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Toward the exact WKB analysis of discrete Painlevé equations

Dedicated to Professor Takahiro Kawai and Professor Hikosaburo Komatsu on the occasion of their seventieth and eightieth birthdays

By

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Abstract

In this note we report our recent research project on the exact WKB analysis of additive discrete Painlevé equations associated with continuous Painlevé equations through Bäcklund transformations. To exemplify the effectiveness and usefulness of our approach, we consider two problems; (approximate) invariants and the Stokes phenomena of discrete Painlevé equations. Here we only make an announcement of the results and outline our approach to these problems. Details will be discussed in our forthcoming papers.

§ 1. Introduction

The exact WKB analysis, i.e., WKB analysis based on the Borel resummation technique, for differential equations was initiated by Silverstone [18] and Voros [22], and later developed mainly by the French and Japanese schools ([6], [7], [16] and references cited therein). In this report, to generalize the exact WKB analysis to discrete equations, we discuss the exact WKB analysis of additive discrete Painlevé equations associated with continuous Painlevé equations through Bäcklund transformations.

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Every continuous Painlevé equation except for the first one (PI) admits Bäcklund transformations. It is well known that some discrete Painlevé equations are derived from these Bäcklund transformations. For example, solutions of the second Painlevé equation

$$\text{(PII)} \qquad \frac{d^2u}{dz^2} = 2u^3 + zu + c$$

admit Bäcklund transformations of the form

$$\begin{aligned} \overline{u} = u\Big|_{c\mapsto c+1} &= -u - \frac{c+1/2}{u^2 + du/dz + z/2}, \\ \underline{u} = u\Big|_{c\mapsto c-1} &= -u - \frac{c-1/2}{u^2 - du/dz + z/2} \end{aligned}$$

and, by eliminating du/dz from these two equations, we obtain the following discrete Painlevé equation known as (alt-dPI) (cf. [11]):

$$\text{(alt-dPI)} \qquad \frac{c+1/2}{\overline{u}+u} + \frac{c-1/2}{\underline{u}+u} + 2u^2 + z = 0.$$

The purpose of our research project is to make an asymptotic study of solutions of such discrete Painlevé equations in the framework of the exact WKB analysis. That is, we suitably introduce a large parameter η (i.e., inverse of the semi-classical parameter) into the equations and develop an asymptotic study with respect to the asymptotic parameter η based on the so-called Borel-Laplace method, as was done for continuous Painlevé equations in [14], [4], [15] and [19]. Note that, in the case of (PII) and (alt-dPI), introduction of η is done by scaling of variables

$$\text{(1.1)} \qquad u = \eta^{1/3}\lambda, \quad z = \eta^{2/3}t, \quad c = \eta\zeta$$

and the equations are transformed into the following form after this scaling:

$$\text{(PII)} \qquad \eta^{-2}\frac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \zeta,$$

$$\text{(1.2)} \qquad \overline{\lambda} = \lambda\Big|_{\zeta\mapsto\zeta+\eta^{-1}} = -\lambda - \frac{\zeta+\eta^{-1}/2}{\lambda^2 + \eta^{-1}(d\lambda/dt) + t/2},$$

$$\text{(1.3)} \qquad \underline{\lambda} = \lambda\Big|_{\zeta\mapsto\zeta-\eta^{-1}} = -\lambda - \frac{\zeta-\eta^{-1}/2}{\lambda^2 - \eta^{-1}(d\lambda/dt) + t/2},$$

$$\text{(alt-dPI)} \qquad \frac{\zeta+\eta^{-1}/2}{\overline{\lambda}+\lambda} + \frac{\zeta-\eta^{-1}/2}{\underline{\lambda}+\lambda} + 2\lambda^2 + t = 0.$$

(For the sake of simplicity, we attach the same symbols (PII) and (alt-dPI) to the scaled equations as the original ones.) Roughly speaking, dealing with both continuous

Painlevé equations and discrete Painlevé equations simultaneously (that is, as systems of differential and difference equations) and making full use of the expansion with respect to η with the help of the Borel-Laplace method, we make an asymptotic study of solutions of discrete Painlevé equations.

