in [9]), we see from Lemma 3.1 that $\{u_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in X, provided (3.6) holds. Since X is complete, there exists $u \in X$ such that u_n converges to u in X. Therefore u satisfies the integral equation (2.3), so that u is the C^2-solution of (1.6). Hence, the proof of Theorem 1.1 is completed. □

S 4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We show that the solution to the following integral equation blows up in finite time:

$$u(x,t) = \frac{\varepsilon}{2}\int_{x-t}^{x+t} g(y)dy + \frac{1}{2}\iint_{D(x,t)} \frac{|u(y,s)|^p}{(1+s)^{p-1}} dyds \tag{4.1}$$

for $(x,t) \in \mathbf{R} \times [0,\infty)$. Because, if $u \in C(\mathbf{R} \times [0,\infty)$ is a solution of (4.1), then u satisfies $u(x,t) \geq 0$ for $(x,t) \in \mathbf{R} \times [0,\infty)$ by $g(x) \geq 0$ for all $x \in \mathbf{R}$. Therefore, this u must solve the equation (2.3) with $F(u) = |u|^{p-1}u$ by the uniqueness of solutions to (1.6).

Before proving Theorem 1.2, we prepare some defintions and lemmas. For $T > 0$, we define the following domains:

$$\begin{aligned} \Sigma_0 &= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq 1\}, \\ \Sigma_j &= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq l_j\} \ (j = 1, 2, \cdots), \\ \Sigma_\infty &= \{(x,t) \in [0,\infty) \times [0,T] : t - x \geq 2\}, \end{aligned} \tag{4.2}$$

where

$$l_j = 1 + \sum_{k=1}^{j} 2^{-k} = 2\left(1 - \frac{1}{2^{j+1}}\right) \quad \text{for } j \geq 1. \tag{4.3}$$

Lemma 4.1. ([9], Lemma 3.1) *Let $p > 1$, $c_0 > 0$, and let us define a sequence $\{C_{p,j}\}_{j=1}^{\infty}$ by*

$$\begin{cases} C_{p,j} = \exp\{p^{j-1}(\log(C_{p,1} F_p^{-S_j} E_p^{1/(p-1)})) - \log E_p^{1/(p-1)}\} \ (j \geq 2), \\ C_{p,1} = c_0^p k_p \varepsilon^p, \end{cases} \tag{4.4}$$

where

$$E_p = \begin{cases} \dfrac{(p-1)^2}{2^{p+3}p^2}, \ \textit{if } 1 < p < 3, \\ (2^5 \cdot 3)^{-1}, \textit{if } p = 3, \end{cases} \tag{4.5}$$

$$F_p = \begin{cases} p^2, \textit{if } 1 < p < 3, \\ 6 \quad \textit{if } p = 3, \end{cases} \tag{4.6}$$

$$k_p = \begin{cases} 2^{-(p+2)}, \quad \textit{if } 1 < p < 3, \\ (2^4 \cdot 3)^{-1}, \textit{if } p = 3, \end{cases} \tag{4.7}$$

and

$$S_j = \sum_{i=1}^{j-1} \frac{i}{p^i}. \tag{4.8}$$

Then, we have the following relation:

$$C_{p,j+1} = \frac{C_{p,j}^p E_p}{F_p^j} \quad (j \in \mathbf{N}). \tag{4.9}$$

Since this lemma follows from Lemma 3.1 in [9], if $C_{a,j}$, $F_{p,a}$, $E_{p,a}$ and k_a are replaced by $C_{p,j}$, F_p, E_p, and k_p, respectively, we omit the proof.

Next, we derive a lower bound of the solution to (4.1) which is a first step of our iteration argument (for the proof, see e.g. Lemma 3.2 in [9]).

Lemma 4.2. ([9], Lemma 3.2) *Suppose that the assumptions in Theorem 1.2 are fulfilled. Let $u \in C(\mathbf{R} \times [0,T])$ be the solution of (4.1). Then, u satisfies*

$$u(x,t) \geq \varepsilon c_0 \quad \textit{for} \quad (x,t) \in \Sigma_0, \tag{4.10}$$

where $c_0 = \dfrac{1}{2}\displaystyle\int_{-1}^{1} g(y)dy > 0$.

