

Regularity and b -functions for D -modules

To Prof. Kawai and Prof. Komatsu

By

Yves LAURENT*

Abstract

A holonomic \mathcal{D} -module on a complex analytic manifold admits always a b -function along any submanifold. If the module is regular, it admits also a regular b -function, that is a b -function with a condition on the order of the lower terms of the equation. There is a weaker condition of regularity: regularity along a submanifold. We prove that a module which is regular along a submanifold admits a regular b -function along this submanifold.

Introduction

Let f be a holomorphic function on an open set X of \mathbb{C}^n . A b -function for f is a polynomial $b \in \mathbb{C}[s]$ such that there exists a differential operator P on U with parameter s satisfying an equation:

$$b(s)f(x)^s = P(s, x, D_x)f(x)^{s+1}$$

The generator of the ideal of the b -functions associated to f is usually called the Bernstein-Sato polynomial of f (see[1] for details).

This definition has been extended by Kashiwara [3] to holonomic \mathcal{D} -modules. Let X be a complex manifold and Y a smooth hypersurface. Let \mathcal{D}_X be the sheaf of differential operators on X . Let \mathcal{M} be a holonomic \mathcal{D}_X -module and u a section of \mathcal{M} . A b -function for u along Y is an equation

$$b(tD_t)u = tP(t, x, tD_t, D_x)u$$

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*Institut Fourier Mathématiques, UMR 5582, CNRS/UJF, BP 74, F-38402, St Martin d'Hères Cedex, France.

e-mail: Yves.Laurent@ujf-grenoble.fr

satisfied by u . Here (t, x) are local coordinates of X such that t is an equation for Y . A similar definition exists for submanifolds of X of any codimension.

The b -function is called **regular** if P may be chosen so that its order as a differential operator is not greater than the degree of b .

It has been proved by Kashiwara [3] that a holonomic \mathcal{D} -module admits a b -function along any smooth hypersurface and by Kashiwara-Kawaiï [5] that a *regular* holonomic \mathcal{D} -module admits a *regular* b -function along any smooth hypersurface.

There is another notion of regularity, the regularity of a \mathcal{D} -module along a submanifold. The definition will be given in definitions 1.3 and 3.1. In the case of a hypersurface, this is equivalent to the fact that solutions in the formal completion transversally to the hypersurface are convergent. It has been proved that a regular holonomic module is regular along any submanifold in [6]. It is also a direct consequence of the definition that if a \mathcal{D} -module admits a regular b -function along a submanifold, it is regular along it. We will prove here that a module which is regular along a submanifold Y admits a regular b -function along Y .

The problem is better understood after microlocalization, that is for modules over the sheaf \mathcal{E}_X of microdifferential operators. Regularity and b -function may be defined for a coherent \mathcal{E}_X -module \mathcal{M} along a conic lagrangian submanifold Λ of the cotangent bundle [8]. All definitions are invariant under quantized canonical transformation. Concerning \mathcal{D} -modules, this will prove that the result is true not only for hypersurfaces but may be extended to any submanifold of X .

A regular holonomic \mathcal{E}_X -module is regular along any conic lagrangian submanifold of T^*X . Conversely, in the definition of Kashiwara-Kawaiï [6], a holonomic \mathcal{E}_X -module is regular if it is regular along the smooth part of its characteristic variety.

In the first section of this paper, we briefly recall the definitions of b -functions and state the result for \mathcal{D}_X -modules. In the second section, we study the microlocal case. We define the filtrations and bifiltrations on holonomic \mathcal{E}_X -modules. In the third section, we study the equivalent definitions of regularity along a lagrangian manifold and we prove the existence of a regular b -function as a product of θ^d by the b -function $b(\theta)$.

§ 1. The case of differential equations

In this first section, we briefly recall the definitions of b -functions and regularity in the framework of \mathcal{D}_X -modules.

§ 1.1. Filtrations and b -functions

Let X be a complex analytic manifold, \mathcal{O}_X the sheaf of holomorphic functions on X and \mathcal{D}_X the sheaf of differential operators on X with coefficients in \mathcal{O}_X . Let T^*X

be the cotangent bundle to X with canonical projection $\pi : T^*X \rightarrow X$. Let Y be a submanifold of X .

The sheaf \mathcal{D}_X is provided with two filtrations. The first one is the filtration $(\mathcal{D}_{X,m})_{m \geq 0}$ by the usual order, that is the degree in the derivations. Here, we will denote this filtration by $F\mathcal{D}_X$, i.e. $F_m\mathcal{D}_X = \mathcal{D}_{X,m}$. The corresponding graded ring $gr^F\mathcal{D}_X$ is identified to $\pi_*\mathcal{O}_{[T^*X]}$ the sheaf of holomorphic functions on T^*X with polynomial coefficients in the fibers of π .

The second one is the V-filtration which has been defined by Kashiwara in [4] as:

$$(1.1) \quad V_k\mathcal{D}_X = \{ P \in \mathcal{D}_X|_Y \mid \forall \ell \in \mathbb{Z}, P\mathcal{I}_Y^\ell \subset \mathcal{I}_Y^{\ell-k} \}$$

where \mathcal{I}_Y is the ideal of definition of Y and $\mathcal{I}_Y^\ell = \mathcal{O}_X$ if $\ell \leq 0$.