In this report, to exemplify the effectiveness and usefulness of our approach, we discuss the following two problems: The first one is concerned with an invariant of discrete Painlevé equations. In the case of continuous Painlevé equations, Hamiltonians provide approximate first integrals of their solutions. As a counterpart of this fact, approximate invariants of solutions of discrete Painlevé equations are provided also by Hamiltonians of continuous Painlevé equations. Here, using our approach, we show this fact in the case of the above discrete Painlevé equation (alt-dPI). The second problem is the Stokes phenomena for (alt-dPI) when the independent variable c of (alt-dPI) tends to infinity with the variable z being (arbitrarily) fixed. Applying the exact WKB analysis to linear differential-difference equations ("Lax pair") associated with (PII) and (alt-dPI) through the isomonodromic deformation theory, we can obtain explicit connection formulas that describe Stokes phenomena for (alt-dPI). In this report we only make an announcement of the results for special cases and outline our approach to these problems. (For the second problem we have written a more detailed announcement [12]. This report is considered to be a résumé of [12] for the second problem.) The details for more general cases will be discussed in our forthcoming papers.

The plan of this report is as follows: In Section 2, to prepare the discussion of the above two problems, we construct two kinds of formal solutions of (alt-dPI). Then in Section 3 we discuss the first problem, that is, approximate invariants of solutions of (alt-dPI). In Section 4, after defining the Stokes geometry (i.e., turning points and Stokes curves) of (alt-dPI), we present explicit connection formulas describing Stokes phenomena for (alt-dPI). Finally in Section 5 we discuss some future problems related especially to the second problem.

§ 2. Formal solutions of (alt-dPI)

In what follows we concentrate our consideration on the case of (alt-dPI) associated with (PII) for the sake of definiteness. Most parts of the discussion can be generalized to other discrete Painlevé equations associated with continuous Painlevé equations.

To develop the exact WKB analysis of (alt-dPI), we need its formal solutions. To this aim we first replace the shift operator $\overline{\lambda}$ and $\underline{\lambda}$ in (alt-dPI) by

$$\overline{\lambda} = \sum_{n=0}^{\infty} \frac{\eta^{-n}}{n!} \frac{\partial^n \lambda}{\partial \zeta^n} \quad \text{and} \quad \underline{\lambda} = \sum_{n=0}^{\infty} (-1)^n \frac{\eta^{-n}}{n!} \frac{\partial^n \lambda}{\partial \zeta^n}, \tag{2.1}$$

respectively. This replacement converts (alt-dPI) to an ∞-order differential equation of

WKB type of the form

$$(2\lambda^2+t)\left[2\lambda+\sum_{n=1}^{\infty}\frac{\eta^{-n}}{n!}\frac{\partial^n\lambda}{\partial\zeta^n}\right]\left[2\lambda+\sum_{n=1}^{\infty}(-1)^n\frac{\eta^{-n}}{n!}\frac{\partial^n\lambda}{\partial\zeta^n}\right] + \zeta\left[4\lambda+2\sum_{k=1}^{\infty}\frac{\eta^{-2k}}{(2k)!}\frac{\partial^{2k}\lambda}{\partial\zeta^{2k}}\right]-\eta^{-1}\sum_{k=1}^{\infty}\frac{\eta^{-(2k-1)}}{(2k-1)!}\frac{\partial^{2k-1}\lambda}{\partial\zeta^{2k-1}}=0. \tag{2.2}$$

Then, in parallel to the case of continuous Painlevé equations (cf. [14], [4]), we can construct the following two kinds of formal solutions of (alt-dPI).

§ 2.1. Formal power series solution

Assume that $\lambda=\lambda^{(0)}$ has the following formal power series expansion with respect to η^{-1}:

$$\lambda^{(0)}=\lambda_0(\zeta)+\eta^{-1}\lambda_1(\zeta)+\eta^{-2}\lambda_2(\zeta)+\cdots . \tag{2.3}$$

Then, by substituting (2.3) into (2.2), we readily find that the top order term $\lambda_0(\zeta)$ satisfies an algebraic equation

$$2\lambda_0^3+t\lambda_0+\zeta=0 \tag{2.4}$$

and the lower order terms $\lambda_j(\zeta)$ ($j\geq 1$) are determined in a unique and recursive manner. Thus we obtain the formal power series solution $\lambda^{(0)}$ of (alt-dPI).

Note that (2.4) is the same as the algebraic equation satisfied by the top order term of the formal power series solution of (PII). More generally, we can verify the following

Proposition 2.1. *The formal power series solution* $\lambda^{(0)}$ *of* (alt-dPI) *coincides with the formal power series solution of* (PII).

§ 2.2. Transseries solution

The formal power series solutions constructed above are uniquely determined up to the choice of solutions of (2.4) and contain no free parameters. Besides the formal power series solutions we can construct formal solutions of (alt-dPI) with free parameters, called transseries solutions, in the following way.