Our iteration argument will be done by using the following estimates.

Proposition 4.3. ([10]) *Suppose that the assumptions in Theorem 1.2 are fulfilled. Let $j \in \mathbf{N}$ and let $u \in C(\mathbf{R} \times [0,T])$ be the solution of (4.1). Then, u satisfies*

$$u(x,t) \geq C_{p,j} \left\{ (t-x)^{-(p-1)}(t-x-1)^2 \right\}^{a_j} \quad \textit{if} \quad 1 < p < 3 \tag{4.11}$$

for $(x,t)\in\Sigma_0$, *and*

$$u(x,t)\geq C_{3,j}\left\{\log\left(\frac{t-x}{l_j}\right)\right\}^{a_j}\quad \textit{if}\quad p=3 \tag{4.12}$$

for $(x,t)\in\Sigma_j$, *where* Σ_0 *and* Σ_j *are defined in (4.2). Here* $C_{p,j}$ *is the one in (4.4) with* $c_0=\dfrac{1}{2}\displaystyle\int_{-1}^{1}g(y)dy>0$ *and* a_j *is defined by*

$$a_j=\frac{p^j-1}{p-1}\quad (j\in\mathbf{N}). \tag{4.13}$$

Proof of (4.12). For simplicity, we give the proof of (4.12). We shall show (4.12) by induction, and "slicing method" which was introduced by Agemi & Kurokawa & Takamura [1]. From (4.1), we get

$$u(x,t)\geq\frac{1}{2}\iint_{D(x,t)}\frac{|u(y,s)|^3}{(1+s)^2}dyds\quad \text{in } \mathbf{R}\times[0,\infty). \tag{4.14}$$

Let $(x,t)\in\Sigma_1$. Define

$$D_0(x,t)=\{(y,s)\in D(x,t):1\leq s-y\leq t-x,\ 0\leq y\leq t-x-s\}.$$

Replacing the domain of integration by $D_0(x,t)$ in the integral of (4.14), and changing the variables by

$$\alpha=s+y,\ \beta=s-y, \tag{4.15}$$

we get

$$u(x,t)\geq\frac{c_0^3\varepsilon^3}{4}\int_1^{t-x}d\beta\int_\beta^{t-x}\frac{d\alpha}{\{1+(\alpha+\beta)/2\}^2}\quad \text{in } \Sigma_1.$$

Since

$$1+\frac{\alpha+\beta}{2}\leq 1+\alpha\quad \text{for } \beta\leq\alpha$$

we get

$$u(x,t)\geq\frac{c_0^3\varepsilon^3}{4}\int_1^{t-x}d\beta\int_\beta^{t-x}\frac{d\alpha}{(1+\alpha)^2}\quad \text{in } \Sigma_1.$$

It follows from

$$\int_\beta^{t-x}\frac{d\alpha}{(1+\alpha)^2}=\frac{t-x-\beta}{(1+\beta)(1+t-x)}\geq\frac{1}{2^2}\cdot\frac{t-x-\beta}{(t-x)\beta}$$

for $\beta\geq 1$ and $t-x\geq l_1=3/2>1$, that

$$u(x,t)\geq\frac{c_0^3\varepsilon^3}{2^4(t-x)}\int_1^{t-x}\frac{t-x-\beta}{\beta}d\beta=\frac{c_0^3\varepsilon^3}{2^4(t-x)}\int_1^{t-x}\log\beta d\beta\quad \text{in } \Sigma_1.$$

Since $1 \leq (t-x)/l_1$ for Σ_1, the β-integral is estimated as follows:

$$\int_1^{t-x} \log \beta d\beta \geq \int_{(t-x)/l_1}^{t-x} \log \beta d\beta$$
$$\geq \left(1 - \frac{1}{l_1}\right)(t-x) \log \left(\frac{t-x}{l_1}\right) = \frac{t-x}{3} \log \left(\frac{t-x}{l_1}\right)$$

in Σ_1. Hence from (4.9) we get

$$u(x,t) \geq C_{3,1} \log \left(\frac{t-x}{l_1}\right) \quad \text{in } \Sigma_1.$$

Therefore, (4.12) holds for $j = 1$.