Let $\tau : T_Y X \rightarrow Y$ be the normal bundle to Y in X and $\mathcal{O}_{[T_Y X]}$ the sheaf of holomorphic functions on $T_Y X$ which are polynomial in the fibers of τ . Let $\mathcal{O}_{[T_Y X]}[k]$ be the subsheaf of $\mathcal{O}_{[T_Y X]}$ of homogeneous functions of degree k in the fibers of τ . There are canonical isomorphisms between $\mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}[k]$, between $\bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}$. Hence the graded ring $gr^V\mathcal{D}_X$ associated to the V-filtration on \mathcal{D}_X acts naturally on $\mathcal{O}_{[T_Y X]}$. An easy calculation [15] shows that as a subring of $\mathcal{E}nd(\tau_*\mathcal{O}_{[T_Y X]})$ it is identified to $\tau_*\mathcal{D}_{[T_Y X]}$ the sheaf of differential operators on $T_Y X$ with coefficients in $\mathcal{O}_{[T_Y X]}$.

The Euler vector field θ of $T_Y X$ is the vector field which acts on $\mathcal{O}_{[T_Y X]}[k]$ by multiplication by k . Let ϑ be any differential operator in $V_0\mathcal{D}_X$ whose image in $gr_0^V\mathcal{D}_X$ is θ . Let \mathcal{M} be a coherent \mathcal{D}_X -module and u a section of \mathcal{M} .

Definition 1.1. A polynomial b is a b -function for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{D}_X$ such that $(b(\vartheta) + Q)u = 0$.

Definition 1.2. A polynomial b of degree m is a *regular* b -function for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{D}_X \cap F_m\mathcal{D}_X$ such that $(b(\vartheta) + Q)u = 0$.

The set of b -functions is an ideal of $\mathbb{C}[T]$, if it is not zero we call a generator of this ideal "the" b -function of u along Y .

Remark that the set of regular b -functions for u is not always an ideal of $\mathbb{C}[T]$.

§ 1.2. Regularity

A holonomic module is regular if its formal solutions converge at each point. More precisely, if x is a point of X , $\mathcal{O}_{X,x}$ the ring of germs of \mathcal{O}_X at x and \mathfrak{m} its maximal ideal, let us denote by $\widehat{\mathcal{O}}_{X,x}$ the formal completion of $\mathcal{O}_{X,x}$ for the \mathfrak{m} -topology. The

holonomic \mathcal{D}_X -module \mathcal{M} is *regular* if and only if:

$$\forall j \geq 0, \quad \forall x \in X, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_{X,x})$$

This is the direct generalization of the definition in dimension 1 (Ramis [13]). An algebraic and microlocal definition is given by Kashiwara-Kawai in [6] and they prove that, for a \mathcal{D}_X -module, it is equivalent to this one.

A weaker notion is the regularity along a submanifold. This has been studied in several papers [7],[8],[11]. The definition will be given in section 3 using microcharacteristic varieties. Here we give a more elementary definition for \mathcal{D}_X -modules which is valid only for hypersurfaces.

Let Y be a smooth hypersurface of X , ϑ is the vector field of section 1.1 and \mathcal{M} be a holonomic \mathcal{D}_X -module \mathcal{M} defined in a neighborhood of Y .

Definition 1.3. The holonomic \mathcal{D}_X -module \mathcal{M} is *regular along Y* if any section u of \mathcal{M} is annihilated by a differential operator of the form $\vartheta^N + P + Q$ where P is in $F_{N-1}\mathcal{D}_X \cap V_0\mathcal{D}_X$ and Q is in $F_N\mathcal{D}_X \cap V_{-1}\mathcal{D}_X$

Let us denote by $\widehat{\mathcal{O}}_{X|Y}$ the formal completion of \mathcal{O}_X along Y , that is

$$\widehat{\mathcal{O}}_{X|Y} = \varprojlim_k \mathcal{O}_X / \mathcal{I}_Y^k$$

We proved in [9] and [11] that \mathcal{M} is regular along Y if and only if

$$(1.2) \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \widehat{\mathcal{O}}_{X|Y}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_X)$$

We can still state definition 1.3 and equation (1.2) when Y is not a hypersurface but they are not equivalent. Correct statements will be given in section 2 in the microlocal framework.

It has been proved by Kashiwara in [3], that if \mathcal{M} is a holonomic \mathcal{D}_X -module, there exists a b -function for any section u of \mathcal{M} along any submanifold Y of X . Kashiwara and Kawai proved in [5] that if \mathcal{M} is a regular holonomic \mathcal{D}_X -module, there exists a regular b -function for any section u of \mathcal{M} along any submanifold Y of X .

We will prove in section 3.2 that if \mathcal{M} is a holonomic \mathcal{D}_X -module which is regular along a submanifold Y then it admits a regular b -function along Y .

§ 1.3. Explicit formulas in local coordinates

Consider local coordinates (x_1, \dots, x_p, t) of X such that Y is the hypersurface $Y = \{(x, t) \in X \mid t = 0\}$. Then $T_Y X$ has coordinates $(x_1, \dots, x_p, \tilde{t})$.