Assume that a solution of (alt-dPI) has an exponentially small correction to the formal power series part as follows:

$$\lambda=\lambda^{(0)}+\lambda^{(1)}+\lambda^{(2)}+\cdots , \tag{2.5}$$

where $\lambda^{(0)}$ is a formal power series solution of (alt-dPI) and $\lambda^{(1)}+\cdots$ denotes an exponentially small correction. Then, substituting the expansion (2.5) into (alt-dPI), we find that the subleading part $\mu = \lambda^{(1)}$ satisfies

$$(2(\lambda^{(0)})^2 + t)\Big[(\underline{\lambda^{(0)}} + \lambda^{(0)})(\overline{\mu} + \mu) + (\overline{\lambda^{(0)}} + \lambda^{(0)})(\underline{\mu} + \mu)\Big] \\ + 4\lambda^{(0)}(\overline{\lambda^{(0)}} + \lambda^{(0)})(\underline{\lambda^{(0)}} + \lambda^{(0)})\mu \\ + \zeta(2\mu + \overline{\mu} + \underline{\mu}) - \frac{1}{2}\eta^{-1}(\overline{\mu} - \underline{\mu}) = 0. \tag{2.6}$$

Let us consider a WKB solution of (2.6).

$$\mu = \exp\left(\eta \int^{\zeta} Z_{-1}(\zeta)\,d\zeta\right) \sum_{n=0}^{\infty} \mu_n(\zeta)\eta^{-n}. \tag{2.7}$$

Note that, in view of (2.1),

$$\overline{\mu} = e^{Z_{-1}}(\mu_0 + O(\eta^{-1})) \tag{2.8}$$

holds for (2.7). Hence, by substituting (2.7) into (2.6) and taking the leading part of both sides with respect to η^{-1}, we have the following equation for Z_{-1}:

$$(4\lambda_0^3 + 2t\lambda_0 + \zeta)(e^{Z_{-1}} + e^{-Z_{-1}}) + (24\lambda_0^3 + 4t\lambda_0 + 2\zeta) = 0, \tag{2.9}$$

that is,

$$Z_{-1} = \cosh^{-1}\frac{8\lambda_0^3 - \zeta}{\zeta} = \log\left(\frac{8\lambda_0^3 - \zeta}{\zeta} + \frac{4\lambda_0^2}{\zeta}\sqrt{6\lambda_0^2 + t}\right). \tag{2.10}$$

Once the exponential term (i.e., the phase factor) Z_{-1} is fixed, the amplitude part μ_n ($n = 0, 1, 2, \ldots$) of a WKB solution (2.7) is determined recursively. Furthermore, the remainder part (i.e., $\lambda^{(2)} + \cdots$) of the exponentially small correction of (2.5) is also recursively determined from $\lambda^{(0)}$ and $\mu = \lambda^{(1)}$. Thus we obtain a transseries solution of (alt-dPI) of the form

$$\lambda = \lambda^{(0)} + \eta^{-1/2} c \exp\left(\eta \int^{\zeta} \cosh^{-1}\frac{8\lambda_0^3 - \zeta}{\zeta}\,d\zeta\right) \sum_{n=0}^{\infty} \mu_n(\zeta)\eta^{-n} + \cdots, \tag{2.11}$$

where $c \in \mathbb{C}$ denotes a free parameter.

Recall that a transseries solution can be constructed also for the continuous Painlevé equation (PII) and has the following explicit form:

$$\lambda = \lambda^{(0)} + \eta^{-1/2}\tilde{c} \exp\left(\eta \int^{t} \sqrt{6\lambda_0^2 + t}\,dt\right) \sum_{n=0}^{\infty} \tilde{\mu}_n(\zeta)\eta^{-n} + \cdots. \tag{2.12}$$

(Cf. [13]; It is also regarded as a special case of 2-parameter instanton-type solutions constructed in [4].) Between these two transseries solutions we have the following relation.

Proposition 2.2. *Up to the constants of integration, the following relation holds:*

$$\int^t \sqrt{6\lambda_0^2 + t}\, dt = \int^\zeta \cosh^{-1} \frac{8\lambda_0^3 - \zeta}{\zeta}\, d\zeta. \tag{2.13}$$

In other words,

$$\frac{\partial}{\partial \zeta} \sqrt{6\lambda_0^2 + t} = \frac{\partial}{\partial t} \cosh^{-1} \frac{8\lambda_0^3 - \zeta}{\zeta}. \tag{2.14}$$

Corollary 2.3.

$$\omega_{-1} = \sqrt{6\lambda_0^2 + t}\, dt + \cosh^{-1} \frac{8\lambda_0^3 - \zeta}{\zeta}\, d\zeta \tag{2.15}$$

is a closed 1*-form in* $\mathbb{C}^2_{(t,\zeta)}$.