Assume that (4.12) holds for some $j \in \mathbf{N}$. Let $(x,t) \in \Sigma_{j+1}$. Define

$$D_j(x,t) = \{(y,s) \in D(x,t) : l_j \leq s - y \leq t - x,\ 0 \leq y \leq t - x - s\}$$

for $j \geq 1$. Replacing the domain of integration in (4.14) by $D_j(x,t)$, and making use of (4.15), we have

$$u(x,t) \geq \frac{1}{4} \int_{l_j}^{t-x} d\beta \int_\beta^{t-x} \frac{|u(y,s)|^3 d\alpha}{\{1 + (\alpha+\beta)/2\}^2} \quad \text{in } \Sigma_{j+1}.$$

Noticing that $D_j(x,t) \subset \Sigma_j$ for $(x,t) \in \Sigma_{j+1}$ and putting (4.12) into the integral above, we have

$$u(x,t) \geq \frac{C_{3,j}^3}{4} \int_{l_j}^{t-x} \left\{\log \left(\frac{\beta}{l_j}\right)\right\}^{3a_j} d\beta \int_\beta^{t-x} \frac{d\alpha}{\{1 + (\alpha+\beta)/2\}^2} \quad \text{in } \Sigma_{j+1}.$$

Analogously to the case of $j = 1$, we get

$$u(x,t) \geq \frac{C_{3,j}^3}{2^4(t-x)} \int_{l_j}^{t-x} \left\{\log \left(\frac{\beta}{l_j}\right)\right\}^{3a_j} \frac{(t-x-\beta)}{\beta} d\beta$$

in Σ_{j+1}. Making use of integration by parts in the integral above, we obtain

$$\int_{l_j}^{t-x} \left\{\log \left(\frac{\beta}{l_j}\right)\right\}^{3a_j} \frac{(t-x-\beta)}{\beta} d\beta$$
$$= \frac{1}{3a_j+1} \int_{l_j}^{t-x} \left\{\log \left(\frac{\beta}{l_j}\right)\right\}^{3a_j+1} d\beta$$
$$\geq \frac{1}{3a_j+1} \int_{(t-x)l_j/l_{j+1}}^{t-x} \left\{\log \left(\frac{\beta}{l_j}\right)\right\}^{3a_j+1} d\beta$$
$$\geq \frac{1}{3a_j+1} \left(1 - \frac{l_j}{l_{j+1}}\right)(t-x) \left\{\log \left(\frac{t-x}{l_{j+1}}\right)\right\}^{3a_j+1}$$

in Σ_{j+1}. Recalling the definitions of a_j and l_j, given by (4.13) and (4.3), we have

$$\begin{aligned} &3a_j+1=a_{j+1}\le \frac{3^{j+1}}{2},\\ &1-\frac{l_j}{l_{j+1}}=\frac{l_{j+1}-l_j}{l_{j+1}}=\frac{2^{-(j+1)}}{l_{j+1}}\ge 2^{-(j+2)}. \end{aligned}$$

Making use of (4.9), we get

$$u(x,t)\ge \frac{C_{3,j}^3}{2^5\cdot 3\cdot 6^j}\left\{\log\left(\frac{t-x}{l_{j+1}}\right)\right\}^{a_{j+1}}=C_{3,j+1}\left\{\log\left(\frac{t-x}{l_{j+1}}\right)\right\}^{a_{j+1}}$$

in Σ_{j+1}. Therefore, (4.12) holds for all $j\in\mathbf{N}$. □

End of the proof of Theorem 1.2. Let $u\in C(\mathbf{R}\times[0,T])$ be the solution of the integral equation (4.1). Setting $S=\lim_{j\to\infty}S_j$, we see from (4.8) that $S_j\le S$ for all $j\in\mathbf{N}$. Therefore, (4.4) yields

$$\tag{4.16} \begin{aligned} C_{p,j}&\ge \exp\{p^{j-1}\{\log(C_{p,1}F_p^{-S}E_p^{1/(p-1)})\}-\log E_p^{1/(p-1)}\}\\ &=E_p^{-1/(p-1)}\exp\{p^{j-1}\{\log(C_{p,1}F_p^{-S}E_p^{1/(p-1)})\}\}. \end{aligned}$$

(i) Upper bound of the lifespan in the case of $1<p<3$.