The Euler vector field θ of $T_Y X$ is $\theta = \tilde{t}D_{\tilde{t}}$ and we can choose $\vartheta = tD_t$. Then a b -function is an equation

$$b(tD_t) + tQ(x, t, D_x, tD_t)$$

and it is a regular b -function if the order of Q is less or equal to the degree of b .

The module \mathcal{M} is regular along Y if any section of \mathcal{M} is annihilated by an equation

$$(tD_t)^N + P(x, D_x, tD_t) + tQ(x, t, D_x, tD_t)$$

with P of order less or equal to $N - 1$ and Q of order less or equal to N .

§ 2. Microdifferential equations

In this section, we review basic definitions of filtration, V-filtration and bifiltrations on \mathcal{E}_X -modules. Details may be founded in [1] or [15], and [8].

§ 2.1. V-filtration on microdifferential operators

We denote by \mathcal{E}_X the sheaf of microdifferential operators of [14]. It is filtered by the order, we will denote that filtration by $\mathcal{E}_X = \bigcup F_k \mathcal{E}_X$ and call it the usual filtration.

Let $\mathcal{O}_{(T^*X)}$ be the sheaf of holomorphic functions on T^*X which are finite sums of homogeneous functions in the fibers of $\pi : T^*X \rightarrow X$. The graded ring $gr^F \mathcal{E}_X$ is isomorphic to $\pi_* \mathcal{O}_{(T^*X)}$ [15].

In [8], we extended the definitions of V-filtrations and b -functions to microdifferential equations and lagrangian submanifolds of the cotangent bundle.

Let Λ be a lagrangian conic submanifold of the cotangent bundle T^*X and \mathcal{M}_Λ be a holonomic \mathcal{E}_X -module supported by Λ . A section u of \mathcal{M}_Λ is non degenerate if the ideal of \mathcal{O}_{T^*X} generated by the principal symbols of the microdifferential operators annihilating u is the ideal of definition of Λ . The module \mathcal{M}_Λ is a simple holonomic \mathcal{E}_X -module if it is generated by a non degenerate section u_Λ . Such a module always exists locally [14].

Let $\mathcal{M}_{\Lambda,k} = (F_k \mathcal{E}_X)u_\Lambda$. The V-filtration on \mathcal{E}_X along Λ is defined by:

$$(2.1) \quad V_k \mathcal{E}_X = \{ P \in \mathcal{E}_X|_\Lambda \mid \forall \ell \in \mathbb{Z}, P \mathcal{M}_{\Lambda,\ell} \subset \mathcal{M}_{\Lambda,\ell+k} \}$$

This filtration is independent of the choices of \mathcal{M}_Λ and u_Λ , so it is globally defined [8, Prop. 2.1.1.].

Let $\mathcal{O}_\Lambda[k]$ be the sheaf of holomorphic functions on Λ homogeneous of degree k in the fibers of $\Lambda \rightarrow X$ and $\mathcal{O}_{(\Lambda)} = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_\Lambda[k]$. There is an isomorphism between $\mathcal{M}_{\Lambda,k}/\mathcal{M}_{\Lambda,k-1}$ and $\mathcal{O}_\Lambda[k]$. By this isomorphism the graded ring $gr^V \mathcal{E}_X$ acts on $\mathcal{O}_{(\Lambda)}$ and may be identified to the sheaf $\mathcal{D}_{(\Lambda)}$ of differential operators on Λ with coefficients in $\mathcal{O}_{(\Lambda)}$.

All these definitions are invariant under quantized canonical transformations [8].

The restriction to the zero section T_X^*X of T^*X of the sheaf \mathcal{E}_X is the sheaf \mathcal{D}_X of differential operators. If Λ is the conormal bundle T_Y^*X to a submanifold Y of X then the V-filtration induced on \mathcal{D}_X by the V-filtration of \mathcal{E}_X is the same as the V-filtration defined in section 1.1. The associated graded ring is the restriction to Y of $\mathcal{D}_{(\Lambda)}$ which is the sheaf $\mathcal{D}_{[\Lambda]}$ of differential operators whose coefficients are polynomial in the fibers of $\Lambda \rightarrow X$.

The correspondence between the isomorphism $gr^V \mathcal{D}_X \simeq \mathcal{D}_{[T_Y X]}$ of section 1.1 and the isomorphism $gr^V \mathcal{D}_X \simeq \mathcal{D}_{[T_Y^* X]}$ is given by the partial Fourier transform associated to the duality between the normal bundle $T_Y X$ and the conormal bundle $T_Y^* X$.

Remark 1. Consider local coordinates $(x_1, \dots, x_p, t_1, \dots, t_d)$ of X and the submanifold $Y = \{(x, t) \in X \mid t = 0\}$. Let (x, \tilde{t}) be the corresponding coordinates of $T_Y X$ and (x, τ) the corresponding coordinates of $T_Y^* X$. Let $\theta = \sum_{i=1}^d \tilde{t}_i D_{\tilde{t}_i}$ be the Euler vector field of the fiber bundle $T_Y X$ and θ_Λ be the Euler vector field of $\Lambda = T_Y^* X$.