These results strongly suggest that we should deal with the following system of differential equations instead of discussing (alt-dPI) solely.

$$\begin{cases} \eta^{-2} \dfrac{d^2\lambda}{dt^2} = 2\lambda^3 + t\lambda + \zeta, \\ \overline{\lambda} \left(\overset{\text{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{\eta^{-n}}{n!} \frac{\partial^n \lambda}{\partial \zeta^n} \right) = -\lambda - \dfrac{\zeta + \eta^{-1}/2}{\lambda^2 + \eta^{-1}(d\lambda/dt) + t/2}. \end{cases} \tag{2.16}$$

As a matter of fact, extending Proposition 2.2 to the remainder parts of transseries solutions, we can verify

Proposition 2.4. *The system* (2.16) *has the following transseries solution:*

$$\lambda(t, \zeta, \eta; \alpha) = \lambda^{(0)} + \eta^{-1/2}\alpha\, \lambda^{(1)} + (\eta^{-1/2}\alpha)^2\, \lambda^{(2)} + \cdots, \tag{2.17}$$

where $\lambda^{(0)}$ *is a formal power series solution and* $\lambda^{(k)}$ $(k \geq 1)$ *is of the form*

$$\exp\Big(k\eta \int_{(t_0,\zeta_0)}^{(t,\zeta)} \omega_{-1}\Big) \sum_{n=0}^{\infty} \eta^{-n} \lambda_n^{(k)}(t, \zeta). \tag{2.18}$$

Here ω_{-1} *is a closed* 1*-form given by* (2.15) *and* α *is an infinite series of the form*

$$\alpha = \sum_{l=0}^{\infty} \alpha_l e^{2\pi i l \eta \zeta} \qquad (\alpha_l \in \mathbb{C}). \tag{2.19}$$

Proposition 2.4 is considered as a generalization of Proposition 2.1 to transseries solutions.

Remark. Reflecting the multi-valuedness of (2.10), the infinite series α appears in the description of transseries solutions. It is essentially c in (2.11). Note that, in dealing with transseries solutions, we implicitly assume the added exponential terms are exponentially small, which excludes terms with negative l in (2.19).

§ 3. Approximate invariants for (alt-dPI)

As the discussion in the preceding section suggests, we should consider (alt-dPI) and (PII) simultaneously and it is certainly effective to deal with the system (2.16) in the study of solutions of (alt-dPI). To show the effectiveness of this approach, we discuss an (approximate) invariant of (alt-dPI) and its relationship with the Hamiltonian of (PII) in this section.

Let us start with the following

Proposition 3.1. *Let $K = K(\lambda, \overline{\lambda}, \zeta, t)$ be defined by*

$$K = K(\lambda, \overline{\lambda}, \zeta, t) = \frac{\zeta}{2}\left(-\frac{2\lambda\overline{\lambda}}{\lambda+\overline{\lambda}} + \frac{t}{\lambda+\overline{\lambda}} + \frac{\zeta}{(\lambda+\overline{\lambda})^2}\right). \tag{3.1}$$

Then, if λ is a formal power series solution or a transseries solution of (alt-dPI)*, $K(\lambda, \overline{\lambda}, \zeta, t)$ is preserved modulo $O(\eta^{-1})$ under the shift $\zeta \mapsto \zeta + \eta^{-1}$, that is,*

$$K - \overline{K} \equiv 0 \mod O(\eta^{-1}) \tag{3.2}$$

holds.

Proof. It suffices to verify $K - \underline{K} \equiv 0 \pmod{O(\eta^{-1})}$. Since

$$\underline{K} = \frac{1}{2}(\zeta - \eta^{-1})\left(-\frac{2\lambda\underline{\lambda}}{\lambda+\underline{\lambda}} + \frac{t}{\lambda+\underline{\lambda}} + \frac{\zeta-\eta^{-1}}{(\lambda+\underline{\lambda})^2}\right), \tag{3.3}$$

we have

$$\begin{aligned} K - \underline{K} \equiv \frac{\zeta}{2}\Bigg[&-2\lambda\left(\frac{\overline{\lambda}}{\lambda+\overline{\lambda}} - \frac{\underline{\lambda}}{\lambda+\underline{\lambda}}\right) \\ &+t\left(\frac{1}{\lambda+\overline{\lambda}} - \frac{1}{\lambda+\underline{\lambda}}\right) + \zeta\left(\frac{1}{(\lambda+\overline{\lambda})^2} - \frac{1}{(\lambda+\underline{\lambda})^2}\right)\Bigg] \end{aligned}$$

