RIMS Kôkyûroku Bessatsu
B61 (2017), 1–7

Relation between the hypergeometric function and WKB solutions

By

Takashi AOKI[*], Toshinori TAKAHASHI[**] and Mika TANDA[***]

Abstract

The hypergeometric function has three parameters. In these parameters, a large parameter is introduced so that the three parameters are linear forms of the large parameter. Thus obtained function satisfies the hypergeometric differential equation with the large parameter and the equation has formal solutions called WKB solutions. In this announcement paper, the relation between the hypergeometric function and the Borel sum of WKB solutions is investigated. Proofs and details will be given in our forthcoming paper.

Introduction

It is well known that the hypergeometric differential equation

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0, \tag{0.1}$$

has a solution defined by the series

$$F(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n, \tag{0.2}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. This is called the hypergeometric series. It converges if $|x| < 1$ and defines a multivalued holomorphic

Received May 14, 2016. Revised June 15, 2016. Accepted June 15, 2016.
2010 Mathematics Subject Classification(s): 33C05, 34M60, 34M40.
Key Words: *hypergeometric differential equation, hypergeometric function, WKB solution, Stokes curve.*
The first author has been supported by JSPS KAKENHI No.26400126.
[*]School of Science and Engineering, Kindai University, Higashi-Osaka, Osaka 577-8502, Japan.
[**]Interdisciplinary Graduate School of Science and Engineering, Kindai University, Higashi-Osaka, Osaka 577-8502, Japan
[***]Kansai Medical University, 2-5-1 Shin-machi, Hirakata, Osaka 573-1010, Japan

function in $x \in \mathbb{P} - \{0, 1, \infty\}$ which is called the hypergeometric function. We introduce a large parameter η in (0.1) as

$$a = \alpha_0 + \alpha\eta, \quad b = \beta_0 + \beta\eta, \quad c = \gamma_0 + \gamma\eta, \tag{0.3}$$

where $\alpha_0, \alpha, \beta_0, \beta, \gamma_0, \gamma$ are complex parameters. Then we can construct a formal solution to (0.1) of the form

$$\hat{w} = x^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)} \exp\left(\int S dx\right). \tag{0.4}$$

Here S is a formal solution to the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q \tag{0.5}$$

associated with the linear equation

$$\left(-\frac{d^2}{dx^2} + \eta^2 Q\right)\psi = 0 \tag{0.6}$$

which is obtained from (0.1) by eliminating the first-order term. We call $\hat{w}$ a WKB solution to (0.1). It is known by Koike and Schäfke that under suitable conditions and normalization of the integral, $\hat{w}$ is Borel summable in an appropriate domain. Hence the Borel sum of $\hat{w}$ is an analytic solution to (0.1). Then the following natural question arises: What is the relation between the Borel sum and the hypergeometric function?

For a special case where $\alpha_0 = \beta_0 = 1/2, \gamma_0 = 1$, we gave an answer to this question in [3]. In this article, we present a generalization of it. This is an announcement of our forthcoming paper. Proofs and detailed discussion will be given in the paper.

§ 1. The hypergeometric differential equation with a large parameter

We introduce a large parameter η in the parameters a, b, c of (0.1) as (0.3) and have the following equation:

$$x(1-x)\frac{d^2w}{dx^2} + (\gamma_0+\gamma\eta-(\alpha_0+\beta_0+1+(\alpha+\beta)\eta)x)\frac{dw}{dx} - (\alpha_0+\alpha\eta)(\beta_0+\beta\eta)w = 0. \tag{1.1}$$

We introduce a new unknown function ψ by

$$w = x^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\psi. \tag{1.2}$$

Then ψ satisfies (0.6) for $Q = Q_0 + \eta^{-1}Q_1 + \eta^{-2}Q_2$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2},$$

$$Q_1 = \frac{(\alpha-\beta)(\alpha_0-\beta_0)x^2 + (2(\alpha\beta_0+\alpha_0\beta) - \beta\gamma_0 - \beta_0\gamma - \gamma\alpha_0 - \gamma_0\alpha + \gamma)x + \gamma(\gamma_0-1)}{2x^2(x-1)^2},$$

$$Q_2 = \frac{(\alpha_0-\beta_0+1)(\alpha_0-\beta_0-1)x^2 + 2(2\alpha_0\beta_0 - \beta_0\gamma_0 - \gamma_0\alpha_0 + \gamma_0)x + \gamma_0(\gamma_0-2)}{4x^2(x-1)^2}.$$