The Fourier transform is given by $\tilde{t}_i \mapsto -D_{\tau_i}$ and $D_{\tilde{t}_i} \mapsto \tau_i$. So θ is mapped to $-d - \theta_\Lambda$ where d is the codimension of Y .

§ 2.2. Filtrations on \mathcal{E} -modules

Let \mathcal{M} be a (left) coherent \mathcal{E}_X -module. A F-filtration (resp. a V-filtration) of \mathcal{M} is a filtration compatible with the F-filtration (resp. V-filtration) of \mathcal{E}_X .

A good filtration of \mathcal{M} is a filtration for which there exists locally sections u_1, \dots, u_N of \mathcal{M} and integers (k_1, \dots, k_N) such that $\mathcal{M}_k = \sum_{i=1}^N (F_{k-k_i} \mathcal{E}_X) u_i$. In the same way, a good V-filtration of \mathcal{M} is locally of the form $V_k \mathcal{M} = \sum_{i=1}^N (V_{k-k_i} \mathcal{E}_X) u_i$.

If \mathcal{M} is provided with a good F-filtration, the associated graded module $gr^F \mathcal{M}$ is a coherent module over $gr^F \mathcal{E}_X = \pi_* \mathcal{O}_{(T^* X)}$. Then $\mathcal{O}_{T^* X} \otimes_{\pi^{-1} gr^F \mathcal{E}_X} \pi^{-1} gr^F \mathcal{M}$ is a coherent $\mathcal{O}_{T^* X}$ -module which defines a positive analytic cycle on $T^* X$ independent of the good filtration. This cycle is the *characteristic cycle* of \mathcal{M} and denoted by $\widetilde{Ch}(\mathcal{M})$. Its support is the *characteristic variety* of \mathcal{M} denoted $Ch(\mathcal{M})$, it is equal to the support of the module \mathcal{M} itself.

The characteristic variety of a coherent \mathcal{E}_X -module is a *homogeneous involutive* subvariety of $T^* X$. When its dimension is minimal, that is when it is a lagrangian, the module is *holonomic*.

If \mathcal{M} is provided with a good V-filtration, the graded module $gr^V \mathcal{M}$ is a coherent module over $gr^V \mathcal{E}_X = \mathcal{D}_{(\Lambda)}$ hence $gr^V \mathcal{M}$ is a coherent $\mathcal{D}_{(\Lambda)}$ -module. The characteristic cycle of this module is a positive analytic cycle on $T^* \Lambda$ and its support is involutive. According to [8] and [11], this characteristic cycle is called the *microcharacteristic cycle* of \mathcal{M} of type ∞ and is denoted by $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$. The corresponding microcharacteristic variety is $Ch_{\Lambda(\infty)}(\mathcal{M})$. We have the following fundamental result:

Theorem 2.1 (Theorem 4.1.1 [8]). *The dimension of the characteristic variety of $gr^V \mathcal{M}$ is less or equal to the dimension of the characteristic variety of \mathcal{M} . In particular, $gr^V \mathcal{M}$ is holonomic if \mathcal{M} is holonomic.*

As $gr_k^V \mathcal{E}_X$ is identified to the subsheaf of $\mathcal{D}_{(\Lambda)}$ of differential operators P satisfying $[\theta, P] = kP$, the Euler vector field θ acts on $gr^V \mathcal{M}$, so we may define a morphism Θ by $\Theta = \theta - k$ on $gr_k^V \mathcal{M}$. As $[\theta, P] = +kP$ on $gr_k^V \mathcal{E}_X$, Θ commutes with the action of $gr^V \mathcal{E}_X$, i.e. defines a section of $\mathcal{E}nd_{gr^V \mathcal{E}_X}(gr^V \mathcal{M})$.

Definition 2.2. The set of polynomials b satisfying

$$b(\Theta)gr^V(\mathcal{M}) = 0$$

is an ideal of $\mathbb{C}[T]$. When this ideal is not zero, a generator is called *the b -function of \mathcal{M} for the V -filtration along Λ* .

The b -function depends on the good V -filtration, its roots are shifted by integers by change of V -filtration.

The b -function is invariant under quantized canonical transform. Let Y be a submanifold of X of codimension d . If we want to identify the b -function of definition 2.2 with that of definition 1.1, we have to replace ϑ in definition 1.1 by $-d - \vartheta$ because of remark 1.

Theorem 2.3. *If \mathcal{M} is a holonomic \mathcal{E}_X -module, there exists a b -function for \mathcal{M} along any lagrangian submanifold Λ of T^*X .*

This theorem has been proved first by Kashiwara for \mathcal{D} -modules in [3]. It has been proved for \mathcal{E}_X -module in [8] as a corollary of theorem 2.1. In fact, as $gr^V \mathcal{M}$ is holonomic, $\mathcal{E}nd_{gr^V \mathcal{E}_X}(gr^V \mathcal{M})$ is finite dimensional and the b -function is the minimal polynomial of Θ on $gr^V \mathcal{M}$.