modulo $O(\eta^{-1})$. Hence, in view of

$$\frac{\overline{\lambda}}{\lambda+\overline{\lambda}} - \frac{\underline{\lambda}}{\lambda+\underline{\lambda}} = \left(1 - \frac{\lambda}{\lambda+\overline{\lambda}}\right) - \left(1 - \frac{\lambda}{\lambda+\underline{\lambda}}\right) = -\lambda\left(\frac{1}{\lambda+\overline{\lambda}} - \frac{1}{\lambda+\underline{\lambda}}\right),$$

we obtain

$$\begin{aligned}K - \underline{K} &\equiv \frac{\zeta}{2}\left[(2\lambda^2+t)\left(\frac{1}{\lambda+\overline{\lambda}} - \frac{1}{\lambda+\underline{\lambda}}\right) + \zeta\left(\frac{1}{(\lambda+\overline{\lambda})^2} - \frac{1}{(\lambda+\underline{\lambda})^2}\right)\right] \\ &= \frac{\zeta}{2}\left(\frac{1}{\lambda+\overline{\lambda}} - \frac{1}{\lambda+\underline{\lambda}}\right)\left((2\lambda^2+t) + \zeta\left(\frac{1}{\lambda+\overline{\lambda}} + \frac{1}{\lambda+\underline{\lambda}}\right)\right) \\ &\equiv 0 \mod O(\eta^{-1}).\end{aligned}$$

□

Proposition 3.1 claims that K is an approximate invariant (i.e., invariant modulo $O(\eta^{-1})$) of (alt-dPI).

On the other hand, the continuous Painlevé equation (PII) has an expression as Hamiltonian system

$$\eta^{-1}\frac{d\lambda}{dt} = \frac{\partial H}{\partial \nu}, \quad \eta^{-1}\frac{d\nu}{dt} = -\frac{\partial H}{\partial \lambda} \tag{3.4}$$

with the Hamiltonian

$$H = H(\lambda,\nu,t,\zeta) = \frac{1}{2}\nu^2 - \left(\frac{1}{2}\lambda^4 + \frac{1}{2}t\lambda^2 + \zeta\lambda\right). \tag{3.5}$$

Between the invariant K of (alt-dPI) and the Hamiltonian H of (PII), there exists the following relation:

Proposition 3.2. *Identifying ν with $\eta^{-1}(d\lambda/dt)$, we solve the defining equation (1.2) of the Bäcklund transformation with respect to $\nu = \eta^{-1}(d\lambda/dt)$ as*

$$\nu = \nu(\lambda,\overline{\lambda},\zeta,t) = -\frac{\zeta+\eta^{-1}/2}{\lambda+\overline{\lambda}} - \left(\lambda^2 + \frac{t}{2}\right). \tag{3.6}$$

Then the following relation holds:

$$H(\lambda,\nu(\lambda,\overline{\lambda},\zeta,t),t,\zeta) \equiv K(\lambda,\overline{\lambda},\zeta,t) + \frac{t^2}{8} \qquad \mod O(\eta^{-1}). \tag{3.7}$$

Proof. Substitution of (3.6) into (3.5) immediately entails

$$\begin{aligned}&\text{LHS of (3.7)} \\ &\equiv \frac{1}{2}\left(\frac{\zeta}{\lambda+\overline{\lambda}} + \lambda^2 + \frac{t}{2}\right)^2 - \left(\frac{1}{2}\lambda^4 + \frac{1}{2}t\lambda^2 + \zeta\lambda\right) \\ &= \frac{\zeta^2}{2}\frac{1}{(\lambda+\overline{\lambda})^2} + \frac{\zeta}{\lambda+\overline{\lambda}}(\lambda^2+\frac{t}{2}) + \frac{1}{2}(\lambda^2+\frac{t}{2})^2 - (\frac{1}{2}\lambda^4 + \frac{1}{2}t\lambda^2 + \zeta\lambda)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\zeta}{2}\left[\frac{\zeta}{(\lambda+\overline{\lambda})^2}+\frac{2\lambda^2+t}{\lambda+\overline{\lambda}}-2\lambda\right]+\frac{t^2}{8} \\
&= \frac{\zeta}{2}\left[\frac{\zeta}{(\lambda+\overline{\lambda})^2}+\frac{t-2\lambda\overline{\lambda}}{\lambda+\overline{\lambda}}\right]+\frac{t^2}{8} \\
&= K(\lambda,\overline{\lambda},\zeta,t)+\frac{t^2}{8} \qquad \bmod O(\eta^{-1}).
\end{aligned}
$$

□

Note that $t^2/8$ in the right-hand side of (3.7) is an innocent term when considering the invariant of (alt-dPI). Thus Proposition 3.2 tells us that an approximate invariant of (alt-dPI) is readily obtained from the Hamiltonian and the Bäcklund transformation of (PII).