We assume that the triplet of parameters (α, β, γ) does not belong to $E_0 \cup E_1 \cup E_2$, where E_j $(j = 0, 1, 2)$ are defined by

$$E_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha\,\beta\,\gamma\,(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)(\alpha + \beta - \gamma) = 0\},$$
$$E_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Re}(\gamma - \alpha)\mathrm{Re}(\gamma - \beta) = 0\},$$
$$E_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha - \beta)\mathrm{Re}(\alpha + \beta - \gamma)\mathrm{Re}\,\gamma = 0\}.$$

Then there exist two distinct zeros a_0, a_1 of the quadratic differential $Q_0 dx^2$ which are different from the regular singular points 0, 1, ∞. These zeros are called simple turning points of (0.6). A Stokes curve emanating from a simple turning point a is, by definition, a curve defined by

$$\mathrm{Im} \int_a^x \sqrt{Q_0} dx = 0. \tag{1.3}$$

Under our assumption on parameters, Stokes curves are non-degenerate, that is, there are no Stokes curves which connect turning point(s).

We consider the WKB solutions ([4]) to (0.6):

$$\frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int S_{\mathrm{odd}} dx\right). \tag{1.4}$$

Here S_{odd} denotes the odd-order part of the formal solution $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j$ to (0.5) with respect to $\sqrt{Q_0}$ having the leading term $S_{-1} = \sqrt{Q_0}$.

Let a be a simple turning point of (0.6). We denote by $\psi_\pm$ the WKB solution normalized at a:

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_a^x S_{\mathrm{odd}} dx\right). \tag{1.5}$$

Here the integration is understood as a half of the contour integral of S_{odd} from x on the second sheet of the Riemann surface of $\sqrt{Q_0}$ to just before a on the segment connecting x and a, going around a and back to x on the first sheet (cf. [4]). As is well known, WKB solutions normalized as above are convenient for describing connection formulas concerned with Stokes curves emanating from a.

On the other hand, we can take another natural normalization of WKB solutions. By using the arguments given in Proposition 3.6 of [4], we obtain

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}} = \frac{1}{2}(\gamma_0 - 1 + \gamma\eta). \tag{1.6}$$

Here we take the branch of $\sqrt{Q_0}$ so that

$$\sqrt{Q_0} \sim \frac{\gamma}{2x} \tag{1.7}$$

holds near the origin. We set

$$\rho = \frac{1}{2}(\gamma_0 - 1)\eta^{-1} + \frac{\gamma}{2}. \tag{1.8}$$

WKB solutions to (0.6) normalized at the origin are defined by

$$\varphi_{\pm}^{(0)} := \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_0^x \left(S_{\text{odd}} - \frac{\rho\eta}{x}\right) dx \pm \int_1^x \frac{\rho\eta}{x} dx\right). \tag{1.9}$$

Note that the integral starting from the origin is termwise convergent and defines a formal power series.

These two ways of normalization can be related as follows. Let V denote the integral

$$\int_0^a \left(S_{\text{odd}} - \frac{\rho\eta}{2x}\right) dx + \int_1^a \frac{\rho\eta}{2x} dx \tag{1.10}$$

and call V the Voros coefficient at the origin. Here the paths of integral are taken suitably. Using V, we can relate $\psi_\pm$ and $\varphi_\pm^{(0)}$ by