We take $\varepsilon_1=\varepsilon_1(g,p)>0$ so small that

$$B_1\varepsilon_1^{-(p-1)/(3-p)}\ge 4,$$

where we set

$$B_1=(k_pc_0^p2^{p(p-5)/(p-1)}F_p^{-S}E_p^{1/(p-1)})^{-(p-1)/p(3-p)}>0.$$

Next, for a fixed $\varepsilon\in(0,\varepsilon_1]$, we suppose that T satisfies

$$\tag{4.17} T>B_1\varepsilon^{-(p-1)/(3-p)}\ (\ge 4).$$

Combining (4.16) with (4.11), we have

$$\begin{aligned} u(x,t)\ge\ &E_p^{-1/(p-1)}\exp\{p^{j-1}\{\log(C_{p,1}F_p^{-S}E_p^{1/(p-1)})\}\}\\ &\times\left\{\frac{(t-x-1)^2}{(t-x)^{(p-1)}}\right\}^{(p^j-1)/(p-1)} \end{aligned}$$

in Σ_0. Let $(x,t)=(t/2,t)$ for $t\in[4,T]$. Then $(x,t)\in\Sigma_0$ and $t-x-1\ge(t-x)/2$. Hence we get

$$\begin{aligned} u(t/2,t)\ge\ &(2^{p-5}E_p)^{-1/(p-1)}\exp\{p^{j-1}\{\log(2^{p(p-5)/(p-1)}C_{p,1}F_p^{-S}E_p^{1/(p-1)})\}\}\\ &\times t^{(3-p)(p^j-1)/(p-1)}\\ =\ &(2^{p-5}E_p)^{-1/(p-1)}\exp\{p^{j-1}H_1(t)\}t^{-(3-p)/(p-1)} \end{aligned}$$

for $t \in [4, T]$, where we set

$$H_1(t) = \log\left(\varepsilon^p k_p c_0^p 2^{p(p-5)/(p-1)} F_p^{-S} E_p^{1/(p-1)} t^{p(3-p)/(p-1)}\right).$$

By (4.17) and the definition of B_1, we have $H_1(T) > 0$. Therefore we get $u(T/2, T) \to \infty$ as $j \to \infty$. Hence, (4.17) implies that $T_\varepsilon \le B_1 \varepsilon^{-(p-1)/(3-p)}$ for $0 < \varepsilon \le \varepsilon_1$.

(ii) Upper bound of the lifespan in the case of $p = 3$.

We take $\varepsilon_2 = \varepsilon_2(g) > 0$ so small that

$$B_2 \varepsilon_2^{-2} \ge \log 4,$$

where we set

$$B_2 = (c_0^3 k_3 F_3^{-S} E_3^{1/2})^{-2/3} > 0.$$

Next, for a fixed $\varepsilon \in (0, \varepsilon_2]$, we suppose that T satisfies

$$T > \exp\{2B_2\varepsilon^{-2}\} \ (> 4). \tag{4.18}$$

Combining (4.16) with (4.12), we have

$$u(x,t) \ge E_3^{-1/2} \exp\{3^{j-1}\{\log(\varepsilon^3 c_0^3 k_3 F_3^{-S} E_3^{1/2})\}\} \left\{\log\left(\frac{t-x}{l_j}\right)\right\}^{(3^j-1)/2}$$

in Σ_j. Now, note that $(t/2, t) \in \Sigma_\infty$ for $t \in [4, T]$, where Σ_∞ is defined in (4.2). Since $l_j < 2$ for $j \ge 1$, we get $(t-x)/l_j > (t-x)/2$. Hence we obtain