Similar relations as (3.7) also hold for the other Painlevé equations (PJ) (J $=$ $\mathrm{III}, \mathrm{IV}, \mathrm{V}, \mathrm{VI}$) and, as its consequence, approximate invariants of the associated discrete Painlevé equations are obtained from the Hamiltonians of (PJ). The details will be discussed in our forthcoming paper.

§ 4. Stokes phenomena for (alt-dPI)

As another example showing the effectiveness of our approach, we discuss the Stokes phenomena for (alt-dPI) from the viewpoint of the exact WKB analysis in this section. Here we only explain an outline of our discussion and present some explicit formulas that describe Stokes phenomena for (alt-dPI). For more details we refer the reader to [12].

§ 4.1. Stokes geometry of (alt-dPI)

The transseries solution constructed in Section 2 provides a formal solution of (alt-dPI) for each fixed t. Its phase factor (more precisely, phase factor of $\lambda^{(1)}$ in the ζ-direction) is given by

$$
\begin{aligned}
(4.1) \qquad Z_{-1,(\pm,l)}(\zeta) &:= \mathrm{Cosh}^{-1}\Big(\frac{8\lambda_0^3-\zeta}{\zeta}\Big)+2\pi i l \\
&= \mathrm{Log}\left(\frac{8\lambda_0^3-\zeta}{\zeta}\pm\frac{4\lambda_0^2}{\zeta}\sqrt{6\lambda_0^2+t}\right)+2\pi i l.
\end{aligned}
$$

Here Cosh^{-1} and Log designate the principal branch of $\cosh^{-1}$ and log, respectively, and we use the suffix $(\pm, l)$ ($l \in \mathbb{Z}_{\geq 0}$) to specify the branch of (2.10). This naturally leads to the following definition of the Stokes geometry (i.e., turning points and Stokes curves) of (alt-dPI).

Definition 4.1. (i) A point $\zeta = \widehat{\zeta}$ is said to be a turning point of (alt-dPI) (of type $(*, l) = (*', l')$) if there exist two suffices $(*, l) \neq (*', l')$ for which

$$Z_{-1,(*,l)}(\widehat{\zeta}) = Z_{-1,(*',l')}(\widehat{\zeta}) \tag{4.2}$$

holds.
(ii) A Stokes curve of (alt-dPI) (of type $(*, l) = (*', l')$) is defined by

$$\Im \int_{\widehat{\zeta}}^{\zeta} (Z_{-1,(*,l)} - Z_{-1,(*',l')}) d\zeta = 0, \tag{4.3}$$

where $\widehat{\zeta}$ is a turning point at which (4.2) is satisfied.

For example, Figure 1 shows the Stokes geometry of (alt-dPI) when t is fixed at $t = e^{\pi i/6}$. (Note that, as turning points and Stokes curves are defined through an

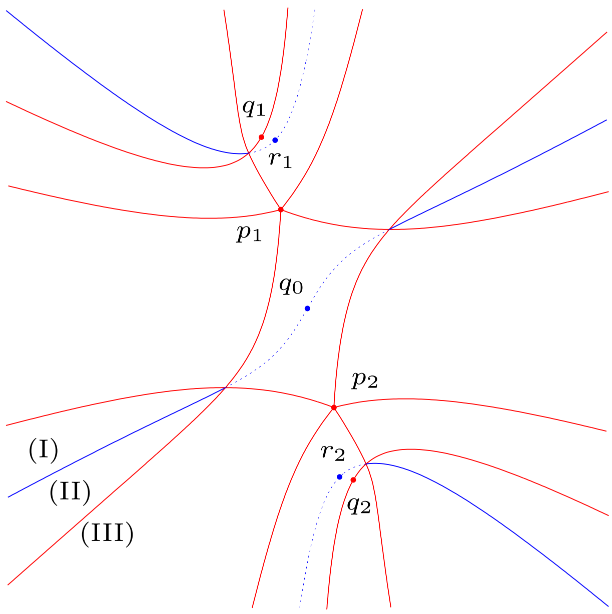

Figure 1. Stokes geometry of (alt-dPI) for $t = e^{\pi i/6}$.

algebraic function λ_0 satisfying (2.4), the Stokes geometry is a geometric object on its Riemann surface. Taking account of this fact, we write Figure 1 on λ_0-plane. In the current situation the Riemann surface of λ_0 can be identified with λ_0-plane since we can take λ_0 as its global coordinate.) When $t = e^{\pi i/6}$, there exist seven turning points p_i, q_j and r_k $(1 \leq i \leq 2, 0 \leq j \leq 2, 1 \leq k \leq 2)$, where p_i, q_j and r_k are of type $((+, l), (-, l))$, $((+, l), (+, l'))$ and $((+, l), (-, l + 2))$, respectively.