$$\varphi_{\pm}^{(0)} = \exp(\pm V)\psi_{\pm}. \tag{1.11}$$

Theorem 1.1. *The Voros coefficient V of the origin has the form*

$$\begin{aligned}
V = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\eta^{1-n}}{n(n-1)} &\left(\frac{B_n(\alpha_0)}{\alpha^{n-1}} + \frac{B_n(\beta_0)}{\beta^{n-1}} + \frac{B_n(\gamma_0-\alpha_0)}{(\gamma-\alpha)^{n-1}} + \frac{B_n(\gamma_0-\beta_0)}{(\gamma-\beta)^{n-1}}\right. \\
&\left. - \frac{B_n(\gamma_0) + B_n(\gamma_0 - 1)}{\gamma^{n-1}}\right) - \frac{\eta}{2}\left(\left(\alpha + \left(\alpha_0 - \frac{1}{2}\right)\eta^{-1}\right)\log\alpha\right. \\
&+ \left(\beta + \left(\beta_0 - \frac{1}{2}\right)\eta^{-1}\right)\log\beta - \left(\alpha - \gamma + \left(\alpha_0 - \gamma_0 + \frac{1}{2}\right)\eta^{-1}\right)\log(\alpha - \gamma) \\
&\left. + \left(\gamma - \beta + \left(\gamma_0 - \beta_0 + \frac{1}{2}\right)\eta^{-1}\right)\log(\gamma - \beta) - 2\left(\gamma + (\gamma_0 - 1)\eta^{-1}\right)\log\gamma\right).
\end{aligned}$$

Remark. (i) The choice of the turning point and the paths of integration in (1.10) is corresponding to the choice of the branch of the logarithmic terms.

(ii) In [1], the Voros coefficients for a special case where $\alpha_0 = \beta_0 = 1/2, \gamma_0 = 1$ are obtained. Note that the definition of the Voros coefficient employed here is slightly different from that in [1].

§ 2. Recessive WKB solutions and the hypergeometric function

We assume that the parameters satisfy the condition $0 < \operatorname{Re}\gamma < \operatorname{Re}\alpha < \operatorname{Re}\beta$. Then there are one Stokes curve, four Stokes curves and one Stokes curve flowing respectively into 0, 1 and ∞ (cf. [2]). A typical configuration of the Stokes curves is shown in Figure 1. Let a denote the source turning point of the Stokes curve flowing into the origin. On this Stokes curve, ψ_+ and $\varphi_+^{(0)}$ are recessive WKB solutions in the sense that $\mathrm{Re}\int_a^x \sqrt{Q_0}dx \to -\infty$ as $x \to 0$ on the curve, for we took the branch of $\sqrt{Q_0}$ as (1.7).

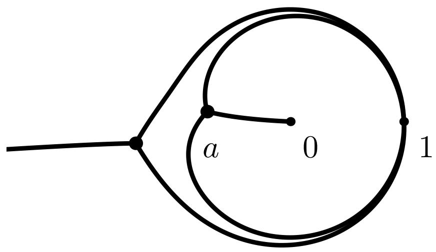

Figure 1

No Stokes phenomena are observed for the recessive WKB solutions when they cross the curve. It follows from [5] that those WKB solutions are Borel summable on the domain surrounded by other two Stokes curves emanating from a except for the origin. Let $\Phi_+^{(0)}$ denote the Borel sum of $\varphi_+^{(0)}$ in this region. Since the characteristic exponents of the origin of (0.6) are $(1 \pm (\gamma\eta + \gamma_0 - 1))/2$ and the recessive solutions have the exponent $(\gamma\eta+\gamma_0)/2$, the function $x^{-\frac{1}{2}(\gamma\eta+\gamma_0)}\Phi_+^{(0)}$ has the origin as a removable singularity. Thus $x^{-\frac{1}{2}(\gamma\eta+\gamma_0)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\Phi_+^{(0)}$ is a holomorphic solution to (1.1). We can evaluate this function at the origin (cf. Theorem 1.2, [3]) and have

Theorem 2.1. *If $0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$, the hypergeometric function and the Borel sum $\Phi_+^{(0)}$ of the recessive WKB solution $\varphi_+^{(0)}$ normalized at the origin are related by*

$$\begin{aligned}(2.1)\quad & F(\alpha_0 + \alpha\eta,\ \beta_0 + \beta\eta, \gamma_0 + \gamma\eta; x) \\ &= \sqrt{\frac{\gamma_0 - 1 + \gamma\eta}{2}}\, x^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\ \Phi_+^{(0)}(x,\eta)\end{aligned}$$

in a neighborhood of the origin.