$$\begin{aligned} u(t/2, t) &\ge E_3^{-1/2} \exp\{3^{j-1}\{\log(\varepsilon^3 c_0^3 k_3 F_3^{-S} E_3^{1/2})\}\} \left\{\log\left(\frac{t}{4}\right)\right\}^{(3^j-1)/2} \\ &= E_3^{-1/2} \exp\{3^{j-1} H_2(t)\} \left\{\log\left(\frac{t}{4}\right)\right\}^{-1/2} \end{aligned}$$

for $t \in [4, T]$, where we set

$$H_2(t) = \log\left\{\varepsilon^3 c_0^3 k_3 F_3^{-S} E_3^{1/2} \left\{\log\left(\frac{t}{4}\right)\right\}^{3/2}\right\}.$$

By (4.18) and the definition of B_2, we have $H_2(T) > 0$. Therefore we get $u(T/2, T) \to \infty$ as $j \to \infty$. Hence, (4.18) implies that $T_\varepsilon \le \exp\{2B_2\varepsilon^{-2}\}$ for $0 < \varepsilon \le \varepsilon_2$. Therefore, the proof of Theorem 1.2 is now completed. □

S 5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First of all, we define the following weighted L^∞ space. For $\gamma > 0$ and $0 < T \le \infty$, we define

$$X_\gamma := \{U \in C([0,\infty) \times [0,T)) : \|U\|_{X_\gamma} < \infty\}, \tag{5.1}$$

$$\|U\|_{X_\gamma} = \sup_{(x,t)\in[0,\infty)\times[0,T)} w_\gamma(x,t)|U(x,t)|,$$

where

$$w_\gamma(x,t) = \frac{(1+x+t)(1+|x-t|)^\gamma}{1+t} \tag{5.2}$$

for $(x,t) \in [0,\infty)^2$.

Next we prepare the following lemmas which play a key role in the proof of Theorem 1.3.

Lemma 5.1. ([10]) *Let $\kappa > 0$. Assume that f, g are odd functions and $(f,g) \in Y_\kappa$. Then there exists a positive constant C_κ such that*

$$\|u^0\|_{X_\kappa} \le C_\kappa \|(f,g)\|_{Y_\kappa}, \tag{5.3}$$

where u^0 is defined in (2.1).

Lemma 5.2. ([10]) *Let $p > 2$. Suppose that f, g are odd functions, and $(f,g) \in Y_\kappa$ with $\kappa > \max\{1/p, (3-p)/(p-1)\}$, and $U \in X_{p-2}$. Then there exist positive constants $C_{p,\kappa}$ and C_p such that*

$$\|\widetilde{L}\left(|u^0|^p\right)\|_{X_{p-2}} \le C_{p,\kappa}\|u^0\|^p_{X_\kappa}, \tag{5.4}$$

$$\|\widetilde{L}\left(|u^0|^{p-1}|U|\right)\|_{X_{p-2}} \le C_{p,\kappa}\|u^0\|^{p-1}_{X_\kappa}\|U\|_{X_{p-2}}, \tag{5.5}$$

$$\|\widetilde{L}\left(|U|^p\right)\|_{X_{p-2}} \le C_p E_{p(p-2)}(T)\|U\|^p_{X_{p-2}}, \tag{5.6}$$

where $E_\sigma(T)$ is the one in (2.7), $\widetilde{L}$ is the one in (2.5), and u^0 is the one in (2.1).

Now, we move on to the proof of Theorem 1.3. In what follows, we consider the following integral equation:

$$U = \widetilde{L}\{|U+U^0|^{p-1}(U+U^0)\} \quad \text{in } [0,\infty)\times[0,T), \tag{5.7}$$

where we set $U^0 = \varepsilon u^0$, with u^0 being the one in (2.1). If $U \in C([0,\infty)\times[0,T))$ is the solution of (5.7), then $u := U + U^0$ satisfies (2.4). Since U^0 exists globally in time, it suffices to examine the lifespan of U.