§ 4.2. Connection formula for (alt-dPI)

On each Stokes curve in Figure 1 it is expected that a Stokes phenomenon occurs with formal power series solutions and transseries solutions of (alt-dPI). In this subsection, taking three regions (I), (II) and (III) specified in Figure 1 for the sake of definiteness, we seek for explicit connection formulas that describe a Stokes phenomenon between Regions (I) and (II) and that between Regions (II) and (III).

For that purpose we employ linear differential-difference equations ("Lax pair") associated with (PII) and (alt-dPI). To be more specific, we use the following system of linear differential-difference equations:

$$\left(\eta^{-2}\frac{\partial^2}{\partial x^2} - Q_{\mathrm{II}}\right)\psi = 0, \tag{4.4}$$

$$\eta^{-1}\frac{\partial\psi}{\partial t} = A_{\mathrm{II}}\,\eta^{-1}\frac{\partial\psi}{\partial x} - \frac{\eta^{-1}}{2}\frac{\partial A_{\mathrm{II}}}{\partial x}\psi, \tag{4.5}$$

$$\overline{\psi} = g_{\mathrm{II}}\,\eta^{-1}\frac{\partial\psi}{\partial x} + f_{\mathrm{II}}\,\psi, \tag{4.6}$$

where Q_{II}, A_{II}, f_{II} and g_{II} are explicitly given by

$$\begin{aligned}
Q_{\mathrm{II}} &= x^4 + tx^2 + 2\zeta x + \nu^2 - (\lambda^4 + t\lambda^2 + 2\zeta\lambda) - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2},\\
A_{\mathrm{II}} &= \frac{1}{2(x-\lambda)},\\
g_{\mathrm{II}} &= \left((2\nu + 2\lambda^2 + t)(x-\lambda)(x-\overline{\lambda})\right)^{-1/2},\\
f_{\mathrm{II}} &= \left(x^2 - \lambda^2 - \nu + \eta^{-1}\frac{1}{2(x-\lambda)}\right) g_{\mathrm{II}}.
\end{aligned}$$

Note that, as is well known, (PII) and (alt-dPI) describe the compatibility condition of the system (4.4)-(4.6) (cf. [10], [11], [17]).

In what follows we take a turning point as an endpoint (t_0, ζ_0) of the integral in the definition (2.18) of a transseries solution $\lambda(t, \zeta, \eta; \alpha)$ and substitute it into the coefficients of the Lax pair (4.4)-(4.6). Then we can confirm that on each Stokes curve of (alt-dPI) some degenerate configuration is observed for the Stokes geometry of the linear equation (4.4) and, as its consequence, the Stokes geometry of (4.4) becomes different according as λ_0 belongs to Region (I), (II) or (III). On the other hand, by applying the exact WKB analysis to the system (4.4)-(4.6), in particular, to the linear equation (4.4), we can explicitly compute the Stokes multipliers of (4.4). Since the computation of Stokes multipliers through the exact WKB analysis heavily depends on the configuration of Stokes curves, the expressions for the Stokes multipliers thus obtained consequently differ according as λ_0 belongs to Region (I), (II) or (III). Thus, if we let $s_k^{(J)}$ $(k = 1, \ldots, 6,\ J = \mathrm{I}, \mathrm{II}, \mathrm{III})$ denote the Stokes multipliers of (4.4) when λ_0 belongs to Region (J) (note that there exist six Stokes multipliers for (4.4)) and further if we assume that two transseries solutions $\lambda(t, \zeta, \eta; \alpha_J)$ in Region (J) and $\lambda(t, \zeta, \eta; \alpha_{J+1})$ in Region $(J + 1)$ define the same analytic solution of (alt-dPI), then the following relations should hold thanks to the "isomonodromy property".

$$s_k^{(J)}(\alpha_J) = s_k^{(J+1)}(\alpha_{J+1}) \qquad (k = 1, 2, \ldots, 6). \tag{4.7}$$

As we mentioned above, the expression of $s_k^{(J)}$ is different from that of $s_k^{(J+1)}$ and hence (4.7) provides a nontrivial constraint on α_J and α_{J+1}. In fact, (4.7) gives explicit connection formulas that describe the Stokes phenomena for transseries solutions of (alt-dPI) between Regions (J) and $(J + 1)$. For the explicit formulas for $s_k^{(J)}$ we refer the reader to [12]. Using the formulas for $s_k^{(J)}$ given in [12], we obtain the following connection formula for (alt-dPI):

Connection formula for (alt-dPI).