§ 2.3. Bifiltration

From the two filtrations on \mathcal{E}_X , we get a bifiltration:

$$(2.2) \quad \forall (k, j) \in \mathbb{Z} \times \mathbb{Z} \quad W_{k,j} \mathcal{E}_X = F_j \mathcal{E}_X \cap V_k \mathcal{E}_X$$

To this bifiltration is associated the bigraded ring:

$$(2.3) \quad gr^W \mathcal{E}_X = \bigoplus_{(k,j)} gr_{k,j}^W \mathcal{E}_X \quad \text{with} \quad gr_{k,j}^W \mathcal{E}_X = W_{k,j} \mathcal{E}_X / (W_{k-1,j} \mathcal{E}_X + W_{k,j-1} \mathcal{E}_X)$$

This bigraded ring is equal to the graded ring of $gr^V \mathcal{E}_X = \mathcal{D}_{(\Lambda)}$ (this latter with the standard filtration). So it is isomorphic to $\pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$, the sheaf of holomorphic

functions on $T^*\Lambda$ which are polynomial in the fibers of $\pi_\Lambda : T^*\Lambda \rightarrow \Lambda$ and sum of homogeneous functions for the second action of \mathbb{C}^* on $T^*\Lambda$. "Second action" means action induced on $T^*\Lambda$ by the action of \mathbb{C}^* on Λ .

Let \mathcal{M} be a (left) coherent \mathcal{E}_X -module. A good bifiltration of \mathcal{M} is a bifiltration compatible with the bifiltration of \mathcal{E}_X which is locally generated by local sections (u_1, \dots, u_N) of \mathcal{M} . This means that there are integers (k_1, \dots, k_N) and (j_1, \dots, j_N) such that $W_{k,j}\mathcal{M} = \sum_{\nu=1}^N W_{k-k_\nu, j-j_\nu} \mathcal{E}_X u_\nu$.

The bigraded module associated to the bifiltration is defined by

$$(2.4) \quad gr^W \mathcal{M} = \bigoplus_{(k,j)} gr_{k,j}^W \mathcal{M} \quad \text{with} \quad gr_{k,j}^W \mathcal{M} = W_{k,j}\mathcal{M} / (W_{k-1,j}\mathcal{M} + W_{k,j-1}\mathcal{M})$$

Proposition 2.4. *The bigraded module associated to a good bifiltration of a coherent \mathcal{E}_X -module is a coherent $gr^W \mathcal{E}_X = \pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$ module.*

The bifiltration induced on a coherent submodule by a good bifiltration is still a good bifiltration.

This result may be found in [2] or [8], let us briefly give a proof.

We denote by \mathcal{E}_V the sheaf $gr_0^V \mathcal{E}_X$ provided with the filtration induced by that of \mathcal{E}_X (cf [6]).

Lemma 2.5. *Outside of the zero section of Λ , there is an equivalence between the following data:*

- a good bifiltration of \mathcal{M}
- a coherent sub- \mathcal{E}_V -module of \mathcal{M} which generates \mathcal{M} and is provided with a good filtration.

Proof. Outside of its zero section, we may transform Λ into the conormal to a hypersurface by a canonical transformation, that is in local coordinates:

$$\Lambda = \{ (x, t, \xi, \tau) \in T^*X \mid t = 0, \xi = 0, \tau \neq 0 \}$$

Then the microdifferential operator D_t is invertible. It is an element of $W_{1,1}\mathcal{E}_X$.

Consider a good bifiltration $W\mathcal{M}$ of \mathcal{M} which is locally generated by sections u_1, \dots, u_N of \mathcal{M} . Multiplying them by suitable powers of D_t we may assume that they belong to

$$\mathcal{N} = \bigcup_{j \in \mathbb{Z}} W_{0,j}\mathcal{M}$$

Then \mathcal{N} is the sub- \mathcal{E}_V -module of \mathcal{M} generated by (u_1, \dots, u_N) and, by [6, Prop. 1.1.10.], \mathcal{N} is an \mathcal{E}_V -coherent submodule of \mathcal{M} . The sections (u_1, \dots, u_N) define a good filtration of \mathcal{N} and $\mathcal{M} = \mathcal{E}_X \otimes \mathcal{N}$.

Conversely, let \mathcal{N} be coherent sub- \mathcal{E}_V -module of \mathcal{M} which generates \mathcal{M} and is provided with a good filtration. The good filtration is given locally by sections (u_1, \dots, u_N) which define a good bifiltration of \mathcal{M} . \square

Proof of proposition 2.4. From [7, §2.6.] (see also [2]), we know that the filtration of \mathcal{E}_V is "a good noetherian filtration" that is a zariskian filtration in the notation of [15]. Then the results of [15] show that good \mathcal{E}_V -filtrations induce good \mathcal{E}_V -filtrations on submodules and that the graded ring associated to a good \mathcal{E}_V -filtration is a coherent $gr\mathcal{E}_V$ -module. The result may be then deduce from the previous lemma out of the zero section of Λ .