We shall construct a solution of the integral equation (5.7) in X_{p-2} with $p > 2$ under suitable assumption on T such as (5.10) below. Define a sequence of functions $\{U_n\}_{n\in\mathbf{N}} \subset X_{p-2}$ by

$$U_n = \widetilde{L}\left\{|U_{n-1}+U_0|^{p-1}(U_{n-1}+U_0)\right\}, \quad U_0 = U^0 (= \varepsilon u^0), \tag{5.8}$$

We take $\varepsilon_0 = \varepsilon_0(f, g, p, \kappa) > 0$ so small that

$$p2^{p+3} C_{p,\kappa} C_\kappa^{p-1} \|(f, g)\|_{Y_\kappa}^{p-1} \varepsilon_0 \leq 1. \tag{5.9}$$

For a fixed $\varepsilon \in (0, \varepsilon_0]$, we take $T > 0$ such that

$$2^{p^2+p+2} p^p C_p C_{p,\kappa}^{p-1} C_\kappa^{p(p-1)} \|(f, g)\|_{Y_\kappa}^{p(p-1)} E_{p(p-2)}(T) \varepsilon^{p(p-1)} \leq 1. \tag{5.10}$$

Analogously to the proof of Theorem 1.2 in [9] (see p.16 in [9]), we see from Lemma 5.1 and Lemma 5.2 that $\{U_n\}_{n\geq 2}$ is a Cauchy sequence in X_{p-2}, provided (5.9) and (5.10) hold. Since X_{p-2} is complete, there exists $U \in X$ such that $\{U_n\}_{n\in\mathbf{N}}$ converges to U in X_{p-2}. Therefore U satisfies the integral equation (5.7). Hence, the proof of Theorem 1.3 is completed. □

Remark. Analogously to the proof of Theorem 1.3, one can get a lower bound of the lifespan in the case of $1 < p \leq 2$. In fact, we have only to change the weight function by

$$w(x,t) = \begin{cases} \dfrac{(1+x+t)^{p-1}}{1+t}, & \text{if } 1 < p < 2, \\ \dfrac{1+x+t}{1+t} \left(\log \dfrac{1+x+t}{1+|x-t|} \right)^{-1}, & \text{if } p = 2, \end{cases}$$

for $(x, t) \in [0, \infty)^2$.

S 6. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We show that the solution to the following integral equation blows up in finite time:

$$u(x,t) = \varepsilon u^0(x,t) + \frac{1}{2} \iint_{\widetilde{D}(x,t)} \frac{|u(y,s)|^{p-1} u(y,s)}{(1+s)^{p-1}} dyds \tag{6.1}$$

for $(x, t) \in [0, \infty)^2$, where u^0 is the one in (2.1).

First of all, we prepare some definitions and lemmas. For $T > 0$ and $\delta \in (0, 1)$, we define the following domains:

$$\begin{aligned} &\Gamma = \{(x,t) \in [0,\infty)^2 : 0 \leq x - t \leq \delta/2,\ x + t \geq \delta\}, \\ &\widetilde{\Sigma_j} = \{(x,t) \in [0,\infty) \times [0,T] : x \geq t - x \geq \widetilde{l_j}\}\ (j = 1, 2, \cdots), \\ &\widetilde{\Sigma_\infty} = \{(x,t) \in [0,\infty) \times [0,T] : x \geq t - x \geq 2\}, \end{aligned} \tag{6.2}$$

where

$$\begin{cases} \widetilde{l_1} = 1 \\ \widetilde{l_j} = \widetilde{l_1} + \displaystyle\sum_{k=1}^{j-1} 2^{-k} = 2 - \frac{1}{2^{j-1}} \quad \text{for } j \geq 2. \end{cases} \tag{6.3}$$