Suppose that the transseries solutions $\lambda(t, \zeta, \eta; \alpha_J)$ in Region (J) $(J = \mathrm{I}, \mathrm{II}, \mathrm{III})$ define the same analytic solution of (alt-dPI). *Then the following relations hold among the free parameters α_J.*

$$\alpha_{\mathrm{II}} = \alpha_{\mathrm{I}}\,(1 + e^{2\pi i\eta\zeta}), \tag{4.8}$$

$$\alpha_{\mathrm{III}} = \alpha_{\mathrm{II}} + \frac{i}{2\sqrt{\pi}}\, e^{2\pi i\eta\zeta}. \tag{4.9}$$

Remark. In terms of the coefficients $\alpha_{J,l}$ of the expression (2.19) of α_J, the formulas (4.8) and (4.9) can be expressed also as

$$\alpha_{\mathrm{II},l} = \alpha_{\mathrm{I},l} + \alpha_{\mathrm{I},l-1}, \tag{4.10}$$

$$\alpha_{\mathrm{III},l} = \alpha_{\mathrm{II},l} + \frac{i}{2\sqrt{\pi}}\,\delta_{l1} \tag{4.11}$$

$(l = 0, 1, 2, \ldots)$, where δ_{jk} denotes Kronecker's delta and $\alpha_{\mathrm{I},-1} = 0$.

Remark. The relation (4.9) between α_{II} and α_{III} is the same as the connection formula for Stokes phenomena of the continuous first Painlevé equation (PI) discussed in [19]. Similarly, the relation (4.8) between α_{I} and α_{II} is the same as the connection formula for parametric Stokes phenomena of the continuous second Painlevé equation (PII) studied by Iwaki [9]. It is interesting that both Stokes phenomena of (PI) type and those of (PII) type are observed with the discrete Painlevé equation (alt-dPI).

§ 5. Related future problems

In this report we have discussed two problems: (approximate) invariants and Stokes phenomena of the discrete Painlevé equation (alt-dPI). A key point of our approach is to consider both equations (alt-dPI) and (PII) simultaneously and we hope the discussions so far successfully exemplify the effectiveness of our approach. However, to complete these analyses, in particular, to understand the structure of Stokes phenomena for (alt-dPI) more thoroughly, we have still several problems to be studied.

First, we have obtained the explicit connection formulas (4.8) and (4.9) for transseries solutions of (alt-dPI) in this report, but transseries solutions are not general solutions of (alt-dPI), that is, they do not contain sufficiently many free parameters. Judging from our experiences with continuous Painlevé equations (cf., e.g., [19], [20]), we have to investigate Stokes phenomena for the so-called "instanton-type solutions". A typical way of constructing instanton-type solutions is to employ the multiple-scale analysis. This was done in [4] for Painlevé equations and later extended to higher order Painlevé equations in [1], [2], [21]. In the current situation, as (alt-dPI) is considered to be an ∞-order differential equation by the replacement (2.1), we need to deal with a system of two differential equations, one of which is of ∞-order (in the variable ζ) and the other of which is of second order (in the variable t). It is an interesting problem to apply the multiple-scale analysis to such a system of (alt-dPI) and (PII) to obtain its instanton-type solutions.

Second, in this report we only discuss Stokes phenomena for (alt-dPI) on some particular Stokes curves and do not discuss Stokes phenomena on the whole Stokes curves. To understand the structure of Stokes phenomena for (alt-dPI) globally, we again have to take into account the fact that (alt-dPI) is considered as an ∞-order equation. For example, it is observed in Figure 1 that there are several crossing points of Stokes curves for (alt-dPI). Such crossing points of Stokes curves are often observed for higher order ordinary differential equations with a large parameter and "new Stokes curves" and "virtual turning points", introduced respectively by [5] and [3], appear in connection with crossing points of Stokes curves (cf. [8] for more details of new Stokes curves and virtual turning points). Thus, to obtain the global understanding of Stokes phenomena, we also need to analyze the global structure of new Stokes curves and virtual turning points of (alt-dPI). We believe (alt-dPI) provides a good example of ∞-order nonlinear differential equations with intriguing Stokes geometry.

Finally, it goes without saying that it is very important to extend the analysis of Stokes phenomena for (alt-dPI) reported in Section 4 to the other Painlevé equations and the associated discrete Painlevé equations. We hope we can discuss these problems in our forthcoming articles.

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