We get the result on the zero section of Λ by adding a dummy variable as usual (cf [6, §A1]). \square

Let $\mathcal{O}_{T^*\Lambda}$ be the sheaf of holomorphic functions on $T^*\Lambda$. As $gr^W \mathcal{M}$ is a $\pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$ coherent module, $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{[T^*\Lambda]}} \pi_{\Lambda}^{-1} gr^W \mathcal{M}$ is a coherent $\mathcal{O}_{T^*\Lambda}$ -module. Its support is independent of the good bifiltration. It is called the microcharacteristic variety of type $(\infty, 1)$ and is denoted by $Ch_{\Lambda(\infty,1)}(\mathcal{M})$. This microcharacteristic variety is an involutive bihomogeneous subvariety of $T^*\Lambda$ [7],[8].

Let us remark that the same microcharacteristic variety has been defined by Teresa Monteiro Fernandes in [2], where it is denoted by $C_{\Lambda}(\mathcal{M})$, and in [7] as the support of the tensor product of the module by a sheaf of 2-microdifferential operators.

§ 2.4. Local coordinates

Assume that the conic lagrangian manifold Λ is the conormal to a submanifold Y of X . (By a canonical transformation we may always transform Λ into a conormal).

Let $(x_1, \dots, x_p, t_1, \dots, t_d)$ be local coordinates of X such that $Y = \{(x, t) \in X \mid t = 0\}$. Then $\Lambda = \{(x, t, \xi, \tau) \in T^*X \mid t = 0, \xi = 0\}$ and $T^*\Lambda$ has coordinates (x, τ, x^*, τ^*) .

The Euler vector field of Λ is $\theta = \sum_{i=1}^N \tau_i D_{\tau_i}$ and its principal symbol is the function $\varphi = \sum_{i=1}^N \tau_i \tau_i^* = \langle \tau, \tau^* \rangle$. The hypersurface S_{Λ} of $T^*\Lambda$ defined by the equation φ is a canonical hypersurface which will be considered in next section.

For the W -bifiltration, the operator x_i ($i = 1, \dots, p$) is of bi-order $(0, 0)$, while t_j ($j = 1, \dots, d$) is of bi-order $(-1, 0)$, D_{x_i} is of bi-order $(0, 1)$ and D_{t_j} is of bi-order $(1, 1)$.

The operator ϑ may be chosen here as $\sum t_j D_{t_j}$, it is of bi-order $(0, 1)$.

§ 3. Regularity along a lagrangian conic submanifold

§ 3.1. Equivalent definitions of regularity

The Euler vector field θ_{Λ} of Λ is a differential operator on Λ , its characteristic variety is a canonical subvariety of $T^*\Lambda$ which will be denoted by S_{Λ} . As in section 2.1,

\mathcal{M}_Λ is a simple holonomic \mathcal{E}_X -module supported by Λ .

Definition 3.1. Let Λ be a lagrangian conic submanifold of T^*X and \mathcal{M} be a holonomic \mathcal{E}_X -module. The module \mathcal{M} is **regular along** Λ if and only if it satisfies the following equivalent properties:

- i) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in S_Λ .
- ii) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is lagrangian.
- iii) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is equal to $Ch_{\Lambda(\infty)}(\mathcal{M})$
- iv) $\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{M}_\Lambda) = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_\Lambda)$

Here \mathcal{E}_X^∞ is the sheaf of microdifferential operators of infinite order.

If Λ is the conormal bundle to a submanifold Y of X , we may take $\mathcal{M}_\Lambda = \mathcal{C}_{Y|X}$ the sheaf of holomorphic microfunctions of [14]. Then (iv) is reformulated as:

$$(iv)' \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X}) = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X}^\infty)$$

Let us now briefly remain how the equivalence between the items of definition 3.1 is proved.

By theorem 2.1, if \mathcal{M} is holonomic, $gr^V \mathcal{M}$ is holonomic hence its characteristic variety $Ch_{\Lambda(\infty)}(\mathcal{M})$ is lagrangian. So if $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is equal to $Ch_{\Lambda(\infty)}(\mathcal{M})$ it is lagrangian that is (iii) \Rightarrow (ii).

The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is defined by the coherent $\pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$ -module $gr^W \mathcal{M}$ hence is bihomogeneous, that is homogeneous in the fibers of $T^*\Lambda$ and homogeneous for the action of \mathbb{C}^* induced by the action of \mathbb{C}^* on Λ . By section 4.3. of [8], any lagrangian bihomogeneous submanifold of $T^*\Lambda$ is contained in S_Λ . In fact, if Σ is lagrangian the canonical 2-form Ω of $T^*\Lambda$ vanishes on Σ and if it is bihomogeneous the vector fields v_1 and v_2 associated to the two actions of \mathbb{C}^* are tangent to Σ . Hence $\Omega(v_1, v_2)$ vanishes on Σ and an easy calculation of [8] shows that $\Omega(v_1, v_2)$ is an equation of S_Λ . So (ii) \Rightarrow (i).

Finally, the main parts of the result are (i) \Rightarrow (iv) which is given by corollary 4.4.2. of [10] and (iv) \Rightarrow (iii) given by theorem 2.4.2 of [11].

§ 3.2. Regular b -function

Let Λ be a conic lagrangian submanifold of the cotangent bundle T^*X . As before, θ_Λ is the Euler vector field of Λ and ϑ_Λ is a microdifferential operator in $V_0 \mathcal{E}_X$ whose image in $gr_0^V \mathcal{E}_X$ is θ_Λ .