Let $p>1$, $c_1>0$, and let us define a sequence $\{\widetilde{C_{p,j}}\}_{j=1}^{\infty}$ by

$$\begin{cases} \widetilde{C_{p,j}} = \exp\{p^{j-1}(\log(\widetilde{C_{p,1}}\widetilde{F_p}^{-S_j}\widetilde{E_p}^{1/(p-1)})) - \log \widetilde{E_p}^{1/(p-1)}\} \ (j \geq 2), \\ \widetilde{C_{p,1}} = c_1^p \widetilde{k_p}\varepsilon^p, \end{cases} \tag{6.4}$$

where S_j is the one in (4.8), and we put

$$\widetilde{E_p} = \begin{cases} \dfrac{(p-1)^2}{2^2 3^{p-1}(p+1)^2}, \text{ if } 1<p<1+\sqrt{2}, \\ \dfrac{p-1}{2^3 3^{p-1}}, \qquad \text{if } p=1+\sqrt{2}, \end{cases} \tag{6.5}$$

$$\widetilde{F_p} = \begin{cases} p^2, \text{ if } 1<p<1+\sqrt{2}, \\ 2p \ \text{ if } p=1+\sqrt{2}, \end{cases} \tag{6.6}$$

$$\widetilde{k_p} = \delta \cdot 2^{p-4} 5^{-(p-1)}. \tag{6.7}$$

We note that $\{\widetilde{C_{p,j}}\}_{j=1}^{\infty}$ satisfies the following relation:

$$\widetilde{C_{p,j+1}} = \frac{\widetilde{C_{p,j}}^{\,p}\widetilde{E_p}}{\widetilde{F_p}^{\,j}} \quad (j \in \mathbf{N}). \tag{6.8}$$

Making use of comparison argument of Keller [6], we get the positivity of the solution of (6.1).

Lemma 6.1. ([6], [10]) *Suppose that the assumptions in Theorem 1.4 are fulfilled. Let $u \in C([0,\infty)^2)$ be the solution of (6.1). Then, u satisfies*

$$u(x,t) > 0 \quad \textit{for } (x,t) \in (0,\infty)^2.$$

Next, we derive a lower bound of the solution to (6.1) which is a first step of our iteration argument.

Lemma 6.2. ([10]) *Suppose that the assumptions in Theorem 1.4 are fulfilled. Let $u \in C([0,\infty)\times[0,T])$ be the solution of (6.1). Then, u satisfies*

$$u(x,t) \geq \varepsilon c_1 \quad \textit{for} \quad (x,t) \in \Gamma, \tag{6.9}$$

where $c_1 = \dfrac{1}{2}\displaystyle\int_{\delta/2}^{\delta}\{f(y)+g(y)\}dy > 0.$

Our iteration argument will be done by using the following estimates.

Proposition 6.3. ([10]) *Suppose that the assumptions in Theorem 1.4 are fulfilled. Let $j \in \mathbf{N}$ and let $u \in C([0,\infty) \times [0,T])$ be the solution of (6.1). Then, u satisfies*

$$u(x,t) \geq \widetilde{C_{p,j}} \frac{(t-x-1)^{b_j}}{(t-x)^{d_j}} \quad \text{if} \quad 1 < p < 1+\sqrt{2} \tag{6.10}$$

for $(x,t) \in \widetilde{\Sigma_1}$, and

$$u(x,t) \geq \frac{\widetilde{C_{p,j}}}{(t-x)^{p-2}} \left\{ \log \left(\frac{t-x}{\widetilde{l_j}} \right) \right\}^{f_j} \quad \text{if} \quad p = 1+\sqrt{2} \tag{6.11}$$

for $(x,t) \in \widetilde{\Sigma_j}$, where $\widetilde{\Sigma_j}$ is defined in (6.2). Here $\widetilde{C_{p,j}}$ is the one in (6.4) with $c_1 = \dfrac{1}{2}\displaystyle\int_{\delta/2}^{\delta} \{f(y)+g(y)\}dy > 0$ and b_j, d_j and f_j are defined by

$$b_j = \frac{\{p^{j-1}(p+1)-2\}}{p-1} \quad (j \in \mathbf{N}), \tag{6.12}$$

$$d_j = p^j - 1 \quad (j \in \mathbf{N}), \tag{6.13}$$

$$f_j = \frac{p^{j-1}-1}{p-1} \quad (j \in \mathbf{N}). \tag{6.14}$$

Analogously to the proof of Theorem 1.2, we get the desired conclusion by Proposition 6.3. Therefore, the proof of Theorem 1.4 is completed. □

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