Remark. The above relation itself holds under the assumption $\mathrm{Re}\,\gamma > 0$. General discussion will be given in our forthcoming article.

It is convenient to use WKB solutions normalized at a turning point for analyzing connection formulas and global properties of the Borel sums.

Theorem 2.2. *If $0 < \mathrm{Re}\,\gamma < \mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$, the Voros coefficient V is Borel* *summable and the Borel sum $\mathcal{V}$ is given by*

$$(2.2)\quad \mathcal{V} = \frac{1}{2}\log\left(\frac{\Gamma(\gamma_0+\gamma\eta)\Gamma(\gamma_0-1+\gamma\eta)\Gamma(\alpha_0-\gamma_0+1(\alpha-\gamma)\eta)}{2\pi\Gamma(\alpha_0+\alpha\eta)} \times \frac{\Gamma(\beta_0-\gamma_0+1+(\beta-\gamma)\eta)}{\Gamma(\beta_0+\beta\eta)}\right).$$

Combining Theorems 2.1 and 2.2, we have

Theorem 2.3. *If $0<\operatorname{Re}\gamma<\operatorname{Re}\alpha<\operatorname{Re}\beta$, the hypergeometric function and the Borel sum Ψ_+ of the recessive WKB solution ψ_+ normalized at the turning point a are related by*

$$(2.3)\quad F(\alpha_0+\alpha\eta,\beta_0+\beta\eta,\gamma_0+\gamma\eta;x) = Cx^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\Psi_+(x,\eta)$$

in a neighborhood of the origin. Here C is a constant defined by

$$(2.4)\qquad C = \frac{\big(\Gamma(\alpha_0-\gamma_0+1+(\alpha-\gamma)\eta)\Gamma(\beta_0-\gamma_0+1+(\beta-\gamma)\eta)\big)^{\frac{1}{2}}\Gamma(\gamma_0+\gamma\eta)}{2\sqrt{\pi}\big(\Gamma(\alpha_0+\alpha\eta)\Gamma(\beta_0+\beta\eta)\big)^{\frac{1}{2}}}.$$

Watson's lemma tells us that the Borel sum of a WKB solution has the WKB solution as an asymptotic expansion. Hence we have

Theorem 2.4. *If $0<\operatorname{Re}\gamma<\operatorname{Re}\alpha<\operatorname{Re}\beta$, the hypergeometric function has an asymptotic expansion*

$$(2.5)\quad F(\alpha_0+\alpha\eta,\ \beta_0+\beta\eta,\gamma_0+\gamma\eta;x) \sim Cx^{-\frac{1}{2}(\gamma_0+\gamma\eta)}(1-x)^{-\frac{1}{2}(\alpha_0+\beta_0-\gamma_0+1+(\alpha+\beta-\gamma)\eta)}\psi_+(x,\eta)$$

as $\eta\to\infty$ when x is close to the origin. Here the constant C is given in Theorem 2.3.

Remark. For x far from the origin, we can apply exact WKB theoretic connection formulas to (2.3) (cf. [4]) and obtain the relation between the hypergemetric function and WKB solutions as well as asymptotic formulas.

References

[1] Aoki, T. and Tanda, M., Parametric Stokes phenomena of the Gauss hypergeometric differrential equation with a large parameter, *J. Math. Soc. Japan* **68** (2016), 1099–1132.

[2] Aoki, T., Takahashi, T. and Tanda, M., Characterization of Stokes graphs and Voros coefficients of hypergeometric differential equations with a large parameter, *RIMS Kôkyûroku Bessatsu* **B40** (2013), 147–162.

[3] Aoki, T., Takahashi, T. and Tanda, M., The hypergeometric function and WKB solutions, *RIMS Kôkyûroku Bessatsu* **B57** (2016), 61–68.

[4] Kawai, T. and Takei, Y., *Algebraic Analysis of Singular Perturbation Theory*, Translation of Mathematical Monographs, vol. 227, AMS, 2005.

[5] Koike, T. and Schäfke, R. On the Borel summability of WKB solutions of Schrödinger equations with polynomial potential and its application, *private communication and in preparation for the publication in RIMS Kôkyûroku Bessatsu.*