Definition 3.2. Let \mathcal{M} be a coherent \mathcal{E}_X -module with a good bifiltration $W\mathcal{M}$ relative to Λ . A *regular b -function* for $W\mathcal{M}$ is a polynomial b such that:

$$\forall (k, j) \in \mathbb{Z} \times \mathbb{Z}, \quad b(\vartheta_\Lambda - k)W_{k,j}\mathcal{M} \subset W_{k-1,j+n}\mathcal{M}$$

where n is the degree of b .

if u is a section of \mathcal{M} , a *regular b -function* for u is a regular b -function for the canonical bifiltration of the module $\mathcal{N} = \mathcal{E}_X u$ that is $W_{k,j}\mathcal{N} = (W_{k,j}\mathcal{E}_X)u$.

Remark that if $W\mathcal{M}$ is a good bifiltration, we get a good V -filtration by setting:

$$V_k\mathcal{M} = \bigcup_{j \in \mathbb{Z}} W_{k,j}\mathcal{M}$$

So, considering non regular b -functions, we may define a b -function for W to be a b -function for the corresponding V -filtration.

Proposition 3.3. *Let \mathcal{M} be a holonomic \mathcal{E}_X -module which a good bifiltration admitting a regular b -function, then \mathcal{M} is regular along Λ .*

Indeed, the existence of a regular b -function implies immediately the point (i) of definition 3.1.

By the following theorem and its corollary, the converse is true which gives another condition equivalent to the regularity along Λ .

Theorem 3.4. *Let \mathcal{M} be holonomic \mathcal{E}_X -module regular along a the conic lagrangian manifold Λ . Any section u of \mathcal{M} admits locally a regular b -function.*

If $b(\theta)$ is the b -function of u , there exists an integer d such that $\theta^d b(\theta)$ is a regular b -function for u .

Proof. Let θ be the Euler vector field of Λ and ϑ an operator of \mathcal{E}_X whose image in $gr^V \mathcal{E}_X$ is θ . By definition, the principal symbol φ of θ is an equation of the canonical hypersurface S_Λ of $T^*\Lambda$.

If \mathcal{M} is regular along Λ , the submodule $\mathcal{N} = \mathcal{E}_X u$ is regular along Λ . We consider on \mathcal{N} the canonical bifiltration, that is $W_{k,j}\mathcal{N} = (W_{k,j}\mathcal{E}_X)u$. By definition 3.1 of regularity, $Ch_\Lambda(\infty,1)(\mathcal{N})$ is contained in S_Λ hence $gr^W \mathcal{N}$ is annihilated by a power of φ . As the operator ϑ is of bi-order $(0,1)$, there is an integer q such that

$$\vartheta^q W_{k,j}\mathcal{N} \subset W_{k-1,j+q}\mathcal{N} + W_{k,j+q-1}\mathcal{N}$$

Hence there is an operator $P \in W_{-1,q}\mathcal{E}_X$ and an operator $Q \in W_{0,q-1}\mathcal{E}_X$ such that

$$\vartheta^q u = Pu + Qu$$

We have $(\vartheta^q - P)^2 u = Q(\vartheta^q - P)u + [\vartheta^q - P, Q]u = Q^2 u + [\vartheta^q - P, Q]u$. The commutator $[\vartheta^q - P, Q]$ belongs to $W_{0,2(q-1)}\mathcal{E}_X$ as well as Q^2 so $(\vartheta^q - P)^2 u = Q_2 u$ with $Q_2 \in W_{0,2(q-1)}\mathcal{E}_X$. By induction we find that $(\vartheta^q - P)^r u = Q_r u$ with $Q_r \in W_{0,r(q-1)}\mathcal{E}_X$.

Let b be the b -function of u . If p is the degree of b , there is some integer r such that $b(\vartheta)W_{0,j}\mathcal{N} \subset W_{-1,j+p+r}\mathcal{N}$.

We get

$$b(\vartheta)(\vartheta^q - P)^r u \subset b(\vartheta)W_{0,rq-r}\mathcal{N} \subset W_{-1,p+rq}\mathcal{N}$$

The operator $(\vartheta^q - P)^r$ is equal to $\vartheta^{rq} - P_r$ where P_r belongs to $W_{-1,rq}\mathcal{E}_X$. hence $b(\vartheta)\vartheta^{rq}u \subset W_{-1,p+rq}\mathcal{N}$ and as $p+rq$ is the degree of $b(\vartheta)\vartheta^{rq}$, this is a regular b -function for u . \square

Remark 2. It is clear in the proof that $\theta^d b(\theta)$ may be replaced by $(\theta - \alpha)^d b(\theta)$ for any complex number α .

Corollary 3.5. *Let \mathcal{M} be holonomic \mathcal{E}_X -module regular along the conic lagrangian manifold Λ . Any good bifiltration $W\mathcal{M}$ relative to Λ admits locally a regular b -function.*

If $b(\theta)$ is the b -function associated to this bifiltration, there exists an integer d such that $\theta^d b(\theta)$ is a regular b -function.

Proof. Let (u_1, \dots, u_N) be local generators of the bifiltration. By definition, there are integers $(\lambda_1, \dots, \lambda_N)$ and (ν_1, \dots, ν_N) such that

$$W_{kj}\mathcal{M} = \sum_{i=1}^N W_{k-\lambda_i, j-\nu_i} \mathcal{E}_X u_i$$

If $b(\theta)$ is the b -function of the bifiltration, $b(\theta - \lambda_i)$ is a b -function for the section u_i . By theorem 3.4 and remark 2, there exists an integer d_i such that $(\theta - \lambda_i)^{d_i} b(\theta - \lambda_i)$ is a regular b -function for u_i . Let d be the maximum of d_i for $i = 1, \dots, N$.

As remarked in section 2.2, $[\theta, P] = +kP$ on $gr_k^V \mathcal{E}_X$ hence if P is an operator of $W_{k-\lambda_i, j-\nu_i} \mathcal{E}_X$ we have $\vartheta P = P(\vartheta + k - \lambda_i) + R$ with R in $W_{k-\lambda_i-1, j-\nu_i} \mathcal{E}_X$.

If u is a section of $W_{kj}\mathcal{M}$ we have $u = \sum P_i u_i$ with P_i in $W_{k-\lambda_i, j-\nu_i} \mathcal{E}_X$ and $(\theta - k)^d b(\theta - k)$ is a regular b -function for u . \square

§ 3.3. Application to \mathcal{D}_X -modules

Previous results have been established for \mathcal{E}_X -modules on the whole of T^*X including the zero section that is also for \mathcal{D}_X -modules.

If x is a point of the zero section X of T^*X , a conic lagrangian submanifold of T^*X defined in a neighborhood of x is the conormal T_Y^*X to a submanifold Y of X . Let $\mathcal{B}_{Y|X}^\infty = \mathcal{H}_Y^1(\mathcal{O}_X)$ be the cohomology of \mathcal{O}_X and $\mathcal{B}_{Y|X} = \mathcal{H}_{[Y]}^1(\mathcal{O}_X)$ the corresponding algebraic cohomology.

If Y has codimension 1 we have:

$$\mathcal{C}_{Y|X}^\infty / \mathcal{C}_{Y|X} = \pi^{-1}(\mathcal{B}_{Y|X}^\infty / \mathcal{B}_{Y|X}) \quad \text{with} \quad \pi : T_Y^*X \rightarrow Y$$

So if \mathcal{M} is a holonomic \mathcal{D}_X -module condition (iv) of definition 3.1 is equivalent to

$$(iv)'' \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}^\infty)$$

and by [11] this is equivalent to formula 1.2.

If Y has codimension greater than 1, the situation is different. By a canonical transformation, the situation is microlocally equivalent to the situation of codimension 1. But in the definition 3.1 we have to consider points of $\dot{T}_Y^*X = T_Y^*X - Y$ the complementary of the zero section in the conormal bundle. Regularity is then local on \dot{T}_Y^*X . The equivalence between algebraic and analytic definitions that is between conditions (i) to (iii) in definition 3.1 and condition (iv) have to be stated locally on \dot{T}_Y^*X

Let $\pi : \dot{T}_Y^*X \rightarrow Y$ and U an open set of Y . If a \mathcal{D}_X -module is regular along \dot{T}_Y^*X globally on $\pi^{-1}U$ we have

$$\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X})|_U = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}^\infty)|_U$$

but the converse is not true.

§ 3.4. A counterexample

In [12, Example 1], T. Oaku gave a simple example which shows that the regular b -function may be different from the b -function.

Let $X = \mathbb{C}^2$ and Y the hypersurface $Y = \{(x, t) \in X \mid t = 0\}$. Consider the \mathcal{D}_X -module $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ where \mathcal{I} is the ideal of \mathcal{D}_X generated by the two operators $P = D_t^2$ and $Q = D_t + D_x^2$.

This module is nonzero as the function 1 is a solution. Its characteristic variety is the zero section of the cotangent bundle T^*X hence it is isomorphic as a \mathcal{D}_X -module to a power of the sheaf \mathcal{O}_X of holomorphic functions on X . So, this module is regular holonomic.

Let u be the class of the operator 1 in \mathcal{M} . As u is annihilated by $tQ = tD_t + tD_x^2$, the b -function of u along Y is $\vartheta = tD_t$. But this not a regular b -function and a regular b -function for u along Y is given by $t^2P = t^2D_t^2 = tD_t(tD_t - 1) = \vartheta(\vartheta - 1)$.

Remark however that \mathcal{M} is isomorphic to $(\mathcal{O}_X)^4$. The function 1 in \mathcal{O}_X is annihilated by D_t hence its b -function is $\vartheta = tD_t$ and is a regular b -function. So \mathcal{M} is generated by four sections for which the regular b -function is equal to the b -function ϑ , that is there is on \mathcal{M} a bifiltration for which the b -function is a regular b -function.

We can see also on this example that the analytic cycle defined by the graded ring of a good bifiltration depend on the good bifiltration. This situation is very different from the case of filtrations where good filtrations define additive objects as analytic cycles which are independent of the good filtration [15, Prop. 1.3.1].

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